SECOND ANSWER.

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1. Definition of Kuranishi structure

Definition 1.1. ([FOOO, Definition A1.1]) A *Kuranishi neighborhood* of p in X is a quintuple $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ such that:

(1) V_p is a smooth manifold of finite dimension, which may or may not have boundary or corner.
(2) E_p is a real vector space of finite dimension.
(3) Γ_p is a finite group acting smoothly and effectively[1] on V_p and has a linear representation on E_p.
(4) s_p is a Γ_p equivariant smooth map $V_p \to E_p$.
(5) ψ_p is a homeomorphism from $s_p^{-1}(0)/\Gamma_p$ to a neighborhood of p in X.

We put $U_p = V_p/\Gamma_p$ and says that U_p is a Kuranishi neighborhood. We sometimes say that V_p is a Kuranishi neighborhood by a slight abuse of terminology.

We call $E_p \times V_p \to V_p$ an *obstruction bundle* and s_p a *Kuranishi map*. For $x \in V_p$, denote by $(\Gamma_p)_x$ the isotropy subgroup at x, i.e.,

$$(\Gamma_p)_x = \{\gamma \in \Gamma_p \mid \gamma x = x\}.$$

Let o_p be a point in V_p with $s_p(o_p) = 0$ and $\psi_p([o_p]) = p$. We will assume that o_p is fixed by all elements of Γ_p.

Definition 1.2. ([FOOO, Definition A1.3]) Let $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ and $(V_q, E_q, \Gamma_q, \psi_q, s_q)$ be a pair of Kuranishi neighborhoods of $p \in X$ and $q \in \psi_p(s_p^{-1}(0)/\Gamma_p)$, respectively. We say a triple $(\hat\phi_{pq}, \phi_{pq}, h_{pq})$ a *coordinate change* if

(1) h_{pq} is an injective homomorphism $\Gamma_q \to \Gamma_p$.
(2) $\phi_{pq} : V_{pq} \to V_p$ is an h_{pq} equivariant smooth embedding from a Γ_q invariant open neighborhood V_{pq} of o_q to V_p, such that the induced map $\overline{\phi}_{pq} : U_{pq} \to U_p$ is injective.
(3) $(\hat\phi_{pq}, \phi_{pq})$ is an h_{pq} equivariant embedding of vector bundles $E_q \times V_{pq} \to E_p \times V_p$. Here and hereafter $\underline{\phi}_{pq} : U_{pq} \to U_q$ is a map induced by ϕ_{pq} and $U_{qp} = V_{qp}/\Gamma_p$.
(4) $\hat\phi_{pq} \circ s_q = s_p \circ \phi_{pq}$. Here and hereafter we sometimes regard s_p as a section $s_p : V_p \to E_p \times V_p$ of trivial bundle $E_p \times V_p \to V_p$.
(5) $\psi_q = \psi_p \circ \underline{\phi}_{pq}$ on $(s_q^{-1}(0) \cap V_{pq})/\Gamma_q$.
(6) The map h_{pq} restricts to an isomorphism $(\Gamma_q)_x \to (\Gamma_p)_{\phi_{pq}(x)}$ for any $x \in V_{pq}$. Here
$$(\Gamma_q)_x = \{\gamma \in \Gamma_\epsilon \mid \gamma x = x\}.$$

[1]We *always* assume orbifold to be effective.

We also assume the following condition[2]. $d_{\rm fiber}s_p$ (the differential of the Kuranishi map in the normal direction of the normal bundle) induces a bundle isomorphism

$$N_{V_{pq}}V_p \cong \frac{E_p \times V_{pq}}{\hat\phi_{pq}(E_q \times V_{pq})} \tag{1.1}$$

as Γ_q-equivariant bundles on $V_{pq} \cap s_q^{-1}(0)$.

Definition 1.3. ([FOOO, Definition A1.5]) A *Kuranishi structure* on X assigns a Kuranishi neighborhood $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ for each $p \in X$ and a coordinate change $(\hat\phi_{pq}, \phi_{pq}, h_{pq})$ for each $q \in \psi_p(s_p^{-1}(0)/\Gamma_p)$ such that the following holds.

(1) $\dim V_p - \operatorname{rank} E_p$ is independent of p.[3]
(2) If $r \in \psi_q((V_{pq} \cap s_q^{-1}(0))/\Gamma_q)$, $q \in \psi_p(s_p^{-1}(0)/\Gamma_p)$ then there exists $\gamma^\alpha_{pqr} \in \Gamma_p$ for each connected component $(\phi_{qr}^{-1}(V_{pq}) \cap V_{qr} \cap V_{pr})_\alpha$ of $\phi_{qr}^{-1}(V_{pq}) \cap V_{qr} \cap V_{pr}$ such that

$$h_{pq} \circ h_{qr} = \gamma^\alpha_{pqr} \cdot h_{pr} \cdot (\gamma^\alpha_{pqr})^{-1}, \quad \phi_{pq} \circ \phi_{qr} = \gamma^\alpha_{pqr} \cdot \phi_{pr}, \quad \hat\phi_{pq} \circ \hat\phi_{qr} = \gamma^\alpha_{pqr} \cdot \hat\phi_{pr}.$$

Here the first equality holds on $(\phi_{qr}^{-1}(V_{pq}) \cap V_{qr} \cap V_{pr})_\alpha$ and the second equality holds on $E_r \times (\phi_{qr}^{-1}(V_{pq}) \cap V_{qr} \cap V_{pr})_\alpha$.

A space X with Kuranishi structure is called *Kuranishi space*.

Definition 1.4. Consider the situation of Lemma 1.3. Let Y be a topological space. A family $\{f_p\}$ of Γ_p-equivariant continuous maps $f_p : V_p \to Y$ is said to be a *strongly continuous map* if

$$f_p \circ \phi_{pq} = f_q$$

on V_{pq}. A strongly continuous map induces a continuous map $f : X \to Y$. We will ambiguously denote $f = \{f_p\}$ when the meaning is clear.

When Y is a smooth manifold, a strongly continuous map $f : X \to Y$ is defined to be smooth if all $f_p : V_p \to Y$ are smooth. We say that it is *weakly submersive* if each of f_p is a submersion.

2. Definition of good coordiate system.

Definition 2.1. An orbifold is a special case of Kuranishi space where all the obstruction bundles are trivial.

Remark 2.2. We include paracompact case for orbifold. The modification of the definition seems obvious and so are omitted. We use only the compact case for general Kuranishi space. (Though we can define paracompact case.)

Definition 2.3. Let X, Y be an orbifold and $F : X \to Y$ be a continous map. F is said to be an *embedding* of orbifold if F is a homeomorphism to the image and the following conditions are satisfied for each $q \in X$ and $p = F(q) \in Y$.

(1) There exists an open subset $V^X_{pq} \subset V^X_q$ of the chart of q that is Γ^X_q equivariant and containing o_q.
(2) There exists a smooth embedding $F_q : V^X_{pq} \to V^Y_p$ of manifolds.
(3) There exist a group isomorphism $h_{pq} : \Gamma^X_q \to \Gamma^Y_p$ such that F_q is h_{pq} equivariant.

[2]This condition is written as a condition for tangent bundle to exist in [FOOO, Definition A1.14]. 16 years of experience shows that Kuranishi structure without tangenet bundle is not useful at all. So we include it in the condition of Kuranishi structure.

[3]This follows from (1.1) under suitable connectivity assumption.

(4) The map h_{pq} restricts to an isomorphism $(\Gamma_q^X)_x \to (\Gamma_p^Y)_{\phi_{pq}(x)}$ for any $x \in V_{pq}^X$.

(5) F_q induces the map $F|_{V_{pq}^X/\Gamma_q^X} : V_{pq}^X/\Gamma_q^X \to V_p^Y/\Gamma_p^Y$.

An embedding of orbifold is said to be a *diffeomorphism* if it has an inverse that is an embedding.

Two embedding of orbifold is said to be the same if they coincides set theoretically.

Remark 2.4. We use only embedding of orbifold and no other maps between them.

We omit the definition of vector bundle over orbifold. ([FOn] Definition 2.7. It is called orbibundle there.) The section of vector bundle is defined in [FOn] page 942 line 3. We also omit the definition of embedding of vector bundle over orbifold.

Hereafter we put

$$\mathcal{U}_{\mathfrak{p}} = \psi_{\mathfrak{p}}(s_{\mathfrak{p}}^{-1}(0)/\Gamma_{\mathfrak{p}}). \tag{2.2}$$

We modify the definition of good coordinate system in [FOOO, Lemma 6.3] as follows.

Definition 2.5. Let X be a space with Kuranishi structure. A *good coordinate system* on it consists of partially ordered set $(\mathfrak{P}, \le)$ of finite order, and $(U_{\mathfrak{p}}, E_{\mathfrak{p}}, \psi_{\mathfrak{p}}, s_{\mathfrak{p}})$ for each $\mathfrak{p} \in \mathfrak{P}$, with the following data.

(1) $U_{\mathfrak{p}}$ is an orbifold of finite dimension, which may or may not have boundary or corner.

(2) $E_{\mathfrak{p}}$ is a real vector bundle over $U_{\mathfrak{p}}$.

(3) s_p is a section of $E_{\mathfrak{p}} \to U_{\mathfrak{p}}$.

(4) $\psi_{\mathfrak{p}}$ is a homeomorphism from $s_{\mathfrak{p}}^{-1}(0)$ to an open set of X.

(5) If $\mathfrak{q} \le \mathfrak{p}$, then there exists

$$(U_{\mathfrak{p}\mathfrak{q}}, \underline{\hat\phi}_{\mathfrak{p}\mathfrak{q}}, \underline{\phi}_{\mathfrak{p}\mathfrak{q}})$$

where :

(a) $U_{\mathfrak{p}\mathfrak{q}}$ is an open subset of $U_{\mathfrak{q}}$ such that

$$\psi_{\mathfrak{q}}(U_{\mathfrak{p}\mathfrak{q}} \cap s_{\mathfrak{q}}^{-1}(0)) = \psi_{\mathfrak{p}}((U_{\mathfrak{p}} \cap s_{\mathfrak{p}}^{-1}(0))) \cap \psi_{\mathfrak{q}}((U_{\mathfrak{q}} \cap s_{\mathfrak{q}}^{-1}(0))), \tag{2.3}$$

(b) $\underline{\phi}_{\mathfrak{p}\mathfrak{q}} : U_{\mathfrak{p}\mathfrak{q}} \to U_{\mathfrak{p}}$ is an embedding of orbifolds.

(c) $\underline{\hat\phi}_{\mathfrak{p}\mathfrak{q}}$ is an embedding of vector bundles $E_{\mathfrak{q}}|_{U_{\mathfrak{p}\mathfrak{q}}} \to E_{\mathfrak{p}}$ over $\underline{\phi}_{\mathfrak{p}\mathfrak{q}}$.

(d) $\underline{\hat\phi}_{\mathfrak{p}\mathfrak{q}} \circ s_{\mathfrak{q}} = s_{\mathfrak{p}} \circ \underline{\phi}_{\mathfrak{p}\mathfrak{q}}$, $\qquad \psi_{\mathfrak{q}} = \psi_{\mathfrak{p}} \circ \underline{\phi}_{\mathfrak{p}\mathfrak{q}}$.

(e) $d_{\rm fiber} s_{\mathfrak{p}}$ induces an isomorphism of vector bundles [4] at $s_{\mathfrak{q}}^{-1}(0) \cap U_{\mathfrak{p}\mathfrak{q}}$.

$$N_{U_{\mathfrak{p}\mathfrak{q}}} U_{\mathfrak{p}} \cong \frac{\underline{\hat\phi}_{\mathfrak{p}\mathfrak{q}}^{*} E_{\mathfrak{p}}}{(E_{\mathfrak{q}})|_{U_{\mathfrak{p}\mathfrak{q}}}} \tag{2.4}$$

(6) If $\mathfrak{r} \le \mathfrak{q} \le \mathfrak{p}$, $\psi_{\mathfrak{p}}(s_{\mathfrak{p}}^{-1}(0)) \cap \psi_{\mathfrak{q}}(s_{\mathfrak{q}}^{-1}(0)) \cap \psi_{\mathfrak{r}}(s_{\mathfrak{r}}^{-1}(0)) \ne \emptyset$, we have

$$\underline{\phi}_{\mathfrak{p}\mathfrak{q}} \circ \underline{\phi}_{\mathfrak{q}\mathfrak{r}} = \underline{\phi}_{\mathfrak{p}\mathfrak{r}}, \quad \underline{\hat\phi}_{\mathfrak{p}\mathfrak{q}} \circ \underline{\hat\phi}_{\mathfrak{q}\mathfrak{r}} = \underline{\hat\phi}_{\mathfrak{p}\mathfrak{r}}.$$

Here the first equality holds on $\underline{\phi}_{\mathfrak{q}\mathfrak{r}}^{-1}(U_{\mathfrak{p}\mathfrak{q}}) \cap U_{\mathfrak{q}\mathfrak{r}} \cap U_{\mathfrak{p}\mathfrak{r}}$, and the second equality holds on $(E_{\mathfrak{r}})|_{(\underline{\phi}_{\mathfrak{q}\mathfrak{r}}^{-1}(U_{\mathfrak{p}\mathfrak{q}}) \cap U_{\mathfrak{q}\mathfrak{r}} \cap U_{\mathfrak{p}\mathfrak{r}})}$.

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[4] We omit the definition of normal bundle of embedding. In our definition of embedding the definition is easy.

(7)
$$\bigcup_{\mathfrak p\in\mathfrak P}\psi_{\mathfrak p}(s_{\mathfrak p}^{-1}(0)) = X.$$
(8) If $\psi_{\mathfrak p}(s_{\mathfrak p}^{-1}(0))\cap\psi_{\mathfrak q}(s_{\mathfrak q}^{-1}(0))\neq\emptyset$, either $\mathfrak p\le\mathfrak q$ or $\mathfrak q\le\mathfrak p$ holds.
(9) The Conditions 2.6, 2.8, 2.10 and 2.11 below hold.

Condition 2.6 (Joyce)**.** Suppose $\mathfrak p\ge\mathfrak q\ge\mathfrak r$.
$$\underline{\phi}_{\mathfrak{pq}}(U_{\mathfrak{pq}})\cap\underline{\phi}_{\mathfrak{pr}}(U_{\mathfrak{pr}}) = \underline{\phi}_{\mathfrak{pr}}(\underline{\phi}_{\mathfrak{qr}}^{-1}(U_{\mathfrak{pq}})\cap U_{\mathfrak{pr}}).$$

Lemma 2.7. *Condition 2.6 is equivalent to the following:*

If $\mathfrak p\ge\mathfrak q\ge\mathfrak r$ and $x\in U_{\mathfrak{pr}}$, $y\in U_{\mathfrak{pq}}$ with $\underline{\phi}_{\mathfrak{pr}}(x)=\underline{\phi}_{\mathfrak{pq}}(y)$, then $\underline{\phi}_{\mathfrak{qr}}(x)\in U_{\mathfrak{pq}}$ and $\underline{\phi}_{\mathfrak{pq}}(\underline{\phi}_{\mathfrak{qr}}(x))=y$.

Proof. The including $\supseteq$ in Condition 2.6 is always obvious. '$x\in U_{\mathfrak{pr}}$, $y\in U_{\mathfrak{pq}}$ with $\underline{\phi}_{\mathfrak{pr}}(x)=\underline{\phi}_{\mathfrak{pq}}(y)$' is the left hand side of Condition 2.6. '$\underline{\phi}_{\mathfrak{qr}}(x)\in U_{\mathfrak{pq}}$ and $\underline{\phi}_{\mathfrak{pq}}(\underline{\phi}_{\mathfrak{qr}}(x))=y$' is the right hand side of it. □

Condition 2.8. If $\bigcap_{i\in I}U_{\mathfrak p_i\mathfrak q}\neq\emptyset$ then $\bigcap_{i\in I}\mathcal U_{\mathfrak p_i\mathfrak q}\neq\emptyset$. If $\bigcap_{i\in I}\underline{\phi}_{\mathfrak{pq}_i}(U_{\mathfrak{pq}_i})\neq\emptyset$ then $\bigcap_{i\in I}\mathcal U_{\mathfrak{pq}_i}\neq\emptyset$.

Here and hafter we put
$$\mathcal U_{\mathfrak{pq}} = \psi_{\mathfrak q}(s_{\mathfrak q}^{-1}(0)\cap U_{\mathfrak{pq}}). \tag{2.5}$$
Condtion 2.8 and Definition 2.5 (8) implies the following:

Lemma 2.9. *Suppose $\mathfrak q\le\mathfrak p_j$ for $j=1,\dots,J$ and $\bigcap_{i\in I}U_{\mathfrak p_i\mathfrak q}\neq\emptyset$. Then the set $\{\mathfrak q\}\cup\{\mathfrak p_j\mid j=1,\dots,J\}$ are linearly ordered. (Namely for each $\mathfrak r,\mathfrak s\in\{\mathfrak q\}\cup\{\mathfrak p_j\mid j=1,\dots,J\}$ at least one of $\mathfrak r\ge\mathfrak s$ or $\mathfrak s\ge\mathfrak r$ holds.)*

The proof is omitted.

Condition 2.10. Suppose $U_{\mathfrak{pr}}\cap U_{\mathfrak{qr}}\neq\emptyset$ or $\hat{\underline{\phi}}_{\mathfrak{qr}}^{-1}(U_{\mathfrak{pq}})\neq\emptyset$. We also assume $\mathfrak p\ge\mathfrak q\ge\mathfrak r$. Then we have
$$\hat{\underline{\phi}}_{\mathfrak{qr}}^{-1}(U_{\mathfrak{pq}}) = U_{\mathfrak{pr}}\cap U_{\mathfrak{qr}}.$$

Condition 2.11. The map $U_{\mathfrak{pq}}\to U_{\mathfrak p}\times U_{\mathfrak q}$ defined below is proper.
$$x\mapsto(\hat{\underline{\phi}}_{\mathfrak{pq}}(x),x). \tag{2.6}$$

The existence of good coordinate system is proved in Section 4.

Lemma 2.12. *The following $\sim$ is an equivalence relation.*

Let $x\in U_{\mathfrak p}$ and $y\in U_{\mathfrak q}$. We say $x\sim y$ if and only if
(1) $x=y$.
(2) $\mathfrak p\ge\mathfrak q$ *and* $\underline{\phi}_{\mathfrak{pq}}(x)=y$.
(3) $\mathfrak q\ge\mathfrak p$ *and* $\underline{\phi}_{\mathfrak{qp}}(y)=x$.

Proof. Only transitivity is nontrivial. Let $x_1\sim x_2$, $x_2\sim x_3$, $x_i\in U_{\mathfrak p_i}$.

Suppose $\mathfrak p_1\le\mathfrak p_2\le\mathfrak p_3$. Condition 2.10 implies $U_{\mathfrak p_3\mathfrak p_1}\cap U_{\mathfrak p_2\mathfrak p_1} = (\underline{\phi}_{\mathfrak p_2\mathfrak p_1})^{-1}U_{\mathfrak p_3\mathfrak p_2}$. Therefore Definition 2.5 (6) implies
$$x_3 = \underline{\phi}_{\mathfrak p_3\mathfrak p_2}(x_2) = \underline{\phi}_{\mathfrak p_3\mathfrak p_2}\underline{\phi}_{\mathfrak p_2\mathfrak p_1}(x_1) = \underline{\phi}_{\mathfrak p_3\mathfrak p_1}(x_1).$$

Namely $x_3 \sim x_1$. The case $\mathfrak{p}_1 \ge \mathfrak{p}_2 \ge \mathfrak{p}_3$ is similar.

Suppose $\mathfrak{p}_1 \ge \mathfrak{p}_2 \le \mathfrak{p}_3$. Condition 2.8 and Definition 2.5 (8) implies either $\mathfrak{p}_1 \ge \mathfrak{p}_3$ or $\mathfrak{p}_1 \le \mathfrak{p}_3$. Let us assume $\mathfrak{p}_1 \le \mathfrak{p}_3$. Then Condition 2.10 implies $x_2 \in (\underline{\phi}_{\mathfrak{p}_1\mathfrak{p}_2})^{-1}U_{\mathfrak{p}_3\mathfrak{p}_1}$. Then Definition 2.5 (6) implies

$$\underline{\phi}_{\mathfrak{p}_3\mathfrak{p}_1}(x_1) = \underline{\phi}_{\mathfrak{p}_3\mathfrak{p}_1}(\underline{\phi}_{\mathfrak{p}_1\mathfrak{p}_2}(x_2)) = x_3.$$

Namely $x_1 \sim x_3$. The case $\mathfrak{p}_1 \ge \mathfrak{p}_3$ is similar.

Let us assume $\mathfrak{p}_1 \le \mathfrak{p}_2 \ge \mathfrak{p}_3$. By Condition 2.8 we have either $\mathfrak{p}_1 \le \mathfrak{p}_3$ or $\mathfrak{p}_1 \ge \mathfrak{p}_3$. Then Condition 2.6 implies $x_3 = \underline{\phi}_{\mathfrak{p}_3\mathfrak{p}_1}(x_1)$ or $x_1 = \underline{\phi}_{\mathfrak{p}_1\mathfrak{p}_3}(x_3)$, as required. □

Definition 2.13. We define $U(X)$ as follows. We take disjoint union

$$\tilde U(X) = \bigcup_{\mathfrak{p}\in\mathfrak{P}} U_{\mathfrak{p}}.$$

$\sim$ defines an equivalence relation. $U(X)$ is the set of $\sim$ equivalence classes. We define a quotient topology on it.

The map $\Pi_{\mathfrak{p}} : U_{\mathfrak{p}} \to U(X)$ sends an element of U_p to its equialence class.

Lemma 2.14. *$U(X)$ is Hausdorff.*

Proof. Let $x_i, y_i, x, y \in \tilde U(X)$. Assume $[x_i] \sim [y_i]$ and $\lim_{i\to\infty}[x_i] = [x]$, $\lim_{i\to\infty}[y_i] = [y]$. It suffices to show $x = y$. We may assume $x_i \ne y_i$ By taking a subsequence if necessary and exchanging x_i with y_i, we may assume $x_i \in U_{\mathfrak{pq}} \in U_{\mathfrak{q}}$ and $y_i \in U_{\mathfrak{p}}$ and $\lim_{i\to\infty} x_i = x \in U_{\mathfrak{q}}$, $\lim_{i\to\infty} y_i = y \in U_{\mathfrak{p}}$. Then Condition 2.11 implies that x_i converges in $U_{\mathfrak{pq}}$. Therefore

$$y = \lim_{i\to\infty} y_i = \lim_{i\to\infty} \underline{\phi}_{\mathfrak{pq}}(x_i) = \underline{\phi}_{\mathfrak{pq}}(x).$$

The lemma follows easily. □

We put

$$s^{-1}(0) = \bigcup_{\mathfrak{p}} \Pi_{\mathfrak{p}}(s_{\mathfrak{p}}^{-1}(0) \cap U_{\mathfrak{p}}). \tag{2.7}$$

The maps $\psi_{\mathfrak{p}}$ for various $\mathfrak{p}$ induce a map

$$\psi : s^{-1}(0) \to X. \tag{2.8}$$

Definition 2.5 (4),(7) and (2.3) imply that (2.8) is a bijection. Let

$$I : X \to U(X)$$

be its imverse.

Lemma 2.15. *I is a homeomorphism to its image.*

Proof. X is compact. $U(X)$ is Hausdorff. Moreover I is continuous and injective. □

Lemma 2.16. *Suppose we have a good coordinate system. Then there exists an open subsets $U'_{\mathfrak{p}} \subset U_{\mathfrak{p}}$ and $U'_{\mathfrak{pq}} \subset U_{\mathfrak{pq}}$ such that the restriction to U'_p and U'_{pq} gives a good coordinate system, and $U'_{\mathfrak{p}}$ and $U'_{\mathfrak{pq}}$ are relatively compact in $U_{\mathfrak{p}}$ and $U_{\mathfrak{pq}}$, respectively.*

Proof. We take an open subset $U'_{\mathfrak p} \subset U_{\mathfrak p}$ for each $\mathfrak p$ that is relatively compact in $U_{\mathfrak p}$ and

$$\bigcup_{\mathfrak p \in \mathfrak P} \psi_{\mathfrak p}(s_{\mathfrak p}^{-1}(0) \cap U'_{\mathfrak p}) = X.$$

We may choose it so that

$$\bigcap_{i \in I} \mathcal U_{\mathfrak p_i} \ne \emptyset \ \Leftrightarrow\ \bigcap_{i \in I} \mathcal U'_{\mathfrak p_i} \ne \emptyset \tag{2.9}$$

We put

$$U'_{\mathfrak p\mathfrak q} = U_{\mathfrak p\mathfrak q} \cap U'_{\mathfrak q} \cap \underline{\phi}_{\mathfrak p\mathfrak q}^{-1}(U'_{\mathfrak p}). \tag{2.10}$$

Condition 2.11 implies that $U'_{\mathfrak p\mathfrak q}$ is relatively compact. It is straightforward to check that they satisfy the conditions in Definition 2.5. □

Remark 2.17. (1) If $\mathcal K_{\mathfrak p}$ is a compact subset of $\mathcal U_{\mathfrak p}$ for each p then we may choose $U'_{\mathfrak p}$ etc. in Proposition 2.16 so that $\mathcal U'_{\mathfrak p}$ contains $\mathcal K_{\mathfrak p}$.

(2) On the other hand, we may choose $U'_{\mathfrak p}$ as small as we want as far as the condition $\bigcup_{\mathfrak p \in \mathfrak P} \mathcal U'_{\mathfrak p} = X$. In fact at the begining of the proof we take $U'_{\mathfrak p}$ so that this is satisfied and do not need to change it.

We define $U'(X)$ from $U'_{\mathfrak p}$ and $U'_{\mathfrak p\mathfrak q}$ in the same way as Definition 2.13.

Definition 2.18. We define $\mathfrak I_{U(X)U'(X)} : U'(X) \to U(X)$ by sending the $\sim$-equivalence class $[x]$ of $x \in \tilde U'(X)$ to the equivalence class of $x \in \tilde U(X)$ in $U(X)$.

By (2.10) we find that if $\tilde x \sim \tilde y$ in $\tilde U(X)$ for $\tilde x, \tilde y \in \tilde U'(X)$ then $\tilde x \sim \tilde y$ in $\tilde U'(X)$. Therefore $\mathfrak I_{U(X)U'(X)}$ is injective.

Lemma 2.19. *$\mathfrak I_{U(X)U'(X)}$ is a homeomorphism to its image.*

The proof is omitted.

Lemma 2.20. *Let $x = \psi_{\mathfrak p}(\tilde x) \in X$, $\tilde x \in s_{\mathfrak p}^{-1}(0) \subset U'_{\mathfrak p}$. Then there exists a neighborhood $\mathfrak O_{\mathfrak p}(x)$ of $\tilde x$ in $U'_{\mathfrak p}$ such that*

1. *$\Pi_{\mathfrak p} : \mathfrak O_{\mathfrak p}(x) \to \Pi_{\mathfrak p}(\mathfrak O_{\mathfrak p}(x))$ is a homeomorphism.*
2. *$\Pi_{\mathfrak p}(\mathfrak O_{\mathfrak p}(x))$ is an open subset of $\bigcup_{\mathfrak q \le \mathfrak p} \Pi_{\mathfrak q}(U_{\mathfrak q})$.*

Proof. Choose $\mathfrak O_{\mathfrak p}(x) \subset U_{\mathfrak p}$ so that it is relatively compact in $U_{\mathfrak p}$.

Clearly $\Pi_{\mathfrak p} : \mathfrak O_{\mathfrak p}(x) \to U'(X)$ is injective and continuous. It extends so to the closure of $\mathfrak O_{\mathfrak p}(x)$ that is compact. (1) follows.

We prove (2). Let $\mathfrak q \le \mathfrak p$. It suffices to show

$$\Pi_{\mathfrak q}^{-1}(\Pi_{\mathfrak p}(\mathfrak O_{\mathfrak p}(x)))$$

is open in $U'_{\mathfrak q}$. In fact

$$\underline{\phi}_{\mathfrak p\mathfrak q}^{-1}(\mathfrak O_{\mathfrak p}(x)) \cap U'_{\mathfrak p\mathfrak q} = \underline{\phi}_{\mathfrak p\mathfrak q}^{-1}(\mathfrak O_{\mathfrak p}(x)) \cap \Big(U_{\mathfrak p\mathfrak q} \cap U'_{\mathfrak q} \cap \underline{\phi}_{\mathfrak p\mathfrak q}^{-1}(U'_{\mathfrak q})\Big) = \Pi_{\mathfrak q}^{-1}(\Pi_{\mathfrak p}(\mathfrak O_{\mathfrak p}(x))).$$

(Here $\underline{\phi}_{\mathfrak p\mathfrak q}$ is $\underline{\phi}_{\mathfrak p\mathfrak q} : U'_{\mathfrak p\mathfrak q} \to U'_{\mathfrak p}$.) □

Remark 2.21. This lemma holds for $U(X)$ also.

The next lemma plays a key role in the next section to show basic properties of the virtual fundamental chain.

Lemma 2.22. *For any $x \in X$ there exists $\mathfrak q_1, \dots, \mathfrak q_m \in \mathfrak P$, $\mathfrak q_1 \le \dots \le \mathfrak q_m$ and open sets $\Omega_{\mathfrak q_i}(x) \subset U'_{\mathfrak q_{i+1}\mathfrak q_i}$ with the following properties.*

(1) $x \in \mathcal{U}'_{\mathfrak{q}_i}$ *for* $i = 1, \dots, m$.
(2) $\psi_{\mathfrak{q}_1}^{-1}(x) \in \Omega_{\mathfrak{q}_1}(x)$.
(3) $\psi_{\mathfrak{q}_i}^{-1}(x) \in \overline{\Omega}_{\mathfrak{q}_i}(x) \setminus \Omega_{\mathfrak{q}_i}(x)$ *for* $i > 1$. *Here the closure is taken in* $U_{\mathfrak{q}_i}$.
(4) *The map* $\Pi_{\mathfrak{q}_i} : \Omega_{\mathfrak{q}_i}(x) \to U'(X)$ *is a homeomorphism to its image.*
(5) *The union of the images of* $\Pi_{\mathfrak{q}_i} : \Omega_{\mathfrak{q}_i}(x) \to U'(X)$ *is an open neighborhood of* $I(x)$.
(6) $\dim U_{\mathfrak{q}_1} < \dim U_{\mathfrak{q}_i}$ *for* $i \ne 1$.

Proof. By Lemma 2.9 there exists a maximal $\mathfrak{q}_1$ such that $x \in \mathcal{U}'_{\mathfrak{q}}$.

Sublemma 2.23. *There exists* $\mathfrak{q}_m \in \mathfrak{P}$ *such that it is maximal in the set* $\{\mathfrak{q} \in \mathfrak{P} \mid x \in \overline{\mathcal{U}}'_{\mathfrak{q}}\}$.

Proof. Let $\mathfrak{q}, \mathfrak{q}' \in \{\mathfrak{q} \in \mathfrak{P} \mid x \in \overline{\mathcal{U}}'_{\mathfrak{q}}\}$. Since the closure of $\mathcal{U}'_{\mathfrak{q}}$ is contained in $\mathcal{U}_{\mathfrak{q}}$, it follows that $\mathcal{U}_{\mathfrak{q}} \cap \mathcal{U}_{\mathfrak{q}'} \ne \emptyset$. Therefore by (2.9) $\mathcal{U}'_{\mathfrak{q}} \cap \mathcal{U}'_{\mathfrak{q}'} \ne \emptyset$. The sublemma follows from Definition 2.5 (8). □

By Sublemma 2.23 we can take $\mathfrak{q}_1, \dots, \mathfrak{q}_m$ such that $\mathfrak{q}_1 \le \dots \le \mathfrak{q}_m$ and

$$\{\mathfrak{q} \mid x \in \overline{\mathcal{U}}_{\mathfrak{q}}\} \cap \{\mathfrak{q} \mid \dim U_{\mathfrak{q}} > \dim U_{\mathfrak{q}_1}\} = \{\mathfrak{q}_i \mid i = 2, \dots, m\}.$$

Then we can use Lemma 2.20 to find required $\Omega_{\mathfrak{q}_i}(x)$. (We do not need $\mathfrak{q}_i$ ($i \ne 1$) with $\dim U_{\mathfrak{q}_1} = \dim U_{\mathfrak{q}_i}$ since x is in (the interior of) $\Omega_{\mathfrak{q}_1}(x)$.) □

We put

$$\mathfrak{U}(x) = \bigcup_{i=1,\dots,m} \Pi_{\mathfrak{q}_i}(\Omega_{\mathfrak{q}_i}(x)). \tag{2.11}$$

By (5) $\mathfrak{U}(x)$ is an open neighborhood of x in $U'(X)$.

3. Virtual fundamental chain.

To start the construction of perturbation and virtual fundamental chain, we shrink $U_{\mathfrak{p}}$ using Lemma 2.16 as follows.

First we take an extension of the subbundle $\hat{\underline{\phi}}_{\mathfrak{p}\mathfrak{q}}(E_{\mathfrak{q}}|_{U'_{\mathfrak{p}\mathfrak{q}}})$ of $E_{\mathfrak{p}}|\underline{\phi}_{\mathfrak{p}\mathfrak{q}}(U'_{\mathfrak{p}\mathfrak{q}})$ to its neighborhood in U_p. We also fix a splitting

$$E_{\mathfrak{p}} = E_{\mathfrak{q}} \oplus E_{\mathfrak{q}}^{\perp} \tag{3.12}$$

on a neighborhood of $\underline{\phi}_{\mathfrak{p}\mathfrak{q}}(U'_{\mathfrak{p}\mathfrak{q}})$. We can take such an extension of the subbundle and splitting since $U'_{\mathfrak{p}\mathfrak{q}}$ is a relatively compact open subset of $U_{\mathfrak{p}\mathfrak{q}}$ that is a suborbifold of $U_{\mathfrak{p}}$.

Using splitting (3.12) the normal differential

$$d_{\text{fiber}} s_{\mathfrak{q}} : N_{U'_{\mathfrak{p}\mathfrak{q}}} U'_{\mathfrak{p}} \to \frac{E_{\mathfrak{p}}}{\hat{\underline{\phi}}_{\mathfrak{p}\mathfrak{q}}(E_{\mathfrak{q}})} \tag{3.13}$$

is defined. (Note without fixing the splitting $d_{\text{fiber}} s_{\mathfrak{q}}$ is well-defined only at $s_{\mathfrak{q}}^{-1}(0) \cap U_{\mathfrak{p}\mathfrak{q}}$.)

By assumption, (3.13) is an isomorphism on $s_{\mathfrak{q}}^{-1}(0) \cap U'_{\mathfrak{p}\mathfrak{q}}$. We take an open neighborhood $\mathfrak{W}'_{\mathfrak{p}\mathfrak{q}}$ of $s_{\mathfrak{q}}^{-1}(0) \cap U'_{\mathfrak{p}\mathfrak{q}}$ so that (3.13) is an isomorphism on $\mathfrak{W}'_{\mathfrak{p}\mathfrak{q}}$.

We take $U''_{\mathfrak{q}}$ for each $\mathfrak{q}$ so that

$$U''_{\mathfrak{q}} \cap U'_{\mathfrak{p}\mathfrak{q}} \subset \mathfrak{W}'_{\mathfrak{p}\mathfrak{q}}$$

for each $\mathfrak{p}$. (We can take such $U''_{\mathfrak{q}}$ by Remark 2.17 (2).) Thus from now on we may assume that (3.13) is an isomorphism on $U_{\mathfrak{p}\mathfrak{q}}$.

We start with this $U_{\mathfrak{p}}$, $U_{\mathfrak{p}\mathfrak{q}}$ and repeat the construction of the last section. Namely we take $U_{\mathfrak{p}}^{(n)}, U_{\mathfrak{p}\mathfrak{q}}^{(n)}$ such that

(1) The conclusion of Lemma 2.16 is satisfied when we replace $U_{\mathfrak{p}}, U_{\mathfrak{p}\mathfrak{q}}$ by $U_{\mathfrak{p}}^{(n-1)}, U_{\mathfrak{p}\mathfrak{q}}^{(n-1)}$ and $U'_{\mathfrak{p}}, U'_{\mathfrak{p}\mathfrak{q}}$ by $U_{\mathfrak{p}}^{(n)}, U_{\mathfrak{p}\mathfrak{q}}^{(n)}$.
(2) The conclusion of Lemmas 2.7-2.22 hold for $U_{\mathfrak{p}}^{(n)}, U_{\mathfrak{p}\mathfrak{q}}^{(n)}$.
(3) $U_{\mathfrak{p}}^{(1)}, U_{\mathfrak{p}\mathfrak{q}}^{(1)}$ is $U'_{\mathfrak{p}}, U'_{\mathfrak{p}\mathfrak{q}}$.

Let $U^{(n)}(X)$ be the space obtained from $U_{\mathfrak{p}}^{(n)}, U_{\mathfrak{p}\mathfrak{q}}^{(n)}$ as in Definition 2.13.

Let us consider the good coordinate sytem $(U_{\mathfrak{p}}, E_{\mathfrak{p}}, \psi_{\mathfrak{p}}, s_{\mathfrak{p}})$ $\mathfrak{p} \in \mathfrak{P}$ of our Kuranishi structure. We put $\mathfrak{P} = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_N\}$, where $\mathfrak{p}_i < \mathfrak{p}_j$ only if $i < j$. Let $\#\mathfrak{P} = N$. We take $n = 10N^2$ and consider $U_{\mathfrak{p}}^{(n)}, U_{\mathfrak{p}\mathfrak{q}}^{(n)}$ as above.

Proposition 3.1. *For each $\epsilon > 0$, there exists a system of multisections $s_{\epsilon,\mathfrak{p}}$ on $U_{\mathfrak{p}}^{(n)}$ for $\mathfrak{p} \in \mathfrak{P}$ with the following properties.*

(1) *$s_{\epsilon,\mathfrak{p}}$ is transversal to 0.*
(2) *$s_{\epsilon,\mathfrak{p}} \circ \underline{\phi}_{\mathfrak{p}\mathfrak{q}} = \hat{\underline{\phi}}_{\mathfrak{p}\mathfrak{q}} \circ s_{\epsilon,\mathfrak{q}}$.*
(3) *The derivative of (arbitrary branch of) $s_{\epsilon,\mathfrak{p}}$ induces an isomorphism*
$$N_{U_{\mathfrak{p}\mathfrak{q}}} U_{\mathfrak{p}} \cong \frac{\hat{\underline{\phi}}_{\mathfrak{p}\mathfrak{q}}^{*} E_{\mathfrak{p}}}{(E_{\mathfrak{q}})|_{U_{\mathfrak{p}\mathfrak{q}}}} \tag{3.14}$$
that coincides with the isomorphism (3.13).
(4) *The C^0 distance of $s_{\epsilon,\mathfrak{p}}$ from $s_{\mathfrak{p}}$ is smaller than ϵ.*

Proof. [5] We will construct a perturbation $s_{\epsilon,\mathfrak{p}}^{k}$ on $U_{\mathfrak{p}}^{(10k^2)}$ for $\mathfrak{p} = \mathfrak{p}_1, \dots, \mathfrak{p}_k$ satisfying (1)-(4) above, by upward induction on k.

We describe the step to construct $s_{\epsilon,\mathfrak{p}_k}$. So we fix k here. We identify the image $\underline{\phi}_{\mathfrak{p}_k\mathfrak{p}_i}(U_{\mathfrak{p}_k\mathfrak{p}_i}^{(m)})$ in $U_{\mathfrak{p}_k}^{(m)}$ with $U_{\mathfrak{p}_k\mathfrak{p}_i}^{(m)}$. (Here $i < k$.) We put
$$\mathcal{N}_k^i = \bigcup_{j=i}^{k-1} \mathcal{N}^i_{U_{\mathfrak{p}_k\mathfrak{p}_j}^{(10(k-1)^2+10(k-i))}} U_{\mathfrak{p}_k}^{(10(k-1)^2+10(k-i))} \tag{3.15}$$
and will define $s_{\epsilon,\mathfrak{p}_k}^{i}$ by downward induction on i. Here the open subset $\mathcal{N}^i_{U_{\mathfrak{p}_k\mathfrak{p}_j}^{(10(k-1)^2+10(k-i))}} U_{\mathfrak{p}_k}^{(10(k-1)^2+10(k-i))}$ is a tubular neighborhood of $U_{\mathfrak{p}_k\mathfrak{p}_j}^{(10(k-1)^2+10(k-i))}$. We assume the closure of
$$\mathcal{N}^i_{U_{\mathfrak{p}_k\mathfrak{p}_j}^{(10(k-1)^2+10(k-i))}} U_{\mathfrak{p}_k}^{(m+1)}$$
is compact in
$$\mathcal{N}^{i+1}_{U_{\mathfrak{p}_k\mathfrak{p}_j}^{(10(k-1)^2+10(k-i-1))}} U_{\mathfrak{p}_k}^{(m)}.$$

Let us start the induction for $i = k-1$. We have embedding $U_{\mathfrak{p}_k\mathfrak{p}_{k-1}}^{(10(k-1)^2)} \to U_{\mathfrak{p}_k}^{(10(k-1)^2)}$. We take its tubular neighborhood $\mathcal{N}^{k-1}_{U_{\mathfrak{p}_k\mathfrak{p}_{k-1}}^{(10(k-1)^2)}} U_{\mathfrak{p}_k}^{(10(k-1)^2)}$. We also

[5]The argument below is one written in [Fu]. (If shorter proof is preferable for the reader please read page 3-4 of [Fu].)

have $s^{k-1}_{\epsilon,\mathfrak{p}_k}$ on $U^{(10(k-1)^2+10(k-i-1))}_{\mathfrak{p}_k\mathfrak{p}_{k-1}}$. We have already taken a splitting

$$E_{\mathfrak{p}_k} = E_{\mathfrak{p}_{k-1}} \oplus E^{\perp}_{\mathfrak{p}_{k-1}}$$

on $U^{(10(k-1)^2+10(k-i-1))}_{\mathfrak{p}_k\mathfrak{p}_{k-1}}$. (Here we identify $E_{\mathfrak{p}_{k-1}}$ with its image by $\hat\phi_{\mathfrak{p}_k\mathfrak{p}_{k-1}}$.) We extended the bundles $E_{\mathfrak{p}_{k-1}}$, $E^{\perp}_{\mathfrak{p}_{k-1}}$ to $\mathcal{N}^{k-1}_{U^{(10(k-1)^2)}_{\mathfrak{p}_k\mathfrak{p}_{k-1}}} U^{(10(k-1)^2)}_{\mathfrak{p}_k}$.[6]

We extend $s_{\epsilon,\mathfrak{p}_{k-1}}$ to $\mathcal{N}^{k-1}_{U^{(10(k-1)^2)}_{\mathfrak{p}_k\mathfrak{p}_{k-1}}} U^{(10(k-1)^2)}_{\mathfrak{p}_k}$ so that it is $\epsilon/2$ close to s_ϵ and coincides with one already defined at the zero section of the normal bundle for the first component $E_{\mathfrak{p}_{k-1}}$ and take the component of the Kuranishi map $s_{\mathfrak{p}_k}$ for the second component. Transversality condition is obviously satified. (2)(3)(4) also hold by construction.

Now we go to the inductive step to construct $s^{i}_{\epsilon,\mathfrak{p}_k}$ assuming we have $s^{i+1}_{\epsilon,\mathfrak{p}_k}$.

We consider the embedding

$$U^{(10(k-1)^2+10(k-i)-10)}_{\mathfrak{p}_k\mathfrak{p}_j} \to U^{(10(k-1)^2+10(k-i)-10)}_{\mathfrak{p}_k},$$

for $j \ge i+1$. We identify $U^{(10(k-1)^2+10(k-i))-10}_{\mathfrak{p}_k\mathfrak{p}_j}$ with the image of embedding and take tubular neighbofhood

$$\mathcal{N}^{i+1}_{U^{(10(k-1)^2+10(k-i)-10)}_{\mathfrak{p}_k\mathfrak{p}_j}} U^{(10(k-1)^2+10(k-i)-10)}_{\mathfrak{p}_k}. \tag{3.16}$$

Note we already have our section $s^{i+1}_{\epsilon,\mathfrak{p}_k}$ on the intersection of (3.16) and $\mathcal{N}^i_k$.

We next apply the same argument as the first step to obtain a $s'^{i}_{\epsilon,\mathfrak{p}_k}$ on

$$\mathcal{N}^{i+1}_{U^{(10(k-1)^2+10(k-i)-9)}_{\mathfrak{p}_k\mathfrak{p}_i}} U^{(10(k-1)^2+10(k-i)-9)}_{\mathfrak{p}_k}.$$

We take a smooth function

$$\chi : \bigcup_{j=i}^{k-1} \mathcal{N}^{i+1}_{U^{(10(k-1)^2+10(k-i)-8)}_{\mathfrak{p}_k\mathfrak{p}_j}} U^{(10(k-1)^2+10(k-i)-9)}_{\mathfrak{p}_k} \to [0,1]$$

of compact support such that $\chi = 1$ on

$$\bigcup_{j=i+1}^{k-1} \mathcal{N}^{i}_{U^{(10(k-1)^2+10(k-i)-8)}_{\mathfrak{p}_k\mathfrak{p}_j}} U^{(10(k-1)^2+10(k-i)-8)}_{\mathfrak{p}_k}. \tag{3.17}$$

We put:

$$s^i_{\epsilon,k} = \chi s^{i+1}_{\epsilon,\mathfrak{p}_k} + (1-\chi) s'^{i}_{\epsilon,\mathfrak{p}_k}. \tag{3.18}$$

Remark 3.2. The sum of multisection is a bit delicate to define. In our case $s^{i+1}_{\epsilon,\mathfrak{p}_k}$ and $s'^{i}_{\epsilon,\mathfrak{p}_k}$ is defined by extending the multisection to the tubular neighborhood. This process does not change the number of branches. (Namely it is extended branch-wise.) So though these two do not coincide as sections, each branch of $s^{i+1}_{\epsilon,\mathfrak{p}_k}$ has a corresponding branch of $s'^{i}_{\epsilon,\mathfrak{p}_k}$. So we can apply the formula (3.18) branch-wise.

We restrict (3.18) to

$$\bigcup_{j=i}^{k-1} \mathcal{N}^{i}_{U^{(10(k-1)^2+10(k-i)-7)}_{\mathfrak{p}_k\mathfrak{p}_j}} U^{(10(k-1)^2+10(k-i)-7)}_{\mathfrak{p}_k}.$$

[6] We can do so by taking $\mathcal{N}^{k-1}_{U^{(10(k-1)^2)}_{\mathfrak{p}_k\mathfrak{p}_{k-1}}} U^{(10(k-1)^2)}_{\mathfrak{p}}$ small.

(3.18) satisfies the properties (1)(2)(3)(4) obviously.

Remark 3.3. We use Lemma 2.9 here for the consistency of tubular neighborhood. We may use the Mather's compatible system of tubular neighborhoods [Ma]. However the present situation is much simpler because of Lemma 2.9. So the compatibility of tubular neighborhoods is obvious.

It conicides with $s^{i+1}_{\epsilon,\mathfrak{p}_k}$ on the overlapped part because we take $\chi = 1$ on (3.17).

We continue the induction upto $i = 1$. Then we have a required multisection $s^1_{\epsilon,\mathfrak{p}_k}$ on

$$\mathcal{N}^0_k = \bigcup_{j=1}^{k-1} \mathcal{N}^i_{U^{(10k^2-10)}_{\mathfrak{p}_k\mathfrak{p}_j}} U^{(10k^2-10)}_{\mathfrak{p}_k}.$$

$(10(k-1)^2 + 10(k-1) < 10k^2 - 10.)$ By [FOn, Lemma 3.14] we can extend it to $U^{(10k^2-9)}_{\mathfrak{p}_k}$ so that it satisfies (1) and it coincides with $s^{i+1}_{\epsilon,\mathfrak{p}_k}$ on

$$\bigcup_{j=1}^{k-1} \mathcal{N}^i_{U^{(10k^2-9)}_{\mathfrak{p}_k\mathfrak{p}_j}} U^{(10k^2-9)}_{\mathfrak{p}_k}.$$

Therefore (1)(2)(3)(4) are satisfied. The proof of Proposition 3.1 is complete. □

We thus have constructed our perturbation that is a multisection $s_{\epsilon,\mathfrak{p}}$. To obtain virtual fundamental chain and prove its basic properties we need to restict it to an appropriate neighborhood of the union of zero sets of $s_{\mathfrak{p}}$ and study the properties of $s_{\epsilon,\mathfrak{p}}$ there.

We already shrunk our good coordinate system several times. We shrink again below. Let us denote by $U_{\mathfrak{p}}$, $U(X)$ etc. the good coordinate system and a space obtained from it when we proved Proposition 3.1. (In other words $U_{\mathfrak{p}} = U^{(N)}_{\mathfrak{p}}$ in the notation we used during the proof of Proposition 3.1.) During the discussion in the rest of this section we restart numbering the shrinked good coordinate system and will write again $U^{(m)}_{\mathfrak{p}}$. The following is the key lemma. Note the closure of $\mathfrak{I}_{U^{(1)}(X)U^{(2)}(X)}(U^{(2)}(X))$ is compact metrizable space. We take and fix a metric on it. Then by (2.19) we can fix a metric on $U^{(m)}(X)$ for all m so that the maps $\mathfrak{I}_{U^{(m')}(X)U^{(m)}(X)}$ preserves metric for $m' > m \geq 2$.

For a metric space Z and $C \subset Z$ we put

$$B_\epsilon(C; Z) = \{z \in Z \mid d(z, C) < \epsilon\}. \tag{3.19}$$

Lemma 3.4. *We may choose $U^{(2)}_{\mathfrak{p}}$, $U^{(3)}_{\mathfrak{p}}$ and $\delta = 0 > 0$, so that*

$$\begin{aligned} &B_\delta(I(X), U^{(2)}(X))) \cap \bigcup_{\mathfrak{p}_i\in\mathfrak{P}} \Pi_{\mathfrak{p}_i}(\overline{s}^{-1}_{\epsilon,\mathfrak{p}_i}(0) \cap U^{(2)}(X)) \\ &= \mathfrak{I}_{U^{(2)}(X)U^{(3)}(X)} \left(B_\delta(I(X), U^{(3)}(X))) \cap \bigcup_{\mathfrak{p}_i\in\mathfrak{P}} \Pi_{\mathfrak{p}_i}(\overline{s}^{-1}_{\epsilon,\mathfrak{p}_i}(0) \cap U^{(3)}(X)) \right). \end{aligned} \tag{3.20}$$

Proof. For each $x \in X$ we take its neighborhood $\mathfrak{U}^{(2)}(x)$ as in (2.11). We choose a neighborhood O_x of x in $U^{(2)}(X)$ and δ_1 so that

$$O_x \cap \Pi_{\mathfrak{q}_1}(\Omega^{(2)}_{\mathfrak{q}_1}(x)) = O_x \cap \overline{\Pi_{\mathfrak{q}_1}(\Omega^{(2)}_{\mathfrak{q}_1}(x))}. \tag{3.21}$$

and

$$O_x \cap \mathfrak{U}^{(2)}(x) \supset B_{\delta_1}(I(x); U^{(2)}(X)).$$

(Note x is in the interior of $\Omega^{(2)}_{\mathfrak{q}_1}(x)$. So we can take O_x small so that (3.21) is satisfied.)

Sublemma 3.5. *We may take $U^{(3)}$ and $\delta_3 > 0$ so that*

$$\begin{aligned}&O_x \cap \mathfrak{U}^{(2)}(x) \cap B_{\delta_3}(I(x), U^{(2)}(X)) \cap \bigcup_{\mathfrak{p}_i \in \mathfrak{P}} \Pi_{\mathfrak{p}_i}(\overline{s}^{-1}_{\epsilon,\mathfrak{p}_i}(0)) \\ &\subset \mathfrak{I}_{U^{(2)}(X)U^{(3)}(X)}\left(B_{\delta_3}(I(X), U^{(3)}(X))) \cap \bigcup_{\mathfrak{p}_i \in \mathfrak{P}} \Pi_{\mathfrak{p}_i}(\overline{s}^{-1}_{\epsilon,\mathfrak{p}_i}(0) \cap U^{(3)}(X))\right).\end{aligned}$$

for each x holds for sufficiently small ϵ.

Proof. We have

$$U^{(2)}(X) \cap O_x = \bigcup_{i=1,\dots,m} \Pi_{\mathfrak{q}_i}(\Omega^{(2)}_{\mathfrak{q}_i}(x)) \cap O_x$$

By (3.21) we have

$$O_x \cap \Pi_{\mathfrak{q}_1}(\Omega^{(2)}_{\mathfrak{q}_1}(x)) = O_x \cap \overline{\Pi_{\mathfrak{q}_1}(\Omega^{(2)}_{\mathfrak{q}_1}(x))}.$$

Therefore we may choose $U^{(3)}_{\mathfrak{p}}$ close enough to $U^{(2)}_{\mathfrak{p}}$ so that

$$O_x \cap \Pi_{\mathfrak{q}_1}(\Omega^{(2)}_{\mathfrak{q}_1}(x)) \subset \mathfrak{I}_{U^{(2)}(X)U^{(3)}(X)}(\Pi_{\mathfrak{q}_1}(\Omega^{(3)}_{\mathfrak{q}_1}(x))).$$

On the other hand, in a sufficietly small tubular neighborhood $\mathcal{N}_{U_{\mathfrak{q}_1}} U_{\mathfrak{q}_i}$ $(i > 2)$ the zero set $s^{-1}_{\epsilon,q_i}(0)$ is contained in the subset $U_{q_1} \subset \mathcal{N}_{U_{\mathfrak{q}_1}} U_{\mathfrak{q}_i}$. This is a consequence of Proposition 3.1 (3).

We can choose $\delta_3 > 0$ sufficiently small so that $O_x \cap \mathfrak{U}^{(2)}(x) \cap B_{\delta_3}(x, U^{(2)}(X))$ is contained in this tubular neighborhood. (We may choose δ_3 independent of x since X is compact.) The sublemma follows. □

We find finitely many $x_i \in I(X)$, $i = 1, \dots, I$ and $\delta > 0$, such that

$$\bigcup_{i=1}^{I} O_{x_i} \cap \mathfrak{U}^{(2)}(x_i) \cap B_{\delta_3}(I(x_i), U^{(2)}(X)) \supset B_{\delta}(I(X); U^{(2)}(X)) \tag{3.22}$$

(3.22) implies that

$$\begin{aligned}&\bigcup_{i=1}^{I}\left(O_{x_i} \cap \mathfrak{U}^{(2)}(x_i) \cap B_{\delta_3}(x_i, U^{(2)}(X)) \cap \bigcup_{\mathfrak{p}_i \in \mathfrak{P}} \Pi_{\mathfrak{p}_i}(\overline{s}^{-1}_{\epsilon,\mathfrak{p}_i}(0))\right) \\ &\supset B_{\delta}(I(X), U^{(2)}(X))) \cap \bigcup_{\mathfrak{p}_i \in \mathfrak{P}} \Pi_{\mathfrak{p}_i}(\overline{s}^{-1}_{\epsilon,\mathfrak{p}_i}(0) \cap U^{(2)}(X)).\end{aligned}$$

Therefore Sublemma 3.5 implies that the left hand side of (3.20) is contained in the right hand side. The inclusion of the other direction is obvious. □

Lemma 3.6.

$$B_{\delta}(I(X), U^{(3)}(X))) \cap \bigcup_{\mathfrak{p}_i \in \mathfrak{P}} \Pi_{\mathfrak{p}_i}(\overline{s}^{-1}_{\epsilon,\mathfrak{p}_i}(0) \cap U^{(3)}(X)) \tag{3.23}$$

is compact if $\epsilon > 0$ is sufficiently small.

Proof. Lema 3.4 implies

$$\begin{aligned} & B_\delta(I(X), U^{(3)}(X))) \cap \bigcup_{\mathfrak{p}_i \in \mathfrak{P}} \Pi_{\mathfrak{p}_i}(\overline{s}_{\epsilon,\mathfrak{p}_i}^{-1}(0) \cap U^{(3)}(X)) \\ & = B_\delta(I(X), \overline{U}^{(3)}(X))) \cap \bigcup_{\mathfrak{p}_i \in \mathfrak{P}} \Pi_{\mathfrak{p}_i}(\overline{s}_{\epsilon,\mathfrak{p}_i}^{-1}(0) \cap \overline{U}^{(3)}(X)) \end{aligned} \tag{3.24}$$

(We take above closure in $U^{(2)}(X)$.) We remark $U^{(3)}(X)$ is a relatively compact subspace of $U^{(2)}(X)$.) The right hand side of (3.24) is clearly compact. □

Hereafter we fix δ and write (3.23) by $s_\epsilon^{-1}(0)_\delta$. It is a subspace of $U^{(3)}(X)$.

Lemma 3.7.

$$\lim_{\epsilon \to 0} \mathfrak{I}_{U^{(2)}(X)U^{(3)}(X)}(s_\epsilon^{-1}(0)_\delta) \subseteq B_\delta(I(X), U^{(2)}(X))) \cap \bigcup_{\mathfrak{p}_i \in \mathfrak{P}} \Pi_{\mathfrak{p}_i}(\overline{s}_{\mathfrak{p}_i}^{-1}(0) \cap U^{(2)}(X)).$$

Here the convergence is by Hausdorff distance.

Proof. This is a consequence of the next sublemma applied to the right hand side of (3.24) chartwise and branchwise. □

Sublemma 3.8. *Let $E \to Z$ be a vector bundle on a compact metric space Z and s its subsection. Suppose s_ϵ be a family of sections converges to s. Then*

$$\lim_{\epsilon \to 0} s_\epsilon^{-1}(0) \subseteq s^{-1}(0).$$

Proof. Let $\rho > 0$. We put

$$\epsilon(\rho) = \inf\{|s(x)| \mid x \in Z \setminus B_\rho(s^{-1}(0); Z)\}.$$

Clearly

$$s_\epsilon^{-1}(0) \subseteq B_\rho(s^{-1}(0); X)$$

if $\epsilon < \epsilon(\rho)$. The compactness of Z implies $\epsilon(\rho) > 0$. □

Lemma 3.9. $s_\epsilon^{-1}(0)_\delta$ *has a triangluation.*

This is proved in [FOn, Lemma 6.9].

Suppose we have a strongly continuous map $f = \{f_\mathfrak{p} \mid \mathfrak{p} \in \mathfrak{P}\}$ to a topological space Z and our Kuranishi space X is oriented. ([FOOO, A1.17].) Then we can put weight to each simplex of top dimension in $s_\epsilon^{-1}(0)_\delta$. and $f_*[X]$. ([FOn, (6.10)].) That is a singular chain of Z (with rational coefficient.) We consider the case our Kuranishi structure has no boundary.

Lemma 3.10. *We can put the weight on each of the top dimensional simplices so that $f_*([X])$ is a cycle.*

Proof. It suffices to prove that for each $x \in s_\epsilon^{-1}(0)_\delta$ the zero set of each brach of $s_{\epsilon,\mathfrak{p}}$ is a smooth manifold in a neighborhood of x. This is again a consequence of the proof of Lemma 3.4 as follows. Suppose $x \in O_{x_i} \cap \mathfrak{U}^{(2)}(x_i)$. Then we proved that $s_\epsilon^{-1}(0)_\delta$ in a neighborhood of x coincides with the zero set of $s_{\epsilon,\mathfrak{q}_1}$. On the other hand each brance of $s_{\epsilon,\mathfrak{q}_1}$ is transversal to 0 by Proposition 3.1. □

4. Existence of good coordinate system.

The purpose of this section is to prove the next theorem.

Theorem 4.1. *Let X be a compact metrizable space with Kuranishi structure in the sense of Definition 1.3. Then X has a good coordinate system in the sense of Definition 2.5. They are compatible in the following sense.*

Definition 4.2. Let X be a space with Kuranishi structure. A good coordinate system is said to be *compatible* with this Kuranishi structure if the following holds.

Let $(U'_{\mathfrak p}, E'_{\mathfrak p}, s'_{\mathfrak p}, \psi'_{\mathfrak p})$ be a chart of the given good coordinate system X. Then for each $q \in \psi'_{\mathfrak p}(\tilde q)$, $\tilde q \in U'_{\mathfrak p} \cap (s'_{\mathfrak p})^{-1}(0)$ there exists $(\underline{\hat\phi}_{\mathfrak p q}, \underline{\phi}_{\mathfrak p q})$ such that

(1) $\underline{\phi}_{\mathfrak p q} : U_{\mathfrak p q} \to U'_{\mathfrak p}$ is an embedding of orbifold. Here $V_{\mathfrak p q}$ is a Γ_q invariant open neighborhood of o_q and $U_{\mathfrak p q} = V_{\mathfrak p q}/\Gamma_q$.
(2) $\underline{\phi}_{\mathfrak p q}([o_q]) = q$.
(3) $\underline{\hat\phi}_{\mathfrak p q}$ is an embedding of vector bundles $E_q|_{U_{\mathfrak p q}} \to E'_{\mathfrak p}$.
(4) $\underline{\hat\phi}_{\mathfrak p q} \circ s_q = s'_{\mathfrak p} \circ \underline{\phi}_{\mathfrak p q}$.
(5) $\psi_q = \psi'_{\mathfrak p} \circ \underline{\phi}_{\mathfrak p q}$ on $s_q^{-1}(0) \cap U_{\mathfrak p q}$.
(6) $d_{\rm fiber} s_{\mathfrak p}$ induces an isomorphism of vector bundles at $s_q^{-1}(0) \cap U_{\mathfrak p q}$.

$$N_{U_{\mathfrak p q}} U_{\mathfrak p} \cong \frac{\underline{\hat\phi}^*_{\mathfrak p q} E_{\mathfrak p}}{(E_q)|_{U_{\mathfrak p q}}}$$

(7) If $r \in \psi_q(U_{\mathfrak p q} \cap s_q^{-1}(0))$, $q \in \psi'_{\mathfrak p}((s'_{\mathfrak p})^{-1}(0))$ then

$$\underline{\phi}_{\mathfrak p q} \circ \phi_{qr} = \underline{\phi}_{\mathfrak p r}, \quad \underline{\hat\phi}_{\mathfrak p q} \circ \underline{\hat\phi}_{qr} = \underline{\hat\phi}_{\mathfrak p r}.$$

Here the first equality holds on $\underline{\phi}_{qr}^{-1}(U_{\mathfrak p q}) \cap U_{qr} \cap U_{\mathfrak p r}$ and the second equality holds on $(E_r)|_{\underline{\phi}_{qr}^{-1}(U_{\mathfrak p q}) \cap U_{qr} \cap U_{\mathfrak p r}}$.

(8) Suppose $\mathfrak o \ge \mathfrak p$ and the coordinate change of good coordinate system is given by $(U'_{\mathfrak o \mathfrak p}, \underline{\hat\phi}'_{\mathfrak o \mathfrak p}, \underline{\phi}'_{\mathfrak o \mathfrak p})$. Let $q \in \psi'_{\mathfrak p}(s_{\mathfrak p}^{-1}(0) \cap U'_{\mathfrak o \mathfrak p})$. Then we have

$$\underline{\phi}'_{\mathfrak o \mathfrak p} \circ \underline{\phi}_{\mathfrak p q} = \underline{\phi}_{\mathfrak o q}, \quad \underline{\hat\phi}'_{\mathfrak o \mathfrak p} \circ \underline{\hat\phi}_{\mathfrak p q} = \underline{\hat\phi}_{\mathfrak o q}.$$

Here the first equality holds on $\underline{\phi}_{\mathfrak p q}^{-1}(U'_{\mathfrak o \mathfrak p}) \cap U_{\mathfrak p q} \cap U_{\mathfrak o q}$, and the second equality holds on $E_q|_{\underline{\phi}_{\mathfrak p q}^{-1}(U'_{\mathfrak o \mathfrak p}) \cap U_{\mathfrak p q} \cap U_{\mathfrak o q}}$.

Proof of Theorem 4.1. The proof is one given in [Fu, pages 5-7 and 11-12]. We explain it in more detail below.

Any point $p \in X$ has well defined $\dim U_p$. We put $d_p = \dim U_p$. We put

$$X(\mathfrak d) = \{p \mid d_p = \mathfrak d\}.$$

The first part of the proof is to construct an orbifold (plus obstruction bundle etc.) that is a 'neighborhood' of a compact subset of $X(\mathfrak d)$. Let us define such a notion precisely.

Definition 4.3. Let K_* be a compact subset of $X(\mathfrak d)$. A *pure orbifold neighborhood* of K_* is (U_*, E_*, s_*, ψ_*) such that the following holds.

(1) U_* is a $\mathfrak d$-dimensional orbifold.

(2) E_* is a vector bundle.[7] Its rank is $\mathfrak{d} - \dim X$. (Here $\dim X$ is a dimension of X as a Kuranishi space.)
(3) s_* is a section of E_*.
(4) $\psi_* : s_*^{-1}(0) \to X$ is a homeomorphism to an neighborhood $\mathcal{U}_*$ of K_* in $X(\mathfrak{d})$.

We also assume the following compatibility condition with Kuranishi structure of X. For any $p \in \psi_*(s_*^{-1}(0)) \subset X$ there exists $(U_{*p}, \underline{\hat\phi}_{*p}, \underline{\phi}_{*p})$ such that

(a) U_{*p} is an open neighborhood of $[o_p]$ in U_p.
(b) $\underline{\phi}_{*p} : U_{*p} \to U_*$ is an embedding of orbifold.
(c) $\underline{\hat\phi}_{*p} : E_p|_{U_{*p}} \to E_*$ is an embedding of vector bundle that cover $\underline{\phi}_{*p}$.
(d) $\underline{\hat\phi}_{*p} \circ s_p = s_* \circ \underline{\phi}_{*p}$, on U_{*p}
(e) $\psi_p = \psi_* \circ \underline{\phi}_{*p}$ on $s_p^{-1}(0) \cap U_{*p}$.
(f) The restriction of ds_* to the normal direction induces an isomorphism

$$N_{U_{*p}} U_* \cong \frac{\underline{\hat\phi}_{*p}^{*} E_*}{E_p|_{U_{*p}}} \tag{4.25}$$

as vector bundles on the orbifold U_{*p} at $s_p^{-1}(0)$..
(g) If $q \in \psi_p(s_p^{-1}(0) \cap U_{*p})$ then

$$\underline{\phi}_{*p} \circ \underline{\phi}_{pq} = \underline{\phi}_{pq}, \quad \underline{\hat\phi}_{*p} \circ \underline{\hat\phi}_{pq} = \underline{\hat\phi}_{*q}.$$

Here the first equality holds on $\underline{\hat\phi}_{pq}^{-1}(U_{*p}) \cap U_{pq} \cap U_{*q}$ and the second equality holds on $E_q|_{\underline{\hat\phi}_{pq}^{-1}(U_{*p}) \cap U_{pq} \cap V_{*q})}$.

Hereafter we put

$$\mathcal{U}_* = \psi(s_*^{-1}(0)). \tag{4.26}$$

The goal of the first part of the proof of Theorem 4.1 is to prove the following.

Proposition 4.4. *For any compact subset K of $X(\mathfrak{d})$ there exists its pure orbifold neighborhood.*

Proof. We cover K by $\mathcal{U}_{p_i}$, where $p_i \in K$ and

$$\psi_{p_i}(s_{p_i}^{-1}(0)) = \mathcal{U}_{p_i}.$$

There exist compact subsets K_i of $\mathcal{U}_{p_i}$ such that the union of K_i contains K. Thus to prove Propositiont 4.4 it suffices to prove the following lemma. □

Lemma 4.5. *Let K_1, K_2 be compact subsets of $X(\mathfrak{d})$. Suppose K_1 and K_2 have pure orbifold neighborhoods. Then $K_1 \cup K_2$ has pure orbifold neighborhoods.*

Proof. Let (U_i, E_i, s_i, ψ_i) be a pure orbifold neighborhood of K_i. We denote the map $\underline{\phi}_{*p}$ the open set U_{*p} etc. for (U_i, E_i, s_i, ψ_i) by $\underline{\phi}_{ip}$, U_{ip} etc. (Namely we replace $*$ by $i \in \{1, 2\}$.) The open subset $\mathcal{U}_i$ is as in (4.26).

Let $q \in K_1 \cap K_2$. We take an open subset $U_{12;q}$ such that

$$o_q \in U_{12;q} \subset U_{1q} \cap U_{2q} \subset U_q \tag{4.27}$$

and

$$\mathcal{U}_{12;q} = \mathcal{U}_{1q} \cap \mathcal{U}_{2q} \subset X. \tag{4.28}$$

[7] Here and hereafter vector bundle is in the sense of orbifold.

Here $\mathcal{U}_{iq} = \psi_q(\overline{s}_q^{-1}(0) \cap U_{iq})$. We take $q_1, \dots, q_I$ such that

$$K_1 \cap K_2 \subseteq \bigcup_{i=1}^{I} \mathcal{U}_{12;q_i}.$$

We take relatively compact open subset $U^-_{12;q}$ in $U_{12;q}$ such that

$$K_1 \cap K_2 \subseteq \bigcup_{i=1}^{I} \mathcal{U}^-_{12;q_i}.$$

By a standard arguement in general topology we can choose it so that the following holds.

Condition 4.6. If $\mathcal{U}_{12;q_i} \cap \mathcal{U}_{12;q_{i'}} \neq \emptyset$, then $\mathcal{U}_{12;q_i} \cap \mathcal{U}_{12;q_{i'}} \cap X(\mathfrak{o}) \neq \emptyset$.

We assume the same condition for $\{\mathcal{U}^-_{12;q_i}\}$.

For each $r \in K_1 \cap K_2$ we take an open subset U^0_r of U_r containing o_r such that

Condition 4.7. (1) $U^0_r \subset U_{1r}$ and $U^0_r \subset U_{2r}$.

(2) If $\underline{\phi}_{1r}(U^0_r) \cap \overline{\underline{\phi}_{1q_i}(U^-_{12;q_i})} \neq \emptyset$ then $U^0_r \subset U_{q_i r} \cap \underline{\phi}_{q_i r}^{-1}(U_{12;q_i})$.

(3) If $\underline{\phi}_{2r}(U^0_r) \cap \overline{\underline{\phi}_{2q_i}(U^-_{21;q_i})} \neq \emptyset$ then $U^0_r \subset U_{q_i r} \cap \underline{\phi}_{q_i r}^{-1}(U_{12;q_i})$.

We choose $r_1, \dots, r_J \in K_1 \cap K_2$ such that

$$\bigcup_{j=1}^{J} \mathcal{U}^0_{r_j} \supset K_1 \cap K_2. \tag{4.29}$$

We put

$$\begin{aligned} U^{(1)}_{21} &= \bigcup_{i=1}^{I} \bigcup_{j=1}^{J} \left(\underline{\phi}_{1r_j}(U^0_{r_j}) \cap \underline{\phi}_{1q_i}(U^-_{12;q_i}) \right) \subset U_1, \\ U^{(1)}_{12} &= \bigcup_{i=1}^{I} \bigcup_{j=1}^{J} \left(\underline{\phi}_{2r_j}(U^0_{r_j}) \cap \underline{\phi}_{2q_i}(U^-_{12;q_i}) \right) \subset U_2. \end{aligned} \tag{4.30}$$

They are open sets orbifolds and so are orbifolds. We remark

$$U^{(1)}_{21} \supset \psi_1^{-1}(K_1 \cap K_2), \qquad U^{(1)}_{12} \supset \psi_2^{-1}(K_1 \cap K_2). \tag{4.31}$$

Lemma 4.8. *There exists a diffeomorphism of orbifolds $\underline{\phi}_{21} : U^{(1)}_{21} \to U^{(1)}_{12}$ with the following properties.*

(1) *If $x = \underline{\phi}_{1r_j}(\tilde{x}_j)$ then*

$$\underline{\phi}_{21}(x) = \underline{\phi}_{2r_j}(\tilde{x}_j). \tag{4.32}$$

(2) *There exists a bundle isomorphism*

$$\hat{\underline{\phi}}_{21} : E_1|_{U^{(1)}_{21}} \to E_2|_{U^{(1)}_{12}}$$

over $\underline{\phi}_{21}$. On the fiber of $x = \underline{\phi}_{1r_j}(\tilde{x}_j)$ we have

$$\hat{\underline{\phi}}_{21} = \hat{\underline{\phi}}_{2r_j} \circ \hat{\underline{\phi}}_{1r_j}^{-1}. \tag{4.33}$$

(3) *On $U^{(1)}_{21}$ we have:*

$$s_2 \circ \underline{\phi}_{21} = \hat{\underline{\phi}}_{21} \circ s_1. \tag{4.34}$$

(4) *On* $s_1^{-1}(0) \cap U_{21}^{(1)}$, *we have:*

$$\psi_2 \circ \underline{\phi}_{21} = \psi_1. \tag{4.35}$$

Proof. (1) Note the right hand side of (4.32) is well-defined because of Condition 4.7 (1). So to define $\underline{\phi}_{21}$ it suffices to show that the right hand side of (4.32) is independent of j. Suppose

$$x = \underline{\phi}_{1r_j}(\tilde{x}_j) = \underline{\phi}_{1r_{j'}}(\tilde{x}_{j'}) \in \underline{\phi}_{1q_i}(U_{q_i}^-).$$

By Condition 4.7 (2) we have $\tilde{x}_j \in U_{q_ir_j}$, $\tilde{x}_{j'} \in U_{q_ir_{j'}}$ and $\underline{\phi}_{q_ir_j}(\tilde{x}_j) \in U_{12;q_i}$, $\underline{\phi}_{q_ir_{j'}}(\tilde{x}_{j'}) \in U_{12;q_i}$. Since

$$\underline{\phi}_{1q_i}(\underline{\phi}_{q_ir_j}(\tilde{x}_j)) = x = \underline{\phi}_{1q_i}(\underline{\phi}_{q_ir_{j'}}(\tilde{x}_{j'}))$$

it follows that

$$\underline{\phi}_{q_ir_j}(\tilde{x}_j) = \underline{\phi}_{q_ir_{j'}}(\tilde{x}_{j'}).$$

Therefore

$$\underline{\phi}_{2r_j}(\tilde{x}_j) = \underline{\phi}_{2q_i}(\underline{\phi}_{q_ir_j}(\tilde{x}_j)) = \underline{\phi}_{2q_i}(\underline{\phi}_{q_ir_{j'}}(\tilde{x}_{j'})) = \underline{\phi}_{2r_{j'}}(\tilde{x}_{j'}),$$

as required.

We thus defined $\underline{\phi}_{21}$. We can define $\underline{\phi}_{12}$ in a similar way. It is easy to see $\underline{\phi}_{21} \circ \underline{\phi}_{12}$ and $\underline{\phi}_{12} \circ \underline{\phi}_{21}$ are identity map. Therefore $\underline{\phi}_{21}$ is an isomorphism.

(2) We define $\hat{\underline{\phi}}_{21}$ by (4.33). We can prove that it is well-defined and is an isomorphism in the same way as the proof of (1).

(3) (4) (4.34) follows from (4.32) and (4.33). (4.35) follows from (4.32). □

We now use the maps $\underline{\phi}_{21}$ etc. to glue U_1 and U_2. In order to obtain a Hausdorff space after glueing we need to shrink them as follows. (The argument to do so is the same as Section 2 so we are slightly sketchy here.) Let $U_1^0 \subset U_1$ and $U_2^0 \subset U_2$ be relatively compact open subsets such that

$$\psi_1(s_1^{-1}(0) \cap U_1^0) \supset K_1, \qquad \psi_2(s_2^{-1}(0) \cap U_2^0) \supset K_2.$$

Let $W_{21} \subset U_{21}^{(1)}$ be a relatively compact open subset such that

$$W_{21} \supset s_1^{-1}(0) \cap \overline{(U_1^0 \cap \underline{\phi}_{21}^{-1}(U_2^0))}. \tag{4.36}$$

We put

$$U_{21} = W_{21} \cap U_1^0 \cap \underline{\phi}_{21}^{-1}(U_2^0), \quad U_{12} = \underline{\phi}_{21}(U_{21}).$$

We put

$$U^+ = (U_1^0 \cup U_2^0)/\sim$$

where $\sim$ is defined as follows $x \sim y$ if and only if one of the following holds.

(1) $x = y$.
(2) $x \in \overline{U}_{21} \cap U_1^0$. $y = \underline{\phi}_{21}(x) \in U_2^0$.
(3) $x \in \overline{U}_{12} \cap U_2^0$. $y = \underline{\phi}_{12}(x) \in U_1^0$.

Lemma 4.9. U^+ *is Hausdorff.*

This is because we take closure $\overline{U}_{21}$ in the definition of $\sim$ (2)(3).

Let

$$A_{21} = (\overline{W}_{21} \setminus W_{21}) \cap U_1^0 \cap \underline{\phi}_{21}^{-1}(U_2^0).$$

and A the closure of the image of A_{21} in U.

Lemma 4.10. *A does not intersect with the image of the sets*

$$\{x \in U_1^0 \mid s_1(x) = 0\} \cup \{x \in U_2^0 \mid s_2(x) = 0\} \tag{4.37}$$

in U^+.

Proof. We denote by $s^{-1}(0)$ the image of (4.37). Let $x_i \in A_{21}$. Suppose $\lim_{i\to\infty}[x_i] = [x]$. We will prove that $[x] \notin s^{-1}(0)$.

Since U_1^0 is relatively compact in U_1, we may assume that x_i converges in U_1. Let y be the limit.

Since $x_i \notin W_{21}$ and W_{21} is open it follows that $y \notin W_{21}$. On the other hand $y \in \overline{(U_1^0 \cap \underline{\phi}_{21}^{-1}(U_2^0))}$. Therefore by (4.36) $s_1(y) \neq 0$.

If $x \in U_1^0$ then $x = y$ and we are done.

Suppose $x \in U_2^0$. Then $x = \lim_{i\to\infty} \underline{\phi}_{21}(x_i)$. Since $\underline{\phi}_{21}$ is continuous on U_1 we have $x = \underline{\phi}_{21}(y)$. It implies $s_2(x) \neq 0$. The proof of Lemma 4.10 is complete. □

Let $\Pi_i : U_i^0 \to U^+$ be the obvious map. We put

$$U = U^+ \setminus \{\Pi_1(x) \mid x \in \Pi_1^{-1}(U) \setminus U_{21}, s_1(x) = 0, \psi_1(x) \in \psi_2(s_2^{-1}(0))\} \setminus A. \tag{4.38}$$

Lemma 4.10 implies that U is an open subset of U^+. Moreover U^+ has an orbifold chart outside A. Therefore U is an orbifold.

Let $s_U^{-1}(0)$ be the intersection of the image of (4.37) and U. There exists a map $\psi : s_U^{-1}(0) \to X$ induced by ψ_1 and ψ_2. Its image contains $K_1 \cup K_2$. It is injective by (4.38).

We use Lemma 4.8 (2)(3)(4) to obtain other data on U. It is straightforward to check that they satisfy the conditions of Definition 4.3. The proof of Proposition 4.4 and Lemma 4.5 are complete. □

We thus completed the first part of the proof of Theorem 4.1 and enter the second part.

Let

$$\mathfrak{D} = \{\mathfrak{d} \in \mathbb{Z}_{>0} \mid X(\mathfrak{d}) \neq \emptyset\}.$$

$(\mathfrak{D}, \leq)$ is an ordered set. A subset $D \subset \mathfrak{D}$ is said to be an *ideal* if

$$\mathfrak{d} \in D, \mathfrak{d}' \geq \mathfrak{d} \quad \Rightarrow \quad \mathfrak{d}' \in D.$$

For $D \subset \mathfrak{D}$ we put

$$X(D) = \bigcup_{\mathfrak{d}\in D} X(\mathfrak{d}).$$

$X(D)$ is a closed subset of X if D is an ideal.

Definition 4.11. Let $D \subset \mathfrak{D}$ is an ideal. A *mixed orbifold neighborhood* of $X(D)$ is given by $U_{\mathfrak{d}}$, $\mathcal{K}_{\mathfrak{d}}$ for $\mathfrak{d} \in D$ and $\underline{\phi}_{\mathfrak{d}'\mathfrak{d}}$, $\underline{\hat{\phi}}_{\mathfrak{d}'\mathfrak{d}}$ for $\mathfrak{d}, \mathfrak{d}' \in D$ with $\mathfrak{d} < \mathfrak{d}'$. Moreover we have $U(D)$ and $\mathfrak{I}_{\mathfrak{d}}$. We assume they have the following properties.

(1) $\mathcal{K}_{\mathfrak{d}}$ is a compact subset of $X(\mathfrak{d})$.

(2) $U_{\mathfrak{d}}$ is a pure orbifold neighborhood of $\mathcal{K}_{\mathfrak{d}}$. (We write $\underline{\phi}_{\mathfrak{d}p}$ etc. instead of $\underline{\phi}_{*p}$ etc. for the structure maps of $U_{\mathfrak{d}}$. Namely we replace $*$ by $\mathfrak{d}$.)

(3) Let $\psi_{\mathfrak{d}} : U_{\mathfrak{d}} \cap s_{\mathfrak{d}}^{-1}(0) \to X$ be as in Definition 4.3 (4). Then we have

$$\mathcal{K}_{\mathfrak{d}} \supset X(\mathfrak{d}) \setminus \bigcup_{\mathfrak{d}'>\mathfrak{d}} \psi_{\mathfrak{d}'}\left(U_{\mathfrak{d}'} \cap s_{\mathfrak{d}'}^{-1}(0)\right). \tag{4.39}$$

(4) $U_{\mathfrak{d}'\mathfrak{d}}$ is an open neighborhood of
$$\psi_{\mathfrak{d}}^{-1}\left(\mathcal{K}_{\mathfrak{d}} \cap \psi_{\mathfrak{d}'}\left(U_{\mathfrak{d}'} \cap s_{\mathfrak{d}'}^{-1}(0)\right)\right)$$
in $U(\mathfrak{d})$ and
$$\underline{\phi}_{\mathfrak{d}'\mathfrak{d}} : U_{\mathfrak{d}'\mathfrak{d}} \to U_{\mathfrak{d}'}$$
is an embedding of orbifold.

(5)
$$\hat{\underline{\phi}}_{\mathfrak{d}'\mathfrak{d}} : E_{\mathfrak{d}}|_{U_{\mathfrak{d}'\mathfrak{d}}} \to E_{\mathfrak{d}'}$$
is an embedding of vector bundle that covers $\underline{\phi}_{\mathfrak{d}'\mathfrak{d}}$, Here $E_{\mathfrak{d}} \to U_{\mathfrak{d}}$ is a vector bundle that is a part of the structure of pure orbifold neighborhood $U_{\mathfrak{d}}$.

(6)
$$s_{\mathfrak{d}'} \circ \underline{\phi}_{\mathfrak{d}'\mathfrak{d}} = \hat{\underline{\phi}}_{\mathfrak{d}'\mathfrak{d}} \circ s_{\mathfrak{d}}$$
holds on $U_{\mathfrak{d}'\mathfrak{d}}$.

(7)
$$\psi_{\mathfrak{d}'} \circ \underline{\phi}_{\mathfrak{d}'\mathfrak{d}} = \psi_{\mathfrak{d}}$$
holds on $U_{\mathfrak{d}'\mathfrak{d}} \cap s_{\mathfrak{d}}^{-1}(0)$.

(8) The restriction of $ds_{\mathfrak{d}'}$ to the normal direction induces an isomorphism
$$N_{U_{\mathfrak{d}'\mathfrak{d}}} U_{\mathfrak{d}'} \cong \frac{E_{\mathfrak{d}'}}{\hat{\phi}_{\mathfrak{d}'\mathfrak{d}}(E_{\mathfrak{d}}|_{U_{\mathfrak{d}'\mathfrak{d}}})} \tag{4.40}$$
as vector bundles on $U_{\mathfrak{d}'\mathfrak{d}} \cap s_{\mathfrak{d}}^{-1}(0)$.

(9) If $p \in \psi_{\mathfrak{d}}(U_{\mathfrak{d}'\mathfrak{d}} \cap s_{\mathfrak{d}}^{-1}(0)) \subset \psi_{\mathfrak{d}'}(U_{\mathfrak{d}'} \cap s_{\mathfrak{d}'}^{-1}(0))$ then we have
$$\underline{\phi}_{\mathfrak{d}'\mathfrak{d}} \circ \underline{\phi}_{\mathfrak{d}p} = \underline{\phi}_{\mathfrak{d}'p}$$
on $U_{\mathfrak{d}'p} \cap \underline{\phi}_{\mathfrak{d}p}^{-1}(U_{\mathfrak{d}'\mathfrak{d}})$. Moreover we have
$$\hat{\underline{\phi}}_{\mathfrak{d}'\mathfrak{d}} \circ \hat{\underline{\phi}}_{\mathfrak{d}p} = \hat{\underline{\phi}}_{\mathfrak{d}'p}$$
on $E_p|_{U_{\mathfrak{d}'p} \cap \underline{\phi}_{\mathfrak{d}p}^{-1}(U_{\mathfrak{d}'\mathfrak{d}})}$.

(10) The space $U(D)$ is Hausdorff and metrizable. $\Pi_{\mathfrak{d}} : U_{\mathfrak{d}} \to U(D)$ is a homeomorphism onto its image. We have
$$U_{\mathfrak{d}'\mathfrak{d}} = \Pi_{\mathfrak{d}}^{-1}\Pi_{\mathfrak{d}'}(U_{\mathfrak{d}'}) \tag{4.41}$$
and
$$\Pi_{\mathfrak{d}'} \circ \underline{\phi}_{\mathfrak{d}'\mathfrak{d}} = \Pi_{\mathfrak{d}}. \tag{4.42}$$
on $U_{\mathfrak{d}'\mathfrak{d}}$. Moreover
$$U(D) = \bigcup_{\mathfrak{d} \in D} I_{\mathfrak{d}}(U_{\mathfrak{d}}).$$
We call $U(D)$ the *total space* of our mixed orbifold neighborhood.

(11) We define a subset $s_D^{-1}(0)$ of $U(D)$ by $s_D^{-1}(0) = \bigcup_{\mathfrak{d} \in D} \Pi_{\mathfrak{d}}(s_{\mathfrak{d}}^{-1}(0))$ We define $\psi_D : s_D^{-1}(0) \to X$ such that $\psi_D \circ \Pi_{\mathfrak{d}} = \psi_{\mathfrak{d}}$ on $s_{\mathfrak{d}}^{-1}(0) \subset U_{\mathfrak{d}}$. (This is well-defined by (7).) We require that
$$\psi_D : s_D^{-1}(0) \to X$$
is a homeomorphism onto a neighborhood of $X(D)$ in X.

Note (4.39) implies

$$X(D) \subset \bigcup_{\mathfrak{d}\in D} \psi_{\mathfrak{d}}\left(s_{\mathfrak{d}}^{-1}(0)\right). \tag{4.43}$$

We also have the following:

Lemma 4.12. *If $U(D)'$ is an open subset of $U(D)$ such that*

$$U(D)' \supset \psi_D^{-1}(X(D)) \cap \bigcup_{\mathfrak{d}\in D} \Pi_{\mathfrak{d}}(s_{\mathfrak{d}}^{-1}(0)).$$

Then there exists mixed orbifold neighborhood of $X(D)$ such that its total space is the above $U(D)'$.

Proof. We put $U'_{\mathfrak{d}} = U_{\mathfrak{d}} \cap \Pi_{\mathfrak{d}}^{-1}(U(D)')$, $U'_{\mathfrak{d}'\mathfrak{d}} = \Pi_{\mathfrak{d}}^{-1}(\Pi_{\mathfrak{d}'}(U_{\mathfrak{d}'}) \cap U(D)')$. We define $E'_{\mathfrak{d}}$ and various maps by restricting ones of $U(D)$. It is straightforward to check that they satisfies the required properties (1)-(11) of Definition 4.11. □

The goal of the second part of the proof of Theorem 4.1 is to prove the following:

Proposition 4.13. *For any ideal D there exists a mixed orbifold neighborhood of $X(D)$.*

Proof. The proof is by an induction on $\#D$. If $\#D = 1$ then $D = \{\mathfrak{d}\}$ with $\mathfrak{d}$ maximal in $\mathfrak{D}$. We put $\mathcal{K}_{\mathfrak{d}} = X(\mathfrak{d})$ that is compact. We use Proposition 4.4 and obtain $U_{\mathfrak{d}}$ a pure orbifold neighborhood of $\mathcal{K}_{\mathfrak{d}} = X(\mathfrak{d})$. The proposition is proved in this case.

Suppose we proved the proposition for all D' with $\#D' < \#D$. We will prove it for D. Let $\mathfrak{d}_0$ be an element of D that is minimal. We put $D' = D \setminus \{\mathfrak{d}_0\}$. D' is an ideal. So we have a mixed orbifold neighborhood of $X(D')$. We denote it by $U_{\mathfrak{d}}^{(1)}$, $\mathcal{K}_{\mathfrak{d}}^{(1)}$, $\underline{\phi}_{\mathfrak{d}'\mathfrak{d}}$ etc. (Here $\mathfrak{d}, \mathfrak{d}' \in D'$.)

Let $\mathcal{K}_{\mathfrak{d}_0}^{(1)}$ be a compact subset of $X(\mathfrak{d}_0)$ such that

$$\bigcup_{\mathfrak{d}\in D'} \psi_{\mathfrak{d}}^{(1)}(U_{\mathfrak{d}}^{(1)} \cap (s_{\mathfrak{d}}^{(1)})^{-1}(0)) \supset \overline{(X(\mathfrak{d}_0) \setminus \mathcal{K}_{\mathfrak{d}_0}^{(1)})}. \tag{4.44}$$

We apply Proposition 4.4 to $\mathcal{K}_{\mathfrak{d}_0}^{(1)}$ to obtain $U_{\mathfrak{d}_0}^{(1)}$. The main part of the proof is to glue $U_{\mathfrak{d}_0}^{(1)}$ with $X(D')$ to obtain the required mixed orbifold neighborhood of $X(D)$. The construction is similar to the proof of Lemma 4.8. The detail follows.

Let $U^{(2)}(D')$ be a relatively compact open subset of $U^{(1)}(D')$ satisfying

$$U^{(2)}(D') \supset \psi_{D'}^{-1}(X(D'))$$

We may choose it sufficiently close to $U^{(1)}(D')$ such that

$$\bigcup_{\mathfrak{d}\in D'} \psi_{\mathfrak{d}}^{(1)}(U_{\mathfrak{d}}^{(2)} \cap (s_{\mathfrak{d}}^{(1)})^{-1}(0)) \supset \overline{(X(\mathfrak{d}_0) \setminus \mathcal{K}_{\mathfrak{d}_0}^{(1)})}. \tag{4.45}$$

Here $U_{\mathfrak{d}}^{(2)}$ is obtained from $U^{(2)}(D')$ as in the proof of Lemma 4.12.

Let $U_{\mathfrak{d}_0}^{(2)} \subset U_{\mathfrak{d}_0}^{(1)}$ be a relatively compact open subset containing $(\psi_{\mathfrak{d}_0}^{(1)})^{-1}(\mathcal{K}_{\mathfrak{d}_0}^{(1)})$. We may choose $U_{\mathfrak{d}_0}^{(2)}$ and $U^{(2)}(D')$ such that

$$\psi_{\mathfrak{d}_0}^{(1)}((s_{\mathfrak{d}_0}^{(1)})^{-1}(0) \cap U_{\mathfrak{d}_0}^{(2)}) \cup \psi_{D'}^{(1)}((s_{D'}^{(1)})^{-1}(0) \cap U^{(2)}(D')) \supset X(\mathfrak{d}_0). \tag{4.46}$$

We put

$$\begin{aligned}\mathcal{L}_{\mathfrak{d}_0} = X(\mathfrak{d}_0) &\cap \psi^{(1)}_{\mathfrak{d}_0}((s^{(1)}_{\mathfrak{d}_0})^{-1}(0) \cap U^{(2)}_{\mathfrak{d}_0}) \\ &\cap \psi^{(1)}_{D'}((s^{(1)}_{D'})^{-1}(0) \cap U^{(2)}(D')).\end{aligned} \tag{4.47}$$

For each $q \in \mathcal{L}_{\mathfrak{d}_0}$ we take an open neighborhood U^0_q in U_q that satisfies the following conditions.

Condition 4.14. (1) If $\mathfrak{d} > \mathfrak{d}_0$, $q \in \overline{\psi^{(1)}_{\mathfrak{d}}((s^{(1)}_{\mathfrak{d}})^{-1}(0) \cap U^{(2)}_{\mathfrak{d}})} \cap \mathcal{L}_{\mathfrak{d}_0}$, then $U^0_q \subset U^{(1)}_{\mathfrak{d}q}$. Here $U^{(1)}_{\mathfrak{d}q}$ is as in Definition 4.3 (a) for the pure orbifold neighborhood $U^{(1)}_{\mathfrak{d}}$.

(2) $U^0_q \subset U^{(1)}_{\mathfrak{d}_0 q}$, where $U^{(1)}_{\mathfrak{d}_0 q}$ is as in Definition 4.3 (a) for the pure orbifold neighborhood $U^{(1)}_{\mathfrak{d}_0}$.

We take finitely many $q_i \in \mathcal{L}_{\mathfrak{d}_0}$, $i = 1, \dots, I$ such that the following condition holds.

Condition 4.15. For any $\mathfrak{d} \in D'$ we have:

$$\bigcup_{\{i | q_i \in \overline{\psi^{(1)}_{\mathfrak{d}}((s^{(1)}_{\mathfrak{d}})^{-1}(0) \cap U^{(2)}_{\mathfrak{d}})} \cap \mathcal{L}_{\mathfrak{d}_0}\}} \mathcal{U}^0_{q_i} \supset \overline{\psi^{(1)}_{\mathfrak{d}}((s^{(1)}_{\mathfrak{d}})^{-1}(0) \cap U^{(2)}_{\mathfrak{d}})} \cap \mathcal{L}_{\mathfrak{d}_0}.$$

Here $\mathcal{U}^0_{q_i} = \psi^{(1)}_{q_i}((s^{(1)}_{q_i})^{-1}(0) \cap U^0_{q_i})$.

Since

$$\mathcal{L}_{\mathfrak{d}_0} \subset \bigcup_{\mathfrak{d} \in D'} \overline{\psi^{(1)}_{\mathfrak{d}}((s^{(1)}_{\mathfrak{d}})^{-1}(0) \cap U^{(2)}_{\mathfrak{d}})}$$

Condition 4.15 implies

$$\bigcup_{i=1}^{I} \mathcal{U}^0_{q_i} \supset \mathcal{L}_{\mathfrak{d}_0}. \tag{4.48}$$

We may assume that $\{\mathcal{U}^0_{q_i}\}$ satisfies Condition 4.6.

We next take a relatively compact open subset $U^{0-}_{q_i}$ of $U^0_{q_i}$ such that the following holds.

Condition 4.16. For any $\mathfrak{d} \in D'$, we have:

$$\bigcup_{\{i | q_i \in \overline{\psi^{(1)}_{\mathfrak{d}}((s^{(1)}_{\mathfrak{d}})^{-1}(0) \cap U^{(2)}_{\mathfrak{d}})} \cap \mathcal{L}_{\mathfrak{d}_0}\}} \mathcal{U}^{0-}_{q_i} \supset \overline{\psi^{(1)}_{\mathfrak{d}}((s^{(1)}_{\mathfrak{d}})^{-1}(0) \cap U^{(2)}_{\mathfrak{d}})} \cap \mathcal{L}_{\mathfrak{d}_0}.$$

Here $\mathcal{U}^{0-}_{q_i} = \psi^{(1)}_{q_i}((s^{(1)}_{q_i})^{-1}(0) \cap U^{0-}_{q_i})$.

We also assume that $\{\mathcal{U}^{0-}_{q_i}\}$ satisfies Condition 4.6.

For $r \in \mathcal{L}_{\mathfrak{d}_0}$ we take an open neighborhood U^0_r of o_r in U_r with the following properties.

Condition 4.17. (1) If $\mathfrak{d} > \mathfrak{d}_0$ and $r \in \overline{\psi^{(1)}_{\mathfrak{d}}((s^{(1)}_{\mathfrak{d}})^{-1}(0) \cap U^{(2)}_{\mathfrak{d}})} \cap \mathcal{L}_{\mathfrak{d}_0}$ then $U^0_r \subset U^{(1)}_{\mathfrak{d}r}$.

(2) $U^0_r \subset U^{(1)}_{\mathfrak{d}_0 r}$.

(3) If $\underline{\phi}^{(1)}_{\mathfrak{d}_0 r}(U^0_r) \cap \overline{\underline{\phi}^{(1)}_{\mathfrak{d}_0 q_i}(U^{0-}_{q_i})} \neq \emptyset$ then $U^0_r \subset U^{(1)}_{q_i r} \cap (\underline{\phi}^{(1)}_{q_i r})^{-1}(U^0_{q_i})$.

(4) If $\underline{\phi}^{(1)}_{\mathfrak{d}r}(U^0_r) \cap \overline{\underline{\phi}^{(1)}_{\mathfrak{d}q_i}(U^{0-}_{q_i})} \neq \emptyset$ then $U^0_r \subset U^{(1)}_{q_i r} \cap (\underline{\phi}^{(1)}_{q_i r})^{-1}(U^0_{q_i})$.

We choose $r_1, \dots, r_J$ such that the following holds for each $\mathfrak{d} \in D'$.

$$\bigcup_{\{j | r_j \in \overline{\psi_{\mathfrak{d}}^{(1)}((s_{\mathfrak{d}}^{(1)})^{-1}(0) \cap U_{\mathfrak{d}}^{(2)})} \cap \mathcal{L}_{\mathfrak{d}_0}\}} \mathcal{U}_{r_j}^0 \supset \overline{\psi_{\mathfrak{d}}^{(1)}((s_{\mathfrak{d}}^{(1)})^{-1}(0) \cap U_{\mathfrak{d}}^{(2)})} \cap \mathcal{L}_{\mathfrak{d}_0}. \tag{4.49}$$

We now put

$$\begin{aligned} U_{\mathfrak{d}\mathfrak{d}_0}^{(1)} = & \bigcup_{\{j | r_j \in \overline{\psi_{\mathfrak{d}}^{(1)}((s_{\mathfrak{d}}^{(1)})^{-1}(0) \cap U_{\mathfrak{d}}^{(2)})} \cap \mathcal{L}_{\mathfrak{d}_0}\}} \\ & \bigcup_{\{i | q_i \in \overline{\psi_{\mathfrak{d}}^{(1)}((s_{\mathfrak{d}}^{(1)})^{-1}(0) \cap U_{\mathfrak{d}}^{(2)})\}} \cap \mathcal{L}_{\mathfrak{d}_0}\}} \underline{\phi}_{\mathfrak{d}_0 r_j}^{(1)}(U_{r_j}^0) \cap \underline{\phi}_{\mathfrak{d}_0 q_i}^{(1)}(U_{q_i}^{0-}). \end{aligned} \tag{4.50}$$

This is an open subset of $U_{\mathfrak{d}_0}^{(1)}$. Since $U_{\mathfrak{d}_0}^{(1)}$ is an orbifold its open subset $U_{\mathfrak{d}\mathfrak{d}_0}^{(1)}$ is also an orbifold. We remark that

$$\psi_{\mathfrak{d}_0}^{(1)}(U_{\mathfrak{d}\mathfrak{d}_0}^{(1)} \cap (s_{\mathfrak{d}_0}^{(1)})^{-1}(0)) \supset \mathcal{L}(\mathfrak{d}_0) \cap \psi_{\mathfrak{d}}^{(1)}((s_{\mathfrak{d}}^{(1)})^{-1}(0) \cap U_{\mathfrak{d}}^{(2)}). \tag{4.51}$$

Lemma 4.18. *There exists an embedding of orbifolds $\underline{\phi}_{\mathfrak{d}\mathfrak{d}_0}^{(1)} : U_{\mathfrak{d}\mathfrak{d}_0}^{(1)} \to U_{\mathfrak{d}}^{(1)}$ with the following properties.*

(1) *If $x = \underline{\phi}_{\mathfrak{d}_0 r_j}^{(1)}(\tilde{x}_j)$ then*

$$\underline{\phi}_{\mathfrak{d}\mathfrak{d}_0}^{(1)}(x) = \underline{\phi}_{\mathfrak{d} r_j}^{(1)}(\tilde{x}_j). \tag{4.52}$$

(2) *There exists a embedding of vector bundles*

$$\hat{\underline{\phi}}_{\mathfrak{d}\mathfrak{d}_0}^{(1)} : E_{\mathfrak{d}_0}^{(1)}|_{U_{\mathfrak{d}\mathfrak{d}_0}^{(1)}} \to E_{\mathfrak{d}}^{(1)}$$

that covers $\underline{\phi}_{\mathfrak{d}\mathfrak{d}_0}^{(1)}$.

(3) *If $\mathfrak{d} > \mathfrak{d}' > \mathfrak{d}_0$ then*

$$\underline{\phi}_{\mathfrak{d}\mathfrak{d}_0}^{(1)} = \underline{\phi}_{\mathfrak{d}\mathfrak{d}'}^{(1)} \circ \underline{\phi}_{\mathfrak{d}'\mathfrak{d}_0}^{(1)}$$

on $(\underline{\phi}_{\mathfrak{d}'\mathfrak{d}_0}^{(1)})^{-1}(U_{\mathfrak{d}\mathfrak{d}'}^{(1)}) \cap U_{\mathfrak{d}\mathfrak{d}_0}^{(1)}$ and

$$\hat{\underline{\phi}}_{\mathfrak{d}\mathfrak{d}_0}^{(1)} = \hat{\underline{\phi}}_{\mathfrak{d}\mathfrak{d}'}^{(1)} \circ \hat{\underline{\phi}}_{\mathfrak{d}'\mathfrak{d}_0}^{(1)}$$

on $E_{\mathfrak{d}_0}^{(1)}|_{(\underline{\phi}_{\mathfrak{d}'\mathfrak{d}_0}^{(1)})^{-1}(U_{\mathfrak{d}\mathfrak{d}'}^{(1)}) \cap U_{\mathfrak{d}\mathfrak{d}_0}^{(1)}}$.

(4) *We have*

$$s_{\mathfrak{d}}^{(1)} \circ \underline{\phi}_{\mathfrak{d}\mathfrak{d}_0}^{(1)} = \hat{\underline{\phi}}_{\mathfrak{d}\mathfrak{d}_0}^{(1)} \circ s_{\mathfrak{d}_0}^{(1)}$$

on $U_{\mathfrak{d}\mathfrak{d}_0}^{(1)}$.

(5) *We have*

$$\psi_{\mathfrak{d}}^{(1)} \circ \underline{\phi}_{\mathfrak{d}\mathfrak{d}_0}^{(1)} = \psi_{\mathfrak{d}_0}^{(1)}$$

on $U_{\mathfrak{d}\mathfrak{d}_0}^{(1)} \cap (\overline{s}_{\mathfrak{d}_0}^{(1)})^{-1}(0)$.

(6) *The restriction of $ds_{\mathfrak{d}}^{(1)}$ to the normal direction induces an isomorphism*

$$N_{U_{\mathfrak{d}\mathfrak{d}_0}} U_{\mathfrak{d}} \cong \frac{E_{\mathfrak{d}}}{\hat{\phi}_{\mathfrak{d}\mathfrak{d}_0}(E_{\mathfrak{d}_0}|_{U_{\mathfrak{d}\mathfrak{d}_0}})} \tag{4.53}$$

as vector bundles on $U_{\mathfrak{d}\mathfrak{d}_0} \cap s_{\mathfrak{d}_0}^{-1}(0)$.

Proof. The proof is similar to the proof of Lemma 4.8.

Note the right hand side of (4.52) is well defined because of Condition 4.17 (1). We first show that the right hand side of (4.52) is independent of j. Suppose

$$x = \underline{\phi}^{(1)}_{\mathfrak{d}_0 r_j}(\tilde x_j) = \underline{\phi}^{(1)}_{\mathfrak{d}_0 r_{j'}}(\tilde x_{j'}) \in \underline{\phi}^{(1)}_{\mathfrak{d}_0 q_i}(U^{0-}_{q_i}).$$

Then by Condition 4.17 (3) we have $\tilde x_j \in U^{(1)}_{q_i r_j}$, $\tilde x_{j'} \in U^{(1)}_{q_i r_{j'}}$ and $\underline{\phi}^{(1)}_{q_i r_j}(\tilde x_j) \in U^0_{q_i}$, $\underline{\phi}^{(1)}_{q_i r_{j'}}(\tilde x_{j'}) \in U^0_{q_i}$. Since

$$\underline{\phi}^{(1)}_{\mathfrak{d}_0 q_i}(\underline{\phi}^{(1)}_{q_i r_j}(\tilde x_j)) = x = \underline{\phi}^{(1)}_{\mathfrak{d}_0 q_i}(\underline{\phi}^{(1)}_{q_i r_{j'}}(\tilde x_{j'}))$$

it follows that

$$\underline{\phi}^{(1)}_{q_i r_j}(\tilde x_j) = \underline{\phi}^{(1)}_{q_i r_{j'}}(\tilde x_{j'}).$$

Therefore

$$\underline{\phi}^{(1)}_{\mathfrak{d} r_j}(\tilde x_j) = \underline{\phi}^{(1)}_{\mathfrak{d} q_i}(\underline{\phi}^{(1)}_{q_i r_j}(\tilde x_j)) = \underline{\phi}^{(1)}_{\mathfrak{d} q_i}(\underline{\phi}^{(1)}_{\mathfrak{d} r_{j'}}(\tilde x_{j'})) = \underline{\phi}^{(1)}_{\mathfrak{d} r_{j'}}(\tilde x_{j'})$$

as required. We remark that $\underline{\phi}^{(1)}_{\mathfrak{d}_0 r_j}$ is an open embedding of orbifolds. Therefore $\underline{\phi}^{(1)}_{\mathfrak{d}\mathfrak{d}_0}$ defined by (4.52) is an embedding of orbifolds.

The proof of (2) is similar. Then the proofs of (3)-(6) are straightforward. □

We put

$$U^{(2)}_{\mathfrak{d}\mathfrak{d}_0} = (\underline{\phi}^{(1)}_{\mathfrak{d}\mathfrak{d}_0})^{-1}(U^{(2)}_{\mathfrak{d}}) \cap U^{(2)}_{\mathfrak{d}_0}. \tag{4.54}$$

Let $\underline{\phi}^{(2)}_{\mathfrak{d}\mathfrak{d}_0}$ and $\underline{\hat\phi}^{(2)}_{\mathfrak{d}\mathfrak{d}_0}$ be the restrictions of $\underline{\phi}^{(1)}_{\mathfrak{d}\mathfrak{d}_0}$ and $\underline{\hat\phi}^{(1)}_{\mathfrak{d}\mathfrak{d}_0}$ to $U^{(2)}_{\mathfrak{d}\mathfrak{d}_0}$ and $E_{\mathfrak{d}_0}|_{U^{(2)}_{\mathfrak{d}\mathfrak{d}_0}}$.

We define

$$U^{(2)}(D) = \bigcup_{\mathfrak{d}\in D} U^{(2)}_{\mathfrak{d}} / \sim . \tag{4.55}$$

Here $\sim$ is defined as follows.

$x \sim y$ if and only if one of the following holds.

(1) $x = y$.
(2) $x \in U^{(2)}_{\mathfrak{d}'}$, $y \in U^{(2)}_{\mathfrak{d}'\mathfrak{d}} \subset U^{(2)}_{\mathfrak{d}}$, $x = \underline{\phi}^{(2)}_{\mathfrak{d}'\mathfrak{d}}(y)$.
(3) $y \in U^{(2)}_{\mathfrak{d}'}$, $x \in U^{(2)}_{\mathfrak{d}'\mathfrak{d}} \subset U^{(2)}_{\mathfrak{d}}$, $y = \underline{\phi}^{(2)}_{\mathfrak{d}'\mathfrak{d}}(x)$.

We define $\Pi_{\mathfrak{d}} : U^{(2)}_{\mathfrak{d}} \to U^{(2)}(D)$ by sending an element to its equivalence class. Those data satisfy (1)-(11) of Definition 4.11 except the Hausdorff-ness of $U^{(2)}(D)$ and injectivity of $\psi^{(2)}_D$.

We again use a similar trick as in the last part of the proof of Proposition 4.4 to modify $U^{(2)}_{\mathfrak{d}}$ etc. as follows.

We remark that we have a continuous map

$$\underline{\phi}^{(2)}_{D'\mathfrak{d}_0} : U^{(2)}_{D'\mathfrak{d}_0} \to U^{(2)}(D')$$

from $U^{(2)}_{D'\mathfrak{d}_0} = \bigcup_{\mathfrak{d}\in D'} \Pi^{(2)}_{\mathfrak{d}}(U^{(2)}_{\mathfrak{d}\mathfrak{d}_0})$ such that

$$\underline{\phi}^{(2)}_{D'\mathfrak{d}_0} = \Pi^{(2)}_{\mathfrak{d}} \circ \underline{\phi}^{(2)}_{\mathfrak{d}\mathfrak{d}_0}$$

holds on $U^{(2)}_{\mathfrak{d}\mathfrak{d}_0}$.

Note $U^{(2)}(D)$ can also be written as

$$U^{(2)}(D) = (U^{(2)}(D') \cup U^{(2)}_{\mathfrak{d}_0})/\sim .$$

where $x \sim y$ if and only if one of the following holds.

(1) $x = y$.
(2) $x \in U^{(2)}_{D'}$, $y \in U^{(2)}_{D'\mathfrak{d}_0} \subset U^{(2)}_{\mathfrak{d}_0}$, $x = \underline{\phi}^{(2)}_{D'\mathfrak{d}_0}(y)$.
(3) $y \in U^{(2)}_{D'}$, $x \in U^{(2)}_{D'\mathfrak{d}_0} \subset U^{(2)}_{\mathfrak{d}_0}$, $y = \underline{\phi}^{(2)}_{D'\mathfrak{d}_0}(x)$.

We also remark that $U^{(2)}(D')$ is already Hausdorff by induction hypothesis.

We take relatively compact subset $U^{(3)}(D')$ of $U^{(2)}(D')$ such that

$$U^{(3)}(D') \supset X(D')$$

and a relatively compact subset $U^{(3)}_{\mathfrak{d}_0}$ of $U^{(2)}_{\mathfrak{d}_0}$ such that

$$U^{(3)}_{\mathfrak{d}_0} \supset (\psi^{(1)}_{\mathfrak{d}_0})^{-1}(\mathcal{K}^{(1)}_{\mathfrak{d}_0}).$$

We take $U^{(3)}(D')$ and $U^{(3)}_{\mathfrak{d}_0}$ such that

$$\psi^{(2)}_D\left(U^{(3)}(D') \cap (s^{(2)}_{D'})^{-1}(0)\right) \cup \psi^{(2)}_{\mathfrak{d}_0}\left(U^{(3)}_{\mathfrak{d}_0} \cap (s^{(2)}_{\mathfrak{d}_0})^{-1}(0)\right) \supset X(D).$$

We take $W_{D'\mathfrak{d}_0} \subset U^{(2)}_{D'\mathfrak{d}_0}$ such that

$$W_{D'\mathfrak{d}_0} \supset s^{-1}_{\mathfrak{d}_0}(0) \cap \overline{(U^{(3)}_{\mathfrak{d}_0} \cap (\underline{\phi}^{(2)}_{D\mathfrak{d}_0})^{-1}(U^{(3)}(D')))}$$

and set

$$U^{(3)}_{D'\mathfrak{d}_0} = W_{D'\mathfrak{d}_0} \cap (U^{(3)}_{\mathfrak{d}_0} \cap (\underline{\phi}^{(2)}_{D\mathfrak{d}_0})^{-1}(U^{(3)}(D'))).$$

We now define $U^{(3)}(D)^+$ by

$$U^{(3)}(D)^+ = (U^{(3)}(D')) \cup U^{(3)}_{\mathfrak{d}_0})/\sim$$

where $x \sim y$ if and only if one of the following holds.

(1) $x = y$.
(2) $x \in U^{(3)}(D')$, $y \in \overline{U}^{(3)}_{D'\mathfrak{d}_0} \subset U^{(3)}_{\mathfrak{d}_0}$, $x = \underline{\phi}^{(2)}_{D'\mathfrak{d}}(y)$.
(3) $y \in U^{(3)}(D')$, $x \in \overline{U}^{(3)}_{D'\mathfrak{d}_0} \subset U^{(3)}_{\mathfrak{d}_0}$, $y = \underline{\phi}^{(2)}_{D'\mathfrak{d}}(x)$.

Lemma 4.19. *$U^{(3)}(D)^+$ is Hausdorff.*

The proof is the same as the proof of Lemma 4.9 and so is omitted.

We put

$$A_{D'\mathfrak{d}_0} = (\overline{W}_{D'\mathfrak{d}_0} \setminus W_{D'\mathfrak{d}_0}) \cap (U^{(3)}_{\mathfrak{d}_0} \cap (\underline{\phi}^{(2)}_{D\mathfrak{d}_0})^{-1}(U^{(3)}(D'))) \subset U^{(3)}_{\mathfrak{d}_0}.$$

Let A be the closure of the image of $A_{D'\mathfrak{d}_0}$ in $U(D)$.

Let $(s^{(3)}_D)^{-1}(0)$ be the image of the union of $(s^{(2)}_{D'})^{-1}(0) \cap U^{(3)}(D'))$ and $(s^{(2)}_{\mathfrak{d}_0})^{-1}(0) \cap U^{(2)}_{\mathfrak{d}_0}$ in $U^{(3)}(D)^+$.

Lemma 4.20. *$(s^{(3)}_D)^{-1}(0) \cap A = \emptyset$.*

The proof is the same as the proof of Lemma 4.10 and is omitted.

We put

$$\begin{aligned} U(D) =&U^{(3)}(D)^+ \\ &\setminus \Big(A\cup \Pi^{(3)}_{D\mathfrak{d}_0}(\{x \in U^{(3)}_{\mathfrak{d}_0} \setminus U^{(3)}_{D'\mathfrak{d}_0} \mid \overline{s}_{\mathfrak{d}_0}(0) = 0,\ \psi_{\mathfrak{d}_0}(x) \in \mathcal{U}^{(3)}_{D'}\})\Big) \end{aligned} \tag{4.56}$$

where $\Pi^{(3)}_{D\mathfrak{d}_0} : U^{(3)}_{\mathfrak{d}_0} \to U^{(3)}(D)^+$ is a map that send an element to its equivalence class.

Lemma 4.20 implies that $U(D)$ is an open subset of $U^{(3)}(D)^+$.

Lemma 4.21. *$\psi^{(2)}_D$ induces an injective map*

$$\psi_D : (s^{(3)}_D)^{-1}(0) \cap U(D) \to X.$$

This is immediate from definition.

Lemma 4.22. *ψ_D is a homeomorphism to the image.*

The proof is the same as the proof of Lemma 2.15 and is omitted. The proof of Proposition 4.13 is complete. □

Lemma 4.23. *We may choose $U(D)$ so that the following holds in addition. Let $\mathfrak{d}_k > \mathfrak{d}_0$. If*

$$\bigcap_{k=1}^{K} \Pi_{\mathfrak{d}}(U_{\mathfrak{d}_k}) \cap \Pi_{\mathfrak{d}'}(U_{\mathfrak{d}_0}) \ne \emptyset$$

then

$$\bigcap_{k=1}^{K} \Pi_{\mathfrak{d}}(U_{\mathfrak{d}_k} \cap s_{\mathfrak{d}}^{-1}(0)) \cap \Pi_{\mathfrak{d}_0}(U_{\mathfrak{d}'} \cap s_{\mathfrak{d}_0}^{-1}(0)) \ne \emptyset.$$

Proof. We will modify $U(D)$ so that it satisfies this addional condition by induction on $\#D$.

The inductive step is as follows. We take $\mathfrak{d}_0 \in D$ that is minimal in D. We put $D' = D \setminus \{\mathfrak{d}_0\}$

We modify $U_{\mathfrak{d}_0}$ so that the conclusion of the lemma holds by induction on K. We assume the conclusion of the lemma holds for $K \le K_0 - 1$. We consider the case of K_0. Let

$$C = \{\{\mathfrak{d}_1, \dots, \mathfrak{d}_{K_0}\} \mid (4.57), \mathfrak{d}_i \text{ are all different.}\}$$

$$\bigcap_{k=1}^{K_0} \Pi_{\mathfrak{d}}(U_{\mathfrak{d}_k} \cap s_{\mathfrak{d}}^{-1}(0)) \cap \Pi_{\mathfrak{d}_0}(U_{\mathfrak{d}'} \cap s_{\mathfrak{d}_0}^{-1}(0)) = \emptyset. \tag{4.57}$$

We replace $U_{\mathfrak{d}_0}$ by

$$U_{\mathfrak{d}_0} \setminus \bigcup_{\{\mathfrak{d}_1,\dots,\mathfrak{d}_{K_0}\}\in C} \bigcap_{k=1}^{K_0} \overline{U}_{\mathfrak{d}_k\mathfrak{d}_0}.$$

The induction works. □

Now we are in the position to complete the proof of Theorem 4.1. We apply Proposition 4.13 to obtain a mixed orbifold neighborhood of $X(\mathfrak{D}) = X$. We put $\mathfrak{P} = \mathfrak{D} \subset \mathbb{Z}_{>0}$. The order is $\le$. For $\mathfrak{d} \in \mathfrak{D} = \mathfrak{P}$, we have $U_{\mathfrak{d}}$, $E_{\mathfrak{d}}$, $s_{\mathfrak{d}}$, $\psi_{\mathfrak{d}}$ by Definition 4.11 (2)(3). Let us check Definition 2.5 (1)-(9).

Definition 2.5 (1)-(4) follows from Definition 4.11 (2)(3). Definition 2.5 (5)(6) follows from Definition 4.11 (4) - (8). Definition 2.5 (7) follows from Definition 4.11 (11). Definition 2.5 (8) is obvious since $\mathfrak{P} \subset \mathbb{Z}_{>0}$. Condition 2.6 in Definition 2.5 (9) follows from Definition 4.11 (10) (11). Conditions 2.8 and 2.10 in Definition 2.5 (9) follow from Lemma 4.23 and (4.41). Condition 2.11 follows from 2.5 (9) especially Hausdorff-ness of $U(X)$.

The proof of Theorem 4.1 is now complete. □

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