THIRD+FOURTH ANSWER

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Contents

This is an answer to question 4.

In the questions posted on March 14, question 4 concerns only the case of moduli space of pseudo-holomorphic curves of genus 0 with one marked point and the homology class is primitive. So there is no bubble. If the question is only on this particular case, it seems to us that there is nothing more to reply than what we wrote on March 21. (Surjectivity, injectivity, smoothness etc. that is mentioned in March 23's post is an immediate consequence of the implicit function theorem, which is certainly a standard result in this case.) On the other hand, in the post on March 23, 'gluing' is mentioned. (Line 7 of the paragraph starting Q4.) This is contradictory. So we gave up replying the question word by word but explain the construction of Kuranishi structure on the moduli space of pseudo-holomorphic curves in general.

Our construction of Kuranishi charts does not use Fredholm theory at infinity.

We do not understand what means 'slicing', the word that appeared in the post on March 23.

There is a well-established technique to find the moduli space as a manifold with boundary in certain situation. It was used by Donaldson in gauge theory (in his first paper [D1] to show that 1 instanton moduli of ASD connections on 4 manifold M with $b_2^+ = 0$ has M as a boundary.) In this method we take some parameter (that is the degree of concentration of the curvature in the case of ASD equation and the parameter T in the situation of Section 1 below). We consider the submanifold where that parameter T is large, say T_0. We throw away everything where $T > T_0$. Then the part $T = T_0$ becomes the boundary of the 'moduli space' we obtain. It was more detailed in a book by Freed and Uhlenbeck [FU] in the gauge theory case. Abouzaid used this technique in his paper [Ab] about exotic spheres in T^*S^n, including the case of corners. At least as far as the results in [FOn1] are concerned we can use this technique since we need to study moduli space of virtual dimension 0 and 1 only to prove all the results in [FOn1]. In other words we can use something like Theorem 1.10 for large and fixed T, but does not need to estimate the T derivative or study the bahavior of the moduli space at $T = \infty$. The reason is as follows. In case we consider codimension 2 or higher codimension corners, then since the virtual dimension of the moduli space is 1 or 0, the restriction to that corner has negative virtual dimension. So after generic multivalued perturbation the zero set on the corner becomes empty. So all we need is to extend multivalued perturbation. (The C^0 extention is enough for this purpose.) For codimension 1 boundary and the case of moduli space of virtual dimension 1, after generic perturbation we have isolated zero of the perturbed moduli space. So, for large T_0, Theorem 1.10 or its analogue implies that the zero on the 'boundary $T = T_0$' corresponds one to one to the zero at the actual boundary ($T = \infty$). So we do not need to see carefully what happens in a neighborhood of the set $T = \infty$. (All we need is to extend this given perturbation at $T = T_0$ to the inside.) This argument is good enough to establish all the results in [FOn1].

As we mentioned explicitly in [FOn1, page 978 line 13] our argument there, in analytic points, is basically the same as in [MS]. (Let us remark however the proof of 'surjectivity' that is written in [FOn1, Section 14] is slightly different from one in [MS].) So the novelity of [FOn1] does *not* lie in the analytic point but in the general strategy, that is

(1) To define some general notion of 'spaces' that contain various moduli spaces of pseudo-holomorphic curves as examples and work out transversality issue in that abstract setting.
(2) Use multivalued abstract perturbation, that we call multisection.

When we go beyond that and prove results such as those we had proved in [FOOO1], we need to study the moduli spaces of higher virtual dimension and study chain level intersection theory. In that case we are not sure whether the above mentioned technique is enough. (It may work. But we did not think enough about it.) It is not the way we had taken in [FOOO1].

Our method in [FOOO1] was using exponential decay estimate ([FOOO1, Lemma A1.59]) and use $s = 1/T$ as the coordinate on the normal direction to the stratum to define smooth coordinate of the Kuranishi structure. We refer [FOOO1, Subsection A1.4] and [FOOO1, Subection 7.1.2] where this construction is written.

Below, we provide more details of the way how to use alternating method to construct smooth chart at infinity following the argument in [FOOO1, Subsection A1.4].

1. A simple case

1.1. **Setting.**

We will describe the general case in Section 2. To simplify the notation and clarify the main analytic point of the proof we prove the case where we glue holomorphic maps from two stable bordered Riemann surfaces to (X, L) in this section.

Let Σ_i be a bordered Riemann surface with one end. ($i = 1, 2$.) We identify their ends as follows.

$$\begin{aligned} \Sigma_1 &= K_1 \cup ((-5T, \infty) \times [0, 1]), \\ \Sigma_2 &= ((-\infty, 5T) \times [0, 1]) \cup K_2. \end{aligned} \tag{1.1}$$

Here K_i are compact and $\pm\infty$ are the ends. We put

$$\Sigma_T = K_1 \cup ((-5T, 5T) \times [0, 1]) \cup K_2. \tag{1.2}$$

We use τ for the coordinate of the factors $(-5T, \infty)$, $(-\infty, 5T)$, or $(-5T, 5T)$ and t for the coordinate of the second factor $[0, 1]$.

Let X be a symplectic manifold with compatible (or tame) almost complex structure and L be its Lagrangian submanifold.

Let

$$u_i : (\Sigma_i, \partial\Sigma_i) \to (X, L), \qquad i = 1, 2$$

be pseudo-holomorphic maps of finite energy. Then, by the removable singularity theorem that is now standard, we have asymptotic value

$$\lim_{\tau\to\infty} u_1(\tau, t) \in L \tag{1.3}$$

and

$$\lim_{\tau\to-\infty} u_2(\tau, t) \in L. \tag{1.4}$$

The limits (1.3) and (1.4) are independent of t.

We assume that the limit (1.3) coincides with (1.4) and denote it by $p_0 \in L$.

We fix a coordinate of X and of L in a neighborhood of p_0. So a trivialization of the tangent bundle TX and TL in a neighborhood of p_0 is fixed. Hereafter we assume the following:

$$\mathrm{Diam}(u_1([-5T, \infty) \times [0, 1])) \le \epsilon_1, \qquad \mathrm{Diam}(u_2((-\infty, 5T] \times [0, 1])) \le \epsilon_1. \tag{1.5}$$

The maps u_i determine homology classes $\beta_i = [u_i] \in H_2(X, L)$.

We take $K_i^{\rm obst}$ a compact subset of the interior of K_i and take

$$E_i \subset \Gamma(K_i^{\rm obst}; u_i^*TX \otimes \Lambda^{0,1}) \tag{1.6}$$

a finite dimensional linear subspace consisting of smooth sections supported in $K_i^{\rm obst}$.

For simplicity we also fix a complex structure of the source Σ_i. The version where it can move will be discussed later. We also assume that Σ_i equipped with marked points $\vec z_i$ is stable. The process to add marked points to stabilize it will be discussed later also. Let

$$D_{u_i}\overline\partial : L^2_{m+1,\delta}((\Sigma_i, \partial\Sigma_i); u_i^*TX, u_i^*TL) \to L^2_{m,\delta}(\Sigma_i; u_i^*TX \otimes \Lambda^{01}) \tag{1.7}$$

be the linearization of the Cauchy-Riemann equation. Here we define the weighted Sobolev space we use as follows.

Definition 1.1. ([FOOO1, Section 7.1.3])[1] Let $L^2_{m+1,loc}((\Sigma_i, \partial\Sigma_i); u_i^*TX; u_i^*TL)$ be the set of the sections s of u_i^*TX which is locally of L^2_{m+1}-class, (Namely its differential up to order $m+1$ is of L^2 class. Here m is sufficiently large, say larger than 10.) We also assume $s(z) \in u_i^*TL$ for $z \in \partial\Sigma_i$.

The weighted Sobovel space $L^2_{m+1,\delta}((\Sigma_i, \partial\Sigma_i); u_i^*TX, u_i^*TL)$ is the set of all pairs (s, v) of elements s of $L^2_{m+1,loc}((\Sigma_i, \partial\Sigma_i); u_i^*TX; u_i^*TL)$ and $v \in T_{p_0}L$, (here $p_0 \in L$ is the point (1.3) or (1.4)) such that

$$\sum_{k=0}^{m+1} \int_{\Sigma_i \setminus K_i} e^{\delta|\tau \pm 5T|} |\nabla^k(s - {\rm Pal}(v))|^2 < \infty, \tag{1.8}$$

where ${\rm Pal} : T_{p_0}X \to T_{u_i(\tau,t)}X$ is defined by the trivialization we fixed right after (1.4). (Here $\pm$ is $+$ for $i = 1$ and $-$ for $i = 2$.) The norm is defined as the sum of (1.8), the norm of v and the L^2_{m+1} norm of s on K_i. (See (1.26).)

$L^2_{m,\delta}(\Sigma_i; u_i^*TX \otimes \Lambda^{01})$ is defined similarly without boundary condition and with out v. (See (1.28).)

When we define $D_{u_i}\overline\partial$ we forget v component and use s only.

Remark 1.2. The positive number δ is chosen as follows. (1.3) and a standard estimate imply that there exists $\delta_1 > 0$ such that

$$\left| \frac{d}{d\tau} u_i \right|_{C^k} (\tau, t) < C_k e^{-\delta_1|\tau|} \tag{1.9}$$

for any k. We choose δ smaller than $\delta_1/10$. (1.9) implies

$$(D_{u_i}\overline\partial)({\rm Pal}(v)) < C_k e^{-\delta_1|\tau|/10}.$$

Therefore (1.7) is defined and bounded.

It is a standard fact that (1.7) is Fredholm.

We work under the following assumption.

Assumption 1.3.

$$D_{u_i}\overline\partial : L^2_{m+1,\delta}((\Sigma_i, \partial\Sigma_i); u_i^*TX, u_i^*TL) \to L^2_{m,\delta}(\Sigma_i; u_i^*TX \otimes \Lambda^{01})/E_i \tag{1.10}$$

[1] In [FOOO1] L^p_1 space is used in stead of L^2_m space.

is surjective. Moreover the following (1.12) holds. Let $(D_{u_i}\overline{\partial})^{-1}(E_i)$ be the kernel of (1.10). We define

$$Dev_{i,\infty} : L^2_{m+1,\delta}((\Sigma_i, \partial\Sigma_i); u_i^*TX, u_i^*TL) \to T_{p_0}L \tag{1.11}$$

by

$$Dev_{i,\infty}(s, v) = v.$$

Then

$$Dev_{1,\infty} - Dev_{2,\infty} : (D_{u_1}\overline{\partial})^{-1}(E_1) \oplus (D_{u_2}\overline{\partial})^{-1}(E_2) \to T_{p_0}L \tag{1.12}$$

is surjective.

Let us start stating the result. Let

$$u' : (\Sigma_T, \partial\Sigma_T) \to (X, L) \tag{1.13}$$

be a smooth map. We consider the following condition depending $\epsilon > 0$.

Condition 1.4. (1) $u'|_{K_i}$ is ϵ-close to $u_i|_{K_i}$ in C^1 sense.
(2) The diameter of $u'([-5T, 5T] \times [0, 1])$ is smaller than ϵ.

We take ϵ_2 sufficiently small compared to the 'injectivity radius' of X so that the next definition makes sense.[2] For u' satisfying Condition 1.4 for $\epsilon < \epsilon_2$:

$$I_{u'} : E_i \to \Gamma(\Sigma_T; (u')^*TX \otimes \Lambda^{01})$$

is the complex linear part of the parallel translation along the short geodesic (between $u_i(z)$ and $u'(z)$. Here $z \in K_i^{\rm obst}$). We put

$$E_i(u') = I_{u'}(E_i). \tag{1.14}$$

The equation we study is

$$\overline{\partial}u' \equiv 0, \qquad \text{mod } E_1(u') \oplus E_2(u'). \tag{1.15}$$

Remark 1.5. In the actual construction of Kuranishi structure, we take several u_i's and take E_i's for each of them. Then in place of $E_1(u') \oplus E_2(u')$ we take sum of finitely many of them. Here we simplify the notation. There is no difference between the proof of Theorem 1.10 and the corresonding result in case we take several such u_i's and E_i's. See [Fu2, pages 4-5] and Section 2.

Theorem 1.10 describes all the solutions of (1.15). To state this precisely we need a bit more notations.

We consider the following condition for $u_i' : (\Sigma_i, \partial\Sigma_i) \to (X, L)$.

Condition 1.6. (1) $u_i'|_{K_i}$ is ϵ-close to $u_i|_{K_i}$ in C^1 sense.
(2) The diameter of $u_1'([-5T, \infty) \times [0, 1])$, (resp. $u_2'((-\infty, 5T]) \times [0, 1])$) is smaller than ϵ.

[2]More precisely, we assume that

$$\{(x, y) \in X \times X \mid d(x, y) < \epsilon_2\} \subset \mathrm{E}(\{(x, v) \in TX \mid |v| < \epsilon\}),$$

where $\mathrm{E} : \{(x, v) \in TX \mid |v| < \epsilon\} \to X$ is induced by an exponential map of certain connection of TX. See (1.30).

Then we define

$$I_{u'_i} : E_i \to \Gamma(\Sigma_i; (u'_i)^*TX \otimes \Lambda^{01})$$

by using the parallel transport in the same way as $I_{u'_T}$. (This makes sense if u'_i satisfies Condtion 1.6 for $\epsilon < \epsilon_2$.) We put

$$E_i(u'_i) = I_{u'_i}(E_i). \tag{1.16}$$

So we can define an equation

$$\overline{\partial} u'_i \equiv 0, \qquad \text{mod } E_i(u'_i). \tag{1.17}$$

Definition 1.7. The set of solutions of equation (1.17) with finite energy and satisfying Condition 1.6 for $\epsilon = \epsilon_2$ is denoted by $\mathcal{M}^{E_i}((\Sigma_i, \vec z_i); \beta_i)_{\epsilon_2}$. Here β_i is the homology class of u_i.

Remark 1.8. In the usual story of pseudo-holomorphic curve, we identify u_i and u'_i if there exists a biholomorphic map $v : (\Sigma_i, \vec z_i) \to (\Sigma_i, \vec z_i)$ such that $u'_i = u_i \circ v$. In our situation where Σ_i has no sphere or disk bubble and has nontrivial boundary with at least one boundary marked points (that is $\tau = \pm\infty$), such v is necessary the identity map. Namely Σ_i has no nontrivial automorphism.

The surjectivity of (1.11), (1.12) and the implicit function theorem imply that if ϵ_2 is small then there exists a finite dimensional vector space $\tilde V_i$ and its neighborhood V_i of 0 such that

$$\mathcal{M}^{E_i}((\Sigma_i, \vec z_i); \beta_i)_{\epsilon_2} \cong V_i.$$

Since we assume that Σ_i is nonsigular the group $\text{Aut}((\Sigma_i, \vec z_i), u_i)$ is trivial. (In the case when there is a sphere bubble, the automorphism group can be nontrivial. That case will be discussed later.)

For any $\rho_i \in V_i$ we denote by $u_i^{\rho_i} : (\Sigma_i, \partial\Sigma_i) \to (X, L)$ the corresponding solution of (1.17).

We have an evaluation map

$$\text{ev}_{i,\infty} : \mathcal{M}^{E_i}((\Sigma_i, \vec z_i); \beta_i)_{\epsilon_2} \to L$$

that is smooth. Namely

$$\text{ev}_{i,\infty}(u'_i) = \lim_{\tau \to \pm\infty} u'_i(\tau, t).$$

(Here $\pm = +$ for $i = 1$ and $-$ for $i = 2$.)[3] We consider the fiber product:

$$\mathcal{M}^{E_1}((\Sigma_1, \vec z_1); \beta_1)_{\epsilon_2} \times_L \mathcal{M}^{E_2}((\Sigma_2, \vec z_2); \beta_2)_{\epsilon_2}. \tag{1.18}$$

The surjectivity of (1.12) implies that this fiber product is transversal so is

$$V_1 \times_L V_2.$$

And an element of $V_1 \times_L V_2$ is written as $\rho = (\rho_1, \rho_2)$.

Definition 1.9. Let $\beta = \beta_1 + \beta_2$. We denote by $\mathcal{M}^{E_1+E_2}((\Sigma_T, \vec z); \beta)_\epsilon$ the set of solutions of (1.15) satisfying the Condition 1.4 with $\epsilon_2 = \epsilon$.

Theorem 1.10. *For each sufficiently small ϵ_3 and sufficiently large T, there exist ϵ_1, ϵ_2 and a map*

$$\text{Glu}_T : \mathcal{M}^{E_1}((\Sigma_1, \vec z_1); \beta_1)_{\epsilon_2} \times_L \mathcal{M}^{E_2}((\Sigma_2, \vec z_2); \beta_2)_{\epsilon_2} \to \mathcal{M}^{E_1+E_2}((\Sigma_T, \vec z); \beta)_{\epsilon_1}$$

that is a diffeomorphism to its image. The image contains $\mathcal{M}^{E_1+E_2}((\Sigma_T, \vec z); \beta)_{\epsilon_3}$.

[3]This is a consequence of the fact that u_i is pseudo-holomorphic outside a compact set and has finite energy.

The result about exponential decay estimate of this map is in Subsection 1.4. (Theorem 1.34.)

1.2. **Proof of Theorem 1.10 : 1 - Bump function and weighted Sobolev norm.** The proof of Theorem 1.10 was given in [FOOO1, Section 7.1.3]. The exponential decay estimate of the solution was proved in [FOOO1, Section A1.4] together with a slightly modified version of the proof of Theorem 1.10. Here we follow the proof of [FOOO1, Section A1.4] and give its more detail. As mentioned there the origin of the proof is Donaldson's paper [D2], and its Bott-Morse version in [Fu1].

We first introduce certain bump functions. First let $\mathcal{A}_T \subset \Sigma_T$ and $\mathcal{B}_T \subset \Sigma_T$ be the domains defined by

$$\mathcal{A}_T = [-T-1, -T+1] \times [0,1], \qquad \mathcal{B}_T = [T-1, T+1] \times [0,1].$$

We may regard $\mathcal{A}_T, \mathcal{B}_T \subset \Sigma_i$. The third domain is

$$\mathcal{X} = [-1,1] \times [0,1] \subset \Sigma_T.$$

We may also regard $\mathcal{X} \subset \Sigma_i$.

Let $\chi_{\mathcal{A}}^{\leftarrow}$, $\chi_{\mathcal{A}}^{\rightarrow}$ be smooth functions on $[-5T, 5T] \times [0,1]$ such that

$$\chi_{\mathcal{A}}^{\leftarrow}(\tau,t) = \begin{cases} 1 & \tau < -T-1 \\ 0 & \tau > -T+1. \end{cases} \tag{1.19}$$

$$\chi_{\mathcal{A}}^{\rightarrow} = 1 - \chi_{\mathcal{A}}^{\leftarrow}.$$

We define

$$\chi_{\mathcal{B}}^{\leftarrow}(\tau,t) = \begin{cases} 1 & \tau < T-1 \\ 0 & \tau > T+1. \end{cases} \tag{1.20}$$

$$\chi_{\mathcal{B}}^{\rightarrow} = 1 - \chi_{\mathcal{B}}^{\leftarrow}.$$

We define

$$\chi_{\mathcal{X}}^{\leftarrow}(\tau,t) = \begin{cases} 1 & \tau < -1 \\ 0 & \tau > 1. \end{cases} \tag{1.21}$$

$$\chi_{\mathcal{X}}^{\rightarrow} = 1 - \chi_{\mathcal{X}}^{\leftarrow}.$$

We extend these functions to Σ_T and Σ_i $(i = 1,2)$ so that they are locally constant outside $[-5T, 5T] \times [0,1]$. We denote them by the same symbol.

We next introduce weighted Sobolev norms and their local versions for sections on Σ_T or Σ_i as follows.

We define $e_{i,\delta} : \Sigma_i \to [1,\infty)$ of C^∞ class as follows.

$$e_{1,\delta}(\tau,t) \begin{cases} = e^{\delta|\tau+5T|} & \text{if } \tau > 1-5T \\ = 1 & \text{on } K_1 \\ \in [1,10] & \text{if } \tau < 1-5T \end{cases} \tag{1.22}$$

$$e_{2,\delta}(\tau,t) \begin{cases} = e^{\delta|\tau-5T|} & \text{if } \tau < 5T-1 \\ = 1 & \text{on } K_2 \\ \in [1,10] & \text{if } \tau > 5T-1 \end{cases} \tag{1.23}$$

We also define $e_{T,\delta} : \Sigma_T \to [1,\infty)$ as follows:

$$e_{T,\delta}(\tau,t)\begin{cases} = e^{\delta|\tau-5T|} & \text{if } 1<\tau<5T-1\\ = e^{\delta|\tau+5T|} & \text{if } -1>\tau>1-5T\\ =1 & \text{on } K_1\cup K_2\\ \in[1,10] & \text{if } |\tau-5T|<1 \text{ or } |\tau+5T|<1\\ \in [e^{5T\delta}/10, e^{5T\delta}] & \text{if } |\tau|<1.\end{cases} \tag{1.24}$$

The weighted Sobolev norm we use for $L^2_{m,\delta}(\Sigma_i; u_i^*TX\otimes\Lambda^{01})$ is

$$\|s\|^2_{L^2_{m,\delta}} = \sum_{k=0}^{m}\int_{\Sigma_i} e_{i,\delta}|\nabla^k s|^2\mathrm{vol}_{\Sigma_i}. \tag{1.25}$$

For $(s,v)\in L^2_{m+1,\delta}((\Sigma_i,\partial\Sigma_i); u_i^*TX, u_i^*TL)$ we define

$$\begin{aligned}\|(s,v)\|^2_{L^2_{m+1,\delta}} = &\sum_{k=0}^{m+1}\int_{K_i}|\nabla^k s|^2\mathrm{vol}_{\Sigma_i}\\ &+\sum_{k=0}^{m+1}\int_{\Sigma_i\setminus K_i} e_{i,\delta}|\nabla^k(s-\mathrm{Pal}(v))|^2\mathrm{vol}_{\Sigma_i}+\|v\|^2.\end{aligned} \tag{1.26}$$

We next define a weighted Sobolev norm for the sections on Σ_T. Let

$$s\in L^2_{m+1}((\Sigma_T,\partial\Sigma_T); u^*TX, u^*TL).$$

Since we take m large, s is continuous. So $s(0,1/2)\in T_{u(0,1/2)}X$ is well defined. There is a canonical trivialization of TX in a neighborhood of p_0 that we fixed right after (1.4). We use it to define Pal below. We put

$$\begin{aligned}\|s\|^2_{L^2_{m+1,\delta}} = &\sum_{k=0}^{m+1}\int_{K_1}|\nabla^k s|^2\mathrm{vol}_{\Sigma_1}+\sum_{k=0}^{m+1}\int_{K_2}|\nabla^k s|^2\mathrm{vol}_{\Sigma_2}\\ &+\sum_{k=0}^{m+1}\int_{[-5T,5T]\times[0,1]} e_{T,\delta}|\nabla^k(s-\mathrm{Pal}(s(0,1/2)))|^2\mathrm{vol}_{\Sigma_i}\\ &+\|s(0,1/2)\|^2.\end{aligned} \tag{1.27}$$

For

$$s\in L^2_m((\Sigma_T,\partial\Sigma_T); u^*TX\otimes\Lambda^{01})$$

we define

$$\|s\|^2_{L^2_{m,\delta}} = \sum_{k=0}^{m}\int_{\Sigma_T} e_{T,\delta}|\nabla^k s|^2\mathrm{vol}_{\Sigma_1}. \tag{1.28}$$

These norms were used in [FOOO1, Section 7.1.3].

For a subset W of Σ_i or Σ_T we define $\|s\|_{L^2_{m,\delta}(W\subset\Sigma_i)}$, $\|s\|_{L^2_{m,\delta}(W\subset\Sigma_T)}$ by restricting the domain of the integration (1.28) or (1.27) to W.

Let $(s_j,v_j)\in L^2_{m+1,\delta}((\Sigma_i,\partial\Sigma_i); u_i^*TX, u_i^*TL)$ for $j=1,2$. We define the inner product among them by:

$$\begin{aligned}\langle\!\langle (s_1,v_1),(s_2,v_2)\rangle\!\rangle_{L^2_\delta} = &\int_{\Sigma_i\setminus K_i}(s_1-\mathrm{Pal}v_1, s_2-\mathrm{Pal}v_2)\\ &+\int_{K_i}(s_1,s_2)+(v_1,v_2).\end{aligned} \tag{1.29}$$

We also use an exponential map. (The same map was used in [FOOO1, pages 410-411].) We take a diffeomorphism

$$\mathrm{E} = (\mathrm{E}_1, \mathrm{E}_2) : \{(x, v) \in TX \mid |v| < \epsilon\} \to X \times X \tag{1.30}$$

to its image such that

$$\mathrm{E}_1(x, v) = x, \qquad \left.\frac{d\mathrm{E}_2(x, tv)}{dt}\right|_{t=0} = v$$

and

$$\mathrm{E}(x, v) \in L \times L, \qquad \text{for } x \in L,\ v \in T_x L.$$

Furthermore we may take it so that

$$\mathrm{E}(x, v) = (x, x + v) \tag{1.31}$$

on a neighborhood of p_0.

To find such E, we take a linear connection ∇ (that may not be a Levi-Civita conneciton of a Riemannian metric) of TX such that TL is parallel with respect to ∇. We then use geodesic with respect to ∇ to define an exponential map. We then define E such that $t \mapsto \mathrm{E}_2(x, tv)$ is a geodesic with initial direction v. Note that we may take ∇ so that in a neighborhood of p_0 it coincides with the standard trivial connection with respect the coordinate we fixed. (1.31) follows.

1.3. **Proof of Theorem 1.10 : 2 - Gluing by alternating method.** Let us start with

$$u^\rho = (u_1^{\rho_1}, u_2^{\rho_2}) \in \mathcal{M}^{E_1}((\Sigma_1, \vec{z}_1); \beta_1)_{\epsilon_2} \times_L \mathcal{M}^{E_2}((\Sigma_2, \vec{z}_2); \beta_2)_{\epsilon_2}.$$

Here $\rho_i \in V_i$ and the corresponding map $(\Sigma_i, \partial\Sigma_i) \to (X, L)$ is denoted by $u_i^{\rho_i}$. Let $\rho = (\rho_1, \rho_2)$. We put

$$p^\rho = \lim_{\tau\to\infty} u_1^{\rho_1}(\tau, t) = \lim_{\tau\to-\infty} u_2^{\rho_2}(\tau, t).$$

Preglueing:

Definition 1.11. We define

$$u^\rho_{T,(0)} = \begin{cases} \chi_{\mathcal{B}}^{\leftarrow}(u_1^{\rho_1} - p^\rho) + \chi_{\mathcal{A}}^{\rightarrow}(u_2^{\rho_2} - p^\rho) + p^\rho & \text{on } [-5T, 5T] \times [0, 1] \\ u_1^{\rho_1} & \text{on } K_1 \\ u_2^{\rho_2} & \text{on } K_2. \end{cases} \tag{1.32}$$

Note that we use the coordinate of the neighborhood of p_0 to define the sum in the first line.

Step 0-3:

Lemma 1.12. *If $\delta < \delta_1/10$ then there exists $\mathfrak{e}^\rho_{i,T,(0)} \in E_i$ such that*

$$\|\overline{\partial} u^\rho_{T,(0)} - \mathfrak{e}^\rho_{1,T,(0)} - \mathfrak{e}^\rho_{2,T,(0)}\|_{L^2_{m,\delta}} < C_{1,m} e^{-\delta T}. \tag{1.33}$$

Moreover

$$\|\mathfrak{e}^\rho_{i,T,(0)}\|_{L^2_m(K_i)} < \epsilon_{4,m}. \tag{1.34}$$

Here $\epsilon_{4,m}$ is a positive number which we may choose arbitraly small by taking V_i to be a sufficiently small neighborhood of zero in $\tilde V_i$.

Moreover $\mathfrak{e}^\rho_{i,T,(0)}$ is independent of T.

Proof. We put

$$\mathfrak{e}_{i,T,(0)} = \overline{\partial} u_i^{\rho} \in E_i.$$

Then by definition the support of $\overline{\partial} u^{\rho}_{T,(0)} - \mathfrak{e}^{\rho}_{1,T,(0)} - \mathfrak{e}^{\rho}_{2,T,(0)}$ is in $[-5T, 5T] \times [0,1]$. Moreover it is estimated as (1.33). □

Step 0-4:

Definition 1.13. We put

$$\begin{aligned}\mathrm{Err}^{\rho}_{1,T,(0)} &= \chi^{\leftarrow}_{\mathcal{X}}(\overline{\partial} u^{\rho}_{T,(0)} - \mathfrak{e}^{\rho}_{1,T,(0)}),\\ \mathrm{Err}^{\rho}_{2,T,(0)} &= \chi^{\rightarrow}_{\mathcal{X}}(\overline{\partial} u^{\rho}_{T,(0)} - \mathfrak{e}^{\rho}_{2,T,(0)}).\end{aligned}$$

We regard them as elements of the weighted Sobolev spaces $L^2_{m,\delta}((\Sigma_1, \partial\Sigma_1); (u_1^{\rho})^*TX \otimes \Lambda^{01})$ and $L^2_{m,\delta}((\Sigma_2, \partial\Sigma_2); (u_2^{\rho})^*TX \otimes \Lambda^{01})$ respectively. (We extend them by 0 outside a compact set.)

Step 1-1: We first cut $u^{\rho}_{T,(0)}$ and extend to obtain maps $\hat u^{\rho}_{i,T,(0)} : (\Sigma_i, \partial\Sigma_i) \to (X, L)$ $(i = 1, 2)$ as follows. (This map is used to set the linearized operator (1.36).)

$$\begin{aligned}&\hat u^{\rho}_{1,T,(0)}(z)\\ &= \begin{cases} \chi^{\leftarrow}_{\mathcal{B}}(\tau - T, t) u^{\rho}_{T,(0)}(\tau, t) + \chi^{\rightarrow}_{\mathcal{B}}(\tau - T, t) p^{\rho} & \text{if } z = (\tau, t) \in [-5T, 5T] \times [0,1]\\ u^{\rho}_{T,(0)}(z) & \text{if } z \in K_1\\ p^{\rho} & \text{if } z \in [5T, \infty) \times [0,1].\end{cases}\\ &\hat u^{\rho}_{2,T,(0)}(z)\\ &= \begin{cases} \chi^{\rightarrow}_{\mathcal{A}}(\tau + T, t) u^{\rho}_{T,(0)}(\tau, t) + \chi^{\leftarrow}_{\mathcal{A}}(\tau + T, t) p^{\rho} & \text{if } z = (\tau, t) \in [-5T, 5T] \times [0,1]\\ u^{\rho}_{T,(0)}(z) & \text{if } z \in K_2\\ p^{\rho} & \text{if } z \in (-\infty, -5T] \times [0,1].\end{cases}\end{aligned} \tag{1.35}$$

Let

$$\begin{aligned} D_{\hat u^{\rho}_{i,T,(0)}}\overline{\partial} : L^2_{m+1,\delta}((\Sigma_i, \partial\Sigma_i); &(\hat u^{\rho}_{i,T,(0)})^*TX, (\hat u^{\rho}_{i,T,(0)})^*TL)\\ &\to L^2_{m,\delta}(\Sigma_i; (\hat u^{\rho}_{i,T,(0)})^*TX \otimes \Lambda^{01})\end{aligned} \tag{1.36}$$

be the linearization of the Cauchy-Riemann equation.

Lemma 1.14. *We put $E_i = E_i(\hat u^{\rho}_{i,T,(0)})$. We have*

$$\mathrm{Im}(D_{\hat u^{\rho}_{i,T,(0)}}\overline{\partial}) + E_i = L^2_{m,\delta}(\Sigma_i; (\hat u^{\rho}_{i,T,(0)})^*TX \otimes \Lambda^{01}). \tag{1.37}$$

Moreover

$$\mathrm{Dev}_{1,\infty} - \mathrm{Dev}_{2,\infty} : (D_{\hat u^{\rho}_{1,T,(0)}}\overline{\partial})^{-1}(E_1) \oplus (D_{\hat u^{\rho}_{2,T,(0)}}\overline{\partial})^{-1}(E_2) \to T_{p^{\rho}}L \tag{1.38}$$

is surjective.

Proof. Since $\hat u^{\rho}_{i,T,(0)}$ is close to u_i in exponential order, this is a consequence of Assumption 1.3. □

Note that $E_i(u'_i)$ actually depends on u'_i. So to obtain a linearized equation of (1.15) we need to take into account of that effect. Let $\Pi_{E_i(u'_i)}$ be the projection to $E_i(u'_i)$ with respect to the L^2 norm. Namely we put

$$\Pi_{E_i(u'_i)}(A) = \sum_{a=1}^{\dim E_i} \langle\langle A, \mathbf{e}_{i,a}(u'_i)\rangle\rangle_{L^2(K_1)}\mathbf{e}_{i,a}(u'_i), \tag{1.39}$$

where $\mathbf{e}_{i,a}$, $a = 1, \dots, \dim E_i(u'_i)$ is an orthonormal basis of $E_i(u'_i)$ which are supported in K_i.

We put

$$(D_{u'_i}E_i)(A, v) = \frac{d}{ds}(\Pi_{E_i(\mathrm{E}(u'_i,sv))}(A))|_{s=0}. \tag{1.40}$$

Here $v \in \Gamma((\Sigma_i, \partial\Sigma_i), (u'_i)^*TX, (u'_i)^*TL)$. (Then $\mathrm{E}(u'_i, sv)$ is a map $(\Sigma_i, \partial\Sigma_i) \to (X, L)$ defined in (1.30).)

Remark 1.15. We use an isomorphism

$$\Gamma(\Sigma_i; \mathrm{E}(u'_i, sv)^*TX \otimes \Lambda^{01}) \cong \Gamma(\Sigma_i; (u'_i)^*TX \otimes \Lambda^{01}) \tag{1.41}$$

to define the right hand side of (1.40). The map (1.41) is defined as follows. Let $z \in \Sigma_i$. We have a path $r \mapsto \mathrm{E}(u'_i(z), rsv(z))$ joining $u'_i(z)$ to $\mathrm{E}(u'_i, sv)(z)$. We use a connection ∇ such that TL is parallel to define a parallel transport along this path. Its complex linear part defines an isomorphism (1.41).

We note that the same isomorphism (1.41) is used also to define $D_{u'_i}\overline{\partial}$. Namely

$$(D_{u'_i}\overline{\partial})(v) = \frac{d}{ds}(\overline{\partial}\mathrm{E}(u'_i, sv))|_{s=0}$$

where the right hand side is defined by using (1.41).

We put

$$\Pi^{\perp}_{E_i(u'_i)}(A) = A - \Pi_{E_i(u'_i)}(A).$$

The equation (1.17) is equivalent to the following

$$\Pi^{\perp}_{E_i(u'_i)}\overline{\partial}u'_i = 0. \tag{1.42}$$

We calculate the linearization

$$\left.\frac{\partial}{\partial s}\Pi^{\perp}_{E_i(\mathrm{E}(u'_i,sV))}\overline{\partial}\mathrm{E}(u'_i, sV))\right|_{s=0}$$

to obtain the linearized equation:

$$D_{u'_i}\overline{\partial}(V) - (D_{u'_i}E_i)(\overline{\partial}u'_i, V) \equiv 0 \mod E_i(u'_i). \tag{1.43}$$

We note that

$$\overline{\partial}\hat{u}^{\rho}_{i,T,(0)} - \mathfrak{e}^{\rho}_{i,T,(0)}$$

is exponentially small. So we use the operator

$$V \mapsto D_{\hat{u}^{\rho}_{i,T,(0)}}\overline{\partial}(V) - (D_{\hat{u}^{\rho}_{i,T,(0)}}E_i)(\mathfrak{e}^{\rho}_{i,T,(0)}, V) \tag{1.44}$$

as an approximation of the linearlization of (1.42).

Lemma 1.16. *We put* $E_i = E_i(\hat u^\rho_{i,T,(0)})$. *We have*

$$\operatorname{Im}(D_{\hat u^\rho_{i,T,(0)}}\overline\partial - (D_{\hat u^\rho_{i,T,(0)}}E_i)(\mathfrak e^\rho_{i,T,(0)},\cdot)) + E_i = L^2_{m,\delta}(\Sigma_i;(\hat u^\rho_{i,T,(0)})^*TX\otimes\Lambda^{01}). \tag{1.45}$$

Moreover

$$\begin{aligned} D\mathrm{ev}_{1,\infty} - D\mathrm{ev}_{2,\infty} : &(D_{\hat u^\rho_{1,T,(0)}}\overline\partial - (D_{\hat u^\rho_{1,T,(0)}}E_1)(\mathfrak e^\rho_{1,T,(0)},\cdot))^{-1}(E_1)\\ &\oplus (D_{\hat u^\rho_{2,T,(0)}}\overline\partial - (D_{\hat u^\rho_{2,T,(0)}}E_2)(\mathfrak e^\rho_{2,T,(0)},\cdot))^{-1}(E_2) \to T_{p^\rho}L \end{aligned} \tag{1.46}$$

is surjective.

Proof. (1.34) implies that $(D_{\hat u^\rho_{1,T,(0)}}E_1)(\mathfrak e^\rho_{1,T,(0)},\cdot)$ is small in operator norm. The lemma follows from Lemma 1.14. □

Remark 1.17. Note that (1.34) is proved by taking V_i in a small neighborhood of 0 (in $\tilde V_i$) with respect to the C^m norm. (Note $V_i \subset \mathcal M^{E_i}((\Sigma_i,\vec z_i);\beta_i)_{\epsilon_2}$ and V_i consists of smooth maps.) However we can take V_i that is independent of m and the conclusion of Lemma 1.16 holds for m. In fact the elliptic regularity implies that if the conclusion of Lemma 1.16 holds for some m then it holds for all $m' > m$. (The inequality (1.34) holds for that particular m only. However this inequality is used to show Lemma 1.16 only.)

We consider

$$\begin{aligned} &\operatorname{Ker}(D\mathrm{ev}_{1,\infty} - D\mathrm{ev}_{2,\infty})\\ \cap &\Big((D_{\hat u^\rho_{1,T,(0)}}\overline\partial - (D_{\hat u^\rho_{1,T,(0)}}E_1)(\mathfrak e^\rho_{1,T,(0)},\cdot)))^{-1}(E_1)\\ &\oplus (D_{\hat u^\rho_{2,T,(0)}}\overline\partial - (D_{\hat u^\rho_{2,T,(0)}}E_2)(\mathfrak e^\rho_{2,T,(0)},\cdot))^{-1}(E_2)\Big). \end{aligned} \tag{1.47}$$

This is a finite dimensional subspace of

$$\operatorname{Ker}(D\mathrm{ev}_{1,\infty} - D\mathrm{ev}_{2,\infty}) \cap \bigoplus_{i=1}^2 L^2_{m+1,\delta}((\Sigma_i,\partial\Sigma_i);(\hat u^\rho_{i,T,(0)})^*TX,(\hat u^\rho_{i,T,(0)})^*TL) \tag{1.48}$$

consisting of smooth sections.

Definition 1.18. We denote by $\mathfrak H(E_1,E_2)$ the intersection of the L^2 orthogonal complement of (1.47) with (1.48). Here the L^2 inner product is defined by (1.29).

Definition 1.19. We define $(V^\rho_{T,1,(1)}, V^\rho_{T,2,(1)}, \Delta p^\rho_{T,(1)})$ as follows.

$$\begin{aligned} (D_{\hat u^\rho_{i,T,(0)}}\overline\partial)(V^\rho_{T,i,(1)}) - (D_{\hat u^\rho_{i,T,(0)}}E_i)(\mathfrak e^\rho_{i,T,(0)}, V^\rho_{T,i,(1)})&\\ + \mathrm{Err}^\rho_{i,T,(0)} \in E_i(\hat u^\rho_{i,T,(0)}).& \end{aligned} \tag{1.49}$$

$$D\mathrm{ev}_\infty(V^\rho_{T,1,(1)}) = D\mathrm{ev}_{-\infty}(V^\rho_{T,2,(1)}) = \Delta p^\rho_{T,(1)}. \tag{1.50}$$

Moreover

$$((V^\rho_{T,1,(1)},\Delta p^\rho_{T,(1)}),(V^\rho_{T,2,(1)},\Delta p^\rho_{T,(1)})) \in \mathfrak H(E_1,E_2).$$

Lemma 1.16 implies that such $(V^\rho_{T,1,(1)}, V^\rho_{T,2,(1)}, \Delta p^\rho_{T,(1)})$ exists and is unique.

Lemma 1.20. *If* $\delta < \delta_1/10$, *then*

$$\|(V^\rho_{T,i,(1)},\Delta p^\rho_{T,(1)})\|_{L^2_{m+1,\delta}(\Sigma_i)} \le C_{2,m}e^{-\delta T}, \qquad |\Delta p^\rho_{T,(1)}| \le C_{2,m}e^{-\delta T}. \tag{1.51}$$

This is immediate from construction and the uniform boundedness of the right inverse of $D_{\hat u^\rho_{i,T,(0)}}\overline{\partial} - (D_{\hat u^\rho_{i,T,(0)}}E_i)(\mathfrak{e}^\rho_{i,T,(0)},\cdot)$.

Step 1-2: We use $(V^\rho_{T,1,(1)}, V^\rho_{T,2,(1)}, \Delta p^\rho_{T,(1)})$ to find an approximate solution $u^\rho_{T,(1)}$ of the next level.

Definition 1.21. We define $u^\rho_{T,(1)}(z)$ as follows. (Here E is as in (1.30).)

(1) If $z \in K_1$, we put

$$u^\rho_{T,(1)}(z) = \mathrm{E}(\hat u^\rho_{1,T,(0)}(z), V^\rho_{T,1,(1)}(z)). \tag{1.52}$$

(2) If $z \in K_2$, we put

$$u^\rho_{T,(1)}(z) = \mathrm{E}(\hat u^\rho_{2,T,(0)}(z), V^\rho_{T,2,(1)}(z)). \tag{1.53}$$

(3) If $z = (\tau,t) \in [-5T,5T]\times[0,1]$, we put

$$\begin{aligned} u^\rho_{T,(1)}(\tau,t) =& \chi_{\mathcal{B}}^{\leftarrow}(\tau,t)(V^\rho_{T,1,(1)}(\tau,t) - \Delta p^\rho_{T,(1)}) \\ &+ \chi_{\mathcal{A}}^{\rightarrow}(\tau,t)(V^\rho_{T,2,(1)}(\tau,t) - \Delta p^\rho_{T,(1)}) + u^\rho_{T,(0)}(\tau,t) + \Delta p^\rho_{T,(1)}. \end{aligned} \tag{1.54}$$

We recall that $\hat u^\rho_{1,T,(0)}(z) = u^\rho_{T,(0)}(z)$ on K_1 and $\hat u^\rho_{2,T,(0)}(z) = u^\rho_{T,(0)}(z)$ on K_2.

Step 1-3: Let $0<\mu<1$. We fix it throughout the proof.

Lemma 1.22. *There exists δ_2 such that for any $\delta<\delta_2$, $T>T(\delta,m,\epsilon_{5,m})$ there exists $\mathfrak{e}^\rho_{i,T,(1)} \in E_i$ with the following properties.*

$$\|\overline{\partial} u^\rho_{T,(1)} - (\mathfrak{e}^\rho_{1,T,(0)} + \mathfrak{e}^\rho_{1,T,(1)}) - (\mathfrak{e}^\rho_{2,T,(0)} + \mathfrak{e}^\rho_{2,T,(1)})\|_{L^2_{m,\delta}} < C_{1,m}\mu\epsilon_{5,m}e^{-\delta T}.$$

(Here $C_{1,m}$ is the constant given in Lemma 1.12.) Moreover

$$\|\mathfrak{e}^\rho_{i,T,(1)}\|_{L^2_m(K_i)} < C_{3,m}e^{-\delta T}. \tag{1.55}$$

Proof. The existence of $\mathfrak{e}^\rho_{i,T,(1)}$ satisfying

$$\|\overline{\partial} u^\rho_{T,(1)} - (\mathfrak{e}^\rho_{1,T,(0)} + \mathfrak{e}^\rho_{1,T,(1)}) - (\mathfrak{e}^\rho_{2,T,(0)} + \mathfrak{e}^\rho_{2,T,(1)})\|_{L^2_{m,\delta}(K_1\cup K_2\subset\Sigma_T)} < C_{1,m}\mu\epsilon_{5,m}e^{-\delta T}/10$$

is a consequene of the fact that (1.43) is the linearized equation of (1.42) and the estimate (1.51). More explicitly we can prove it by a routine calculation as follows. We first estimate on K_1. We have:

$$\begin{aligned} &\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(0)}, V^\rho_{T,1,(1)})) \\ &= \overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(0)},0)) + \int_0^1 \frac{\partial}{\partial s}\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(0)}, sV^\rho_{T,1,(1)}))ds \\ &= \overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(0)},0)) + (D_{\hat u^\rho_{1,T,(0)}}\overline{\partial})(V^\rho_{T,1,(1)}) \\ &\qquad + \int_0^1 ds\int_0^s \frac{\partial^2}{\partial r^2}\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(0)}, rV^\rho_{T,1,(1)}))dr. \end{aligned} \tag{1.56}$$

We remark

$$\begin{aligned} &\left\|\int_0^1 ds\int_0^s \frac{\partial^2}{\partial r^2}\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(0)}, rV^\rho_{T,1,(1)}))dr\right\|_{L^2_m(K_1)} \\ &\le C_{3,m}\|V^\rho_{T,1,(1)}\|^2_{L^2_{m+1,\delta}} \le C_{4,m}e^{-2\delta T}. \end{aligned} \tag{1.57}$$

We have

$$\begin{aligned}
&\Pi^{\perp}_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))}\\
&\quad= \Pi^{\perp}_{E_1(\hat u^{\rho}_{1,T,(0)})} + \int_0^1 \frac{\partial}{\partial s}\Pi^{\perp}_{E_i(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},sV^{\rho}_{T,1,(1)}))} ds\\
&\quad= \Pi^{\perp}_{E_1(\hat u^{\rho}_{1,T,(0)})} - (D_{\hat u^{\rho}_{1,T,(0)}}E_1)(\cdot, V^{\rho}_{T,1,(1)})\\
&\qquad\qquad + \int_0^1 ds \int_0^s \frac{\partial^2}{\partial r^2}\Pi^{\perp}_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},rV^{\rho}_{T,1,(1)}))} dr.
\end{aligned}\tag{1.58}$$

We can estimate the third term of the right hand side of (1.58) in the same way as in (1.57).

On the other hand, (1.56) implies that

$$\left\|\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)}, V^{\rho}_{T,1,(1)})) - \mathfrak{e}^{\rho}_{1,T,(0)}\right\|_{L^2_m(K_1)} \le C_{6,m}e^{-\delta T}. \tag{1.59}$$

Therefore, using (1.58) and (1.51), we have

$$\begin{aligned}
&\Big\|\Pi^{\perp}_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))}\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)}, V^{\rho}_{T,1,(1)}))\\
&- \Pi^{\perp}_{E_1(\hat u^{\rho}_{1,T,(0)},0)}\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)}, V^{\rho}_{T,1,(1)}))\\
&- \Pi^{\perp}_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))}(\mathfrak{e}^{\rho}_{1,T,(0)}) + \Pi^{\perp}_{E_1(\hat u^{\rho}_{1,T,(0)},0)}(\mathfrak{e}^{\rho}_{1,T,(0)})\Big\|_{L^2_m(K_1)} \le C_{7,m}e^{-2\delta T}.
\end{aligned}\tag{1.60}$$

Therefore using (1.58) we have:

$$\begin{aligned}
&\|\Pi^{\perp}_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))}\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)}, V^{\rho}_{T,1,(1)}))\\
&- \Pi^{\perp}_{E_1(\hat u^{\rho}_{1,T,(0)},0)}\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)}, V^{\rho}_{T,1,(1)}))\\
&+ (D_{\hat u^{\rho}_{1,T,(0)}}E_1)(\mathfrak{e}^{\rho}_{1,T,(0)}, V^{\rho}_{T,1,(1)})\|_{L^2_m(K_1)} \le C_{8,m}e^{-2\delta T}.
\end{aligned}\tag{1.61}$$

By (1.49) and Definition 1.13, we have:

$$\begin{aligned}
&\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)}, 0)) + (D_{\hat u^{\rho}_{1,T,(0)}}\overline{\partial})(V^{\rho}_{T,1,(1)})\\
&\qquad - (D_{\hat u^{\rho}_{1,T,(0)}}E_1)(\mathfrak{e}^{\rho}_{1,T,(0)}, V^{\rho}_{T,1,(1)}) \in E_1(\hat u^{\rho}_{1,T,(0)})
\end{aligned}\tag{1.62}$$

on K_1.

(1.61) and (1.62) imply

$$\begin{aligned}
&\|\Pi^{\perp}_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))}\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)}, V^{\rho}_{T,1,(1)}))\\
&- \Pi^{\perp}_{E_1(\hat u^{\rho}_{1,T,(0)},0)}\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)}, V^{\rho}_{T,1,(1)}))\\
&+ \Pi^{\perp}_{E_1(\hat u^{\rho}_{1,T,(0)},0)}\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)}, 0))\\
&+ \Pi^{\perp}_{E_1(\hat u^{\rho}_{1,T,(0)},0)}(D_{\hat u^{\rho}_{1,T,(0)}}\overline{\partial})(V^{\rho}_{T,1,(1)})\|_{L^2_m(K_1)} \le C_{9,m}e^{-2\delta T}.
\end{aligned}\tag{1.63}$$

Combined with (1.56) and (1.57), we have

$$\begin{aligned}
&\|\Pi^{\perp}_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))}(\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)}, V^{\rho}_{T,1,(1)})))\|_{L^2_m(K_1)}\\
&\le C_{10,m}e^{-2\delta T} \le C_{1,m}e^{-\delta T}\epsilon_{5,m}\mu/10,
\end{aligned}\tag{1.64}$$

for $T > T_m$ if we choose T_m so that $C_{10,m}e^{-\delta T_m} < C_{1,m}\epsilon_{5,m}\mu/10$.

It follows from (1.59) and (1.64) that

$$\|\Pi_{E_1(\mathrm{E}(\hat u^\rho_{1,T,(0)},V^\rho_{T,1,(1)}))}(\overline\partial(\mathrm{E}(\hat u^\rho_{1,T,(0)},V^\rho_{T,1,(1)})) - \mathfrak{e}^\rho_{1,T,(0)}\|_{L^2_m(K_1)} \le C_{11,m}e^{-\delta T}.$$

Then (1.55) follows, by selecting

$$\mathfrak{e}^\rho_{1,T,(1)} = \Pi_{E_1(\mathrm{E}(\hat u^\rho_{1,T,(0)},V^\rho_{T,1,(1)}))}(\overline\partial(\mathrm{E}(\hat u^\rho_{1,T,(0)},V^\rho_{T,1,(1)}) - \mathfrak{e}^\rho_{1,T,(0)}).$$

The estimate on K_2 is the same.

Let us estimate $\overline\partial u^\rho_{T,(1)}$ on $[-T+1,T-1]\times[0,1]$. The inequality

$$\|\overline\partial u^\rho_{T,(1)}\|_{L^2_{m,\delta}([-T+1,T-1]\times[0,1]\subset\Sigma_T)} < C_{1,m}\mu\epsilon_{5,m}e^{-\delta T}/10$$

is also a consequence of the fact that (1.43) is the linearized equation of (1.42) and the estimate (1.51). (Note the bump functions $\chi^{\leftarrow}_{\mathcal B}$ and $\chi^{\rightarrow}_{\mathcal A}$ are $\equiv 1$ there.) On $\mathcal A_T$ we have

$$\overline\partial u^\rho_{T,(1)} = \overline\partial(\chi^{\rightarrow}_{\mathcal A}(V^\rho_{T,2,(1)} - \Delta p^\rho_{T,(1)}) + V^\rho_{T,1,(1)} + u^\rho_{T,(0)}). \tag{1.65}$$

Note

$$\begin{aligned}\|\overline\partial(\chi^{\rightarrow}_{\mathcal A}(V^\rho_{T,2,(1)} - \Delta p^\rho_{T,(1)})\|_{L^2_m(\mathcal A_T)} &\le C_{3,m}e^{-6T\delta}\|V^\rho_{T,2,(1)} - \Delta p^\rho_{T,(1)}\|_{L^2_{m+1,\delta}(\mathcal A_T\subset\Sigma_2)}\\ &\le C_{12,m}e^{-7T\delta}.\end{aligned}$$

The first inequality follows from the fact the weight function $e_{2,\delta}$ is around $e^{6T\delta}$ on $\mathcal A_T$. The second inequality follows from (1.51). On the other hand the weight function $e_{T,\delta}$ is around $e^{4T\delta}$ at $\mathcal A_T$.[4] Therefore

$$\|\overline\partial(\chi^{\rightarrow}_{\mathcal A}(V^\rho_{T,2,(1)} - \Delta p^\rho_{T,(1)}))\|_{L^2_{m,\delta}(\mathcal A_T\subset\Sigma_T)} \le C_{13,m}e^{-3T\delta}. \tag{1.66}$$

Note

$$\mathrm{Err}^\rho_{2,T,(0)} = 0$$

on $\mathcal A_T$. Using this in the same way as we did on K_1 we can show

$$\|\overline\partial(V^\rho_{T,1,(1)} + u^\rho_{T,(0)})\|_{L^2_{m,\delta}(\mathcal A_T\subset\Sigma_T)} \le C_{1,m}e^{-\delta T}\epsilon_{5,m}\mu/20 \tag{1.67}$$

for $T > T_m$. Therefore by taking T large we have

$$\|\overline\partial u^\rho_{T,(1)}\|_{L^2_{m,\delta}(\mathcal A_T\subset\Sigma_T)} < C_{1,m}\mu\epsilon_{5,m}e^{-\delta T}/10. \tag{1.68}$$

(Note that the almost complex structure may not be integrable. So the almost complex structure may not be constant with respect to the flat metric we are taking in the neighborhood of p_0. However we can still deduce (1.68) from (1.67) and (1.66).)

The estimate on $\mathcal B_T$ and on $([-5T,-T-1]\cup[T+1,5T])\times[0,1]$ are similar. The proof of Lemma 1.22 is complete. □

Step 1-4:

Definition 1.23. We put

$$\begin{aligned}\mathrm{Err}^\rho_{1,T,(1)} &= \chi^{\leftarrow}_{\mathcal X}(\overline\partial u^\rho_{T,(1)} - (\mathfrak{e}^\rho_{1,T,(0)} + \mathfrak{e}^\rho_{1,T,(1)})),\\ \mathrm{Err}^\rho_{2,T,(1)} &= \chi^{\rightarrow}_{\mathcal X}(\overline\partial u^\rho_{T,(1)} - (\mathfrak{e}^\rho_{2,T,(0)} + \mathfrak{e}^\rho_{2,T,(1)})).\end{aligned}$$

[4]This drop of the weight is the main part of the idea. It was used in [FOOO1, page 414]. See [FOOO1, Figure 7.1.6].

We regard them as elements of the weighted Sobolev spaces $L^2_{m,\delta}(\Sigma_1;(\hat u^\rho_{1,T,(1)})^*TX\otimes\Lambda^{01})$ and $L^2_{m,\delta}(\Sigma_2;(\hat u^\rho_{2,T,(1)})^*TX\otimes\Lambda^{01})$ respectively. (We extend them by 0 outside a compact set.)

We put $p^\rho_{(1)} = p^\rho + \Delta p^\rho_{T,(1)}$.

We now come back to the Step 2-1 and continue. In other words, we will prove the following by induction on κ.

$$\left\|(V^\rho_{T,i,(\kappa)},\Delta p^\rho_{T,(\kappa)})\right\|_{L^2_{m+1,\delta}(\Sigma_i)} < C_{2,m}\mu^{\kappa-1}e^{-\delta T}, \tag{1.69}$$

$$\left\|\Delta p^\rho_{T,(\kappa)}\right\| < C_{2,m}\mu^{\kappa-1}e^{-\delta T}, \tag{1.70}$$

$$\left\|u^\rho_{T,(\kappa)} - u^\rho_{T,(0)}\right\|_{L^2_{m+1,\delta}(\Sigma_T)} < C_{14,m}e^{-\delta T}, \tag{1.71}$$

$$\left\|\mathrm{Err}^\rho_{i,T,(\kappa)}\right\|_{L^2_{m,\delta}(\Sigma_i)} < C_{1,m}\epsilon_{5,m}\mu^{\kappa}e^{-\delta T}, \tag{1.72}$$

$$\left\|\mathfrak{e}^\rho_{i,T,(\kappa)}\right\|_{L^2_m(K_i^{\rm obst})} < C_{15,m}\mu^{\kappa-1}e^{-\delta T}, \quad \text{for } \kappa\ge 1. \tag{1.73}$$

Remark 1.24. The left hand side of (1.71) is defined as follows. We define $\mathfrak{u}^\rho_{T,(\kappa)}$ by $u^\rho_{T,(\kappa)} = \mathrm{E}(u^\rho_{T,(\kappa-1)},\mathfrak{u}^\rho_{T,(\kappa)})$. Then the left hand side of (1.71) is

$$\|\mathfrak{u}^\rho_{T,(\kappa)}\|_{L^2_{m+1,\delta}((\Sigma_T,\partial\Sigma_T);(u^\rho_{T,(\kappa-1)})^*TX,(u^\rho_{T,(\kappa-1)})^*TL)}.$$

More precisely the claim we will prove is: for any $\epsilon_{5,m}$ we can choose T_m so that (1.69) and (1.70) imply (1.72) and (1.73) for given $T>T_m$, and we can choose $\epsilon_{5,m}$ so that (1.72) and (1.73) for κ implies (1.69) and (1.70) for $\kappa+1$. (It is easy to see that (1.69) and (1.70) imply (1.71).)

Below we describe Steps κ-1,…,κ-4.

Step κ-1:

We first cut $u^\rho_{T,(\kappa-1)}$ and extend to obtain maps $\hat u^\rho_{i,T,(\kappa-1)} : (\Sigma_i,\partial\Sigma_i)\to(X,L)$ $(i=1,2)$ as follows.

$$\hat u^\rho_{1,T,(\kappa-1)}(z) = \begin{cases} \chi_{\mathcal B}^{\leftarrow}(\tau-T,t)u^\rho_{T,(\kappa-1)}(\tau,t) + \chi_{\mathcal B}^{\rightarrow}(\tau-T,t)p^\rho_{(\kappa-1)} & \text{if } z=(\tau,t)\in[-5T,5T]\times[0,1] \\ u^\rho_{T,(\kappa-1)}(z) & \text{if } z\in K_1 \\ p^\rho_{T,(\kappa-1)} & \text{if } z\in[5T,\infty)\times[0,1]. \end{cases}$$

$$\hat u^\rho_{2,T,(\kappa-1)}(z) = \begin{cases} \chi_{\mathcal A}^{\rightarrow}(\tau+T,t)u^\rho_{T,(\kappa-1)}(\tau,t) + \chi_{\mathcal A}^{\leftarrow}(\tau+T,t)p^\rho_{(\kappa-1)} & \text{if } z=(\tau,t)\in[-5T,5T]\times[0,1] \\ u^\rho_{T,(\kappa-1)}(z) & \text{if } z\in K_2 \\ p^\rho_{T,(\kappa-1)} & \text{if } z\in(-\infty,-5T]\times[0,1]. \end{cases} \tag{1.74}$$

Let

$$\begin{aligned} D_{\hat u^\rho_{i,T,(\kappa-1)}}\overline\partial : L^2_{m+1,\delta}((\Sigma_i,\partial\Sigma_i);&(\hat u^\rho_{i,T,(\kappa-1)})^*TX,(\hat u^\rho_{i,T,(\kappa-1)})^*TL) \\ &\to L^2_{m,\delta}(\Sigma_i;(\hat u^\rho_{i,T,(\kappa-1)})^*TX\otimes\Lambda^{01}). \end{aligned} \tag{1.75}$$

Lemma 1.25. *We have*

$$\mathrm{Im}(D_{\hat{u}^{\rho}_{i,T,(\kappa-1)}}\overline{\partial}) + E_i = L^2_{m,\delta}(\Sigma_i; (\hat{u}^{\rho}_{i,T,(\kappa-1)})^*TX \otimes \Lambda^{01}). \tag{1.76}$$

Moreover

$$\mathrm{Dev}_{1,\infty} - \mathrm{Dev}_{2,\infty} : (D_{\hat{u}^{\rho}_{1,T,(0)}}\overline{\partial})^{-1}(E_1) \oplus (D_{\hat{u}^{\rho}_{2,T,(0)}}\overline{\partial})^{-1}(E_2) \to T_{p^{\rho}_{T,(\kappa-1)}}L \tag{1.77}$$

is surjective.

Proof. Since $\hat{u}^{\rho}_{i,T,(\kappa-1)}$ is close to u_i in exponential order, this is a consequence of Assumption 1.3. □

We denote

$$(\mathfrak{se})^{\rho}_{i,T,(\kappa-1)} = \sum_{a=0}^{\kappa-1} \mathfrak{e}^{\rho}_{i,T,(a)}. \tag{1.78}$$

Lemma 1.26. *We have*

$$\begin{aligned}&\mathrm{Im}(D_{\hat{u}^{\rho}_{i,T,(\kappa-1)}}\overline{\partial} - (D_{\hat{u}^{\rho}_{i,T,(\kappa-1)}}E_i)((\mathfrak{se})^{\rho}_{i,T,(\kappa-1)},\cdot)) + E_i \\ &= L^2_{m,\delta}(\Sigma_i; (\hat{u}^{\rho}_{i,T,(\kappa-1)})^*TX \otimes \Lambda^{01}).\end{aligned} \tag{1.79}$$

Moreover

$$\begin{aligned}&\mathrm{Dev}_{1,\infty} - \mathrm{Dev}_{2,\infty} \\ &:(D_{\hat{u}^{\rho}_{1,T,(\kappa-1)}}\overline{\partial} - (D_{\hat{u}^{\rho}_{1,T,(\kappa-1)}}E_1)((\mathfrak{se})^{\rho}_{1,T,(\kappa-1)},\cdot)))^{-1}(E_1) \\ &\oplus (D_{\hat{u}^{\rho}_{2,T,(\kappa-1)}}\overline{\partial} - (D_{\hat{u}^{\rho}_{2,T,(\kappa-1)}}E_2)((\mathfrak{se})^{\rho}_{2,T,(\kappa-1)},\cdot))^{-1}(E_2) \to T_{p^{\rho}_{T,(\kappa-1)}}L\end{aligned} \tag{1.80}$$

is surjective.

Proof.

$$\left\Vert \sum_{a=0}^{\kappa-1} \mathfrak{e}^{\rho}_{i,T,(a)} \right\Vert_{L^2_m(K_i)} < \epsilon_{4,m} + C_{15,m}\frac{e^{-\delta T}}{1-\mu}. \tag{1.81}$$

imply that $(D_{\hat{u}^{\rho}_{1,T,(0)}}E_1)(\mathfrak{e}^{\rho}_{1,T,(0)},\cdot)$ is small in operator norm. The lemma follows from Lemma 1.25. □

Note that Remark 1.17 still applies to Lemma 1.26.

Definition 1.27. We define $(V^{\rho}_{T,1,(\kappa)}, V^{\rho}_{T,2,(\kappa)}, \Delta p^{\rho}_{T,(\kappa)})$ as follows.

$$\begin{aligned}D_{\hat{u}^{\rho}_{i,T,(\kappa-1)}}(V^{\rho}_{T,i,(\kappa)}) - (D_{\hat{u}^{\rho}_{i,T,(\kappa-1)}}E_i)((\mathfrak{se})^{\rho}_{i,T,(\kappa-1)}, V^{\rho}_{T,i,(\kappa)}) \\ + \mathrm{Err}^{\rho}_{i,T,(\kappa-1)} \in E_i(\hat{u}^{\rho}_{i,T,(\kappa-1)}).\end{aligned} \tag{1.82}$$

$$\mathrm{Dev}_{1,\infty}(V^{\rho}_{T,1,(\kappa)}) = \mathrm{Dev}_{2,\infty}(V^{\rho}_{T,2,(\kappa)}) = \Delta p^{\rho}_{T,(\kappa)}. \tag{1.83}$$

We also require

$$((V^{\rho}_{T,1,(\kappa)}, \Delta p^{\rho}_{T,(\kappa)}), (V^{\rho}_{T,2,(\kappa)}, \Delta p^{\rho}_{T,(\kappa)})) \in \mathfrak{H}(E_1,E_2). \tag{1.84}$$

Lemma 1.26 implies that such $(V^{\rho}_{T,1,(\kappa)}, V^{\rho}_{T,2,(\kappa)}, \Delta p^{\rho}_{T,(\kappa)})$ exists and is unique.

Remark 1.28. Note in (1.84) we use the same space $\mathfrak{H}(E_1,E_2)$ as in Definition 1.19. We may use the orthogonal complement of

$$\mathrm{Ker}(\mathrm{Dev}_{1,\infty} - \mathrm{Dev}_{2,\infty}) \cap \bigoplus_{i=1}^{2}(D_{\hat{u}^{\rho}_{i,T,(\kappa-1)}}\overline{\partial} - (D_{\hat{u}^{\rho}_{i,T,(\kappa-1)}}E_i)((\mathfrak{se})^{\rho}_{i,T,(\kappa-1)},\cdot))^{-1}(E_i)$$

instead. The reason why we use the same space as one in Definition 1.19 here is that then a calculation we need to do for the exponential decay estimate of T derivative becomes a bit shorter. Since $\hat u^\rho_{i,T,(\kappa)}$ is sufficiently close to $\hat u^\rho_{i,T,(0)}$, the unique existence of $(V^\rho_{T,1,(\kappa)}, V^\rho_{T,2,(\kappa)}, \Delta p^\rho_{T,(\kappa)})$ satisfying (1.82) - (1.84) holds by (1.81).

Lemma 1.29. *If $\delta < \delta_1/10$ and $T > T(\delta, m)$, then*

$$\begin{aligned} &\|(V_{T,i,(\kappa)}, \Delta p^\rho_{T,(\kappa)})\|_{L^2_{m+1,\delta}(\Sigma_i)} \le C_{2,m}\mu^{\kappa-1}e^{-\delta T},\\ &|\Delta p^\rho_{T,(\kappa)}| \le C_{2,m}\mu^{\kappa-1}e^{-\delta T}. \end{aligned} \tag{1.85}$$

Proof. This follows from uniform boundedness of the inverse of (1.79) together with the $\kappa-1$ version of Lemma 1.22. (That is Lemma 1.31.) □

This lemma implies (1.69) and (1.70).

Step κ-2: We use $(V^\rho_{T,1,(\kappa)}, V^\rho_{T,2,(\kappa)}, \Delta p^\rho_{T,(\kappa)})$ to find an approximate solution $u^\rho_{T,(\kappa)}$ of the next level.

Definition 1.30. We define $u^\rho_{T,(\kappa)}(z)$ as follows.

(1) If $z \in K_1$, we put

$$u^\rho_{T,(\kappa)}(z) = \mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)}(z), V^\rho_{T,1,(\kappa)}(z)). \tag{1.86}$$

(2) If $z \in K_2$, we put

$$u^\rho_{T,(\kappa)}(z) = \mathrm{E}(\hat u^\rho_{2,T,(\kappa-1)}(z), V^\rho_{T,2,(\kappa)}(z)). \tag{1.87}$$

(3) If $z = (\tau, t) \in [-5T, 5T] \times [0,1]$, we put

$$\begin{aligned} u^\rho_{T,(\kappa)}(\tau,t) =& \chi^{\leftarrow}_{\mathcal B}(\tau,t)(V^\rho_{T,1,(\kappa)}(\tau,t) - \Delta p^\rho_{T,(\kappa)})\\ &+ \chi^{\rightarrow}_{\mathcal A}(\tau,t)(V^\rho_{T,2,(\kappa)}(\tau,t) - \Delta p^\rho_{T,(\kappa)})\\ &+ u^\rho_{T,(\kappa-1)}(\tau,t) + \Delta p^\rho_{T,(\kappa)}. \end{aligned} \tag{1.88}$$

We note that $\hat u^\rho_{1,T,(\kappa-1)}(z) = u^\rho_{T,(\kappa-1)}(z)$ on K_1 and $\hat u^\rho_{2,T,(\kappa-1)}(z) = u^\rho_{T,(\kappa-1)}(z)$ on K_2.

(2.267) is immediate from the definition and (1.69) and (1.70), since $0 < \mu < 1$.

Step κ-3:

Lemma 1.31. *For each $\epsilon_5 > 0$ we have the following. If $\delta < \delta_2$ and $T > T(\delta, m, \epsilon_5)$, then there exists $\mathfrak e^\rho_{i,T,(\kappa)} \in E_i$ such that*

$$\left\| \overline\partial u^\rho_{T,(\kappa)} - \sum_{a=0}^{\kappa} \mathfrak e^\rho_{1,T,(a)} - \sum_{a=0}^{\kappa} \mathfrak e^\rho_{2,T,(a)} \right\|_{L^2_{m,\delta}} < C_{1,m}\mu^\kappa \epsilon_5 e^{-\delta T}.$$

(Here $C_{1,m}$ is as in Lemma 1.12.) Moreover

$$\|\mathfrak e^\rho_{i,T,(\kappa)}\|_{L^2_m(K_i)} < C_{15,m}\mu^{\kappa-1}e^{-\delta T}. \tag{1.89}$$

Proof. The proof is similar to the proof of Lemma 1.22 and proceed as follows. We have:

$$\begin{aligned}
&\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},V^\rho_{T,1,(\kappa)}))\\
&\quad=\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},0))+\int_0^1\frac{\partial}{\partial s}\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},sV^\rho_{T,1,(\kappa)}))ds\\
&\quad=\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},0))+(D_{\hat u^\rho_{1,T,(\kappa-1)}}\overline{\partial})(V^\rho_{T,1,(\kappa)})\\
&\qquad\qquad+\int_0^1 ds\int_0^s\frac{\partial^2}{\partial r^2}\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},rV^\rho_{T,1,(\kappa)}))dr.
\end{aligned}\tag{1.90}$$

We remark

$$\begin{aligned}
&\left\|\int_0^1 ds\int_0^s\frac{\partial^2}{\partial r^2}\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},rV^\rho_{T,1,(\kappa)}))dr\right\|_{L^2_m(K_1)}\\
&\le C_{4,m}\|V^\rho_{T,1,(\kappa)}\|^2_{L^2_{m+1,\delta}}\le C_{5,m}e^{-2\delta T}\mu^{2(\kappa-1)}.
\end{aligned}\tag{1.91}$$

We have

$$\begin{aligned}
&\Pi^\perp_{E_1(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},V^\rho_{T,1,(\kappa)}))}\\
&\quad=\Pi^\perp_{E_1(\hat u^\rho_{1,T,(\kappa-1)})}+\int_0^1\frac{\partial}{\partial s}\Pi^\perp_{E_i(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},sV^\rho_{T,1,(\kappa)}))}ds\\
&\quad=\Pi^\perp_{E_1(\hat u^\rho_{1,T,(\kappa-1)})}-(D_{\hat u^\rho_{1,T,(\kappa-1)}}E_1)(\cdot,V^\rho_{T,1,(\kappa)})\\
&\qquad\qquad+\int_0^1 ds\int_0^s\frac{\partial^2}{\partial r^2}\Pi^\perp_{E_1(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},rV^\rho_{T,1,(\kappa)}))}dr.
\end{aligned}\tag{1.92}$$

We can estimate the third term of the right hand side of (1.92) in the same way as (1.91).

On the other hand, (1.90) implies that

$$\left\|\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},V^\rho_{T,1,(\kappa)}))-\mathfrak{se}^\rho_{1,T,(\kappa-1)}\right\|_{L^2_m(K_1)}\le C_{6,m}e^{-\delta T}\mu^{\kappa-1}.\tag{1.93}$$

Therefore

$$\begin{aligned}
&\|\Pi^\perp_{E_1(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},V^\rho_{T,1,(\kappa)}))}\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},V^\rho_{T,1,(\kappa)}))\\
&-\Pi^\perp_{E_1(\hat u^\rho_{1,T,(\kappa-1)},0)}\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},V^\rho_{T,1,(\kappa)}))\\
&+(D_{\hat u^\rho_{1,T,(\kappa-1)}}E_1)(\mathfrak{se}^\rho_{1,T,(\kappa-1)},V^\rho_{T,1,(\kappa)})\|_{L^2_m(K_1)}\le C_{7,m}e^{-2\delta T}\mu^{\kappa-1}.
\end{aligned}\tag{1.94}$$

By (1.82) we have:

$$\begin{aligned}
&\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},0))+(D_{\hat u^\rho_{1,T,(\kappa-1)}}\overline{\partial})(V^\rho_{T,1,(\kappa)})\\
&\qquad-(D_{\hat u^\rho_{1,T,(\kappa-1)}}E_1)(\mathfrak{se}^\rho_{1,T,(\kappa-1)},V^\rho_{T,1,(\kappa)})\in E_1(\hat u^\rho_{1,T,(\kappa-1)})
\end{aligned}\tag{1.95}$$

on K_1.

Summing up we have

$$\begin{aligned}
&\|\Pi^\perp_{E_1(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},V^\rho_{T,1,(\kappa)}))}(\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},V^\rho_{T,1,(\kappa)})))\|_{L^2_m(K_1)}\\
&\le C_{10,m}e^{-2\delta T}\mu^{\kappa-1}\le C_{1,m}e^{-\delta T}\epsilon_{5,m}\mu^\kappa/10
\end{aligned}\tag{1.96}$$

for $T>T_m$.

It follows from (1.93) that

$$\|\Pi_{E_1(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},V^\rho_{T,1,(\kappa)}))}(\overline\partial(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},V^\rho_{T,1,(\kappa)}))-\mathfrak{se}^\rho_{1,T,(\kappa-1)}\|_{L^2_m(K_1)} \le C_{8,m}e^{-\delta T}\mu^{\kappa-1}.$$

Then (1.89) follows by putting

$$\begin{aligned}\mathfrak e^\rho_{1,T,(\kappa)} &= \Pi_{E_1(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},V^\rho_{T,1,(\kappa)}))}(\overline\partial(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},V^\rho_{T,1,(\kappa)})) - \mathfrak{se}^\rho_{1,T,(\kappa-1)}\\ &\in E_1(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)},V^\rho_{T,1,(\kappa)})) \cong E_1.\end{aligned}$$

Let us estimate $\overline\partial u^\rho_{T,(\kappa)}$ on $[-T,T]\times[0,1]$. The inequality

$$\|\overline\partial u^\rho_{T,(\kappa)}\|_{L^2_{m,\delta}([-T,T]\times[0,1]\subset\Sigma_T)} < C_{1,m}\mu^\kappa\epsilon_{5,m}e^{-\delta T}/10$$

is also a consequence of the fact that (1.43) is the linearized equation of (1.42) and the estimate (1.85). (Note the bump functions $\chi^{\leftarrow}_{\mathcal B}$ and $\chi^{\rightarrow}_{\mathcal A}$ are $\equiv 1$ there.) On $\mathcal A_T$ we have

$$\overline\partial u^\rho_{T,(\kappa)} = \overline\partial(\chi^{\rightarrow}_{\mathcal A}(V^\rho_{T,2,(\kappa)} - \Delta p^\rho_{T,(\kappa)}) + V^\rho_{T,1,(\kappa)} + u^\rho_{T,(\kappa-1)}). \tag{1.97}$$

Note

$$\begin{aligned}\|\overline\partial(\chi^{\rightarrow}_{\mathcal A}(V^\rho_{T,2,(\kappa)} - \Delta p^\rho_{T,(\kappa)}))\|_{L^2_m(\mathcal A_T)} &\le C_{3,m}e^{-6T\delta}\|V^\rho_{T,2,(\kappa)} - \Delta p^\rho_{T,(\kappa)}\|_{L^2_{m+1,\delta}(\mathcal A_T\subset\Sigma_2)}\\ &\le C_{12,m}e^{-7T\delta}\mu^{\kappa-1}.\end{aligned}$$

The first inequality follows from the fact the weight function $e_{2,\delta}$ is around $e^{6T\delta}$ on $\mathcal A_T$. The second inequality follows from (1.85). On the other hand the weight function $e_{T,\delta}$ is around $e^{4T\delta}$ at $\mathcal A_T$.[5] Therefore

$$\|\overline\partial(\chi^{\rightarrow}_{\mathcal A}(V^\rho_{T,2,(\kappa)} - \Delta p^\rho_{T,(\kappa)}))\|_{L^2_{m,\delta}(\mathcal A_T\subset\Sigma_T)} \le C_{13,m}e^{-3T\delta}\mu^{\kappa-1}. \tag{1.98}$$

Note

$$\mathrm{Err}^\rho_{2,T,(\kappa-1)} = 0$$

on $\mathcal A_T$. Therefore in the same way as we did on K_1 we can show

$$\|\overline\partial(V^\rho_{T,1,(\kappa)} + u^\rho_{T,(\kappa-1)})\|_{L^2_{m,\delta}(\mathcal A_T\subset\Sigma_T)} \le C_{1,m}e^{-\delta T}\epsilon_{5,m}\mu^\kappa/20 \tag{1.99}$$

for $T > T_m$. Therefore by taking T large we have

$$\|\overline\partial u^\rho_{T,(\kappa)}\|_{L^2_{m,\delta}(\mathcal A_T\subset\Sigma_T)} < C_{1,m}\mu^\kappa\epsilon_{5,m}e^{-\delta T}/10. \tag{1.100}$$

The estimate on $\mathcal B_T$ and on $([-5T,-T-1]\cup[T+1,5T])\times[0,1]$ are similar. The proof of Lemma 1.31 is complete. □

Step κ-4:

Definition 1.32. We put

$$\begin{aligned}\mathrm{Err}^\rho_{1,T,(\kappa)} &= \chi^{\leftarrow}_{\mathcal X}\left(\overline\partial u^\rho_{T,(\kappa)} - \sum_{a=0}^{\kappa}\mathfrak e^\rho_{1,T,(a)}\right),\\ \mathrm{Err}^\rho_{2,T,(\kappa)} &= \chi^{\rightarrow}_{\mathcal X}\left(\overline\partial u^\rho_{T,(\kappa)} - \sum_{a=0}^{\kappa}\mathfrak e^\rho_{2,T,(a)}\right).\end{aligned}$$

We regard them as elements of the weighted Sobolev spaces $L^2_{m,\delta}(\Sigma_1;(\hat u^\rho_{1,T,(\kappa)})^*TX\otimes\Lambda^{01})$ and $L^2_{m,\delta}(\Sigma_2;(\hat u^\rho_{2,T,(\kappa)})^*TX\otimes\Lambda^{01})$ respectively. (We extend them by 0 outside a compact set.)

[5]This drop of the weight is the main part of the idea. It was used in [FOOO1, page 414]. See [FOOO1, Figure 7.1.6].

We put $p^{\rho}_{(\kappa)} = p^{\rho}_{(\kappa-1)} + \Delta p^{\rho}_{T,(\kappa)}$.
Lemma 1.31 implies (1.72) and (1.73).

We have thus described all the induction steps. For each fixed m there exists T_m such that if $T > T_m$ then

$$\lim_{\kappa\to\infty} u^{\rho}_{T,(\kappa)}$$

coverges in $L^2_{m+1,\delta}$ sense to the solution of (1.15). The limit is automatically of C^∞ class by elliptic reguralty. We have thus constructed the map in Theorem 1.10. We will prove its surjectivity and injectivity in Subsection 1.5 below. Before doing so we prove an exponential decay estimate of its T derivative.

1.4. **Exponential decay of T derivatives.** We first state the result of this subsection. We recall that for T sufficiently large and $\rho = (\rho_1,\rho_2) \in V_1 \times_L V_2$ we have defined $u^{\rho}_{T,(\kappa)}$. We denote its limit by

$$u^{\rho}_{T} = \lim_{\kappa\to\infty} u^{\rho}_{T,(\kappa)} : (\Sigma_T, \partial\Sigma_T) \to (X, L). \tag{1.101}$$

The main result of this subsection is an estimate of T and ρ derivatives of this map. We prepare some notations to state the result.

We change the coordinates of Σ_i and Σ_T as follows. In the last subsection we put

$$\Sigma_1 = K_1 \cup ([-5T,\infty) \times [0,1])$$

and use (τ, t) for the coordinate of $[-5T,\infty) \times [0,1]$. This identification depends on T. So we rewrite it to

$$\Sigma_1 = K_1 \cup ([0,\infty) \times [0,1])$$

and the coordinate for $[0,\infty) \times [0,1]$ is (τ', t) where

$$\tau' = \tau + 5T. \tag{1.102}$$

Similarly we rewrite

$$\Sigma_2 = ((-\infty, 5T] \times [0,1]) \cup K_2$$

to

$$\Sigma_2 = ((-\infty, 0] \times [0,1]) \cup K_2$$

and use the coordinate (τ'', t) where

$$\tau'' = \tau - 5T. \tag{1.103}$$

We may use either (τ', t) or (τ'', t) as the coordinate of $\Sigma_T \setminus (K_1 \cup K_2)$.

Let S be a positive number. We have $K_i \subset \Sigma_T$. We put

$$\begin{aligned} K_1^{+S} &= K_1 \cup ([0,S] \times [0,1]) \subset \Sigma_T, \\ K_2^{+S} &= ([-S,0] \times [0,1]) \cup K_2 \subset \Sigma_T. \end{aligned} \tag{1.104}$$

Here the inclusion $K_1 \cup ([0,S] \times [0,1]) \subset \Sigma_T$ is by using the coordinate τ' and the inclusion $([-S,0] \times [0,1]) \cup K_2 \subset \Sigma_T$ is by using the coordinate τ''.

We may also regard $K_i^{+S} \subset \Sigma_i$. Note that the spaces K_i^{+S} are independent of T, as far as $10T > S$.

We restrict the map u^{ρ}_T to K_i^{+S}. We thus obtain a map

$$\mathrm{Glures}_{i,S} : [T_m, \infty) \times V_1 \times_L V_2 \to \mathrm{Map}_{L^2_{m+1}}((K_i^{+S}, K_i^{+S} \cap \partial\Sigma_i), (X, L))$$

by

$$\begin{cases} \mathrm{Glures}_{1,S}(T,\rho)(x) & = u^\rho_T(x) \qquad x \in K_1 \\ \mathrm{Glures}_{1,S}(T,\rho)(\tau',t) & = u^\rho_T(\tau',t) = u^\rho_T(\tau+5T,t) \end{cases} \tag{1.105}$$

$$\begin{cases} \mathrm{Glures}_{2,S}(T,\rho)(x) & = u^\rho_T(x) \qquad x \in K_2 \\ \mathrm{Glures}_{2,S}(T,\rho)(\tau'',t) & = u^\rho_T(\tau'',t) = u^\rho_T(\tau-5T,t) \end{cases} \tag{1.106}$$

Here $\mathrm{Map}_{L^2_{m+1}}((K_i^{+S}, K_i^{+S}\cap\partial\Sigma_i),(X,L))$ is the space of maps of L^2_{m+1} class (m is sufficiently large, say $m > 10$.) It has a structure of Hilbert manifold in an obvious way. This Hilbert manifold is independent of T. So we can define T derivative of a family of elements of $\mathrm{Map}_{L^2_{m+1}}((K_i^{+S}, K_i^{+S}\cap\partial\Sigma_i),(X,L))$ parametrized by T.

Remark 1.33. The domain and the target of the map $\mathrm{Glures}_{i,S}$ depend on m. However its image actually is in the set of smooth maps. Also none of the constructions of u^ρ_T depends on m. (The proof of the convergence of (1.101) depends on m. So the number T_m depends on m.) Therefore the map $\mathrm{Glures}_{i,S}$ is *independent* of m on the intersection of the domains. Namely the map $\mathrm{Glures}_{i,S}$ constructed by using $L^2_{m_1}$ norm coincides with the map $\mathrm{Glures}_{i,S}$ constructed by using $L^2_{m_2}$ norm on $[\max\{T_{m_1},T_{m_2}\},\infty)\times V_1 \times_L V_2$.

Theorem 1.34. *For each m and S there exist $T(m), C_{16,m,S}, \delta > 0$ such that the following holds for $T > T(m)$ and $n+\ell \le m-10$ and $\ell > 0$.*

$$\left\Vert \nabla_\rho^n \frac{d^\ell}{dT^\ell}\mathrm{Glures}_{i,S}\right\Vert_{L^2_{m+1-\ell}} < C_{16,m,S}e^{-\delta T}. \tag{1.107}$$

Here ∇_ρ^n is the n-th derivative in ρ direction.

Remark 1.35. Theorem 1.34 is basically equivalent to [FOOO1, Lemma A1.58]. The proof below is basically the same as the one in [FOOO1, page 776]. We add some more detail.

Proof. The construction of $u^\rho_{T,(\kappa)}$ was by induction on κ. We divide the inductive step of the construction of $u^\rho_{T,(\kappa+1)}$ from $u^\rho_{T,(\kappa)}$ into two.

- (Part A) Start from $(V^\rho_{T,1,(\kappa)}, V^\rho_{T,2,(\kappa)}, \Delta p^\rho_{T,(\kappa)})$ and end with $\mathrm{Err}^\rho_{1,T,(\kappa)}$ and $\mathrm{Err}^\rho_{2,T,(\kappa)}$. This is step κ-2,κ-3,κ-4.
- (Part B) Start from $\mathrm{Err}^\rho_{1,T,(\kappa)}$ and $\mathrm{Err}^\rho_{2,T,(\kappa)}$ and end with $(V^\rho_{T,1,(\kappa+1)}, V^\rho_{T,2,(\kappa+1)}, \Delta p^\rho_{T,(\kappa+1)})$. This is step $(\kappa+1)$-1.

We will prove the following inequality by induction on κ, under the assumption $T > T(m)$, $\ell > 0$, $n + \ell \le m - 10$.

$$\left\| \nabla_\rho^n \frac{\partial^\ell}{\partial T^\ell}(V^\rho_{T,i,(\kappa)}, \Delta p^\rho_{T,(\kappa)}) \right\|_{L^2_{m+1-\ell,\delta}(\Sigma_i)} < C_{17,m}\mu^{\kappa-1}e^{-\delta T}, \tag{1.108}$$

$$\left\| \nabla_\rho^n \frac{\partial^\ell}{\partial T^\ell}\Delta p^\rho_{T,(\kappa)} \right\| < C_{17,m}\mu^{\kappa-1}e^{-\delta T}, \tag{1.109}$$

$$\left\| \nabla_\rho^n \frac{\partial^\ell}{\partial T^\ell} u^\rho_{T,(\kappa)} \right\|_{L^2_{m+1-\ell,\delta}(K_i^{+5T+1})} < C_{18,m}e^{-\delta T}, \tag{1.110}$$

$$\left\| \nabla_\rho^n \frac{\partial^\ell}{\partial T^\ell} \mathrm{Err}^\rho_{i,T,(\kappa)} \right\|_{L^2_{m-\ell,\delta}(\Sigma_i)} < C_{19,m}\epsilon_{6,m}\mu^{\kappa}e^{-\delta T}, \tag{1.111}$$

$$\left\| \nabla_\rho^n \frac{\partial^\ell}{\partial T^\ell} \mathfrak{e}^\rho_{i,T,(\kappa)} \right\|_{L^2_{m-\ell}(K_i^{\rm obst})} < C_{19,m}\mu^{\kappa-1}e^{-\delta T}. \tag{1.112}$$

More precisely, the claim we will prove is the following: For each $\epsilon_{6,m}$, we can choose $T(m)$ so that (1.108) and (1.109) imply (1.111) and (1.112) for $T > T(m)$, and we can choose $\epsilon_{6,m}$ so that (1.111) and (1.112) for κ implies (1.108) and (1.109) for $\kappa + 1$. (1.110) follows from (1.108) and (1.109).

Remark 1.36. We use L^2_{m+1} norm on K_i^{+5T+1} only in formula (1.110). Note we use coordinate (τ', t) on $K_1^{+5T+1} \setminus K_1$, and (τ'', t) on $K_2^{+5T+1} \setminus K_2$. We remark also that $\Sigma_T = K_1^{+5T+1} \cup K_2^{+5T+1}$.

Remark 1.37. Note that $(V^\rho_{T,i,(\kappa)}, \Delta p^\rho_{T,(\kappa)})$ appearing in (1.108) is an element of the weighted Sobolev space $L^2_{m+1,\delta}((\Sigma_i, \partial\Sigma_i); (\hat u^\rho_{i,T,(\kappa-1)})^*TX, (\hat u^\rho_{i,T,(\kappa-1)})^*TL)$ that depends on T and ρ. To make sense of T and ρ derivatives we identify

$$\begin{aligned} &L^2_{m+1,\delta}((\Sigma_i, \partial\Sigma_i); (\hat u^\rho_{i,T,(\kappa-1)})^*TX, (\hat u^\rho_{i,T,(\kappa-1)})^*TL) \\ &\cong L^2_{m+1,\delta}((\Sigma_i, \partial\Sigma_i); u_i^*TX, u_i^*TL) \end{aligned}$$

as follows. We find V such that $\hat u^\rho_{i,T,(\kappa-1)} = \mathrm{E}(u_i, V)$. We use the parallel transport with respect to the path $r \mapsto \mathrm{E}(u_i, rV)$ and its complex linear part to define this isomorphism. The same remark applies to (1.111) and (1.112).

Remark 1.38. The square of the left hand side of (1.108), in case $i = 1$, is :

$$\begin{aligned} &\left\| \nabla_\rho^n \frac{\partial^\ell}{\partial T^\ell} V^\rho_{T,1,(\kappa)} \right\|^2_{L^2_{m+1-\ell}(K_1)} \\ &+ \sum_{k=0}^{m+1-\ell} \int_{[0,\infty)\times[0,1]} e_{1,T}(\tau, t) \left\| \nabla^k_{\tau',t} \nabla_\rho^n \frac{\partial^\ell}{\partial T^\ell}\big(V^\rho_{T,i,(\kappa)} - \mathrm{Pal}(\Delta p^\rho_{T,(\kappa)})\big) \right\|^2 d\tau' dt. \end{aligned}$$

Note that we apply Remark 1.37 to define T and ρ derivatives in the above formula.

The case $i = 2$ is similar using τ'' coordinate.

(Part A) (See [FOOO1, page 776 paragraph (A) and (B)].)

We assume (1.108) and (1.109).

We find that

(1)

$$\mathrm{Err}^{\rho}_{1,T,(\kappa)}(z) = \Pi_{E_1^{\perp}(\hat u^{\rho}_{1,T,(\kappa-1)})}\overline{\partial}\mathrm{E}(\hat u^{\rho}_{1,T,(\kappa-1)}(z), V^{\rho}_{T,1,(\kappa)}(z)) \tag{1.113}$$

for $z \in K_1$.

(2)

$$\begin{aligned}&\mathrm{Err}^{\rho}_{1,T,(\kappa)}(\tau') \\ =&(1-\chi(\tau'-5T))\overline{\partial}\big(\chi(\tau'-4T)(V^{\rho}_{T,2,(\kappa)}(\tau'-10T,t)-\Delta p^{\rho}_{T,(\kappa)}) \\ &\qquad + V^{\rho}_{T,1,(\kappa)}(\tau',t) + u^{\rho}_{T,(\kappa-1)}(\tau',t)\big),\end{aligned} \tag{1.114}$$

for $(\tau',t) \in [0,\infty)\times[0,1]$. (Note $\tau' = \tau'' + 10T$ and the variable of $V^{\rho}_{T,2,(\kappa)}$ is (τ'',t).)

Here $\chi : \mathbb{R} \to [0,1]$ is a smooth function such that

$$\chi(\tau)\begin{cases} = 0 & \tau < -1 \\ = 1 & \tau > 1 \\ \in [0,1] & \tau \in [-1,1]. \end{cases} \tag{1.115}$$

Note that in Formulas (1.108)-(1.112) the Sobolev norms in the left hand side are $L^2_{m+1-\ell,\delta}(\Sigma_i)$ etc. and are not $L^2_{m+1,\delta}(\Sigma_i)$ etc. The origin of this loss of differentiability (in the sense of Sobolev space) comes from the term $V^{\rho}_{T,2,(\kappa)}(\tau' - 10T)$. In fact, we have

$$\frac{\partial}{\partial T}V^{\rho}_{T_1,2,(\kappa)}(\tau'-10T) = -10\frac{\partial}{\partial \tau''}V^{\rho}_{T_1,2,(\kappa)}(\tau'-10T)$$

for a fixed T_1. Hence $\partial/\partial T$ is continuous as $L^2_{m+1} \to L^2_m$. We remark in (1.108) for $i = 2$ we use the coordinate (τ'',t) on $(-\infty,0]\times[0,1]$ to define T derivative of $V^{\rho}_{T,2,(\kappa)}$.

Taking this fact into acount the proof goes as follows.

We can estimate T and ρ derivative of $\mathrm{Err}^{\rho}_{1,T,(\kappa)}$ on K_1 in the same way as the proof of Lemma 1.31.

Remark 1.39. The fact we use here is that the maps such as $(u,v) \mapsto \mathrm{E}(u,v)$, $(u,v) \to \Pi^{\perp}_{E_i(u)}(v)$ are smooth maps from $L^2_{m+1,loc}\times L^2_{m+1,\delta} \to L^2_{m+1,\delta}$ or $L^2_{m+1,loc}\times L^2_{m,\delta} \to L^2_{m,\delta}$ and $u \to \overline{\partial}u$ is a smooth map $L^2_{m+1,\delta} \to L^2_{m,\delta}$. (Since we assume m sufficiently large this is a well-known fact.) Moreover the map $T \mapsto u^{\rho}_{T,(\kappa-1)}$ and $T \mapsto V^{\rho}_{T,1,(\kappa)}$ are C^{ℓ} maps as a map $[T(m),\infty) \to L^2_{m+1-\ell,\delta}$ with its differential estimated by induction hypothesis (1.110) and (1.108).

We note that $\rho \mapsto u^{\rho}_{T,(\kappa-1)}$ is smooth as a map $V_1 \times_L V_2 \to L^2_{m+1,\delta}$.

The estimates of T and ρ derivatives of (1.114) are as follows.

We first consider the domain $\tau' \in [4T+1,\infty)$. There we have

$$\begin{aligned}\mathrm{Err}^{\rho}_{1,T,(\kappa)}(\tau',t) =&(1-\chi(\tau'-5T))\overline{\partial}(V^{\rho}_{T,2,(\kappa)}(\tau'-10T,t) \\ &+ V^{\rho}_{T,1,(\kappa)}(\tau',t) + u^{\rho}_{T,(\kappa-1)}(\tau',t) - \Delta p^{\rho}_{T,(\kappa)}).\end{aligned} \tag{1.116}$$

By the same calculation as in the proof of Lemma 1.31, (1.116) is equal to

$$\begin{aligned}(1-\chi(\tau'-5T))\int_0^1 ds\int_0^s \frac{\partial^2}{\partial r^2}\overline{\partial}\big(&r(V^{\rho}_{T,2,(\kappa)}(\tau'-10T)-\Delta p^{\rho}_{T,(\kappa)})\\ &+r(V^{\rho}_{T,1,(\kappa)}(\tau',t)-\Delta p^{\rho}_{T,(\kappa)})\\ &+u^{\rho}_{T,(\kappa-1)}(\tau',t)+r\Delta p^{\rho}_{T,(\kappa)}\big)dr.\end{aligned}$$

(Note that we are away from the support of E_i.)[6] Using the fact that $T\mapsto (V^{\rho}_{T,1,(\kappa)}(\tau',t)-\Delta p^{\rho}_{T,(\kappa)})+(V^{\rho}_{T,2,(\kappa)}(\tau'-10T)-\Delta p^{\rho}_{T,(\kappa)})$ and $T\mapsto u^{\rho}_{T,(\kappa-1)}(\tau',t)$ are of C^{ℓ} class as a map to $L^2_{m+1-\ell,\delta}$, we can estimate it to obtain the required estimate (1.111) on this part. We remark $T\mapsto (V^{\rho}_{T,2,(\kappa-1)},\Delta p^{\rho}_{T,(\kappa-1)})$ is C^{ℓ} with exponential decay estimate on T derivatives as a map $[T(m),\infty)\to L^2_{m-\ell+1,\delta}$. This follows from the induction hypothesis as follows.

$$\begin{aligned}&\frac{\partial^{\ell}}{\partial T^{\ell}}\left(V^{\rho}_{T,2,(\kappa)}(\tau'-10T)\right)\Big|_{T=T_1}\\ &=\sum_{\ell_1+\ell_2=\ell}(-10)^{\ell_2}\frac{\partial^{\ell_1}}{\partial T^{\ell_1}}\frac{\partial^{\ell_2}}{(\partial\tau'')^{\ell_2}}V^{\rho}_{T,2,(\kappa)}(\tau'-10T_1).\end{aligned}\tag{1.117}$$

The $L^2_{m+1-\ell,\delta}$-norm of the right hand side can be estimated by (1.108).

We next consider $\tau'\in[0,4T+1]$. There we have

$$\begin{aligned}\mathrm{Err}^{\rho}_{1,T,(\kappa)}(\tau',t)=&\overline{\partial}(\chi(\tau'-4T)(V^{\rho}_{T,2,(\kappa)}(\tau'-10T)-\Delta p^{\rho}_{T,(\kappa)})\\ &+V^{\rho}_{T,1,(\kappa)}(\tau',t)+u^{\rho}_{T,(\kappa-1)}(\tau',t)).\end{aligned}\tag{1.118}$$

Note

$$\overline{\partial}u^{\rho}_{T,(\kappa-1)}(\tau',t))=\mathrm{Err}^{\rho}_{1,T,(\kappa-1)}(\tau',t),$$

there. Therefore we can calculate in the same way as the proof of Lemma 1.31 to find

$$\begin{aligned}&\overline{\partial}(V^{\rho}_{T,1,(\kappa)}(\tau',t)+u^{\rho}_{T,(\kappa-1)}(\tau',t))\\ &=\int_0^1 ds\int_0^s\frac{\partial^2}{\partial r^2}\overline{\partial}(r(V^{\rho}_{T,1,(\kappa)}(\tau',t)-\Delta p^{\rho}_{T,(\kappa)})+u^{\rho}_{T,(\kappa-1)}(\tau',t)+r\Delta p^{\rho}_{T,(\kappa)})dr.\end{aligned}$$

We can again estimate the right hand side by using the fact that the maps $T\mapsto (V^{\rho}_{T,1,(\kappa)}(\tau',t),\Delta p^{\rho}_{T,(\kappa)})$ and $T\mapsto u^{\rho}_{T,(\kappa-1)}(\tau',t)$ are of C^{ℓ} class as a map to $L^2_{m+1-\ell,\delta}$ with estimate (1.110).

Finally we observe that the ratio between weight function of $L^2_{m+1,\delta}(\Sigma_2)$ and of $L^2_{m+1,\delta}(\Sigma_T)$ is $e^{2T\delta}$ on $\tau=-T$ (that is $\tau'=4T$). We use this fact to estimate $\overline{\partial}(\chi(\tau'-4T)(V^{\rho}_{T,2,(\kappa)}(\tau'-10T)-\Delta p^{\rho}_{T,(\kappa)}))$. We thus obtain the required estimate (1.111) for $\mathrm{Err}^{\rho}_{1,T,(\kappa)}$ on $\tau'\in[0,4T+1]$.

We thus obtain an estimate for $\mathrm{Err}^{\rho}_{1,T,(\kappa)}(\tau',t)$.

The estimate of derivatives of $\mathrm{Err}^{\rho}_{2,T,(\kappa)}(\tau',t)$ is similar. Thus we have (1.111).

We note that $\mathfrak{e}^{\rho}_{i,T,(0)}$ is independent of T as an element of E_i. Among $\mathfrak{e}^{\rho}_{i,T,(\kappa)}$'s, the term $\mathfrak{e}^{\rho}_{i,T,(0)}$ is the only one that is not of exponential decay with respect to T.

[6]Note $\overline{\partial}$ is non-constant. So $\overline{\partial}(r(V^{\rho}_{T,2,(\kappa)}(\tau'-10T)-\Delta p^{\rho}_{T,(\kappa)})+r(V^{\rho}_{T,1,(\kappa)}(\tau',t)-\Delta p^{\rho}_{T,(\kappa)})+u^{\rho}_{T,(\kappa-1)}(\tau',t)+r\Delta p^{\rho}_{T,(\kappa)})$ is nonlinear on r.

Once we note this point the rest of the proof of (1.112) is the same as the proof of Lemma 1.31.

We finally prove (1.110). On K_1 we have

$$u^{\rho}_{T,(\kappa)} = \mathrm{E}(u^{\rho}_{T,(\kappa-1)}, V^{\rho}_{1,T,(\kappa)}).$$

So using $\mu < 1$, (1.110) follows from (1.108) on K_1.

On $(\tau', t) \in [0, 5T+1) \times [0,1]$ we have:

$$\begin{aligned}
&u^{\rho}_{T,(\kappa)}(\tau', t)\\
&= V^{\rho}_{T,1,(\kappa)}(\tau', t) + (1 - \chi(\tau' - 4T))(V^{\rho}_{T,2,(\kappa)}(\tau' - 10T, t) - \Delta p^{\rho}_{T,(\kappa)})\\
&\qquad\qquad + u^{\rho}_{T,(\kappa-1)}(\tau', t)\\
&= \sum_{a=1}^{\kappa} V^{\rho}_{T,1,(a)}(\tau', t) + (1 - \chi(\tau' - 4T)) \sum_{a=1}^{\kappa}(V^{\rho}_{T,2,(a)}(\tau' - 10T, t) - \Delta p^{\rho}_{T,(a)})\\
&\qquad\qquad + u^{\rho}_{T,(0)}(\tau', t).
\end{aligned}$$

Then using a calclation similar to (1.117) we have (1.108) on $(\tau', t) \in [0, 5T+1) \times [0,1]$.

Remark 1.40. In [Ab] Abouzaid used L^p_1 norm for the maps u. He then proved that the gluing map is continuous with respect to T (that is S in the notation of [Ab]) but does not prove its differentiability with respect to T. (Instead he used the technique to remove the part of the moduli space with $T > T_0$, as we mentioned at the beginning of this note. This technique certainly works for the purpose of [Ab].) In fact if we use L^p_1 norm instead of L^2_m norm then the left hand side of (1.110) becomes L^p_{-1} norm which is hard to use.

Abouzaid mentioned in [Ab, Remark 5.1] that this point is related to the fact that quotients of Sobolev spaces by the diffeomorphisms in the source are not naturally equipped with the structure of smooth Banach manifold. Indeed in the situation when there is an automorphism on Σ_2, for example Σ_2 is disk with one boundary marked point $(-\infty, t)$, then the T parameter is killed by a part of the automorphism. So the shift of $V^{\rho}_{T,2,(\kappa)}$ by T that appears in the second term of (1.114) will be equivalent to the action of the automorphism group of Σ_2 in such a situation. The shift of T causes the loss of differentiability in the sense of Sobolev space in the formula (1.108) -(1.112). However at the end of the day we can still get the differentiability of C^{∞} order and its exponential decay by using various *wighted* Sobolev spaces with various m simultaneously. (See Remark 1.33 also.)

(Part B) (See [FOOO1, page 776 the paragraph next to (B)].)

We assume (1.108)-(1.112) for κ and will prove (1.108) and (1.109) for $\kappa + 1$. This part is nontrivial only because the construction here is global. (Solving linear equation.) So we first review the set up of the function space that is independent of T.

In Definition 1.18 we defined a function space $\mathfrak{H}(E_1, E_2)$, that is a subspace of (1.48). Since (1.48) is still T dependent we rewrite it a bit. We consider $u^{\rho}_i : (\Sigma_i, \partial\Sigma_i) \to (X, L)$ that is *T-independent*.

The maps $\hat u^{\rho}_{i,T,(\kappa)}$ are close to u^{ρ}_i. (Namely the C^0 distance between them is smaller than injectivity radius of X.) We take a connection of TX so that L is totally geodesic. We use the complex linear part of the parallel transport with respect to this connection, to send

$$\bigoplus_{i=1}^{2} L^2_{m,\delta}((\Sigma_i,\partial\Sigma_i);(u_i^\rho)^*TX,(u_i^\rho)^*TL)$$

to

$$\bigoplus_{i=1}^{2} L^2_{m,\delta}((\Sigma_i,\partial\Sigma_i);(\hat u^\rho_{i,T,(\kappa)})^*TX,(\hat u^\rho_{i,T,(\kappa)})^*TL).$$

Note that $\mathrm{Ker}(D\mathrm{ev}_{1,\infty}-D\mathrm{ev}_{2,\infty})$ is sent to $\mathrm{Ker}(D\mathrm{ev}_{1,\infty}-D\mathrm{ev}_{2,\infty})$ by this map. Therefore we obtain an isomorphism between

$$\mathrm{Ker}(D\mathrm{ev}_{1,\infty}-D\mathrm{ev}_{2,\infty})\cap\bigoplus_{i=1}^{2} L^2_{m,\delta}((\Sigma_i,\partial\Sigma_i);(u_i^\rho)^*TX,(u_i^\rho)^*TL) \tag{1.119}$$

and

$$\mathrm{Ker}(D\mathrm{ev}_{1,\infty}-D\mathrm{ev}_{2,\infty})\cap\bigoplus_{i=1}^{2} L^2_{m,\delta}((\Sigma_i,\partial\Sigma_i);(\hat u^\rho_{i,T,(\kappa)})^*TX,(\hat u^\rho_{i,T,(\kappa)})^*TL). \tag{1.120}$$

In case $\kappa=0$ we send $\mathfrak H(E_1,E_2)$ by this isomorphism to obtain a subspace of (1.119) which we denote by $\mathfrak H(E_1,E_2)$ by an abuse of notation. We send it to the subspace of (1.120) and denote it by $\mathfrak H(E_1,E_2;\kappa,T)$. We thus have an isomorphism

$$I_{1,\kappa,T}:\mathfrak H(E_1,E_2)\to\mathfrak H(E_1,E_2;\kappa,T).$$

We next use the parallel transport in the same way to find an isomorphism

$$I_{2,\kappa,T}:L^2_{m,\delta}(\Sigma_i;(u_i^\rho)^*TX\otimes\Lambda^{01})\to L^2_{m,\delta}(\Sigma_i;(\hat u^\rho_{i,T,(\kappa)})^*TX\otimes\Lambda^{01}).$$

Thus the composition

$$I^{-1}_{2,\kappa,T}\circ\Big(D_{\hat u^\rho_{i,T,(\kappa-1)}}\overline\partial-(D_{\hat u^\rho_{i,T,(\kappa-1)}}E_i)((\mathfrak{se})^\rho_{i,T,(\kappa-1)},\cdot))\Big)\circ I_{1,\kappa,T}$$

defines an operator

$$D_{\kappa,T}:\mathfrak H(E_1,E_2)\to L^2_{m,\delta}(\Sigma_i;(u_i^\rho)^*TX\otimes\Lambda^{01}).$$

Here the domain and the target is independent of T,κ.

Remark 1.41. Note $D_{\hat u^\rho_{i,T,(\kappa-1)}}\overline\partial-(D_{\hat u^\rho_{i,T,(\kappa-1)}}E_i)((\mathfrak{se})^\rho_{i,T,(\kappa-1)},\cdot)$ is the differential operator in (1.43) and (1.44). This differential operator gives the linearization of the right hand side of (1.113).

We next eliminate T,κ dependence of E_i. We consider the finite dimensional subspace:

$$E_i(\hat u^\rho_{i,T,(\kappa)})\subset L^2_{m,\delta}(\Sigma_i;(\hat u^\rho_{i,T,(\kappa)})^*TX\otimes\Lambda^{01}).$$

Let us consider

$$E_{i,(\kappa),T}=I^{-1}_{2,\kappa,T}(E_i(\hat u^\rho_{i,T,(\kappa)}))$$

that may depend on T. However

$$E_{i,(0)}=I^{-1}_{2,\kappa,T}(E_i(\hat u^\rho_{i,T,(0)}))$$

is independent of T since $\hat u^\rho_{i,T,(0)}=u_i^\rho$ on K_i. Let $E^\perp_{i,(0)}$ be the L^2 orthogonal complement of $E_{i,(0)}$ in $L^2_{m,\delta}(\Sigma_i;(\hat u^\rho_{i,T,(\kappa)})^*TX\otimes\Lambda^{01})$.

We have

$$E_{i,(\kappa),T} \oplus E^{\perp}_{i,(0)} = L^2_{m,\delta}(\Sigma_i; (u^{\rho}_i)^*TX \otimes \Lambda^{01}). \tag{1.121}$$

Therefore the inclusion induces an isomorphism

$$E^{\perp}_{i,(0)} \cong L^2_{m,\delta}(\Sigma_i; (u^{\rho}_i)^*TX \otimes \Lambda^{01})/E_{i,(\kappa),T}.$$

We thus obtain

$$\overline{D}_{\kappa,T} : \mathfrak{H}(E_1, E_2) \to E^{\perp}_{i,(0)}. \tag{1.122}$$

The induction hypothesis implies the following:

(1) There exist $C_{20,m}, C_{21,m} > 0$ such that

$$C_{20,m}\|V\|_{L^2_{m+1,\delta}} \le \|\overline{D}_{0,T}(V)\|_{L^2_{m,\delta}} \le C_{21,m}\|V\|_{L^2_{m+1,\delta}}. \tag{1.123}$$

(2)

$$\|\overline{D}_{\kappa,T}(V) - \overline{D}_{0,T}(V)\|_{L^2_{m,\delta}} \le C_{21,m}e^{-\delta T}\|V\|_{L^2_{m+1,\delta}}. \tag{1.124}$$

Moreover

$$\left\|\nabla^n_{\rho}\frac{\partial^{\ell}}{\partial T^{\ell}}\overline{D}_{\kappa,T}(V)\right\|_{L^2_{m-\ell,\delta}} \le C_{22,m}e^{-\delta T}\|V\|_{L^2_{m+1,\delta}}. \tag{1.125}$$

In fact, (1.125) follows from

$$\left\|\nabla^n_{\rho}\frac{\partial^{\ell}}{\partial T^{\ell}}\hat{u}^{\rho}_{i,T,(\kappa)}\right\|_{L^2_{m-\ell}(K_i)} \le C_{23,m}e^{-\delta T}, \tag{1.126}$$

$$\left\|\nabla^n_{\rho}\frac{\partial^{\ell}}{\partial T^{\ell}}\hat{u}^{\rho}_{i,T,(\kappa)}\right\|_{L^2_{m-\ell}([S,S+1]\times[0,1])} \le C_{23,m}e^{-\delta T} \tag{1.127}$$

for any $S \in [0,\infty)$. (See also the Remark 1.42.) Note that the weighted Sobolev norm $\|\nabla^n_{\rho}\frac{\partial^{\ell}}{\partial T^{\ell}}\hat{u}^{\rho}_{i,T,(\kappa)}\|_{L^2_{m-\ell,\delta}(\Sigma_i)}$ can be large because

$$\frac{\partial}{\partial T}\chi^{\leftarrow}_{\mathcal{B}}(\tau - T, t)u^{\rho}_{T,(\kappa-1)}$$

is only estimated by $e^{-3\delta T}$ on the support of $\chi^{\leftarrow}_{\mathcal{B}}(\tau - T, t)$ but the weight $e_{1,\delta}$ is roughly $e^{7T\delta}$ on the support of $\chi^{\leftarrow}_{\mathcal{B}}(\tau - T, t)$. However this does not cause any problem to prove (1.125). In fact the operator $\overline{D}_{\kappa,T}$ is a differential operator whose coefficient depends on $\hat{u}^{\rho}_{i,T,(\kappa)}$. So to estimate the operator norm of its derivatives with respect to the *weighted* Sobolev norm, we only need to estimate the local Sobolev norm without weight of $\hat{u}^{\rho}_{i,T,(\kappa)}$, that is provided by (1.126) and (1.127).

We note that $\overline{D}_{0,T}$ is independent of T. So we write $\overline{D}_0$. Now we have:

$$\begin{aligned}\overline{D}^{-1}_{\kappa,T} &= \Big((1 + (\overline{D}_{\kappa,T} - \overline{D}_0)\overline{D}^{-1}_0)\overline{D}_0\Big)^{-1} \\ &= \overline{D}^{-1}_0\sum_{k=0}^{\infty}(-1)^k((\overline{D}_{\kappa,T} - \overline{D}_0))\overline{D}^{-1}_0)^k.\end{aligned} \tag{1.128}$$

Therefore

$$\left\|\nabla^n_{\rho}\frac{\partial^{\ell}}{\partial T^{\ell}}\overline{D}^{-1}_{\kappa,T}(W)\right\|_{L^2_{m+1-\ell,\delta}} \le C_{24,m}e^{-\delta}\|W\|_{L^2_{m,\delta}} \tag{1.129}$$

for $\ell > 0$ and $\ell + n \le m$. (Here we assume W is T independent.) Since

$$(V^{\rho}_{T,1,(\kappa+1)}, V^{\rho}_{T,2,(\kappa+1)}, \Delta p^{\rho}_{T,(\kappa+1)}) = (I_{1,\kappa,T} \circ \overline{D}^{-1}_{\kappa,T} \circ I^{-1}_{2,\kappa,T})(\mathrm{Err}^{\rho}_{1,T,(\kappa)}, \mathrm{Err}^{\rho}_{2,T,(\kappa)}),$$

(1.111) and (1.129) imply (1.108) and (1.109) for $\kappa+1$.

The proof of Theorem 1.34 is now complete. □

Remark 1.42. Let us add a few more explanation about the proof of (1.124) and (1.125). Especially the relation between two operators $D_{\kappa,T}$ and $\overline{D}_{\kappa,T}$. We consider the direct sum decomposition

$$E_{i,(\kappa),T} \oplus E_{i,(0)}^{\perp} = L^2_{m,\delta}(\Sigma_i;(u_i^{\rho})^*TX \otimes \Lambda^{01}). \tag{1.130}$$

Note that this is not an orthogonal decomposition. We take an isomorphism

$$B_{i,(\kappa),T} : L^2_{m,\delta}(\Sigma_i;(u_i^{\rho})^*TX \otimes \Lambda^{01}) \to L^2_{m,\delta}(\Sigma_i;(u_i^{\rho})^*TX \otimes \Lambda^{01})$$

such that according to the orthogonal decomposition

$$E_{i,(0)} \oplus E_{i,(0)}^{\perp} = L^2_{m,\delta}(\Sigma_i;(u_i^{\rho})^*TX \otimes \Lambda^{01}). \tag{1.131}$$

The restriction $B_{i,(\kappa),T}|_{E_{i,(0)}^{\perp}}$ is the identity map and the restriction $B_{i,(\kappa),T}|_{E_{i,(0)}}$ is the canonical isomorphism

$$A_{i,(\kappa),T} : E_{i,(0)} \to E_{i,(0)}$$

given by the parallel transportation. Namely we put

$$B_{i,(\kappa),T} = A_{i,(\kappa),T} \circ \Pi_{E_{i,(0)}} + \Pi_{E_{i,(0)}^{\perp}}.$$

It is easy to prove

$$\|\overline{B}_{i,(\kappa),T}(V) - V\|_{L^2_{m,\delta}} \le C_{25,m} e^{-\delta T} \|V\|_{L^2_{m,\delta}}. \tag{1.132}$$

Moreover

$$\left\| \nabla_{\rho}^{n} \frac{\partial^{\ell}}{\partial T^{\ell}} B_{i,(\kappa),T}(V) \right\|_{L^2_{m-\ell,\delta}} \le C_{26,m} e^{-\delta T} \|V\|_{L^2_{m,\delta}}. \tag{1.133}$$

Note that

$$C_{i,(\kappa),T} = \Pi_{E_{i,(0)}^{\perp}} \circ B_{i,(\kappa),T}^{-1}$$

is the projection to the second factor in (1.130) and hence

$$\overline{D}_{\kappa,T} = \Pi_{E_{i,(0)}^{\perp}} \circ B_{i,(\kappa),T}^{-1} \circ D_{\kappa,T}. \tag{1.134}$$

We can use (1.132), (1.133) and (1.134) to prove (1.124), (1.125).

1.5. **Surjectivity and injectivity of the gluing map.** In this subsection we prove surjectivity and injectivity of the map Glu_T in Theorem 1.10 and complete the proof of Theorem 1.10.[7] The proof goes along the line of [D1]. (See also [FU].) The surjectivity proof is written in [FOn1, Section 14] and injectivity is proved in the same way. ([FOn1, Section 14] studies the case of pseudo-holomorphic curve without boundary. It however can be adapted easily to the bordered case as we mentioned in [FOOO1, page 417 lines 21-26].) Here we explain the argument in our situation in more detail.

We begin with the following a priori estimate.

[7] Here surjectivity means the second half of the statement of Theorem 1.10, that is 'The image contains $\mathcal{M}^{E_1+E_2}((\Sigma_T,\vec z);\beta)_{\epsilon_3}$.'

Proposition 1.43. ([FOn1, Lemma 11.2]) *There exist* $\epsilon_3, C_{25,m}, \delta_2 > 0$ *such that if* $u : (\Sigma_T, \partial\Sigma_T) \to (X, L)$ *is an element of* $\mathcal{M}^{E_1+E_2}((\Sigma_T, \vec{z}); \beta)_\epsilon$ *for* $0 < \epsilon < \epsilon_3$ *then we have*

$$\left\| \frac{\partial u}{\partial \tau} \right\|_{C^m([\tau-1,\tau+1]\times[0,1])} \le C_{27,m} e^{-\delta_2(5T-|\tau|)}. \tag{1.135}$$

The proof is the same as [FOn1, Lemma 11.2] that is proved in [FOn1, Section 14] and so is omitted.

We also have the following:

Lemma 1.44. $\mathcal{M}^{E_1+E_2}((\Sigma_T, \vec{z}); \beta)_\epsilon$ *is a smooth manifold of dimension* $\dim V_1 + \dim V_2 - \dim L$.

This is a consequence of the implicit function theorem and the index sum formula.

Proof of surjectivity. During this proof we take m sufficiently large and fix it. We will fix ϵ and T_0 during the proof and assume $T > T_0$. (They are chosen so that the discussion below works.) Let $u : (\Sigma_T, \partial\Sigma_T) \to (X, L)$ be an element of $\mathcal{M}^{E_1+E_2}((\Sigma_T, \vec{z}); \beta)_\epsilon$. The purpose here is to show that u is in the image of Glu_T. We define $u_i' : (\Sigma_i, \partial\Sigma_i) \to (X, L)$ as follows. We put $p_0^u = u(0, 0) \in L$.

$$\begin{aligned}
&u_1'(z)\\
&= \begin{cases} \chi_{\mathcal{B}}^{\leftarrow}(\tau - T, t)u(\tau, t) + \chi_{\mathcal{B}}^{\rightarrow}(\tau - T, t)p_0^u & \text{if } z = (\tau, t) \in [-5T, 5T] \times [0, 1] \\ u(z) & \text{if } z \in K_1 \\ p_0^u & \text{if } z \in [5T, \infty) \times [0, 1]. \end{cases}\\
&u_2'(z)\\
&= \begin{cases} \chi_{\mathcal{A}}^{\rightarrow}(\tau + T, t)u(\tau, t) + \chi_{\mathcal{A}}^{\leftarrow}(\tau + T, t)p_0^u & \text{if } z = (\tau, t) \in [-5T, 5T] \times [0, 1] \\ u(z) & \text{if } z \in K_2 \\ p_0^u & \text{if } z \in (-\infty, -5T] \times [0, 1]. \end{cases}
\end{aligned} \tag{1.136}$$

Proposition 1.43 implies

$$\|\Pi_{E_i(u_i')}\overline{\partial} u_i'\|_{L^2_{m,\delta}(\Sigma_i)} \le C_{28,m} e^{-\delta T}. \tag{1.137}$$

Here we take $\delta < \delta_2/10$. On the other hand, by assumption and elliptic regurality we have

$$\|u_i' - u_i\|_{L^2_{m+1,\delta}(\Sigma_i)} \le C_{29,m}\epsilon. \tag{1.138}$$

Therefore by an implicit function theorem we have the following:

Lemma 1.45. *There exists* $\rho_i \in V_i$ *such that*

$$\|u_i' - u_i^{\rho_i}\|_{L^2_{m+1,\delta}(\Sigma_i)} \le C_{30,m} e^{-\delta T}, \tag{1.139}$$

$\rho = (\rho_1, \rho_2) \in V_1 \times_L V_2$, *and*

$$|\rho_i| \le C_{31,m}\epsilon. \tag{1.140}$$

(Note when $\rho_i = 0$, $u_i^{\rho_i} = u_i$.)

By (1.139) we have

$$\|u - u_T^{\rho}\|_{L^2_{m+1,\delta}(\Sigma_T)} \le C_{32,m} e^{-\delta T}. \tag{1.141}$$

Here $u_T^{\rho} = \mathrm{Glu}_T(\rho)$.

We take $V \in \Gamma((\Sigma_T, \partial\Sigma_T); (u^\rho_T)^*TX; (u^\rho_T)^*TL)$ so that

$$u(z) = \mathrm{E}(u^\rho_T(z), V(z)).$$

We define $u^s : (\Sigma_T, \partial\Sigma_T) \to (X, L)$ by

$$u^s(z) = \mathrm{E}(u^\rho_T(z), sV(z)). \tag{1.142}$$

(1.141) implies

$$\|\Pi^\perp_{(E_1+E_2)(u^s)}\overline{\partial}u^s\|_{L^2_{m,\delta}(\Sigma_T)} \le C_{33,m}e^{-\delta T} \tag{1.143}$$

and

$$\left\|\frac{\partial}{\partial s}u^s\right\|_{L^2_{m+1,\delta}(K_i^{+S})} \le C_{34,m}e^{-\delta T} \tag{1.144}$$

for each $s \in [0,1]$.

Lemma 1.46. *If T is sufficiently large, then there exists $\hat u^s : (\Sigma_T, \partial\Sigma_T) \to (X, L)$ ($s \in [0,1]$) with the following properties.*

(1)
$$\overline{\partial}\hat u^s \equiv 0 \mod (E_1 + E_2)(\hat u^s).$$

(2)
$$\left\|\frac{\partial}{\partial s}\hat u^s\right\|_{L^2_{m+1,\delta}(K_i^{+S})} \le 2C_{35,m}e^{-\delta T}. \tag{1.145}$$

(3) $\hat u^s = u^s$ *for* $s = 0, 1$.

Proof. Run the alternating method described in Subsection 1.3 in one parameter family version. Since u^s is already a solution for $s = 0, 1$, it does not change. □

Lemma 1.47. *The map* $\mathrm{Glu}_T : V_1 \times_L V_2 \to \mathcal{M}^{E_1+E_2}((\Sigma_T, \vec z); \beta)_\epsilon$ *is an immersion if T is suffciently large.*

Proof. We consider the composition of Glu_T with

$$\mathcal{M}^{E_1+E_2}((\Sigma_T, \vec z); \beta)_\epsilon \to L^2_{m+1}((K_i^{+S}, K_i^{+S} \cap \partial\Sigma_i), (X, L))$$

defined by restriction. In the case $T = \infty$ this composition is obtained by restriction of maps. By unique continuation, this is certainly an immersion for $T = \infty$. Then Theorem 1.34 implies that it is an immersion for sufficiently large T. □

Now we will prove that

$$A = \{s \in [0,1] \mid \hat u^s \in \text{image of } \mathrm{Glu}_T\}$$

is open and closed. Lemma 1.44 implies that $\mathcal{M}^{E_1+E_2}((\Sigma_T, \vec z); \beta)_\epsilon$ is a smooth manifold and has the same dimension as $V_1 \times_L V_2$. Therefore Lemma 1.47 implies that A is open. The closedness of A follows from (1.145).

Note $0 \in A$. Therefore $1 \in A$. Namely u is in the image of Glu_T as required. □

Proof of injectivity. Let $\rho^j = (\rho^j_1, \rho^j_2) \in V_1 \times_L V_2$ for $j = 0, 1$. We assume

$$\mathrm{Glu}_T(\rho^0) = \mathrm{Glu}_T(\rho^1) \tag{1.146}$$

and

$$\|\rho^j_i\| < \epsilon. \tag{1.147}$$

We will prove that $\rho^0 = \rho^1$ if T is sufficiently large and ϵ is sufficiently small. We may assume that $V_1 \times_L V_2$ is connected and simply connected. Then, we have a path $s \mapsto \rho^s = (\rho^s_1, \rho^s_2) \in V_1 \times_L V_2$ such that

(1) $\rho^s = \rho^j$ for $j = 0, 1$.
(2)
$$\left\|\frac{\partial}{\partial s}\rho^s\right\| \le \Phi_1(\epsilon)$$
where $\lim_{\epsilon\to 0}\Phi_1(\epsilon) = 0$.

We define $V(s) \in \Gamma((\Sigma_T, \partial\Sigma_T); (u_T^{\rho^0})^*TX; (u_T^{\rho^0})^*TL)$ such that
$$u_T^{\rho^s}(z) = \mathrm{E}(u_T^{\rho^0}(z), V(s)(z)).$$
(By (2) $u_T^{\rho^s}(z)$ is C^0-close to $u_T^{\rho^0}(z)$, as $\epsilon \to 0$. Therefore there exists such a unique $V(s)$ if ϵ is small.) Note $V(1) = V(0)$ since $u^{\rho^1} = u^{\rho^0}$. Therefore for $w \in D^2 = \{w \in \mathbb{C} \mid |w| \le 1\}$ there exists $V(w)$ such that

(1) $V(s) = V(w)$ if $w = e^{2\pi\sqrt{-1}s}$.
(2) We put $w = x + \sqrt{-1}y$.
$$\left\|\frac{\partial}{\partial x}V(w)\right\|_{L^2_{m+1,\delta}(\Sigma_T)} + \left\|\frac{\partial}{\partial y}V(w)\right\|_{L^2_{m+1,\delta}(\Sigma_T)} \le \Phi_2(\epsilon) \tag{1.148}$$
where $\lim_{\epsilon\to 0}\Phi_2(\epsilon) = 0$.

We put $u^w(z) = \mathrm{E}(u_T^{\rho^0}(z), V(w)(z))$.

Lemma 1.48. *If T is sufficiently large and ϵ is sufficiently small then there exists $\hat u^w : (\Sigma_T, \partial\Sigma_T) \to (X, L)$ $(s \in [0,1])$ with the following properties.*

(1)
$$\overline{\partial}\hat u^w \equiv 0 \mod (E_1 + E_2)(\hat u^w).$$
(2)
$$\left\|\frac{\partial}{\partial x}\hat u^w\right\|_{L^2_{m+1,\delta}(K_i^{+S})} + \left\|\frac{\partial}{\partial y}\hat u^w\right\|_{L^2_{m+1,\delta}(K_i^{+S})} \le \Phi_3(\epsilon) \tag{1.149}$$
with $\lim_{\epsilon\to 0}\Phi_3(\epsilon) = 0$.
(3) $\hat u^w = u^w$ *for* $w \in \partial D^2$.

Proof. Run the alternating method described in Subsection 1.3 in two parameter family version. □

Lemma 1.49. *If T is sufficiently large and ϵ is sufficiently small, there exists a smooth map $F : D^2 \to V_1 \times_L V_2$ such that*

(1) $\mathrm{Glu}_T(F(w)) = \hat u^w$.
(2) *If $s \in [0,1]$ then we have:*
$$F(e^{2\pi\sqrt{-1}s}) = \rho^s.$$

Proof. Note that $\rho \mapsto \mathrm{Glu}_T(\rho)$ is a local diffeomorphism. So we can apply the proof of homotopy lifting property as follows. Let $D_r^2 = \{z \in \mathbb{C} \mid |z - (r-1)| \le r\}$. We put
$$A = \{r \in [0,1] \mid \exists\, F : D_r^2 \to V_1 \times_L V_2 \text{ satisfying (1) above and } F(-1) = \rho^{1/2}\}.$$
Since $\mathrm{Glu}_T(\rho)$ is a local diffeomorphism, A is open. We can use (1.149) to show closednsss of A. Since $0 \in A$, it follows that $1 \in A$. The proof of Lemma 1.49 is complete. □

The proof of Theorem 1.10 is now complete. □

2. The general case

2.1. **Graph associated to a stable map.** We first recall the definition of the moduli space of (bordered) stable maps of genus zero.

Definition 2.1. Let $\beta \in H_2(X, L; \mathbb{Z})$ and $k, \ell \geq 0$. The *compactified moduli space of pseudo-holomorphic disks with $k+1$ boundary marked points and ℓ interior marked points with boundary condition given by L* that we denote by $\mathcal{M}_{k+1,\ell}(\beta)$ is the set of equivalence classes of $((\Sigma, \vec{z}, \vec{z}^{\text{int}}), u)$, where:

(1) Σ is a bordered semi-stable curve of genus zero with one boundary component $\partial\Sigma$.
(2) $u : (\Sigma, \partial\Sigma) \to (X, L)$ is a pseudo-holomorphic map of homology class β.
(3) $\vec{z} = (z_0, \dots, z_k)$ are boundary marked points. None of them are singular points and they are all distinct. We assume that they respect the cyclic order of $\partial\Sigma$.
(4) $\vec{z}^{\text{int}} = (z_1^{\text{int}}, \dots, z_\ell^{\text{int}})$ are interior marked points of Σ. None of them are singular points and they are all distinct.

We say $((\Sigma, \vec{z}, \vec{z}^{\text{int}}), u)$ is *equivalent* to $((\Sigma', \vec{z}', \vec{z}^{\text{int}\prime}), u')$ if there exists a biholomorphic map $v : \Sigma' \to \Sigma$ such that $u \circ v = u'$ and $v(z_i') = z_i$, $v(z_i^{\text{int}\prime}) = z_i^{\text{int}}$.

Definition 2.2. Let $\alpha \in H_2(X; \mathbb{Z})$ and $\ell \geq 0$. The *compactified moduli space of pseudo-holomorphic sphere with ℓ (interior) marked points* that we denote by $\mathcal{M}_\ell^{\text{cl}}(\alpha)$ is the set of the equivalence classes of $((\Sigma, \vec{z}^{\text{int}}), u)$, where:

(1) Σ is a semi-stable curve of genus zero without boundary.
(2) $u : \Sigma \to X$ is a pseudo-holomorphic map of homology class α.
(3) $\vec{z}^{\text{int}} = (z_1^{\text{int}}, \dots, z_\ell^{\text{int}})$ are marked points of Σ. None of them are singular points and they are all distinct.

We say $((\Sigma, \vec{z}^{\text{int}}), u)$ is *equivalent* to $((\Sigma', \vec{z}^{\text{int}\prime}), u')$ if there exists a biholomorphic map $v : \Sigma' \to \Sigma$ such that $u \circ v = u'$ and $v(z_i^{\text{int}\prime}) = z_i^{\text{int}}$.

The topology of $\mathcal{M}_\ell^{\text{cl}}(\alpha)$ is defined in [FOn1, Definition 10.3] and the topology of $\mathcal{M}_{k+1,\ell}(\beta)$ is defined in [FOOO1, Definition 7.1.42]. (See Definition 2.103.)

It is proved in [FOn1, Theorem 11.1 and Lemma 10.4] that $\mathcal{M}_\ell^{\text{cl}}(\alpha)$ is compact and Hausdorff. $\mathcal{M}_{k+1,\ell}(\beta)$ is also compact and Hausdorff. See [FOOO1, Theorem 7.1.43] and the references therein.

We refer [FOOO1, Section 2.1] for the moduli space $\mathcal{M}_{k+1,\ell}(\beta)$. See also [Liu].

We consider the case when X is a point and denote the moduli space of that case by $\mathcal{M}_{k+1,\ell}$. We call it Deligne-Mumford moduli space. (This is a slight abuse of notation since Deligne-Mumford studied the case when there is no boundary.) We define $\mathcal{M}_\ell^{\text{cl}}$ in the same way.

Theorem 2.3. *$\mathcal{M}_\ell^{\text{cl}}(\alpha)$ has a Kuranishi structure (without boundary) and $\mathcal{M}_{k+1,\ell}(\beta)$ has a Kuranishi structure with corners.*

Remark 2.4. (1) Theorem 2.3 in case of $\mathcal{M}_\ell^{\text{cl}}(\alpha)$ is a special case of [FOn1, Theorem 7.10]. In the case of $\mathcal{M}_{k+1,\ell}(\beta)$, Theorem 2.3 is [FOOO1, Theorem 2.1.29].
(2) In the case of $\mathcal{M}_{k+1,\ell}(\beta)$ we need to describe the way how various moduli spaces with differet k, ℓ, β are related along their boundaries and corners, for the application. See [FOOO1, Proposition 7.1.2] for the precise statement on this point. It is easy to see that the proof we will give in this note implies that version.

Below we give a detailed proof of Theorem 2.3. The proof is based on the proof in [FOn1]. The smoothness of coordinate at infinity is useful especially in the case of $\mathcal{M}_{k+1,\ell}(\beta)$. On that point we follow the method of [FOOO1, Section 7.2 and Appendix A1.4].

Remark 2.5. We discuss the case of genus zero here. We can handle the case of moduli space of pseudo-holomorphic curves with or without boundary and of arbitrary genus and with arbitrary number of boundary components, in the same way. The case of multi Lagrangian submanifolds in pairwise clearn intersection can be also handled in the same way. To slightly simplify the notation we restrict ourselves to the case of disks, that is mainly used in our book [FOOO1] and spheres, that is asked in this google group explicitely. In fact *no* new idea is required for generalization to higher genus etc. as far as the construction of Kuranishi structure concerns.

In a way similar to [FOn1, Section 8], we stratify $\mathcal{M}_{k+1,\ell}(\beta)$ as follows. For each element $\mathfrak{p} = [(\Sigma, \vec z, \vec z^{\rm int}), u]$ of $\mathcal{M}_{k+1,\ell}(\beta)$ we associate $\mathcal{G} = \mathcal{G}_{\mathfrak{p}}$, a graph with some extra data, as follows.

A vertex v of $\mathcal{G}$ corresponds to $\Sigma_{\rm v}$ an irreducible component of Σ. (It is either a disk or a sphere.) We put data $\beta_{\rm v} = [u\vert_{\Sigma_{\rm v}}]$ that is either an element of $H_2(X,L;\mathbb{Z})$ or an element of $H_2(X;\mathbb{Z})$.

To each singular point z of Σ we associate an edge e_z of $\mathcal{G}$. The edge e_z joins two vertices $\mathrm{v}_1, \mathrm{v}_2$ such that $z \in \Sigma_{\mathrm{v}_i}$. Note z can be either boundary or interior singular points. We also denote by $z_{\rm e}$ the singular point of Σ corresponding to the edge e.

For each vertex v we also include the data which marked points are contained in $\Sigma_{\rm v}$.

Definition 2.6. We call a graph $\mathcal{G}$ equipped with some other data described above, the *combinatorial type* of $\mathfrak{p} = [(\Sigma, \vec z, \vec z^{\rm int}), u]$. We denote by $\mathcal{M}_{k+1,\ell}(\beta;\mathcal{G})$ the set of $\mathfrak{p}$ with combinatorial type $\mathcal{G}$.

We write $\overset{\circ}{\mathcal{M}}_{k+1,\ell}(\beta)$ the staratum $\mathcal{M}_{k+1,\ell}(\beta;\mathrm{pt})$, where pt is a graph without edge.[8]

We say that $\mathcal{G}$ is *stable* if corresponding pseudo-holomorphic curve is stable. We say that $\mathcal{G}$ is *source stable* if the marked bordered curve obtained by forgetting the map is stable.

Let $\mathcal{G}$ and $\mathcal{G}'$ be combinatorial types. We say $\mathcal{G} \succ \mathcal{G}'$ if $\mathcal{G}'$ is obtained from $\mathcal{G}$ by iterating the following process finitely many times.

Take an edge e of $\mathcal{G}$. We shrink e and identify two vertices v_1, v_2 contained in e. Let v be the vertex identified to v_1, v_2. We put $\beta_{\rm v} = \beta_{\mathrm{v}_1} + \beta_{\mathrm{v}_2}$. The marked points assigned to v_1 or v_2 will be assigned to v.

Lemma 2.7. *If*

$$\overline{\mathcal{M}_{k+1,\ell}(\beta;\mathcal{G})} \cap \mathcal{M}_{k+1,\ell}(\beta;\mathcal{G}') \neq \emptyset,$$

then $\mathcal{G} \succ \mathcal{G}'$.

[8] $\overset{\circ}{\mathcal{M}}_{k+1,\ell}(\beta)$ is slightly smaller than the 'interior' of $\mathcal{M}_{k+1,\ell}(\beta)$. Namely elements of $\overset{\circ}{\mathcal{M}}_{k+1,\ell}(\beta)$ do not contain any disk or sphere bubble. On the other hand, elements of the interior of $\mathcal{M}_{k+1,\ell}(\beta)$ may contain sphere bubble.

The proof is easy so omitted.

Sometimes we add the following data to $\mathcal{G}$.

(1) Orientation to each of the edge. We call that $\mathcal{G}$ is *oriented* in case we include this data.[9]
(2) The length $T_{\rm e} \in \mathbb{R}_{>0}$ to each of the edges e.

We say an edge e is an *outgoing edge* of its vertex v and *incoming edge* of its vertex v′ if the orientation of e is goes from v to v′. By an abuse of terminology we say v is an *incoming vertex* (resp. outgoing vertex) of the e if e is an *incoming edge* (resp. outgoing edge) of v. [10]

We use the following notation.

$C^0_{\rm d}(\mathcal{G})$ = the set of the vertices that correspond to a disk component.
$C^0_{\rm s}(\mathcal{G})$ = the set of the vertices that correspond to a sphere component.
$C^0(\mathcal{G}) = C^0_{\rm d}(\mathcal{G}) \cup C^0_{\rm s}(\mathcal{G})$.
$C^1_{\rm o}(\mathcal{G})$ = the set of the edges that correspond to a boundary singular point.
$C^1_{\rm c}(\mathcal{G})$ = the set of the edges that correspond to an interior singular point.
$C^1(\mathcal{G}) = C^1_{\rm o}(\mathcal{G}) \cup C^1_{\rm c}(\mathcal{G})$.

Here d,s,o,c indicate disk, sphere, open (string), closed (string), respectively.

We define moduli space of marked stable maps from genus zero curve *without* boundary in the same way. We denote it by $\mathcal{M}^{\rm cl}_\ell(\alpha)$ where $\alpha \in H_2(X;\mathbb{Z})$. ($\ell$ is the number of (interior) marked points.) In the same way we can associate a combinatorial type to it that is a graph $\mathcal{G}$. In this case there is no $C^0_{\rm d}(\mathcal{G})$ or $C^1_{\rm o}(\mathcal{G})$. We define $\mathcal{M}^{\rm cl}_\ell(\alpha;\mathcal{G})$, $\overset{\circ}{\mathcal{M}}{}^{\rm cl}_\ell(\alpha)$, in the same way.

Let us introduce some more notations. Let $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(\beta)$. We put

$$\mathfrak{p} = (\mathfrak{x}, u) = ((\Sigma, \vec z, \vec z^{\rm int}), u).$$

Then we sometimes write $\mathfrak{x} = \mathfrak{x}_{\mathfrak{p}}$, $\Sigma = \Sigma_{\mathfrak{p}} = \Sigma_{\mathfrak{x}}$, $\vec z = \vec z_{\mathfrak{p}} = \vec z_{\mathfrak{x}}$, $\vec z^{\rm int} = \vec z^{\rm int}_{\mathfrak{p}} = \vec z^{\rm int}_{\mathfrak{x}}$. We also write $u = u_{\mathfrak{p}}$. We use a similar notation in case $\mathfrak{p} \in \mathcal{M}^{\rm cl}_\ell(\alpha)$.

Definition 2.8. We put

$$\begin{aligned}\Gamma_{\mathfrak{p}} = \{v : \Sigma_{\mathfrak{p}} \to \Sigma_{\mathfrak{p}} \mid & v \text{ is a biholomorphic map, } v(z_{\mathfrak{p},i}) = z_{\mathfrak{p},i}, \\ & v(z^{\rm int}_{\mathfrak{p},i}) = z^{\rm int}_{\mathfrak{p},i}, u_{\mathfrak{p}} \circ v = u_{\mathfrak{p}}.\}\end{aligned} \tag{2.150}$$

$$\begin{aligned}\Gamma^+_{\mathfrak{p}} = \{v : \Sigma_{\mathfrak{p}} \to \Sigma_{\mathfrak{p}} \mid & v \text{ is a biholomorphic map, } v(z_{\mathfrak{p},i}) = z_{\mathfrak{p},i}, \\ & \exists \sigma \in \mathfrak{S}_\ell \; v(z^{\rm int}_{\mathfrak{p},i}) = z^{\rm int}_{\mathfrak{p},\sigma(i)}, \; u_{\mathfrak{p}} \circ v = u_{\mathfrak{p}}.\}\end{aligned} \tag{2.151}$$

Here $\mathfrak{S}_\ell$ is the group of permutations of $\{1, \dots, \ell\}$.

The assignment $v \mapsto \sigma$ defines a group homomorphism

$$\Gamma^+_{\mathfrak{p}} \to \mathfrak{S}_\ell. \tag{2.152}$$

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[9]Actually in our case of genus 0 with at least one marked point there is a canonical way to orient the edges as follows. We remove $z_{\rm e}$ from Σ. Then there is a component which contains the 0-th boundary marked point (or first interior marked point if $\partial\Sigma = \emptyset$). If v is a vertex contained in e we orient e so that v is inward if and only if the corresponding irreducible component is in the connected component of Σ minus boundary marked points that contains 0-th boudary marked point.

[10]This might be different from the usual meaning of the English word incoming and outgoing.

When $\mathfrak{H}$ is a subgroup of $\mathfrak{S}_\ell$ we denote by $\Gamma_{\mathfrak{p}}^{\mathfrak{H}}$ its inverse image by (2.152). We denote

$$\mathcal{M}_{k+1,\ell}(\beta;\mathfrak{H}) = \mathcal{M}_{k+1,\ell}(\beta)/\mathfrak{H},$$

where $\mathfrak{H}$ acts by permutation of the interior marked points.

In case X is a point we write $\mathcal{M}_{k+1,\ell}(\mathfrak{H})$ and define the groups $\Gamma_{\mathfrak{x}}^{\mathfrak{H}}$, $\Gamma_{\mathfrak{x}}^{+}$ for an element $\mathfrak{x} \in \mathcal{M}_{k+1,\ell}$. Note that in our case of genus zero with at least one boundary marked point, the group $\Gamma_{\mathfrak{x}}$ is trivial. (However this fact is never used in this note.)

We define a similar notion in the case of $\mathcal{M}_\ell^{\rm cl}$ etc.

2.2. **Coordinate around the singular point.** Let us assume that $\mathcal{G}$ is an oriented combinatorial type that is source stable and $\mathfrak{H}$ is a subgroup of $\mathfrak{S}_\ell$. Let $\mathfrak{x} = [\Sigma, \vec z, \vec z^{\rm int}] \in \mathcal{M}_{k+1,\ell}(\mathfrak{H})$ with combinatorial type $\mathcal{G}$. It is well-known that $\mathcal{M}_{k+1,\ell}(\mathfrak{H})$ is an effective orbifold with boundary and corners with its local model $\mathfrak{V}(\mathfrak{x})/\Gamma_{\mathfrak{x}}^{\mathfrak{H}}$. Let us describe this neighborhood in more detail below.

For each ${\rm v} \in C^0_{\rm d}(\mathcal{G})$, the element $\mathfrak{x}$ determines a marked disk $\mathfrak{x}_{\rm v} \in \overset{\circ}{\mathcal{M}}_{k_{\rm v}+1,\ell_{\rm v}}$. Here $k_{\rm v}$ is the sum of the number of edges $\in C^1_{\rm o}(\mathcal{G})$ containing v and the number of boundary marked points assigned to v. $\ell_{\rm v}$ is the sum of the number of edges $\in C^1_{\rm c}(\mathcal{G})$ containing v and the number of interior marked points assigned to v. (In other words the singular points of Σ that is contained in $\Sigma_{\rm v}$ is regarded as a marked point of $\mathfrak{x}_{\rm v}$.)

For each ${\rm v} \in C^0_{\rm s}(\mathcal{G})$, the element $\mathfrak{x}$ determines a marked sphere $\mathfrak{x}_{\rm v} \in \overset{\circ}{\mathcal{M}}{}^{\rm cl}_{\ell_{\rm v}}$ in the same way.

Let $\mathfrak{V}(\mathfrak{x}_{\rm v})/\Gamma_{\mathfrak{x}_{\rm v}}^{\mathfrak{H}}$ be the neighborhood of $\mathfrak{x}_{\rm v}$ in $\mathcal{M}_{k_{\rm v}+1,\ell_{\rm v}}(\mathfrak{H})$ or in $\mathcal{M}^{\rm cl}_{\ell_{\rm v}}(\mathfrak{H})$, respectively, according to whether ${\rm v} \in C^0_{\rm d}(\mathcal{G})$ or ${\rm v} \in C^0_{\rm s}(\mathcal{G})$. The group $\Gamma_{\mathfrak{x}}^{\mathfrak{H}}$ acts on the product $\prod \mathfrak{V}(\mathfrak{x}_{\rm v})$. The quotient

$$\mathfrak{V}(\mathfrak{x};\mathcal{G})/\Gamma_{\mathfrak{x}}^{\mathfrak{H}} = \left(\prod_{{\rm v}\in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{x}_{\rm v})\right)/\Gamma_{\mathfrak{x}}^{\mathfrak{H}}$$

is a neighborhood of $\mathfrak{x}$ in $\mathcal{M}_{k+1,\ell}(\mathcal{G};\mathfrak{H})$.

A neighborhood of $\mathfrak{x}$ in $\mathcal{M}_{k+1,\ell}(\mathfrak{H})$ is identified with

$$\left(\mathfrak{V}(\mathfrak{x};\mathcal{G}) \times \left(\prod_{{\rm e}\in C^1_{\rm o}(\mathcal{G})} (T_{{\rm e},0},\infty]\right) \times \left(\prod_{{\rm e}\in C^1_{\rm c}(\mathcal{G})} ((T_{{\rm e},0},\infty] \times S^1)/\sim\right)\right)/\Gamma_{\mathfrak{x}}^{\mathfrak{H}}. \tag{2.153}$$

Remark 2.9. The equivalence relation $\sim$ in (2.153) is defined as follows. $(T,\theta) \sim (T',\theta')$ if $(T,\theta) = (T',\theta')$ or $T = T' = \infty$.

The action of $\Gamma_{\mathfrak{x}}^{\mathfrak{H}}$ on

$$\left(\prod_{{\rm e}\in C^1_{\rm o}(\mathcal{G})} (T_{{\rm e},0},\infty]\right) \times \left(\prod_{{\rm e}\in C^1_{\rm c}(\mathcal{G})} ((T_{{\rm e},0},\infty] \times S^1)/\sim\right)$$

is by exchanging the factors associated to the edges e and by rotation of the S^1 factors. (See the proof of Lemma 2.17.)

We will define a map from (2.153) to $\mathcal{M}_{k+1,\ell}(\mathfrak{H})$. (See Definition 2.14.) We need to fix a coordinate of Σ around each of the singular point for this purpose. For the sake of consistency with the analytic construction in Section 1, we use cylindrical coordinate.

Definition 2.10. Let

$$\pi : \mathfrak{M}_{\mathfrak{x}_{\rm v}} \to \mathfrak{V}(\mathfrak{x}_{\rm v}) \tag{2.154}$$

be a fiber bundle whose fiber is a two dimensional manifold together with fiberwise complex structure. This fiber bundle is the universal family in the sense of (2) below. We call (2.154) with extra data described below a *universal family with coordinate at infinity* if the following conditions are satsified.

(1) $\mathfrak{M}_{\mathfrak{x}_{\rm v}}$ has a fiberwise biholomorphic $\Gamma^+_{\mathfrak{x}_{\rm v}}$ action and π is $\Gamma^+_{\mathfrak{x}_{\rm v}}$ equivariant.
(2) For $\mathfrak{y} \in \mathfrak{V}(\mathfrak{x}_{\rm v})$ the fiber $\pi^{-1}(\mathfrak{y})$ is biholomorphic to $\Sigma_{\mathfrak{y}}$ minus marked points corresponding to the singular points of $\mathfrak{y}$.
(3) As a part of the data we fix a closed subset $\mathfrak{K}_{\mathfrak{x}_{\rm v}} \subset \mathfrak{M}_{\mathfrak{x}_{\rm v}}$ such that $\pi : \mathfrak{K}_{\mathfrak{x}_{\rm v}} \to \mathfrak{V}(\mathfrak{x}_{\rm v})$ is proper.
(4) We consider the direct product

$$\begin{aligned}\mathfrak{V}(\mathfrak{x}_{\rm v})\times &\bigcup_{\substack{{\rm e}\in C^1_{\rm o}(\mathcal{G})\\ \text{e is an outgoing edge of v}}} (0,\infty)\times[0,1]\\ \cup &\bigcup_{\substack{{\rm e}\in C^1_{\rm o}(\mathcal{G})\\ \text{e is an incoming edge of v}}} (-\infty,0)\times[0,1]\\ \cup &\bigcup_{\substack{{\rm e}\in C^1_{\rm c}(\mathcal{G})\\ \text{e is an outgoing edge of v}}} (0,\infty)\times S^1\\ \cup &\bigcup_{\substack{{\rm e}\in C^1_{\rm c}(\mathcal{G})\\ \text{e is an incoming edge of v}}} (-\infty,0)\times S^1.\end{aligned} \tag{2.155}$$

(Here and hereafter the symbols $\cup$ and $\bigcup$ in (2.155) are the *disjoint* union.)
As a part of the data we fix a diffeomorphism between $\mathfrak{M}_{\mathfrak{x}_{\rm v}} \setminus \mathfrak{K}_{\mathfrak{x}_{\rm v}}$ and (2.155) that commutes with the projection to $\mathfrak{V}(\mathfrak{x}_{\rm v})$ and is a fiberwise biholomorphic map. Moreover the diffeomorphism sends each end corresponding to a singular point $z_{\rm e}$ to the end in (2.155) coprresponding to the edge e.
(5) The diffeomorphism in (4) extends to a fiber preserving diffeomorphism

$$\mathfrak{M}_{\mathfrak{x}_{\rm v}} \cong \mathfrak{V}(\mathfrak{x}_{\rm v}) \times \Sigma_{\mathfrak{x}_{\rm v}}.$$

This diffeomorphism sends each of the interior or boundary marked points of the fiber of $\mathfrak{y}$ to the corresponding marked point of $\{\mathfrak{y}\} \times \Sigma_{\mathfrak{x}_{\rm v}}$. However, this diffeomorphism does *not* preserve fiberwise complex structure. As a part of the data we fix this extension of diffeomorphism.
(6) The action of an element of $\Gamma^+_{\mathfrak{x}_{\rm v}}$ on (2.155) is given by exchanging the factors associated to the edges e and by rotation of the S^1 factors.

Hereafter we sometimes call a *coordinate at infinity* in place of a universal family with coordinate at infinity.

Example 2.11. Let $\mathfrak{x}_{\rm v}$ be S^2 with $\ell + 2$ marked points

$$z_0 = 0, z_1 = \infty, z_2 = 1, \dots, z_{\ell+1} = e^{2\pi\sqrt{-1}(\ell-1)/\ell}.$$

Let $\mathfrak{H} \subset \mathfrak{S}_{\ell+2}$ be the subgroup $\mathfrak{S}_\ell$ consisting of elements that fix z_0, z_1. We assume that z_0 and z_1 correspond to singular points of $\mathfrak{x}$. It is easy to see that $\Gamma_{\mathfrak{x}}^{\mathfrak{H}} = \mathbb{Z}_\ell$. Then $\Sigma_{\mathfrak{x}_{\rm v}} \setminus \{z_0, z_1\} = \mathbb{R} \times S^1$ and the action of $\Gamma_{\mathfrak{x}}^{\mathfrak{H}}$ is given by rotation of the S^1 factors.

Definition 2.12. Suppose we are given a coordinate at infinity for each of $\mathfrak{x}_{\rm v}$ where $\mathfrak{x}_{\rm v}$ corresponds to an irreducible component of $\mathfrak{x}$. We say that they are *invariant under the* $\Gamma_{\mathfrak{x}}^+$*-action* if the following holds.

We define a fiber bundle

$$\pi : \bigodot_{{\rm v}\in C^0(\mathcal{G})} \mathfrak{M}_{\mathfrak{x}_{\rm v}} \to \prod_{{\rm v}\in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{x}_{\rm v}) \tag{2.156}$$

as follows. We take projections $\prod_{{\rm v}\in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{x}_{\rm v}) \to \mathfrak{V}(\mathfrak{x}_{\rm v})$ and pull back the bundle (2.154) by this projection. We thus obtain a fiber bundle over $\prod_{{\rm v}\in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{x}_{\rm v})$. (2.156) is the disjoint union of those bundles over ${\rm v} \in C^0(\mathcal{G})$. In other words the fiber of (2.156) at $(\mathfrak{y}_{\rm v} : {\rm v} \in C^0(\mathcal{G}))$ is a disjoint union of $\mathfrak{y}_{\rm v}$'s.

The fiber bundle (2.156) has a $\Gamma_{\mathfrak{x}_{\rm v}}^+$-action. We consider its restriction to

$$\pi : \bigodot_{{\rm v}\in C^0(\mathcal{G})} (\mathfrak{M}_{\mathfrak{x}_{\rm v}} \setminus \mathfrak{K}_{\mathfrak{x}_{\rm v}}) \to \prod_{{\rm v}\in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{x}_{\rm v}). \tag{2.157}$$

The group $\Gamma_{\mathfrak{x}}^+$ acts on the sum of the second factors of (2.155) by exchanging the factors associated to the edges e and by rotation of the S^1 factors. We require that (2.157) is invariant under this action.

Moreover we assume that the diffeomorphisms in Definition 2.10 (4)(5) are $\Gamma_{\mathfrak{x}}^+$ equivariant.

Now we fix a coordinate at infinity for each of $\mathfrak{x}_{\rm v}$ that is invariant under the $\Gamma_{\mathfrak{x}}^{\mathfrak{H}}$ action. We will use it to define a map from (2.153) to a neighborhood of $\mathfrak{x}$ in $\mathcal{M}_{k+1,\ell}(\mathfrak{H})$ as follows. Let $(\mathfrak{y}_{\rm v} : {\rm v} \in C^0(\mathcal{G}))$ and $\mathfrak{y}_{\rm v} \in \mathfrak{V}(\mathfrak{x}_{\rm v})$. Take a representative $\Sigma_{\mathfrak{y}_{\rm v}}$ of $\mathfrak{y}_{\rm v}$. We put $K_{\mathfrak{y}_{\rm v}} = \Sigma_{\mathfrak{y}_{\rm v}} \cap \mathfrak{K}_{\mathfrak{x}_{\rm v}}$. The coordinate at infinity defines a biholomorphic map between $\bigcup_{{\rm v}\in C^0(\mathcal{G}))} \Sigma_{\mathfrak{y}_{\rm v}} \setminus K_{\rm v}$ and

$$\begin{aligned}
&\bigcup_{\substack{{\rm e}\in C^1_{\rm o}(\mathcal{G}) \\ \text{e is an outgoing edge of v}}} (0,\infty) \times [0,1] \\
\cup &\bigcup_{\substack{{\rm e}\in C^1_{\rm o}(\mathcal{G}) \\ \text{e is an incoming edge of v}}} (-\infty,0) \times [0,1] \\
\cup &\bigcup_{\substack{{\rm e}\in C^1_{\rm c}(\mathcal{G}) \\ \text{e is an outgoing edge of v}}} (0,\infty) \times S^1 \\
\cup &\bigcup_{\substack{{\rm e}\in C^1_{\rm c}(\mathcal{G}) \\ \text{e is an incoming edge of v}}} (-\infty,0) \times S^1.
\end{aligned} \tag{2.158}$$

We write the coordinate of each summand of (2.158) by $(\tau'_{\rm e}, t_{\rm e})$, $(\tau''_{\rm e}, t_{\rm e})$, $(\tau'_{\rm e}, t'_{\rm e})$, $(\tau''_{\rm e}, t''_{\rm e})$ respectively. (Here we identify $S^1 = \mathbb{R}/\mathbb{Z}$ so $t_{\rm e} \in [0,1]$ or $t'_{\rm e}, t''_{\rm e} \in \mathbb{R}/\mathbb{Z}$.)

Now, let $((T_{\rm e}; {\rm e} \in C^1_{\rm o}(\mathcal{G})), ((T_{\rm e}, \theta_{\rm e}); {\rm e} \in C^1_{\rm c}(\mathcal{G}))$ be an element of

$$\left(\prod_{{\rm e}\in C^1_{\rm o}(\mathcal{G})} (T_{{\rm e},0}, \infty] \right) \times \left(\prod_{{\rm e}\in C^1_{\rm c}(\mathcal{G})} ((T_{{\rm e},0}, \infty] \times S^1)/\sim \right). \tag{2.159}$$

(Here $\theta_{\rm e} \in \mathbb{R}/\mathbb{Z}$.)

Definition 2.13. We denote the right hand side of (2.159) by $(\vec{T}^{\rm o}_0, \infty] \times ((\vec{T}^{\rm c}_0, \infty] \times \vec{S}^1)$.

We first consider the case $T_{\rm e} \ne \infty$. We define $\tau_{\rm e}$ for $\rm e \in C^1(\mathcal{G})$ and $t_{\rm e}$ for $\rm e \in C^1_c(\mathcal{G})$ as follows.

$$\begin{aligned} \tau_e &= \tau'_{\rm e} - 5T_{\rm e} = \tau''_{\rm e} + 5T_{\rm e}, &(2.160)\\ t_{\rm e} &= t'_{\rm e} = t''_{\rm e} - \theta_{\rm e}. &(2.161)\end{aligned}$$

We note that (2.160), (2.161) are consistent with the notation of Section 1.4. We consider

$$[-5T_{\rm e}, 5T_{\rm e}] \times [0,1] \tag{2.162}$$

for each $\rm e \in C^1_o(\mathcal{G})$ with coordinate $(\tau_{\rm e}, t_{\rm e})$ and

$$[-5T_{\rm e}, 5T_{\rm e}] \times S^1 \tag{2.163}$$

for each $\rm e \in C^1_c(\mathcal{G})$ with coordinate $(\tau_{\rm e}, t_{\rm e})$.

We now consider the union

$$\begin{aligned}\bigcup_{\rm v \in C^0(\mathcal{G})} K_{\mathfrak{y}_{\rm v}} \cup \bigcup_{\rm e \in C^1_o(\mathcal{G})} [-5T_{\rm e}, 5T_{\rm e}] \times [0,1] \\ \cup \bigcup_{\rm e \in C^1_c(\mathcal{G})} [-5T_{\rm e}, 5T_{\rm e}] \times S^1.\end{aligned} \tag{2.164}$$

(2.160) and (2.161) describe the way how we glue various summands in (2.164) to obtain a bordered Riemann surface, that is nonsingular in our case where $T_e \ne \infty$.

Definition 2.14. We denote by $\overline{\Phi}((\mathfrak{y}_{\rm v}; \rm v \in C^0(\mathcal{G})), (T_{\rm e}; \rm e \in C^1_o(\mathcal{G})), (T_{\rm e}, \theta_{\rm e}); \rm e \in C^1_c(\mathcal{G}))$ the element of $\mathcal{M}_{k+1,\ell}$ represented by the above bordered Riemann surface.

Hereafter we write $\mathfrak{y} = (\mathfrak{y}_{\rm v}; \rm v \in C^0(\mathcal{G}))$, $\vec{T}^{\rm o} = (T_{\rm e}; \rm e \in C^1_o(\mathcal{G}))$, $\vec{T}^{\rm c} = (T_{\rm e}; \rm e \in C^1_c(\mathcal{G}))$, and $\vec{\theta} = (\theta_{\rm e}; \rm e \in C^1_c(\mathcal{G}))$. We put $\vec{T} = (\vec{T}^{\rm o}, \vec{T}^{\rm c})$. We denote $\overline{\Phi}(\mathfrak{y}, \vec{T}^{\rm o}, (\vec{T}^{\rm c}, \vec{\theta})) = \overline{\Phi}(\mathfrak{y}, \vec{T}, \vec{\theta}) \in \mathcal{M}_{k+1,\ell}$.

We next consider the case when some $T_{\rm e} = \infty$. We define a graph $\mathcal{G}'$ as follows : We shrink all the edges e of $\mathcal{G}$ with $T_{\rm e} \ne \infty$. Various data we associate to $\mathcal{G}'$ are induced by the one associated to $\mathcal{G}$ in an obvious way. The element $\overline{\Phi}(\mathfrak{y}, \vec{T}^{\rm o}, (\vec{T}^{\rm c}, \vec{\theta}))$ is contained in $\mathcal{M}_{k+1,\ell}(\mathcal{G}')$. Namely we glue (2.164) to obtain a (noncompact) bordered Riemann surface Σ'. Then we add a finite number of points (each corresponds to the edges with infinite length) to obtain (singular) stable bordered curve $\overline{\Phi}(\mathfrak{y}, \vec{T}^{\rm o}, (\vec{T}^{\rm c}, \vec{\theta}))$ such that $\overline{\Phi}(\mathfrak{y}, \vec{T}^{\rm o}, (\vec{T}^{\rm c}, \vec{\theta}))$ minus singular points is Σ'.

Thus we have defined

$$\overline{\Phi} : \prod_{\rm v \in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{x}_{\rm v}) \times (\vec{T}^{\rm o}_0, \infty] \times ((\vec{T}^{\rm c}_0, \infty] \times \vec{S}^1) \to \mathcal{M}_{k+1,\ell}.$$

We define some terminology below.

Definition 2.15. We call $K_{\mathfrak{y}_{\rm v}}$ as in (2.164) a component of the *core* of $\mathfrak{y}$ or of $\overline{\Phi}(\mathfrak{y}, \vec{T}^{\rm o}, (\vec{T}^{\rm c}, \vec{\theta}))$. Each of the connected component of the second or third term of (2.164) is called a component of the *neck region*. In case T is infinity, there is a domain identified with $([0,\infty) \cup (-\infty, 0]) \times [0,1]$ or with $([0,\infty) \cup (-\infty, 0]) \times S^1$ corresponding to it. We call it also a component of the neck region. The union of all the components of the core and the neck region is $\overline{\Phi}(\mathfrak{y},\vec{T}^{\rm o},(\vec{T}^{\rm c},\vec{\theta}))$ minus singular points.

Remark 2.16. Note that $\mathcal{M}_{k+1,\ell}$ has an $\mathfrak{S}_\ell$ action by permutation of the interior marked points. A local chart of $\mathcal{M}_{k+1,\ell}$ at $\mathfrak{x}$ is of the form $\mathfrak{V}/\Gamma_{\mathfrak{x}}$, and a local chart of $\mathcal{M}_{k+1,\ell}/\mathfrak{S}_\ell$ at $[\mathfrak{x}]$ is of the form $\mathfrak{V}/\Gamma^+_{\mathfrak{x}}$.

Lemma 2.17. *The map $\overline{\Phi}$ is $\Gamma^+_{\mathfrak{x}}$ equivariant.*

Proof. We first define a $\Gamma^+_{\mathfrak{x}}$ action on (2.159). Note an element of $\Gamma^+_{\mathfrak{x}}$ acts on the graph $\mathcal{G}$ in an obvious way. So it determines the way how to exchange the factors of (2.159). The rotation part of the action is defined as follows. By Definition 2.10 (6) we can determine the rotation of the $t_{\rm e}$ coordiante induced by an element of $\Gamma^+_{\mathfrak{x}}$. Therefore by (2.161) the action on $\theta_{\rm e}$ coordinate is determined.

Once we defined $\Gamma^+_{\mathfrak{x}}$ action on (2.159) the equivalence of the map $\overline{\Phi}$ is immediate from definition. □

Note that the space (2.153) has a stratification. (This stratification is induced by the stratification of $(0,\infty]$ that consists of $(0,\infty)$ and $\{\infty\}$. The map $\overline{\Phi}$ respects this stratification and stratification of $\mathcal{M}_{k+1,\ell}$ by $\{\mathcal{M}_{k+1,\ell}(\mathcal{G})\}$. Moreover $\overline{\Phi}$ is continuous and strata-wise smooth. We do not discuss the smooth structure of (2.153) yet. (See Subsection 2.7.)

We remark that the map $\overline{\Phi}$ *depends* on the choice of coordinate at infinity. The next result describes how $\overline{\Phi}$ depends on the choice of coordinate at infinity.

Let

$$\overline{\Phi}_1 : \prod_{{\rm v}\in C^0(\mathcal{G})} \mathfrak{V}^{(1)}(\mathfrak{x}_{\rm v}) \times (\vec{T}^{\rm o}_0,\infty] \times ((\vec{T}^{\rm c}_0,\infty] \times \vec{S}^1) \to \mathcal{M}_{k+1,\ell} \tag{2.165}$$

be the map in Definition 2.14. Suppose

$$\mathfrak{Y}_0 = \overline{\Phi}_1(\mathfrak{y}_0, \vec{T}_{\mathfrak{Y}_0}, \vec{\theta}_{\mathfrak{Y}_0})$$

and $\mathcal{G}_{\mathfrak{Y}_0}$ is the combinatorial type of $\mathfrak{Y}_0$. Note $\mathcal{G}_{\mathfrak{Y}_0}$ is obtained from $\mathcal{G}_{\mathfrak{x}}$ by shrinking several edges. Therefore we may regard

$$C^1(\mathcal{G}_{\mathfrak{Y}_0}) \subseteq C^1(\mathcal{G}_{\mathfrak{x}}).$$

Namely we can canonically identify ${\rm e} \in C^1(\mathcal{G}_{\mathfrak{x}})$ with an element of ${\rm e} \in C^1(\mathcal{G}_{\mathfrak{Y}_0})$ if $T_{\mathfrak{Y}_0,{\rm e}} = \infty$.

We take a coordinate at infinity of $\mathfrak{Y}_0$. By Definition 2.14 it determines an embedding

$$\overline{\Phi}_2 : \prod_{{\rm v}\in C^0(\mathcal{G}_{\mathfrak{Y}_0})} \mathfrak{V}^{(2)}(\mathfrak{Y}_{0,{\rm v}}) \times (\vec{T}^{\rm o}_1,\infty] \times ((\vec{T}^{\rm c}_1,\infty] \times \vec{S}^1) \to \mathcal{M}_{k+1,\ell}. \tag{2.166}$$

Here an element of $(\vec{T}^{\rm o}_1,\infty] \times ((\vec{T}^{\rm c}_1,\infty] \times \vec{S}^1)$ is $((T_{\rm e}; {\rm e} \in C^1_{\rm o}(\mathcal{G}_{\mathfrak{Y}_0}), ((T_{\rm e},\theta_{\rm e}), {\rm e} \in C^1_{\rm c}(\mathcal{G}_{\mathfrak{Y}_0}))$.

We put

$$\overline{\Phi}_{12} = \overline{\Phi}_1^{-1} \circ \overline{\Phi}_2. \tag{2.167}$$

We next define Ψ_{12}. Let $(\mathfrak{z}_{\rm v}) \in \prod_{{\rm v}\in C^0(\mathcal{G}_{\mathfrak{Y}_0})} \mathfrak{V}^{(2)}(\mathfrak{Y}_{0,{\rm v}})$. We denote $\vec{\infty} \in (\vec{T}^{\rm o}_1,\infty] \times ((\vec{T}^{\rm c}_1,\infty] \times \vec{S}^1)$ to be the point whose components are all ∞. Then $\overline{\Phi}_2((\mathfrak{z}_{\rm v}),\vec{\infty})$ has the same combinatorial type $\mathcal{G}_{\mathfrak{Y}_0}$ as $\mathfrak{Y}_0$. We define $\Psi_{12}^{\mathfrak{y}}((\mathfrak{z}_{\rm v})) \in \prod_{{\rm v}\in C^0(\mathcal{G})} \mathfrak{V}^{(1)}(\mathfrak{x}_{\rm v})$ and $\vec{T}', \vec{\theta}'$ by

$$\overline{\Phi}_1^{-1}(\overline{\Phi}_2((\mathfrak{z}_{\rm v}), \vec{\infty})) = (\Psi_{12}^{\mathfrak{y}}((\mathfrak{z}_{\rm v})), \vec{T}', \vec{\theta}').$$

We note that $T'_{\rm e} = \infty$ if ${\rm e} \in C^1(\mathcal{G}_{\mathfrak{Y}_0}) \subset C^1(\mathcal{G}_{\mathfrak{x}})$. Then we put

$$\Psi_{12}((\mathfrak{z}_{\rm v}), \vec{T}, \vec{\theta}) = (\Psi_{12}^{\mathfrak{y}}((\mathfrak{z}_{\rm v})), \vec{T}'', \vec{\theta}'') \tag{2.168}$$

where

$$T''_{\rm e} = \begin{cases} T_{\rm e} & \text{if } {\rm e} \in C^1(\mathcal{G}_{\mathfrak{Y}_0}) \\ T'_{\rm e} & \text{if } {\rm e} \in C^1(\mathcal{G}_{\mathfrak{x}}) \setminus C^1(\mathcal{G}_{\mathfrak{Y}_0}), \end{cases}$$

$$\theta''_{\rm e} = \begin{cases} \theta_{\rm e} & \text{if } {\rm e} \in C^1_{\rm c}(\mathcal{G}_{\mathfrak{Y}_0}) \\ \theta'_{\rm e} & \text{if } {\rm e} \in C^1_{\rm c}(\mathcal{G}_{\mathfrak{x}}) \setminus C^1_{\rm c}(\mathcal{G}_{\mathfrak{Y}_0}). \end{cases}$$

Remark 2.18. If $\mathfrak{Y}_0$ has the same combinatorial type as $\mathfrak{x}$ then Ψ_{12} is the identity map. Note that even in the case $\mathfrak{Y}_0 = \mathfrak{x}$ the map $\overline{\Phi}_{12}$ may not be the identity map since $\overline{\Phi}_j$ depends on the choice of coordinate at infinity.

Let $k_{T,{\rm e}} = 0, 1, \dots,$ $k_{\theta,{\rm e}} = 0, 1, 2, \dots$ and define

$$\frac{\partial^{|\vec{k}_T|}}{\partial T^{\vec{k}_T}} = \prod_{{\rm e}\in C^1(\mathcal{G})} \frac{\partial^{k_{T,{\rm e}}}}{\partial T_{\rm e}^{k_{T,{\rm e}}}}.$$

We define $\frac{\partial^{|\vec{k}_\theta|}}{\partial T^{\vec{k}_\theta}}$ in the same way. We put

$$\vec{k}_T \cdot \vec{T} = \sum_{{\rm e}\in C^1(\mathcal{G})} k_{T,{\rm e}} T_{\rm e}, \quad \vec{k}_\theta \cdot \vec{T}^{\rm c} = \sum_{{\rm e}\in C^1_{\rm c}(\mathcal{G})} k_{\theta,{\rm e}} T_{\rm e}.$$

Proposition 2.19. *In the above situation we have the following inequality for any compact subset $\mathfrak{V}_0(\mathfrak{x}, \mathcal{G})$ of $\mathfrak{V}(\mathfrak{x}, \mathcal{G})$:*

$$\left\| \frac{\partial^{|\vec{k}_T|}}{\partial T^{\vec{k}_T}} \frac{\partial^{|\vec{k}_\theta|}}{\partial \theta^{\vec{k}_\theta}} (\overline{\Phi}_{12} - \Psi_{12}) \right\|_{C^k} \le C_{1,k} e^{-\delta'(\vec{k}_T\cdot\vec{T} + \vec{k}_\theta\cdot\vec{T}^{\rm c})}, \tag{2.169}$$

for $|\vec{k}_T|, |\vec{k}_\theta| \le k$ with $|\vec{k}_T| + |\vec{k}_\theta| \ne 0$, where the left hand sides are C^k norm (as maps on $\mathfrak{y}$) and $\delta' > 0$ depends only on δ and k.

Remark 2.20. The estimates in Proposition 2.19 holds strata-wise. Namely in the situation where some of $T_{\rm e}$ is infinity, we only consider $\vec{k}_T, \vec{k}_\theta$ such that $k_{T,{\rm e}} = k_{\theta,{\rm e}} = 0$ for the edges e with $T_{\rm e} = \infty$.

Remark 2.21. During the proof of Proposition 2.19 and also during various discussions in later subsections, we need metrics of the source and the target to define various norms etc. For this purpose we take a Riemannian metric on X and also a family of metrics of the fibers of (2.154) such that outside $K_{\rm v}$ it coincides with the standard flat metric (via coordinates τ and t). We include it in the data of universal family with coordinate at infinity. Since we use it only to fix norm etc. it is not an important part of that data.

Proposition 2.19 is a generalization of [FOOO1, Lemma A1.59] and will be used for the same purpose later to derive the exponential decay estaimate of the coordinate change of our Kuranishi structure. We suspect Proposition 2.19 is not new. However for completeness sake the proof will be given later in Subsection 3.1.

Remark 2.22. In case $\mathfrak{Y}_0 = \mathfrak{x}$, Proposition 2.19 implies that there exists $\vec{\Delta T}$: $\mathfrak{V}_0(\mathfrak{x},\mathcal{G}) \to \mathbb{R}^{\#C^1(\mathcal{G})}$, $\vec{\Delta\theta} : \mathfrak{V}_0(\mathfrak{x},\mathcal{G}) \to (S^1)^{\#C^1_{\rm c}(\mathcal{G})}$ such that T component (resp. θ component) of $\overline{\Phi}_{21}$ goes to $\vec{T} + \vec{\Delta T}$ (resp. $\vec{\theta}_e + \vec{\Delta\theta}$) in an exponential order as T goes to infinity. (2.169) implies that $\mathfrak{y}$ component of $\overline{\Phi}_{21}$ goes to $\mathfrak{y}$ in exponential order as T goes to infinity.

Proposition 2.19 describes the coordinate change (change of the parametrization) of the moduli space. A coordinate at infinity determines a parametrization of the (bordered) curve itself, since it includes the trivialization of the fiber bundle (2.154). Proposition 2.23 below describes the way how it changes when we change the coordinate at infinity.

Let $\overline{\Phi}_{12} = \overline{\Phi}_1^{-1} \circ \overline{\Phi}_2$ be as in Proposition 2.19 and let $(\mathfrak{y}_j, \vec{T}_j, \vec{\theta}_j)$ $(j = 1, 2)$ be in the domain of $\overline{\Phi}_j$. We assume

$$(\mathfrak{y}_1, \vec{T}_1, \vec{\theta}_1) = \overline{\Phi}_{12}(\mathfrak{y}_2, \vec{T}_2, \vec{\theta}_2). \tag{2.170}$$

Let $\Sigma_{(\mathfrak{y}_j,\vec{T}_j,\vec{\theta}_j)}$ be a curve representing $\overline{\Phi}_j(\mathfrak{y}_j, \vec{T}_j, \vec{\theta}_j)$. It comes with coordinate at infinity. By (2.170) and stability, there exists a *unique* isomorphism

$$\mathfrak{v}_{(\mathfrak{y}_2,\vec{T}_2,\vec{\theta}_2)} : \Sigma_{(\mathfrak{y}_2,\vec{T}_2,\vec{\theta}_2)} \to \Sigma_{(\mathfrak{y}_1,\vec{T}_1,\vec{\theta}_1)} \tag{2.171}$$

of marked curves.

Let $K^{(j)}_{\rm v}$ be the core of $\Sigma_{(\mathfrak{y}_j,\vec{T}_j,\vec{\theta}_j)}$. We take a compact subset $K^{(2)}_{{\rm v},0} \subset K^{(2)}_{\rm v}$ such that

$$\mathfrak{v}_{(\mathfrak{y}_2,\vec{T}_2,\vec{\theta}_2)}(K^{(2)}_{{\rm v},0}) \subset K^{(1)}_{\rm v} \tag{2.172}$$

for sufficiently large $\vec{T}_1$. Note that the sets $K^{(1)}_{\rm v}$ and $K^{(2)}_{{\rm v},0}$ are independent of $(\mathfrak{y}_2, \vec{T}_2, \vec{\theta}_2)$. Let

$$C^k(K^{(2)}_{{\rm v},0}, K^{(1)}_{\rm v})$$

be the space of C^k maps with C^k topology. The restriction of $\mathfrak{v}_{(\mathfrak{y}_2,\vec{T}_2,\vec{\theta}_2)}$ to $K^{(2)}_{{\rm v},0}$ defines an element of it that we denote by

$$\mathrm{Res}(\mathfrak{v}_{(\mathfrak{y}_2,\vec{T}_2,\vec{\theta}_2)}) \in C^k(K^{(2)}_{{\rm v},0}, K^{(1)}_{\rm v}).$$

Proposition 2.23. *There exist $C_{2,k}$, T_k such that for each ${\rm e}_0 \in C^1_{\rm c}(\mathcal{G}_{\mathfrak{y}_2})$ we have*

$$\begin{aligned}
&\left\| \nabla^n_{\mathfrak{y}_2} \frac{\partial^{|\vec{k}_T|}}{\partial T_2^{\vec{k}_T}} \frac{\partial^{|\vec{k}_\theta|}}{\partial \theta_2^{\vec{k}_\theta}} \frac{\partial}{\partial T_{2,{\rm e}_0}} \mathrm{Res}(\mathfrak{v}_{(\mathfrak{y}_2,\vec{T}_2,\vec{\theta}_2)}) \right\|_{C^k} < C_{2,k} e^{-\delta_2 T_{2,{\rm e}_0}}, \\
&\left\| \nabla^n_{\mathfrak{y}_2} \frac{\partial^{|\vec{k}_T|}}{\partial T_2^{\vec{k}_T}} \frac{\partial^{|\vec{k}_\theta|}}{\partial \theta_2^{\vec{k}_\theta}} \frac{\partial}{\partial \theta_{2,{\rm e}_0}} \mathrm{Res}(\mathfrak{v}_{(\mathfrak{y}_2,\vec{T}_2,\vec{\theta}_2)}) \right\|_{C^k} < C_{2,k} e^{-\delta_2 T_{2,{\rm e}_0}},
\end{aligned} \tag{2.173}$$

if each of $T_{2,{\rm e}}$ is greater than T_k and $|\vec{k}_T| + |\vec{k}_\theta| + n \le k$. Here $\vec{T}_2 = (T_{2,{\rm e}}; {\rm e} \in C^1(\mathcal{G}_{\mathfrak{y}_2}))$, $\vec{\theta}_2 = (\theta_{2,{\rm e}}; {\rm e} \in C^1_{\rm c}(\mathcal{G}_{\mathfrak{y}_2}))$.

The first inequality also holds for ${\rm e}_0 \in C^1_{\rm o}(\mathcal{G}_{\mathfrak{y}_2})$.

We note that when all the numbers $T_{2,{\rm e}}$ are ∞, $\overline{\Phi}_2(\mathfrak{y}_2, \vec{T}_2, \vec{\theta}_2)$ has the same combinatorial type as $\mathfrak{Y}_0$. (Note $\overline{\Phi}_2$ gives a coordinate of the Deligne-Mumford moduli space in a neighborhood of $\mathfrak{Y}_0$.) Then, integrating on $T_{2,{\rm e}}$, Proposition 2.23 implies:

Corollary 2.24.

$$
\begin{aligned}
&\left\Vert \nabla_{\mathfrak{y}_2}^n \frac{\partial^{|\vec k_T|}}{\partial T_2^{\vec k_T}} \frac{\partial^{|\vec k_\theta|}}{\partial \theta_2^{\vec k_\theta}} (\mathrm{Res}(\mathfrak{v}_{(\mathfrak{y}_2,\vec T_2,\vec\theta_2)}) - \mathrm{Res}(\mathfrak{v}_{(\mathfrak{y}_2,\vec\infty)}))\right\Vert_{C^k} < C_{3,k} e^{-\delta_2 T_{2,\min}},\\
&\left\Vert \nabla_{\mathfrak{y}_2}^n \frac{\partial^{|\vec k_T|}}{\partial T_2^{\vec k_T}} \frac{\partial^{|\vec k_\theta|}}{\partial \theta_2^{\vec k_\theta}} (\mathrm{Res}(\mathfrak{v}_{(\mathfrak{y}_2,\vec T_2,\vec\theta_2)}) - \mathrm{Res}(\mathfrak{v}_{(\mathfrak{y}_2,\vec\infty)}))\right\Vert_{C^k} < C_{3,k} e^{-\delta_2 T_{2,\min}},
\end{aligned}
\tag{2.174}
$$

if $T_{2,\mathrm{e}} \ge T_{2,\min} > T_k$ *for all* e *and* $|\vec k_T| + |\vec k_\theta| + n \le k$. *Here* $T_{2,\min} = \min(T_{2,\mathrm{e}}; \mathrm{e} \in C^1(\mathcal{G}_{\mathfrak{y}_2}))$.

In later subsections we also use a parametrized version of Propositions 2.19 and 2.23, which we discuss now.

Let Q be a finite dimensional manifold. Suppose we have a fiber bundle

$$
\pi : \tilde{\mathfrak{M}}^{(2)}_{\mathfrak{x}_{\mathrm{v}}} \to Q_{\mathrm{v}} \times \mathfrak{V}(\mathfrak{x}_{\mathrm{v}}) \tag{2.175}
$$

that is a universal family (2.154) when we restrict it to each of $\{\xi\} \times \mathfrak{V}(\mathfrak{x}_{\mathrm{v}})$ for $\xi_{\mathrm{v}} \in Q_{\mathrm{v}}$. We put

$$
Q = \prod_{\mathrm{v}\in C^0(\mathcal{G})} Q_{\mathrm{v}}.
$$

Definition 2.25. A *Q-parametrized family of coordinates at infinity* is a fiber bundle (2.175) and its trivialization so that for each $\xi = (\xi_{\mathrm{v}})$ the restriction to $\{\xi_{\mathrm{v}}\} \times \mathfrak{V}(\mathfrak{x}_{\mathrm{v}})$ gives a coordinate at infinity in the sense of Definition 2.10.

Suppose a Q-parametrized family of coordinate at infinity in the above sense is given. Then we can perform the construction we already described for each ξ and obtain a map

$$
\overline{\Phi}_2 : Q \times \prod_{\mathrm{v}\in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{x}_{\mathrm{v}}) \times (\vec T_0^{\mathrm{o}}, \infty] \times ((\vec T_0^{\mathrm{c}}, \infty] \times \vec S^1) \to \mathcal{M}_{k+1,\ell}. \tag{2.176}
$$

Note that for each $\xi \in Q$ it gives a diffeomorphism to a neighborhood of $\mathfrak{x}$ in $\mathcal{M}_{k+1,\ell}$.

Suppose we have a (unparametrized) coordinate at infinity that is a fiber bundle

$$
\pi : \mathfrak{M}^{(1)}_{\mathfrak{x}_{\mathrm{v}}} \to \mathfrak{V}(\mathfrak{x}_{\mathrm{v}})
$$

equipped with trivialization. It induces an embedding

$$
\overline{\Phi}_1 : \prod_{\mathrm{v}\in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{x}_{\mathrm{v}}) \times (\vec T_0^{\mathrm{o}}, \infty] \times ((\vec T_0^{\mathrm{c}}, \infty] \times \vec S^1) \to \mathcal{M}_{k+1,\ell}.
$$

They induce a map

$$
\begin{aligned}
\overline{\Phi}_{12} : &Q \times \prod_{\mathrm{v}\in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{x}_{\mathrm{v}}) \times (\vec T_0^{\mathrm{o}\prime}, \infty] \times ((\vec T_0^{\mathrm{c}\prime}, \infty] \times \vec S^1)\\
&\to \prod_{\mathrm{v}\in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{x}_{\mathrm{v}}) \times (\vec T_0^{\mathrm{o}}, \infty] \times ((\vec T_0^{\mathrm{c}}, \infty] \times \vec S^1)
\end{aligned}
\tag{2.177}
$$

by the formula:

$$
\overline{\Phi}_1(\overline{\Phi}_{12}(\xi, \mathfrak{y}, \vec T, \vec\theta)) = \overline{\Phi}_2(\xi, \mathfrak{y}, \vec T, \vec\theta).
$$

Here $\vec T_0^{\mathrm{o}\prime}$ and $\vec T_0^{\mathrm{c}\prime}$ are sufficiently large compared with $\vec T_0^{\mathrm{o}}$ and $\vec T_0^{\mathrm{c}}$.

Moreover we have a family of biholomorphic maps:

$$\mathfrak{v}_{(\xi,\mathfrak{y},\vec{T},\vec{\theta})} : \Sigma^{\mathfrak{y},\xi,(2)}_{\vec{T},\vec{\theta}} \to \Sigma^{\mathfrak{y}',(1)}_{\vec{T}',\vec{\theta}'}. \tag{2.178}$$

Here $(\mathfrak{y}', \vec{T}', \vec{\theta}') = \overline{\Phi}_{12}(\xi, \mathfrak{y}, \vec{T}, \vec{\theta})$ and $\Sigma^{\mathfrak{y}',(1)}_{\vec{T}',\vec{\theta}'}$, $\Sigma^{\mathfrak{y},\xi,(2)}_{\vec{T},\vec{\theta}}$ are marked bordered curves representing $\overline{\Phi}_1(\overline{\Phi}_{12}(\xi, \rho, \vec{T}, \vec{\theta}))$ and $\overline{\Phi}_2(\xi, \rho, \vec{T}, \vec{\theta})$, respectively.

Lemma 2.26. *We have $C_{4,k}$, $C_{5,k}$ such that:*

$$\left\| \nabla_{\xi}^{\vec{k}_{\xi}} \frac{\partial^{|\vec{k}_T|}}{\partial T^{\vec{k}_T}} \frac{\partial^{|\vec{k}_\theta|}}{\partial \theta^{\vec{k}_\theta}} (\overline{\Phi}_{12}(\xi, \mathfrak{y}, \vec{T}, \vec{\theta}) - \Psi_{12}(\mathfrak{y}, \vec{T}, \vec{\theta})) \right\|_{C^k} \le C_{4,k} e^{-\delta(\vec{k}_T \cdot \vec{T} + \vec{k}_\theta \cdot \vec{T}^{\rm c})} \tag{2.179}$$

for $|\vec{k}_{\xi}|$, $|\vec{k}_T|, |\vec{k}_\theta| \le k$, if each of $T_{\rm e}$ is greater than T_k. The left hand sides are C^k norm (as functions on $\mathfrak{y}$). Moreover for each ${\rm e}_0 \in C^1_{\rm c}(\mathcal{G}_{\mathfrak{y}_2})$ we have

$$\begin{aligned} \left\| \nabla_{\xi}^{\vec{k}_{\xi}} \nabla_{\mathfrak{y}}^{n} \frac{\partial^{|\vec{k}_T|}}{\partial T^{\vec{k}_T}} \frac{\partial^{|\vec{k}_\theta|}}{\partial \theta^{\vec{k}_\theta}} \frac{\partial}{\partial T_{{\rm e}_0}} {\rm Res}(\mathfrak{v}_{(\xi,\mathfrak{y},\vec{T},\vec{\theta})}) \right\|_{C^k} &< C_{5,k} e^{-\delta_2 T_{{\rm e}_0}}, \\ \left\| \nabla_{\xi}^{\vec{k}_{\xi}} \nabla_{\mathfrak{y}}^{n} \frac{\partial^{|\vec{k}_T|}}{\partial T^{\vec{k}_T}} \frac{\partial^{|\vec{k}_\theta|}}{\partial \theta^{\vec{k}_\theta}} \frac{\partial}{\partial \theta_{{\rm e}_0}} {\rm Res}(\mathfrak{v}_{(\xi,\mathfrak{y},\vec{T},\vec{\theta})}) \right\|_{C^k} &< C_{5,k} e^{-\delta_2 T_{{\rm e}_0}}, \end{aligned} \tag{2.180}$$

if each of $T_{\rm e}$ is greater than T_k and $|\vec{k}_{\xi}| + |\vec{k}_T| + |\vec{k}_\theta| + n \le k$.

The first inequality of (2.180) also holds for ${\rm e}_0 \in C^1_{\rm o}(\mathcal{G}_{\mathfrak{y}_2})$.

Note that (2.179), (2.180) are parametrized versions of Propositions 2.19, 2.23, respectively. For the proof, see Subsection 3.1.

2.3. Stabilization of the source by adding marked points and obstruction bundles.

Let $((\Sigma, \vec{z}, \vec{z}^{\rm int}), u) = (\mathfrak{x}, u) \in \mathcal{M}_{k+1,\ell}(\beta; \mathcal{G})$. We assume that $\mathcal{G}$ is stable but is not source stable. In Section 1 we assumed that the source is stable. In order to carry out analytic detail similar to the one in Section 1 in the general case, we stabilize the source by adding marked points. In other words, we use the method of [FOn1, appendix] for this purpose.[11]

Remark 2.27. We note that the method of [FOn1, appendix] had been used earlier in various places by many people. A nonexhausting list of it is [Wo, Proposition 7.11, Theorem 9.1], [FOn1, appendix], [LT, begining of Section 3 and the proof of Lemma 3.1], [Si, page 395], [FOOO1, page 424], [FOOO2, Section 4.3]. See also [Ru, (3.9)].

We recall:

Definition 2.28. An irreducible component $\mathfrak{x}_{\rm v} = (\Sigma_{\rm v}, \vec{z}_{\rm v}, \vec{z}^{\rm int}_{\rm v})$ of $\mathfrak{x}$ is said to be *unstable*, if and only if one of the following holds:

(1) $\mathfrak{x}_{\rm v} \in \mathcal{M}_{k_{\rm v}+1,\ell_{\rm v}}$ and $k_{\rm v} + 1 + 2\ell_{\rm v} < 3$.
(2) $\mathfrak{x}_{\rm v} \in \mathcal{M}^{\rm cl}_{\ell_{\rm v}}$ and $\ell_{\rm v} < 3$.

There is at least one boundary marked point in case $\mathfrak{x}_{\rm v}$ is a disk ($\mathfrak{x} \in \mathcal{M}_{k+1,\ell}$ and $k+1 > 0$), and at least one interior marked point in case $\mathfrak{x}_{\rm v}$ is a sphere. (This is because it should be attached to a disk or to a sphere.) Note we assume $\ell \ge 1$ in case of $\mathcal{M}^{\rm cl}_{\ell}$.) Therefore there are three cases where $\mathfrak{x}_{\rm v}$ is unstable:

[11] 16 years of experience shows that the method of [FOn1, appendix] is easier to use in various applications than the method of [FOn1, Section 13].

(a) $\mathfrak{x}_{\rm v}$ is a disk. $\mathfrak{x}_{\rm v} \in \mathcal{M}_{k_{\rm v}+1,\ell_{\rm v}}$ and $k_{\rm v} = 0$ or 1. $\ell_{\rm v} = 0$.
(b) $\mathfrak{x}_{\rm v}$ is a sphere. $\mathfrak{x}_{\rm v} \in \mathcal{M}^{\rm cl}_{\ell_{\rm v}}$ and $\ell_{\rm v} = 2$.
(c) $\mathfrak{x}_{\rm v}$ is a sphere. $\mathfrak{x}_{\rm v} \in \mathcal{M}^{\rm cl}_{\ell_{\rm v}}$ and $\ell_{\rm v} = 1$.

Remark 2.29. In the case of higher genus there are some other kinds of irreducible components that are unstable. For example, T^2 without marked points is unstable. We can handle them in the same way. If we consider also $\mathcal{M}^{\rm cl}_0(\alpha)$, then $\mathcal{M}^{\rm cl}_0$ also appears.

Definition 2.30. ([FOn1, Section 13 p989 and appendix p1047]) A *minimal stabilization* is a choice of additional interior marked points, where we put one interior marked point $w_{\rm v}$ of $\Sigma_{\rm v}$ for each $\mathfrak{x}_{\rm v}$ satisfying (a) or (b) above and two interior marked points $w_{{\rm v},1}$, $w_{{\rm v},2}$ for each $\mathfrak{x}_{\rm v}$ satisfying (c) above, so that the following holds.

(1) $w_{\rm v} \notin \vec z^{\rm int}_{\rm v}$. $w_{{\rm v},1}, w_{{\rm v},2} \notin \vec z^{\rm int}_{\rm v}$. They are not singular.
(2) u is an immersion at $w_{\rm v}$, $w_{{\rm v},1}$, $w_{{\rm v},2}$.
(3) Let $v \in \Gamma^+_{(\mathfrak{x},u)}$ such that $v\Sigma_{\rm v} = \Sigma_{\rm v'}$. Suppose $\mathfrak{x}_{\rm v}$ satisfies (a) or (b) above. Then $vw_{\rm v} = v'w_{\rm v'}$ for some $v' \in \Gamma^+_{(\mathfrak{x}_{\rm v'},u)}$. Suppose $\mathfrak{x}_{\rm v}$ satisfies (c) above. Then there exists $v' \in \Gamma^+_{(\mathfrak{x}_{\rm v'},u)}$ such that $vw_{{\rm v},i} = v'w_{{\rm v'},i}$ for $i = 1, 2$.
(4) $w_{{\rm v},1} \ne v'w_{{\rm v},2}$ for any $v' \in \Gamma^+_{(\mathfrak{x}_{\rm v},u)}$.

(We add three marked points in the case of $\mathcal{M}^{\rm cl}_0$.)

Definition 2.31. A *symmetric stabilization* is a choice of additional marked points $\vec w = (w_1, \dots, w_{\ell'}) \in {\rm Int}\,\Sigma$, such that:

(1) $\vec w \cap \vec z^{\rm int} = \emptyset$.
(2) $w_i \ne w_j$ for $i \ne j$.
(3) u is an immersion at each w_i.
(4) $(\Sigma, \vec z, \vec w \cup \vec z^{\rm int})$ is stable.
(5) For each $v \in \Gamma^+_{(\mathfrak{x},u)}$ there exists $\sigma_v \in \mathfrak{S}_{\ell'}$, such that
$$v(w_i) = w_{\sigma_v(i)}.$$

We note that a minimal stabilization induces a symmetric stabilization. Namely we take
$$\begin{aligned}&\{vw_{\rm v} \mid v \in \Gamma^+_{(\mathfrak{x}_{\rm v},u)},\ \mathfrak{x}_{\rm v} \text{ satisfies (a) or (b)}\}\\ &\cup \{vw_{{\rm v},i} \mid v \in \Gamma^+_{(\mathfrak{x}_{\rm v},u)}, i = 1, 2,\ \mathfrak{x}_{\rm v} \text{ satisfies (c)}\}.\end{aligned}$$
Since the notion of symmetric stabilization is more general, we use symmetric stabilization in this note. Symmetric stabilization was used in [FOOO2].

We write
$$\mathfrak{x} \cup \vec w = (\Sigma, \vec z, \vec z^{\rm int} \cup \vec w)$$
when $\mathfrak{x} = (\Sigma, \vec z, \vec z^{\rm int})$.

Remark 2.32. In our genus zero case, Definition 2.31 (4) implies that the automorphism group of $(\Sigma, \vec z, \vec z^{\rm int} \cup \vec w)$ is trivial.[12] So we can define an *injective* homomorphism
$$\sigma : \Gamma_{(\mathfrak{x},u)} \to \mathfrak{S}_{\ell'} \tag{2.181}$$

[12]In the case of higher genus, we may include the triviality of the automorphism as a part of the definition of the symmetric stabilization. If we do so then (2.181) is still an injective homomorphism.

by

$$v(w_i) = w_{\sigma(i)}.$$

(Here $\mathfrak{S}_{\ell'}$ is the symmetric group of order $\ell'!$.) We denote by $\mathfrak{H}_{(\mathfrak{x},u)}$ the image of (2.181). In a similar way we obtain an injective homomorphism

$$\sigma : \Gamma^+_{(\mathfrak{x},u)} \to \mathfrak{S}_\ell \times \mathfrak{S}_{\ell'}. \tag{2.182}$$

We denote its image by $\mathfrak{H}^+_{(\mathfrak{x},u)}$.

We use the notion of symmetric stabilization of $\mathfrak{x} \in \mathcal{M}_{k+1,\ell}(\beta;\mathcal{G})$ to define the notion of obstruction bundle data as follows.

Definition 2.33. An *obstruction bundle data* $\mathfrak{E}_{\mathfrak{p}}$ *centered at*

$$\mathfrak{p} = (\mathfrak{x},u) = ((\Sigma,\vec z,\vec z^{\rm int}),u) \in \mathcal{M}_{k+1,\ell}(\beta;\mathcal{G})$$

is the data satisfying the conditions described below.

(1) A symmetric stabilization $\vec w = (w_1,\dots,w_{\ell'})$ of $(\mathfrak{x},u)$. We denote by $\mathcal{G}_{\vec w\cup\mathfrak{x}}$ the combinatorial type of $\vec w \cup \mathfrak{x}$.

(2) A neighborhood $\mathfrak{V}(\mathfrak{x}_{\rm v}\cup\vec w_{\rm v})$ of $\mathfrak{x}_{\rm v}\cup\vec w_{\rm v} = (\Sigma_{\mathfrak{x}_{\rm v}},\vec z_{\mathfrak{x}_{\rm v}},\vec z^{\rm int}_{\rm v}\cup\vec w_{\rm v})$ in $\overset{\circ}{\mathcal{M}}_{k_{\rm v}+1,\ell_{\rm v}+\ell'_{\rm v}}$ or $\overset{\circ}{\mathcal{M}}{}^{\rm cl}_{\ell_{\rm v}+\ell'_{\rm v}}$. Here $\mathfrak{x}_{\rm v} \in \overset{\circ}{\mathcal{M}}_{k_{\rm v}+1,\ell_{\rm v}}$ or $\in \overset{\circ}{\mathcal{M}}{}^{\rm cl}_{\ell_{\rm v}+\ell'_{\rm v}}$ is an irreducible component of $\mathfrak{x}$ and $\vec w_{\rm v}$ is a part of $\vec w$ that is contained in this irreducible component.

(3) A universal family with coordinate at infinity of $\mathfrak{x}_{\rm v}\cup\vec w_{\rm v}$ defined on $\mathfrak{V}(\mathfrak{x}_{\rm v}\cup\vec w_{\rm v})$. (We use the notation of Definition 2.10.) We assume that it is invariant under the $\Gamma^{\mathfrak{H}^+_{(\mathfrak{x},u)}}_{(\mathfrak{x}\cup\vec w,u)}$ action in the sense we will explain later.

(4) A compact subset $K^{\rm obst}_{\rm v}$ such that $K^{\rm obst}_{\rm v}\times\mathfrak{V}(\mathfrak{x}_{\rm v}\cup\vec w_{\rm v})$ is contained in $\mathfrak{K}_{\mathfrak{x}_{\rm v}}$, which is defined in Definition 2.10 (3). We assume that they are $\Gamma^{\mathfrak{H}^+_{(\mathfrak{x},u)}}_{(\mathfrak{x}\cup\vec w,u)}$ invariant in the sense we will explain later. We call $K^{\rm obst}_{\rm v}$ the *support of the obstruction bundle*.

(5) A $\mathfrak{y}\in\mathfrak{V}(\mathfrak{x}_{\rm v}\cup\vec w_{\rm v})$-parametrized smooth family of finite dimensional complex linear subspaces $E_{\mathfrak{p},{\rm v}}(\mathfrak{y},u)$ of

$$\Gamma_0({\rm Int}\,K^{\rm obst}_{\rm v}; u^*TX\otimes\Lambda^{01}).$$

Here Γ_0 denotes the set of the smooth sections with compact support on the domain $\Sigma_{\mathfrak{y}_{\rm v}}$ induced by $\mathfrak{y}_{\rm v}\in\mathfrak{V}(\mathfrak{x}_{\rm v}\cup\vec w_{\rm v})$. We regard $u:\Sigma_{\mathfrak{x}_{\rm v}}\to X$ also as a map from $\Sigma_{\mathfrak{y}_{\rm v}}$ by using the smooth trivialization of the universal family given as a part of Definition 2.10 (5).

We assume that $\bigoplus_{{\rm v}\in C^0(\mathcal{G})} E_{\rm v}$ is invariant under the $\Gamma^{\mathfrak{H}^+_{\mathfrak{p}}}_{(\mathfrak{x}\cup\vec w,u)}$ action in the sense we will explain later.

(6) For each ${\rm v}\in C^0_{\rm d}(\mathcal{G}_{\mathfrak{p}})$ and $\mathfrak{y}_{\rm v}\in\mathfrak{V}(\mathfrak{x}_{\rm v}\cup\vec w_{\rm v})$ the differential operator

$$\begin{aligned}\overline{D}_u\overline{\partial} :& L^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_{\rm v}},\partial\Sigma_{\mathfrak{y}_{\rm v}}); u^*TX, u^*TL)\\ &\to L^2_{m,\delta}(\Sigma_{\mathfrak{y}_{\rm v}}; u^*TX\otimes\Lambda^{01})/E_{\mathfrak{p},{\rm v}}(\mathfrak{y},u)\end{aligned} \tag{2.183}$$

is surjective. (We define the above weighted Sobolev spaces in the same way as in Subsection 1.2. See Subsection 2.5 for the precise definition in the general case.)

If $\mathrm{v} \in C^0_{\mathrm{s}}(\mathcal{G}_{\mathfrak{p}})$ and $\mathfrak{y}_{\mathrm{v}} \in \mathfrak{V}(\mathfrak{x}_{\mathrm{v}} \cup \vec{w}_{\mathrm{v}})$, the differential operator

$$\begin{aligned} \overline{D}_u\overline{\partial} : & L^2_{m+1,\delta}(\Sigma_{\mathfrak{y}_{\mathrm{v}}}; u^*TX) \\ & \to L^2_{m,\delta}(\Sigma_{\mathfrak{y}_{\mathrm{v}}}; u^*TX \otimes \Lambda^{01})/E_{\mathfrak{p},\mathrm{v}}(\mathfrak{y}, u) \end{aligned} \tag{2.184}$$

is surjective.

(7) The kernels of (2.183) and (2.184) satisfy a transversality property for evaluation maps that is as described in Condition 2.34.

(8) For each $w_i \in \Sigma_{\mathrm{v}}$ we take a codimension 2 submanifold $\mathcal{D}_i$ of X such that $u(w_i) \in \mathcal{D}_i$ and

$$u_* T_{w_i}\Sigma_{\mathrm{v}} + T_{u(w_i)}\mathcal{D}_i = T_{w_i}X.$$

Moreover $\{\mathcal{D}_i\}$ is invariant under the $\Gamma^+_{\mathfrak{p}}$ action in the following sense. Let $v \in \Gamma^+_{\mathfrak{p}}$ and $v(w_i) = w_{\sigma(i)}$ then

$$\mathcal{D}_i = \mathcal{D}_{\sigma(i)}. \tag{2.185}$$

(Note $u(w_i) = u(w_{\sigma(i)})$ since $u \circ v = u$.)

Condition 2.34. Suppose a vertex $\mathrm{v} \in C^0_{\mathrm{d}}(\mathcal{G}_{\mathfrak{p}})$ is contained in an edge $\mathrm{e} \in C^1_{\mathrm{o}}(\mathcal{G}_{\mathfrak{p}})$. Let z_{e} be a singular point of $\Sigma_{\mathfrak{x}}$ corresponding to the edge $\mathrm{e} \in C^1_{\mathrm{o}}(\mathcal{G}_{\mathfrak{p}})$. We define

$$\mathrm{ev}_{\mathrm{v},\mathrm{e}} : L^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_{\mathrm{v}}}, \partial\Sigma_{\mathfrak{y}_{\mathrm{v}}}); u^*TX, u^*TL) \to T_{u(z_{\mathrm{e}})}L \tag{2.186}$$

by $s \mapsto \pm s(z_{\mathrm{e}})$ where we take $+$ if v is an outgoing vertex of e and we take $-$ if v is an incoming vertex of e. If $\mathrm{v} \in C^0_{\mathrm{d}}(\mathcal{G}_{\mathfrak{p}})$ and $\mathrm{e} \in C^1_{\mathrm{c}}(\mathcal{G}_{\mathfrak{p}})$, then we define

$$\mathrm{ev}_{\mathrm{v},\mathrm{e}} : L^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_{\mathrm{v}}}, \partial\Sigma_{\mathfrak{y}_{\mathrm{v}}}); u^*TX, u^*TL) \to T_{u(z_{\mathrm{e}})}X \tag{2.187}$$

by the same formula. In a similar way we define

$$\mathrm{ev}_{\mathrm{v},\mathrm{e}} : L^2_{m+1,\delta}(\Sigma_{\mathfrak{y}_{\mathrm{v}}}; u^*TX) \to T_{u(z_{\mathrm{e}})}X, \tag{2.188}$$

if $\mathrm{e} \in C^1_{\mathrm{c}}(\mathcal{G}_{\mathfrak{p}})$ and $\mathrm{v} \in C^0_{\mathrm{s}}(\mathcal{G}_{\mathfrak{p}})$ is its vertex.

Combining all of (2.186), (2.187), (2.188) we obtain a map:

$$\begin{aligned} \mathrm{ev}_{\mathcal{G}_{\mathfrak{p}}} : & \bigoplus_{\mathrm{v} \in C^0_{\mathrm{d}}(\mathcal{G}_{\mathfrak{p}})} L^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_{\mathrm{v}}}, \partial\Sigma_{\mathfrak{y}_{\mathrm{v}}}); u^*TX, u^*TL) \\ & \oplus \bigoplus_{\mathrm{v} \in C^0_{\mathrm{s}}(\mathcal{G}_{\mathfrak{p}})} L^2_{m+1,\delta}(\Sigma_{\mathfrak{y}_{\mathrm{v}}}; u^*TX) \\ & \to \bigoplus_{\mathrm{e} \in C^1_{\mathrm{o}}(\mathcal{G}_{\mathfrak{p}})} T_{u(z_{\mathrm{e}})}L \oplus \bigoplus_{\mathrm{e} \in C^1_{\mathrm{c}}(\mathcal{G}_{\mathfrak{p}})} T_{u(z_{\mathrm{e}})}X. \end{aligned} \tag{2.189}$$

The condition we require is that the restriction of $\mathrm{ev}_{\mathcal{G}_{\mathfrak{p}}}$ to

$$\bigoplus_{\mathrm{v} \in C^0(\mathcal{G}_{\mathfrak{p}})} \mathrm{Ker}\overline{D}_{u_{\mathrm{v}}}\overline{\partial}$$

is surjective.

Remark 2.35. In [FOOO1] we used Kuranishi structures on $\mathcal{M}_{k+1,\ell}(\beta)$ so that the evaluation maps $\mathrm{ev} : \mathcal{M}_{k+1,\ell}(\beta) \to L^{k+1} \times X^{\ell}$ are weakly submersive. To construct Kuranishi structures satisfying this additional property, we need to require an additional assumption to the obstruction bundle data. Namely we need to assume that the evaluation maps at the marked points

$$\mathrm{ev} : \bigoplus_{\mathrm{v}\in C^0_{\mathrm{d}}(\mathcal{G}_{\mathfrak{p}})} L^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_{\mathrm{v}}}, \partial\Sigma_{\mathfrak{y}_{\mathrm{v}}}); u^*TX, u^*TL) \to \prod_{i=0}^{k} T_{u(z_i)}L \times \prod_{i=1}^{\ell} T_{u(z_i^{\mathrm{int}})}X$$

are also surjective. But we do not include it in the definition here since there are cases we do not assume it.

We next explain the precise meaning of invariace under the action in (3), (4), (5). The invariance in (3) is defined in Definition 2.12. The $\Gamma^{\mathfrak{H}^+_{(\mathfrak{x},u)}}_{(\mathfrak{x}\cup\vec{w},u)}$ action on $\mathfrak{K}_{\mathfrak{x}_{\mathrm{v}}}$ is induced by its action. (See Definition 2.12.) So we require (the totality of) $K^{\mathrm{obst}}_{\mathrm{v}}$ is invariant under this action in (4). To make sense of (5) we define a $\Gamma^{\mathfrak{H}^+_{(\mathfrak{x},u)}}_{(\mathfrak{x}\cup\vec{w},u)}$ action on

$$\bigoplus_{\mathrm{v}\in C^0(\mathcal{G})} \Gamma_0(\mathrm{Int}\, K^{\mathrm{obst}}_{\mathrm{v}}; u^*TX \otimes \Lambda^{01}). \tag{2.190}$$

If $v \in \Gamma^{\mathfrak{H}^+_{(\mathfrak{x},u)}}_{(\mathfrak{x}\cup\vec{w},u)}$ then $v\Sigma_{\mathrm{v}} = \Sigma_{\mathrm{v}'}$ for some v' and $K^{\mathrm{obst}}_{\mathrm{v}'} = vK^{\mathrm{obst}}_{\mathrm{v}}$ by (4). Moreover $u \circ v = u$ holds on Σ_{v}. Therefore we obtain

$$v_* : \Gamma_0(\mathrm{Int}\, K^{\mathrm{obst}}_{\mathrm{v}}; u^*TX \otimes \Lambda^{01}) \cong \Gamma_0(\mathrm{Int}\, K^{\mathrm{obst}}_{\mathrm{v}'}; u^*TX \otimes \Lambda^{01}).$$

They induce a $\Gamma^{\mathfrak{H}^+_{(\mathfrak{x},u)}}_{(\mathfrak{x}\cup\vec{w},u)}$ action on (2.190). Note that this is the case of the action at $\vec{w} \cup \mathfrak{p} = (\vec{w} \cup \mathfrak{x}, u)$. When we move to a nearby point $(\mathfrak{y}, u)$, the situation becomes slightly different, since $v_*\mathfrak{y} = \mathfrak{y}$ holds no longer. We have a smooth trivialization of the bundle (2.154). (Definition 2.10 (5).) Namely we are given a diffeomorphism

$$v : K_{\mathrm{v}}(\mathfrak{y}) \to K_{\mathrm{v}'}(\mathfrak{y})$$

between the cores. (Here we write $K_{\mathrm{v}}(\mathfrak{y})$ in place of K_{v} to include its complex structure.) However this is not a biholomorphic map. On the other hand

$$v : K_{\mathrm{v}}(\mathfrak{y}) \to K_{\mathrm{v}'}(v_*\mathfrak{y})$$

is a biholomorphic map by Definition 2.10 (1). Therefore we still obtain a map

$$\begin{aligned} v_* :&\Gamma_0(\mathrm{Int}\, K^{\mathrm{obst}}_{\mathrm{v}}(\mathfrak{y}); u^*TX \otimes \Lambda^{01}) \\ &\cong \Gamma_0(\mathrm{Int}\, K^{\mathrm{obst}}_{\mathrm{v}'}(v_*\mathfrak{y}); (u\circ v^{-1})^*TX \otimes \Lambda^{01}). \end{aligned} \tag{2.191}$$

Definition 2.33 (5) means

$$v_* \left(E_{\mathfrak{p},\mathrm{v}}(\mathfrak{y}, u)\right) = E_{\mathfrak{p},\mathrm{v}'}(v_*\mathfrak{y}, u \circ v^{-1}) = E_{\mathfrak{p},\mathrm{v}'}(v_*\mathfrak{y}, u)$$

where the map v_* appearing at the beginning of the formula is the map (2.191).

Remark 2.36. The condition (8), especially $u(w_i) \in \mathcal{D}_i$, is assumed only for $\mathfrak{p}$ and $\vec{w}$. For the general point $\mathfrak{V}(\mathfrak{y}_{\mathrm{v}} \cup \vec{w}_{\mathrm{v}})$ this condition is not assumed at this stage. We put this condition only at later step (Subsection 2.6. See also Definition 2.49.) and only to the solutions of the equation.

Lemma 2.37. *For each $\mathfrak{p}$ there exists an obstruction bundle data $\mathfrak{E}_{\mathfrak{p}}$ centered at $\mathfrak{p}$.*

Proof. Existence of symmetric stabilization is obvious. We can find $E_{\mathfrak p,\mathrm v}(\mathfrak p \cup \vec w_{\mathfrak p})$ for $\mathrm v \in C^0(\mathcal G_{\mathfrak p\cup\vec w_{\mathfrak p}})$ satisfying (7), (8) by the unique continuation properties of the linearization of the Cauchy-Rieman equation. We can make them $\Gamma^{\mathfrak H_{\mathfrak p}^+}_{\mathfrak p\cup\vec w_{\mathfrak p}}$ invariant by taking the union of the images of actions. Then we extend them to a small neighborhood of $\mathfrak p\cup\vec w_{\mathfrak p}$ in a way such that (7), (8) are satisfied. We make them $\Gamma^{\mathfrak H_{\mathfrak p}^+}_{\mathfrak p\cup\vec w_{\mathfrak p}}$ invariant by taking average as follows. Let $\mathfrak y = (\mathfrak y_{\mathrm v})$ such that $\mathfrak y_{\mathrm v} \in \mathfrak V(\mathfrak x_{\mathrm v}\cup\vec w_{\mathrm v})$. Using the trivialization of the bundle (2.154) we can define

$$\mathfrak I'_{\mathfrak y} : \bigoplus_{\mathrm v\in C^0(\mathcal G_{\mathfrak p\cup\vec w_{\mathfrak p}})} E_{\mathfrak p,\mathrm v} \to \bigoplus_{\mathrm v\in C^0(\mathcal G)} \Gamma(\Sigma_{\mathfrak y,\mathrm v}; u^*TX\otimes\Lambda^{01}).$$

Note for $v \in \Gamma^{\mathfrak H_{\mathfrak p}^+}_{\mathfrak p\cup\vec w_{\mathfrak p}}$ the equality $v_*\circ\mathfrak I'_{\mathfrak y} = \mathfrak I'_{v\mathfrak y}\circ v_*$ may not be satisfied. However since $v_*\circ\mathfrak I'_{\mathfrak p} = \mathfrak I'_{\mathfrak p}\circ v_*$ we may assume

$$\|v_*\circ\mathfrak I'_{\mathfrak y} - \mathfrak I'_{v\mathfrak y}\circ v_*\|$$

is small by taking $\mathfrak V(\mathfrak x_{\mathrm v}\cup\vec w_{\mathrm v})$ small. Therefore

$$\mathfrak I_{\mathfrak y} = \frac{1}{\#\Gamma_{\mathfrak p\cup\vec w_{\mathfrak p}}} \sum_{v\in\Gamma^{\mathfrak H_{\mathfrak p}^+}_{\mathfrak p\cup\vec w_{\mathfrak p}}} (v^{-1})_*\circ\mathfrak I'_{v\mathfrak y}\circ v_*$$

is injective and close to $\mathfrak I'_{\mathfrak p}$. We hence obtain the required $E_{\mathfrak p}(\mathfrak y)$ by

$$E_{\mathfrak p}(\mathfrak y) = \mathrm{Im}\mathfrak I_{\mathfrak y}.$$

The existence of the codimension 2 submanifolds $\mathcal D_i$ is obvious. □

The obstruction bundle data determines

$$E_{\mathfrak p}(\mathfrak y,u) = \bigoplus_{\mathrm v\in C^0(\mathcal G_{\mathfrak p\cup\vec w_{\mathfrak p}})} E_{\mathfrak p,\mathrm v}(\mathfrak y,u) \subset \bigoplus_{\mathrm v\in C^0(\mathcal G_{\mathfrak p\cup\vec w_{\mathfrak p}})} L^2_{m,\delta}(\Sigma_{\mathfrak y_{\mathrm v}}; u^*TX\otimes\Lambda^{01})$$

for $\mathfrak y\in\mathfrak V(\mathfrak x\cup\vec w)$. This subspace plays the role of (a part of) the obstruction bundle of the Kuranishi structure we will construct. To define our equation and thickened moduli space we need to extend the family of linear subspaces $E_{\mathfrak p}(\cdot)$ so that we associate $E_{\mathfrak p}(\mathfrak q)$ to an object $\mathfrak q$ which is 'close' to $\mathfrak p$. We will define this close-ness below. (This is a generalization of Condition 1.4.)

We use the map

$$\overline\Phi : \prod_{\mathrm v\in C^0(\mathcal G_{\mathfrak p\cup\vec w_{\mathfrak p}})} \mathfrak V(\mathfrak x_{\mathrm v}\cup\vec w_{\mathrm v}) \times (\vec T^{\mathrm o}_0,\infty] \times ((\vec T^{\mathrm c}_0,\infty]\times\vec S^1) \to \mathcal M_{k+1,\ell+\ell'}.$$

(See Definition 2.13.) Let $\mathfrak Y = \overline\Phi(\mathfrak y,\vec T,\vec\theta)$ be an element of $\mathcal M_{k+1,\ell+\ell'}$ that is represented by $(\Sigma_{\mathfrak Y},\vec z_{\mathfrak Y},\vec z^{\mathrm{int}}_{\mathfrak Y}\cup\vec w_{\mathfrak Y})$. By construction (2.164) we have

$$\begin{aligned}\Sigma_{\mathfrak Y} = \bigcup_{\mathrm v\in C^0(\mathcal G_{\mathfrak p\cup\vec w_{\mathfrak p}})} K^{\mathfrak Y}_{\mathrm v} &\cup \bigcup_{\mathrm e\in C^1_{\mathrm o}(\mathcal G_{\mathfrak p\cup\vec w_{\mathfrak p}})} [-5T_{\mathrm e},5T_{\mathrm e}]\times[0,1]\\ &\cup \bigcup_{\mathrm e\in C^1_{\mathrm c}(\mathcal G_{\mathfrak p\cup\vec w_{\mathfrak p}})} [-5T_{\mathrm e},5T_{\mathrm e}]\times S^1.\end{aligned}$$

We called the second and the third summand the neck region. In case $T_{\mathrm e}=\infty$ the product of the union of two half lines and $[0,1]$ or S^1 is also called the neck region. See Definition 2.15.

Definition 2.38. Let $u' : (\Sigma_{\mathfrak{Y}}, \partial\Sigma_{\mathfrak{Y}}) \to (X, L)$ be a smooth map in homology class β. We say that $(\Sigma_{\mathfrak{Y}}, u')$ is *ϵ-close to* $\mathfrak{p}$ with respect to the given obstruction bundle data if the following holds.

(1) Since $\mathfrak{Y} = \overline{\Phi}(\mathfrak{y}, \vec{T}, \vec{\theta})$ the core $K_{\rm v}^{\mathfrak{Y}} \subset \Sigma_{\mathfrak{Y}}$ is identified with $K_{\rm v}^{\mathfrak{y}} \subset \Sigma_{\mathfrak{y}}$. We require
$$|u - u'|_{C^{10}(K_{\rm v}^{\mathfrak{Y}})} < \epsilon \tag{2.192}$$
for each v. (We regard u as a map from $\Sigma_{\mathfrak{y}}$ by using the smooth trivialization of the universal family given as a part of Definition 2.10 (4).)

(2) The map u' is holomorphic on each of the neck region.

(3) The diameter of the u' image of each of the connected component of the neck region is smaller than ϵ.

(4) $T_{\rm e} > \epsilon^{-1}$ for each e.

Remark 2.39. We use metrics of the source and of X to define the left hand side of (2.192). See Remark 2.21.

Remark 2.40. We note that Definition 2.38 is not a definition of topology on certain set. In fact, '$(\Sigma_{\mathfrak{Y}}, u')$ is close to $\mathfrak{p}$' is defined only when $\mathfrak{p}$ is an element of $\mathcal{M}_{k+1,\ell}(\beta)$, but $(\Sigma_{\mathfrak{Y}}, u')$ may not be an element of $\mathcal{M}_{k+1,\ell}(\beta)$.

Even in case $(\Sigma_{\mathfrak{Y}}, u') \in \mathcal{M}_{k+1,\ell}(\beta)$, the fact that $(\Sigma_{\mathfrak{Y}}, u')$ is ϵ-close to $\mathfrak{p}$ does not imply that $\mathfrak{p}$ is ϵ-close $(\Sigma_{\mathfrak{Y}}, u')$. In fact, if $(\Sigma_{\mathfrak{Y}}, u')$ is ϵ-close to $\mathfrak{p}$ then $\mathcal{G}_{\mathfrak{p}} \succ \mathcal{G}_{\mathfrak{Y}}$.

On the other hand, we have the following. If $(\Sigma_{\mathfrak{Y}}, u') \in \mathcal{M}_{k+1,\ell}(\beta)$ and is ϵ_1-close to $\mathfrak{p}$ and if $(\Sigma_{\mathfrak{Y}'}, u'')$ is ϵ_2-close to $(\Sigma_{\mathfrak{Y}}, u')$, then $(\Sigma_{\mathfrak{Y}'}, u'')$ is $\epsilon_1 + o(\epsilon_2)$-close to $\mathfrak{p}$. (Here $\lim_{\epsilon_2 \to 0} o(\epsilon_2) = 0$.)

Let $\mathfrak{Y} = \overline{\Phi}(\mathfrak{y}, \vec{T}, \vec{\theta})$ and $u' : (\Sigma_{\mathfrak{Y}}, \partial\Sigma_{\mathfrak{Y}}) \to (X, L)$ be a smooth map in homology class β such that $(\Sigma_{\mathfrak{Y}}, u')$ is ϵ-close to $\mathfrak{p}$. We assume that ϵ is smaller than the injectivity radius of X. Let ${\rm v} \in C^0(\mathcal{G})$.

Definition 2.41. Suppose that we are given an obstruction bundle data $\mathfrak{E}_{\mathfrak{p}}$ centered at $\mathfrak{p}$. We define a map
$$I^{{\rm v},\mathfrak{p}}_{(\mathfrak{y},u),(\mathfrak{Y},u')} : E_{\mathfrak{p},{\rm v}}(\mathfrak{y}, u) \to \Gamma_0({\rm Int}\, K_{\rm v}^{\rm obst}; (u')^*TX \otimes \Lambda^{01}) \tag{2.193}$$
by using the complex linear part of the parallel transport along the path of the form $t \mapsto {\rm E}(u(z), tv)$, where ${\rm E}(u(z), v) = u'(z)$. (Note this is a short geodesic joining $u(z)$ and $u'(z)$ with respect to the connection which we used to define E.) Here we identify
$$K_{\rm v}^{\rm obst} \subset K_{\rm v} \subset \Sigma_{\mathfrak{y}}, \quad K_{\rm v}^{\rm obst} \subset K_{\rm v} \subset \Sigma_{\mathfrak{Y}}.$$
We write the image of (2.193) by $E_{\mathfrak{p},{\rm v}}(\mathfrak{Y}, u')$.

The map $I^{{\rm v},\mathfrak{p}}_{(\mathfrak{y},u),(\mathfrak{Y},u')}$ is $\Gamma^{\mathfrak{H}^+_{(\mathfrak{x},u)}}_{(\mathfrak{x}\cup\vec{w},u)}$ invariant in the sense of Lemma 2.42 below. Note we have an injective homomorphism $\Gamma^{\mathfrak{H}^+_{(\mathfrak{x},u)}}_{(\mathfrak{x}\cup\vec{w},u)} \to \mathfrak{S}_\ell \times \mathfrak{S}_{\ell'}$ such that the $\Gamma^{\mathfrak{H}^+_{(\mathfrak{x},u)}}_{(\mathfrak{x}\cup\vec{w},u)}$ action on the elements of $\mathfrak{V}(\mathfrak{x}\cup\vec{w})$ is identified with the permutation of the ℓ marked points in $\mathfrak{x}$ and ℓ' marked points $\vec{w}$. (See (2.151).) For $v \in \Gamma^{\mathfrak{H}^+_{(\mathfrak{x},u)}}_{(\mathfrak{x}\cup\vec{w},u)}$ we define $v_*\mathfrak{Y}$ by permuting the marked points of $\mathfrak{Y}$ in the same way. If $(\mathfrak{Y}, u')$ is ϵ-close to $\mathfrak{p}$ then $(v_*\mathfrak{Y}, u')$ is ϵ-close to $\mathfrak{p}$. Let ${\rm v}'$ be the vertex which is mapped from v by v with respect to the $\Gamma^{\mathfrak{H}^+_{(\mathfrak{x},u)}}_{(\mathfrak{x}\cup\vec{w},u)}$ action of $\mathcal{G}$. (See the discussion about Definition 2.33 (5) we gave right above Remark 2.36.) We remark that $v_*\mathfrak{Y} = \overline{\Phi}(v_*\mathfrak{y}, v_*\vec{T}, v_*\vec{\theta})$.

By using diffeomorphism in Definition 2.13, we have a map $v : \mathfrak{y} \to v_*\mathfrak{y}$. Note there exists a map (diffeomorphism) $v : \Sigma_{\mathfrak{y}} \to \Sigma_{\mathfrak{y}}$ that permutes the marked points in the required way. However this map is not holomorphic in general. It becomes biholomorphic as a map $v : \Sigma_{\mathfrak{y}} \to \Sigma_{v_*\mathfrak{y}}$.

Lemma 2.42. *The following diagram commutes.*

$$\begin{array}{ccc} E_{\mathfrak{p},\mathrm{v}}(\mathfrak{y},u) & \xrightarrow{I^{\mathrm{v},\mathfrak{p}}_{(\mathfrak{y},u),(\mathfrak{Y},u')}} & \Gamma_0(\mathrm{Int}\, K^{\mathrm{obst}}_{\mathrm{v}}(\mathfrak{Y}); (u')^*TX \otimes \Lambda^{01}) \\ {\scriptstyle v_*}\downarrow & & \downarrow{\scriptstyle v_*} \\ E_{\mathfrak{p},\mathrm{v}'}(v_*\mathfrak{y}, u\circ v^{-1}) & \xrightarrow{I^{\mathrm{v}',\mathfrak{p}}_{(v_*\mathfrak{y},u),(v_*\mathfrak{Y},u'\circ v^{-1})}} & \Gamma_0(\mathrm{Int}\, K^{\mathrm{obst}}_{\mathrm{v}'}(v_*\mathfrak{Y}); (u'\circ v^{-1})^*TX \otimes \Lambda^{01}) \end{array} \tag{2.194}$$

Here we define v' *by* $v(K_{\mathrm{v}}) = K_{\mathrm{v}'}$.

Proof. The lemma follows from the fact that parallel transport etc. is independent of the enumeration of the marked points. (Note the left vertical arrow is well-defined by Definition 2.33 (5).) □

Corollary 2.43.

$$v_*\left(\bigoplus_{\mathrm{v}\in C^0(\mathcal{G})} E_{\mathfrak{p},\mathrm{v}}(\mathfrak{Y},u')\right) = \bigoplus_{\mathrm{v}\in C^0(\mathcal{G})} E_{\mathfrak{p},\mathrm{v}}(v_*\mathfrak{Y},u'\circ v^{-1}).$$

This is a consequence of Lemma 2.42 and Definition 2.33 (5).

We next show that the Fredholm regularity (Definition 2.33 (6)) and evaluation map transversality (Definition 2.33 (7)) are preserved when we take $(\mathfrak{Y},u')$ that is ϵ-close to $\mathfrak{p}$. (See Proposition 2.48.) To state them precisely we need some preparation.

Let $\mathfrak{Y} = \overline{\Phi}(\mathfrak{y},\vec{T},\vec{\theta})$ be an element of $\mathcal{M}_{k+1,\ell+\ell'}$ that is represented by $(\Sigma_{\mathfrak{Y}},\vec{z}_{\mathfrak{Y}},\vec{z}^{\mathrm{int}}_{\mathfrak{Y}}\cup \vec{w}_{\mathfrak{Y}})$. We denote by $\mathcal{G}_{\mathfrak{Y}}$ the combinatorial type of $\mathfrak{Y}$. (Here $\mathcal{G}_{\mathfrak{y}}$ is the combinatorial type of $\mathfrak{y}$ and $\mathcal{G}_{\mathfrak{Y}}$ is obtained from $\mathcal{G}_{\mathfrak{y}}$ by shrinking the edges e such that $T_{\mathrm{e}} \ne \infty$.) Let $\mathrm{v} \in C^0_{\mathrm{d}}(\mathcal{G}_{\mathfrak{Y}})$. We have a differential operator

$$\begin{aligned} D_{u',\mathrm{v}}\overline{\partial} : L^2_{m+1,\delta}((\Sigma_{\mathfrak{Y}_{\mathrm{v}}},\partial\Sigma_{\mathfrak{Y}_{\mathrm{v}}});(u')^*TX,(u')^*TL) \\ \to L^2_{m,\delta}(\Sigma_{\mathfrak{Y}_{\mathrm{v}}};(u')^*TX\otimes\Lambda^{01}). \end{aligned} \tag{2.195}$$

In case $\mathrm{v} \in C^0_{\mathrm{s}}(\mathcal{G}_{\mathfrak{Y}})$ we have

$$D_{u',\mathrm{v}}\overline{\partial} : L^2_{m+1,\delta}(\Sigma_{\mathfrak{Y}_{\mathrm{v}}};(u')^*TX) \to L^2_{m,\delta}(\Sigma_{\mathfrak{Y}_{\mathrm{v}}};(u')^*TX\otimes\Lambda^{01}). \tag{2.196}$$

Definition 2.44. We say $(\mathfrak{Y},u')$ is *Fredholm regular* with respect to the obstruction bundle data $\mathfrak{E}_{\mathfrak{p}}$ if the sum of the image of (2.195) and $E_{\mathfrak{p},\mathrm{v}}(\mathfrak{Y},u')$ is $L^2_{m,\delta}(\Sigma_{\mathfrak{Y}_{\mathrm{v}}};(u')^*TX\otimes\Lambda^{01})$ and if the sum of the image of (2.196) and $E_{\mathfrak{p},\mathrm{v}}(\mathfrak{Y},u')$ is $L^2_{m,\delta}(\Sigma_{\mathfrak{Y}_{\mathrm{v}}};(u')^*TX\otimes\Lambda^{01})$.

Using this terminology, Definition 2.33 (6) means that $(\mathfrak{x},u)$ is Fredholm regular with respect to the obstruction bundle data $\mathfrak{E}_{\mathfrak{p}}$.

We next define the notion of evaluation map transversality.

Definition 2.45. A *flag* of $\mathcal{G}$ is a pair (v,e) of edges e and its vertex v. Suppose $\mathcal{G}$ is oriented. We say a flag (v,e) is *incoming* if e is an incoming edge. Otherwise it is said *outgoing*. We denote by z_{e} the singular point corresponding to an edge e.

For each flag (v,e) of $\mathcal{G}_{\mathfrak{Y}}$, we define

$$\mathrm{ev}_{\mathrm{v},\mathrm{e}} : L^2_{m+1,\delta}((\Sigma_{\mathfrak{Y}_\mathrm{v}}, \partial\Sigma_{\mathfrak{Y}_\mathrm{v}}); (u')^*TX, (u')^*TL) \to T_{u'(z_\mathrm{e})}L, \tag{2.197}$$

if $\mathrm{v} \in C^0_{\mathrm{d}}(\mathcal{G}_{\mathfrak{Y}})$, $\mathrm{e} \in \mathrm{C}^1_{\mathrm{o}}(\mathcal{G}_{\mathfrak{Y}})$ in the same way as (2.186),

$$\mathrm{ev}_{\mathrm{v},\mathrm{e}} : L^2_{m+1,\delta}((\Sigma_{\mathfrak{Y}_\mathrm{v}}, \partial\Sigma_{\mathfrak{Y}_\mathrm{v}}); (u')^*TX, (u')^*TL) \to T_{u'(z_\mathrm{e})}X, \tag{2.198}$$

if $\mathrm{v} \in C^0_{\mathrm{d}}(\mathcal{G}_{\mathfrak{Y}})$, $\mathrm{e} \in \mathrm{C}^1_{\mathrm{c}}(\mathcal{G}_{\mathfrak{Y}})$ in the same way as (2.187), and

$$\mathrm{ev}_{\mathrm{v},\mathrm{e}} : L^2_{m+1,\delta}(\Sigma_{\mathfrak{Y}_\mathrm{v}}; (u')^*TX) \to T_{u'(z_\mathrm{e})}X, \tag{2.199}$$

if $\mathrm{e} \in C^1_{\mathrm{c}}(\mathcal{G}_{\mathfrak{Y}})$ in the same way as (2.188).

Combining them we obtain

$$\begin{aligned}
\mathrm{ev}_{\mathcal{G}_{\mathfrak{Y}}} : &\bigoplus_{\mathrm{v}\in C^0_{\mathrm{d}}(\mathcal{G}_{\mathfrak{Y}})} L^2_{m+1,\delta}((\Sigma_{\mathfrak{Y}_\mathrm{v}}, \partial\Sigma_{\mathfrak{Y}_\mathrm{v}}); (u')^*TX, (u')^*TL) \\
&\bigoplus_{\mathrm{v}\in C^0_{\mathrm{s}}(\mathcal{G}_{\mathfrak{Y}})} L^2_{m+1,\delta}(\Sigma_{\mathfrak{Y}_\mathrm{v}}; (u')^*TX) \\
&\to \bigoplus_{\mathrm{e}\in C^1_{\mathrm{o}}(\mathcal{G}_{\mathfrak{Y}})} T_{u'(z_\mathrm{e})}L \oplus \bigoplus_{\mathrm{e}\in C^1_{\mathrm{c}}(\mathcal{G}_{\mathfrak{Y}})} T_{u'(z_\mathrm{e})}X.
\end{aligned} \tag{2.200}$$

Definition 2.46. Suppose $(\mathfrak{Y}, u')$ is Fredholm regular with respect to the obstruction bundle data $\mathfrak{E}_{\mathfrak{p}}$. We say that $(\mathfrak{Y}, u')$ is *evaluation map transversal* with respect to the obstruction bundle data $\mathfrak{E}_{\mathfrak{p}}$ if the restriction of (2.200) to the direct sum of the kernels of (2.197), (2.198) and of (2.199) is surjective.

Using this terminology, Definition 2.33 (7) means that $(\mathfrak{x}, u)$ is evaluation map transversal with respect to the obstruction bundle data $\mathfrak{E}_{\mathfrak{p}}$.

Proposition 2.48 below says that Fredholm regularity and evaluation map transversality are preserved if $(\mathfrak{Y}, u')$ is sufficiently close to $\mathfrak{p}$. To state it we need to note the following point.

When we define ϵ-close-ness, we put the condition that the image of each connected component of the neck region has diameter $< \epsilon$. But we did not assume a similar condition for $\mathfrak{p}$ and $\mathfrak{E}_{\mathfrak{p}}$ itself. So in case when this condition is not satisfied for $\mathfrak{p}$, there can not exist any object that is ϵ-close to $\mathfrak{p}$. Especially $\mathfrak{p}$ itself is not ϵ-close to $\mathfrak{p}$.

However, we can always modify the core K_v so that $\mathfrak{p}$ itself becomes ϵ-close to $\mathfrak{p}$ as follows. We take a positive number $R_{(\mathrm{v},\mathrm{e})}$ for each flag of $\mathcal{G}$ and write $\vec{R}$ the totality of such $R_{(\mathrm{v},\mathrm{e})}$. We put

$$\begin{aligned}
K^{+\vec{R}}_\mathrm{v} = K_\mathrm{v} &\cup \bigcup_{\substack{\mathrm{e}\in C^1_{\mathrm{o}}(\mathcal{G}) \\ (\mathrm{v},\mathrm{e}) \text{ is an outgoing flag}}} (0, R_{(\mathrm{v},\mathrm{e})}] \times [0,1] \\
&\cup \bigcup_{\substack{\mathrm{e}\in C^1_{\mathrm{o}}(\mathcal{G}) \\ (\mathrm{v},\mathrm{e}) \text{ is an incoming flag}}} [-R_{(\mathrm{v},\mathrm{e})}, 0) \times [0,1] \\
&\cup \bigcup_{\substack{\mathrm{e}\in C^1_{\mathrm{c}}(\mathcal{G}) \\ (\mathrm{v},\mathrm{e}) \text{ is an outgoing flag}}} (0, R_{(\mathrm{v},\mathrm{e})}] \times S^1 \\
&\cup \bigcup_{\substack{\mathrm{e}\in C^1_{\mathrm{c}}(\mathcal{G}) \\ (\mathrm{v},\mathrm{e}) \text{ is an incoming flag}}} [-R_{(\mathrm{v},\mathrm{e})}, 0) \times S^1.
\end{aligned} \tag{2.201}$$

Definition 2.47. We can define an obstruction bundle data $\mathfrak{E}_{\mathfrak{p}}$ centered at $\mathfrak{p}$ using $K_{\rm v}^{+\vec R}$ in place of $K_{\rm v}$. We call it the obstruction bundle data obtained by *extending the core* and write $\mathfrak{E}_{\mathfrak{p}}^{+\vec R}$. We call (2.201) the *extended core.* (In case we need to specify $\vec R$ we call it the $\vec R$-extended core.) (2.201) is a generalization of (1.104).

Proposition 2.48. *Let $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(\beta)$ and $\mathfrak{E}_{\mathfrak{p}}$ be an obstraction bundle data centered at $\mathfrak{p}$. Then there exist $\epsilon > 0$ and $\vec R$ with the following properties.*

1. *If $(\mathfrak{Y}, u')$ is ϵ-close to $\mathfrak{p}$ with respect to $\mathfrak{E}_{\mathfrak{p}}^{+\vec R}$, then $(\mathfrak{Y}, u')$ is Fredholm regular with respect to $\mathfrak{E}_{\mathfrak{p}}^{+\vec R}$.*
2. *If $(\mathfrak{Y}, u')$ is ϵ-close to $\mathfrak{p}$ with respect to $\mathfrak{E}_{\mathfrak{p}}^{+\vec R}$, then $(\mathfrak{Y}, u')$ is evaluation map transversal with respect to $\mathfrak{E}_{\mathfrak{p}}^{+\vec R}$.*
3. *$\mathfrak{p}$ is ϵ-close to $\mathfrak{p}$ with respect to $\mathfrak{E}_{\mathfrak{p}}^{+\vec R}$.*

Proof. By using the fact that the diameter of the u' image of the connected component of the neck region is small, we can prove an exponential decay estimate of u' on the neck region. This is an analogue of Lemma 1.43 and its proof is the same as the proof of [FOn1, Lemma 11.2]. Then the rest of the proof of (1),(2) is a version of the proof of Mayer-Vietoris principle of Mrowka [Mr]. See [FOOO1, Proposition 7.1.27] or [Fu1, Lemma 8.5]. (3) is obvious. □

So far we have discussed the case of bordered genus zero curve. The case of genus zero curve without boundary is the same so we do not repeat it. [13]

2.4. The differential equation and thickened moduli space.

To construct a Kuranishi neighborhood of each point in our moduli space $\mathcal{M}_{k+1,\ell}(\beta)$ or $\mathcal{M}_{\ell}^{\rm cl}(\alpha)$, we need to assign an obstruction bundle to each point of it. To do so we follow the way we had written in [FOn1, end of the page 1003] and [FOOO1, end of the page 423-middle of page 424]. The outline of the argument is as follows. For each $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(\beta)$ we take an obstruction bundle data $\mathfrak{E}_{\mathfrak{p}}$. We then consider a closed neighborhood $\mathfrak{W}_{\mathfrak{p}}$ of $\mathfrak{p}$ in $\mathcal{M}_{k+1,\ell}(\beta)$ so that its elements together with certain marked points added is $\epsilon_{\mathfrak{p}}$-close to $\mathfrak{p}$ with respect to $\mathfrak{E}_{\mathfrak{p}}^{+\vec R}$. Here we choose $\epsilon_{\mathfrak{p}}$ and $\mathfrak{E}_{\mathfrak{p}}^{+\vec R}$ so that Proposition 2.48 holds. We next take a finite number of $\mathfrak{p}_c \in \mathcal{M}_{k+1,\ell}(\beta)$ such that

$$\bigcup_c \operatorname{Int} \mathfrak{W}_{\mathfrak{p}_c} = \mathcal{M}_{k+1,\ell}(\beta).$$

For $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(\beta)$, we collect all $E_{\mathfrak{p}_c}$ such that $\mathfrak{p}_c$ satisfies $\mathfrak{p} \in \mathfrak{W}_{\mathfrak{p}_c}$. The sum will be the obstruction bundle $\mathcal{E}_{\mathfrak{p}}$ at $\mathfrak{p}$. Now we will describe this process in more detail below.

We first define the subset $\mathfrak{W}_{\mathfrak{p}}$ in more detail. We note that in Definition 2.38, we need $\ell + \ell_{\mathfrak{p}}$ interior marked points to define its ϵ-close-ness to an element $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(\beta)$. (Here $\ell_{\mathfrak{p}}$ is the number of marked points we add as a part of the obstruction bundle data $\mathfrak{E}_{\mathfrak{p}}$.) We start with describing the process of forgetting those $\ell_{\mathfrak{p}}$ marked points.

Definition 2.49. We consider the situation of Definition 2.38. Let $\mathfrak{Y} = \Phi(\mathfrak{y}, \vec T, \vec\theta)$ and let $u' : (\Sigma_{\mathfrak{Y}}, \partial\Sigma_{\mathfrak{Y}}) \to (X, L)$ be a smooth map in the homology class β that is

[13] Higher genus case is also the same.

ϵ-close to $\mathfrak{p}$. We say $(\mathfrak{Y}, u')$ satisfies the *transversal constraint* if for each $w_i \in \vec{w}$ we have

$$u'(w_i) \in \mathcal{D}_{\mathfrak{p},i}. \tag{2.202}$$

Let us explain the notation appearing in the above definition. We have $\vec{w}_{\mathfrak{p}}$, the additional marked points on $\Sigma_{\mathfrak{p}}$ as a part of the obstruction bundle data $\mathfrak{E}_{\mathfrak{p}}$. The element $\mathfrak{y}$ is in a neighborhood $\mathfrak{V}(\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}_{\mathfrak{p}})$. (This neighborhood $\mathfrak{V}(\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}_{\mathfrak{p}})$ is also a part of the date $\mathfrak{E}_{\mathfrak{p}}$.) $(\vec{T}, \vec{\theta})$ is as in Definition 2.14. Thus $\mathfrak{Y} = \overline{\Phi}(\mathfrak{y}, \vec{T}, \vec{\theta})$ is a bordered genus zero curve with $k+1$ boundary and $\ell + \ell_{\mathfrak{p}}$ interior marked points. ($\ell_{\mathfrak{p}}$ is the number of points in $\vec{w}_{\mathfrak{p}}$.) We denote by $w_{\mathfrak{p},i}$ the $(\ell+i)$-th interior marked point. (It is i-th among the additional marked points.) For each $i = 1, \dots, \ell_{\mathfrak{p}}$, we took $\mathcal{D}_{\mathfrak{p},i}$ that is transversal to $u_{\mathfrak{p}}(\Sigma_{\mathfrak{p}})$ at $u_{\mathfrak{p}}(w_i)$ as a part of the data $\mathfrak{E}_{\mathfrak{p}}$.

Lemma 2.50. *For each $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(\beta)$ and an obstruction bundle data $\mathfrak{E}_{\mathfrak{p}}$ centered at $\mathfrak{p}$ there exists $\epsilon_{\mathfrak{p}}$ such that the following holds.*

Let $\mathfrak{q} = (\mathfrak{x}_{\mathfrak{q}}, u_{\mathfrak{q}}) \in \mathcal{M}_{k+1,\ell}(\beta)$. We consider the set of symmetric marking $\vec{w}'_{\mathfrak{p}}$ of $\mathfrak{x}_{\mathfrak{q}}$ with $\#\vec{w}'_{\mathfrak{p}} = \ell_{\mathfrak{p}}$, such that the following holds.

1. *There exists $\mathfrak{y} \in \mathfrak{V}(\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}_{\mathfrak{p}})$ and $(\vec{T}, \vec{\theta}) \in (\vec{T}^{\rm o}_0, \infty] \times ((\vec{T}^{\rm c}_0, \infty] \times \vec{S}^1)$ such that $\mathfrak{x}_{\mathfrak{q}} \cup \vec{w}'_{\mathfrak{p}} = \overline{\Phi}(\mathfrak{y}, \vec{T}, \vec{\theta})$.*
2. *$(\mathfrak{x}_{\mathfrak{q}} \cup \vec{w}'_{\mathfrak{p}}, u_{\mathfrak{q}})$ is $\epsilon_{\mathfrak{p}}$-close to $\mathfrak{p}$.*
3. *$(\mathfrak{x}_{\mathfrak{q}} \cup \vec{w}'_{\mathfrak{p}}, u_{\mathfrak{q}})$ satisfies the transversal constraint.*

Then the set of such $\vec{w}'_{\mathfrak{p}}$ consists of a single $\Gamma_{\mathfrak{p}}$ orbit if it is nonempty. Here we regard $\Gamma_{\mathfrak{p}} \subset \mathfrak{S}_{\ell'}$ by (2.181) and $\Gamma_{\mathfrak{p}}$ acts on the set of $\vec{w}'_{\mathfrak{p}}$'s by permutation.

The proof of Lemma 2.50 is not difficult. We however postpone its proof to Subsection 2.6 where the transversal constraint is studied more systematically.

We are now ready to provide the definition of $\mathfrak{W}_{\mathfrak{p}} \subset \mathcal{M}_{k+1,\ell}(\beta)$.

First for each $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(\beta)$ we take and fix an obstruction bundle data $\mathfrak{E}_{\mathfrak{p}}$. Let $\vec{w}_{\mathfrak{p}}$ be the additional marked points we take as a part of $\mathfrak{E}_{\mathfrak{p}}$. We take $\epsilon_{\mathfrak{p}}$ so that Proposition 2.48 and Lemma 2.50 hold. Moreover we may change $\mathfrak{E}_{\mathfrak{p}}$ if necessary so that Proposition 2.48 holds for $\mathfrak{E}^{+\vec{R}}_{\mathfrak{p}} = \mathfrak{E}_{\mathfrak{p}}$.

Definition 2.51. $\mathfrak{W}^+(\mathfrak{p})$ is the set of all $\mathfrak{q} \in \mathcal{M}_{k+1,\ell}(\beta)$ such that the set of $\vec{w}'_{\mathfrak{p}}$ satisfying (1)-(3) of Lemma 2.50 is nonempty. The constant $\epsilon_{\mathfrak{p}}$ (which is often denoted by $\epsilon_{\mathfrak{p}_c}$ or ϵ_c) is determined later. (See Lemma 2.64 (Remark 2.65), Proposition 2.95, Lemma 2.105, Lemma 2.108, Sublemma 2.109 (Remark 2.110.) See also 2 lines above Definition 2.121.) We note that $\mathfrak{W}^+(\mathfrak{p})$ is open, as we will see in Subsection 2.6. See Remark 2.111.

We choose a compact subset $\mathfrak{W}_{\mathfrak{p}} \subset \mathfrak{W}^+(\mathfrak{p})$ that is a neighborhood of $\mathfrak{p}$. We take $\mathfrak{W}^0_{\mathfrak{p}}$ that is a compact subset of Int $\mathfrak{W}_{\mathfrak{p}}$ and is a neighborhood of $\mathfrak{p}$.

We take a finite set $\{\mathfrak{p}_c \mid c \in \mathfrak{C}\} \subset \mathcal{M}_{k+1,\ell}(\beta)$ such that

$$\bigcup_{c\in\mathfrak{C}} \text{Int } \mathfrak{W}^0_{\mathfrak{p}_c} = \mathcal{M}_{k+1,\ell}(\beta). \tag{2.203}$$

We fix this set $\{\mathfrak{p}_c \mid c \in \mathfrak{C}\}$ in the rest of the construction of the Kuranishi structure. From now on none of the obstruction bundle data at $\mathfrak{p}$ for $\mathfrak{p} \notin \mathfrak{C}$ is used in this note.

Definition 2.52. For $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(\beta)$, we define

$$\mathfrak{C}(\mathfrak{p}) = \{c \in \mathfrak{C} \mid \mathfrak{p} \in \mathfrak{W}_{\mathfrak{p}_c}\}.$$

We also choose additional marked points $\vec w^{\mathfrak p}_c$ of $\mathfrak x_{\mathfrak p}$ for each $c \in \mathfrak C(\mathfrak p)$ such that

(1) There exist $\mathfrak y \in \mathfrak V(\mathfrak x_{\mathfrak p_c} \cup \vec w_{\mathfrak p_c})$ and $(\vec T, \vec\theta) \in (\vec T^{\rm o}_0, \infty] \times ((\vec T^{\rm c}_0, \infty] \times \vec S^1)$ such that $\mathfrak x_{\mathfrak p} \cup \vec w^{\mathfrak p}_c = \overline\Phi(\mathfrak y, \vec T, \vec\theta)$.
(2) $(\mathfrak x_{\mathfrak p} \cup \vec w^{\mathfrak p}_c, u_{\mathfrak p})$ is $\epsilon_{\mathfrak p_c}$-close to $\mathfrak p_c$.
(3) $(\mathfrak x_{\mathfrak p} \cup \vec w^{\mathfrak p}_c, u_{\mathfrak p})$ satisfies the transversal constraint.

Lemma 2.53. *For each $\mathfrak p$ there exists a neighborhood U of it so that if $\mathfrak q \in U$ then*

$$\mathfrak C(\mathfrak q) \subseteq \mathfrak C(\mathfrak p).$$

Proof. The lemma follows from the fact that $\mathfrak W_{\mathfrak p_c}$ is closed. □

We next define an obstruction bundle $\mathcal E_{\mathfrak p}$ for each $\mathfrak p = (\mathfrak x_{\mathfrak p}, u_{\mathfrak p}) \in \mathcal M_{k+1,\ell}(\beta)$. Take $c \in \mathfrak C(\mathfrak p)$. Let $\vec w^{\mathfrak p}_c$ be as in Definition 2.52. By Definition 2.41, the map

$$I^{{\rm v},\mathfrak p_c}_{(\mathfrak y_c,u_c),\mathfrak p \cup \vec w^{\mathfrak p}_c} : E_{\mathfrak p_c,{\rm v}}(\mathfrak y_c, u_c) \to \Gamma_0({\rm Int}\, K^{\rm obst}_{\rm v}; u^*_{\mathfrak p}TX \otimes \Lambda^{01}) \tag{2.204}$$

is defined. Here $\mathfrak x_{\mathfrak p} \cup \vec w^{\mathfrak p}_c = \overline\Phi(\mathfrak y_c, \vec T_c, \vec\theta_c)$ and $\mathfrak y_c \in \mathfrak V(\mathfrak x_{\mathfrak p_c} \cup \vec w_{\mathfrak p_c})$. Note $K^{\rm obst}_{\rm v} \subset K_{\rm v} \subset \mathfrak x_{\mathfrak p_c}$. We have also $K^{\rm obst}_{\rm v} \subset \mathfrak x_{\mathfrak p}$ since $\vec w^{\mathfrak p}_c \cup \mathfrak x_{\mathfrak p} = \overline\Phi(\mathfrak y_c, \vec T_c, \vec\theta_c)$.

Lemma 2.54. *The image $E_c(\mathfrak p)$ of (2.204) depends only on $\mathfrak p \in \mathfrak W^+_{\mathfrak p_c}$ and is independent of the choices of $\vec w^{\mathfrak p}_c$ satisfying Definition 2.52 (1)-(3).*

Proof. This is a consequence of Corollary 2.43 and Lemma 2.50. □

Definition 2.55. We define

$$\mathcal E_{\mathfrak C(\mathfrak p)}(\mathfrak p) = \sum_{c \in \mathfrak C(\mathfrak p)} E_c(\mathfrak p). \tag{2.205}$$

For $\mathfrak A \subset \mathfrak C(\mathfrak p)$ we put

$$\mathcal E_{\mathfrak A}(\mathfrak p) = \sum_{c \in \mathfrak A} E_c(\mathfrak p). \tag{2.206}$$

The defining equation of the thickend moduli space at $\mathfrak p$ is

$$\overline\partial u_{\mathfrak p} \equiv 0 \mod \mathcal E_{\mathfrak C(\mathfrak p)}(\mathfrak p).$$

We need to extend the subspace $\mathcal E_{\mathfrak C(\mathfrak p)}(\mathfrak p)$ to a family of subspaces parametrized by a neighborhood of $\mathfrak p$. Before doing so we need the following.

Lemma 2.56. *By perturbing $E_{\mathfrak p_c}$ (that is a part of the obstruction bundle data $\mathfrak E_{\mathfrak p_c}$) we may assume that*

$$E_c(\mathfrak p) \cap E_{c'}(\mathfrak p) = \{0\},$$

if $c, c' \in \mathfrak C(\mathfrak p)$ and $c \ne c'$.

Proof. The proof will be written in Subsection 3.3. □

Now we start extending the equation (2.205) to an element $\mathfrak q$ in a 'neighborhood' of $\mathfrak p$. We do not yet assume that $\mathfrak q$ satisfies the transversal constraint (Definition 2.49). So to define $E_c(\mathfrak q)$ we need to include $\vec w'_c$ for all $c \in \mathfrak C(\mathfrak p)$ as marked points of $\mathfrak q$. We also take more marked points $\vec w_{\mathfrak p}$ to stabilize $\mathfrak p$ and take corresponding additional marked points $\vec w'_{\mathfrak p}$ on $\Sigma_{\mathfrak q}$. The marked points $\vec w_{\mathfrak p}$ are used to fix the coordinate to perform the gluing construction in subsection 2.5. $\vec w'_c$ is used to define the map (2.204). Thus they have different roles.

A technical point to take care of is the following. We may assume that the ℓ_c components of $\vec w^{\mathfrak p}_c$ are mutually different, for each c. (This is because ℓ_c components of $\vec{w}_{\mathfrak{p}_c}$ are mutually different.) However there is no obvious way to arrange so that $\vec{w}^{\mathfrak{p}}_c \cap \vec{w}^{\mathfrak{p}}_{c'} = \emptyset$ for $c \neq c'$. Note, in the usual stable map compactification, at the point where two or more marked points become coincide, we put the 'phantom bubble' so that they become different points on this bubbled component. For our purpose, the proof becomes simpler when we do *not* put a phantom bubble in case one of the components of $\vec{w}^{\mathfrak{p}}_c$ coincides with one of the components of $\vec{w}^{\mathfrak{p}}_{c'}$ for $c \neq c'$. Taking these points into acount we define $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta,\mathfrak{p})_{\epsilon_0,\vec{T}_0}$ below.

We first review the situation we are working in and prepare some notations. Let $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(\beta)$. We defined $\mathfrak{C}(\mathfrak{p})$ in Definition 2.52. For $c \in \mathfrak{C}(\mathfrak{p})$ we fixed an obstruction bundle data $\mathfrak{E}_{\mathfrak{p}_c}$ centered at $\mathfrak{p}_c$. Additional marked points $\vec{w}_{\mathfrak{p}_c}$ is a part of the data $\mathfrak{E}_{\mathfrak{p}_c}$. We put $\ell_c = \#\vec{w}_c$. We also put $\epsilon_c = \epsilon_{\mathfrak{p}_c}$ where the right hand side is as in Lemma 2.50. As memtioned before we take $\mathfrak{E}_{\mathfrak{p}_c}$ so that Proposition 2.48 holds for $\mathfrak{E}^{+\vec{R}}_{\mathfrak{p}_c} = \mathfrak{E}_{\mathfrak{p}_c}$.

Definition 2.57. A *stabilization data* at $\mathfrak{p}$ is the data as follows.

(1) A symmetric stabilization $\vec{w}_{\mathfrak{p}} = (w_{\mathfrak{p},1},\dots,w_{\mathfrak{p},\ell_{\mathfrak{p}}})$ of $\mathfrak{p}$. Let $\ell_{\mathfrak{p}} = \#\vec{w}_{\mathfrak{p}}$.
(2) For each $w_{\mathfrak{p},i}$ $(i = 1,\dots,\ell_{\mathfrak{p}})$, we take and fix $\mathcal{D}_{\mathfrak{p},i}$ such that it is a codimension two submanifold of X and is transversal to $u_{\mathfrak{p}}$ at $u_{\mathfrak{p}}(w_{\mathfrak{p},i})$. We also assume $u_{\mathfrak{p}}(w_{\mathfrak{p},i}) \in \mathcal{D}_{\mathfrak{p},i}$.
(3) We assume that $\{\mathcal{D}_{\mathfrak{p},i} \mid i = 1,\dots,\ell_{\mathfrak{p}}\}$ is invariant under the $\Gamma_{\mathfrak{p}}$ action in the same sense as in Definition 2.33 (8) (2.185).
(4) A coordinate at infinity of $\mathfrak{p} \cup \vec{w}_{\mathfrak{p}}$.
(5) $\vec{w}_{\mathfrak{p}} \cap \vec{w}^{\mathfrak{p}}_c = \emptyset$ for any $c \in \mathfrak{C}(\mathfrak{p})$.
(6) Let $K^{\rm obst}_{{\rm v},c}$ be the support of the obstruction bundle as in Definition 2.33 (4). (Here ${\rm v} \in C^0(\mathcal{G}_{\mathfrak{p}_c})$.) Since $\mathfrak{x}_{\mathfrak{p}} = \overline{\Phi}(\mathfrak{y},\vec{T},\vec{\theta})$ we may regard $K^{\rm obst}_{{\rm v},c} \subset \Sigma_{\mathfrak{p}}$. We require
$$K^{\rm obst}_{{\rm v},c} \subset \bigcup_{{\rm v}' \in C^0(\mathcal{G}_{\mathfrak{p}})} {\rm Int} K_{{\rm v}'}.$$
Here the right hand side is the core of the coordinate at infinity given by item (4) Definition 2.57.

A stabilization data at $\mathfrak{p}$ is similarly defined as the obstruction bundle data centered at $\mathfrak{p}$. But it does not include $K^{\rm obst}_{\rm v}$ or $E_{\mathfrak{p},{\rm v}}$. The stablization data at $\mathfrak{p}$ has no relation to the obstruction bundle data at $\mathfrak{p}$.[14]

We fix a metric on all the Deligne-Mumford moduli spaces. Let $\mathfrak{V}_{\epsilon_0}(\mathfrak{p} \cup \vec{w}_{\mathfrak{p}})$ be the ϵ_0-neighbhorhood of $\mathfrak{p} \cup \vec{w}_{\mathfrak{p}}$ in $\mathcal{M}_{k+1,\ell+\ell_{\mathfrak{p}}}(\mathcal{G}_{(\mathfrak{p}\cup\vec{w}_{\mathfrak{p}})})$ where $\mathcal{G}_{(\mathfrak{p}\cup\vec{w}_{\mathfrak{p}})}$ is the combinatorial type of $\mathfrak{p} \cup \vec{w}_{\mathfrak{p}}$.

Definition 2.58. [Definition of $\mathfrak{U}_{k+1,(\ell;\ell_{\mathfrak{p}},(\ell_c))}(\beta,\mathfrak{p};\mathfrak{B})_{\epsilon_0,\vec{T}_0}$]. We fix a stabilization data at $\mathfrak{p}$ and an obstruction bundle data centered at $\mathfrak{p}_c$ for each $c \in \mathfrak{C}(\mathfrak{p})$. Let $\mathfrak{B} \subset \mathfrak{C}(\mathfrak{p})$. For each $c \in \mathfrak{G}(\mathfrak{p})$ we chose $\vec{w}^{\mathfrak{p}}_c$ in Definition 2.52.

For $\epsilon_0 > 0$ and $\vec{T}_0 = (\vec{T}^{\rm o}_0, \vec{T}^{\rm c}_0) = (T_{{\rm e},0} : {\rm e} \in C^1(\mathcal{G}_{\mathfrak{p}}))$ we consider the set of all $(\mathfrak{Y}, u', (\vec{w}'_c; c \in \mathfrak{B}))$ such that the following holds for some $\vec{R}$.

[14] In case $\mathfrak{p} = \mathfrak{p}_c$ we have both stabilization data and obstruction bundle data at $\mathfrak{p}$. The notation $\vec{w}_{\mathfrak{p}}$ is used for both structures. They may not be coincide. We use the same symbol for both since this can not cause any confusion and the case $\mathfrak{p} = \mathfrak{p}_c$ does not play a role in our discussion.

(1) There exist $\mathfrak{y} \in \mathfrak{V}_{\epsilon_0}(\mathfrak{p} \cup \vec{w}_{\mathfrak{p}})$, $(\vec{T}, \vec{\theta}) \in (\vec{T}_0^{\rm o}, \infty] \times ((\vec{T}_0^{\rm c}, \infty] \times \vec{S}^1)$ such that

$$\mathfrak{Y} = \overline{\Phi}(\mathfrak{y}, \vec{T}, \vec{\theta}) \in \mathcal{M}_{k+1,\ell+\ell_{\mathfrak{p}}}.$$

(2) u' is ϵ_0-close to $u_{\mathfrak{p}}$ on the extended core $K_{\rm v}^{+\vec{R}}$ of $\Sigma_{\mathfrak{p}}$ in C^{10}-topology. We use the coordinate at infinity of $\mathfrak{p} \cup \vec{w}_{\mathfrak{p}}$ that is included in the stabilization data at $\mathfrak{p}$, to define this C^{10} close-ness.

(3) Moreover we assume that the diameter of the u' image of each neck region of $\Sigma_{\mathfrak{Y}}$ is smaller than ϵ_0. We assume furthermore that u' is pseudo-holomorphic in the neck regions. (The neck region here is the complement of the union of the extended cores $K_{\rm v}^{+\vec{R}}$.)

(4) We write $\mathfrak{Y} = \mathfrak{Y}_0 \cup \vec{w}'_{\mathfrak{p}}(\mathfrak{Y})$ where $\vec{w}'_{\mathfrak{p}}(\mathfrak{Y})$ are $\ell_{\mathfrak{p}}$ marked points that correspond to $\vec{w}_{\mathfrak{p}}$. We assume that $(\mathfrak{Y}_0 \cup \vec{w}'_c, u')$ is ϵ_0-close to $\mathfrak{p} \cup \vec{w}^{\mathfrak{p}}_c$ in the sense of Definition 2.38 after extending the core of $\mathfrak{p} \cup \vec{w}^{\mathfrak{p}}_c$ by $\vec{R}$.

We say that $(\mathfrak{Y}^{(1)}, u'^{(1)}, (\vec{w}_c'^{(1)}; c \in \mathfrak{B}))$ is *weakly equivalent* to $(\mathfrak{Y}^{(2)}, u'^{(2)}, (\vec{w}_c'^{(2)}; c \in \mathfrak{B}))$ if there exists a bi-holomorphic map $v : \mathfrak{Y}^{(1)} \to \mathfrak{Y}^{(2)}$ such that

(a) $u'^{(1)} = u'^{(2)} \circ v$.

(b) $v(w'^{(1)}_{c,i}) = w'^{(2)}_{c,\sigma_c(i)}$, where $\sigma_c \in \mathfrak{S}_{\ell_c}$.

(c) v sends the i-th boundary marked point of $\mathfrak{Y}^{(1)}$ to the i-th boundary marked point of $\mathfrak{Y}^{(2)}$. v sends 1-st,…,ℓ-th interior marked points of $\mathfrak{Y}^{(1)}$ to the corresponding interior marked points of $\mathfrak{Y}^{(2)}$. v sends $\ell+1$,…,$\ell+k$,… $\ell+\ell_{\mathfrak{p}}$-th interior marked points of $\mathfrak{Y}^{(1)}$ to the $\ell+\sigma(1)$,…,$\ell+\sigma(k)$,… $\ell+\sigma(\ell_{\mathfrak{p}})$-th interior marked points of $\mathfrak{Y}^{(2)}$, where $\sigma \in \mathfrak{S}_{\ell_{\mathfrak{p}}}$.

We denote by $\overline{\mathfrak{U}}_{k+1,(\ell;\ell_{\mathfrak{p}},(\ell_c))}(\beta, \mathfrak{p}; \mathfrak{B})_{\epsilon_0,\vec{T}_0}$ the set of all weak equivalence classes of $(\mathfrak{Y}, u', (\vec{w}'_c; c \in \mathfrak{B}))$ satisfying (1)-(4) above. (Here we use the weak equivalence relation defined by (a), (b), (c).)

We say that $(\mathfrak{Y}^{(1)}, u'^{(1)}, (\vec{w}_c'^{(1)}; c \in \mathfrak{B}))$ is *equivalent* to $(\mathfrak{Y}^{(2)}, u'^{(2)}, (\vec{w}_c'^{(2)}; c \in \mathfrak{B}))$ when $\sigma = \sigma_c =$ identity is satsified in (a)-(c) above in addition. Let

$$\mathfrak{U}_{k+1,(\ell;\ell_{\mathfrak{p}},(\ell_c))}(\beta, \mathfrak{p}; \mathfrak{B})_{\epsilon_0,\vec{T}_0}$$

be the set of equivalence classes of this equivalence relation.

Lemma 2.59. *We may choose ϵ_0 sufficiently small so that the following holds. Suppose $(\mathfrak{Y}^{(1)}, u'^{(1)}, (\vec{w}_c'^{(1)}; c \in \mathfrak{B}))$ is weakly equivalent to $(\mathfrak{Y}^{(2)}, u'^{(2)}, (\vec{w}_c'^{(2)}; c \in \mathfrak{B}))$ in the above sense and $\mathfrak{Y}^{(j)} = \overline{\Phi}(\mathfrak{y}^{(j)}, \vec{T}^{(j)}, \vec{\theta}^{(j)}) \in \mathcal{M}_{k+1,\ell+\ell_{\mathfrak{p}}}$. Then we have*

$$(\mathfrak{y}^{(2)}, \vec{T}^{(2)}, \vec{\theta}^{(2)}) = v_*(\mathfrak{y}^{(1)}, \vec{T}^{(1)}, \vec{\theta}^{(1)})$$

for some $v \in \Gamma_{\mathfrak{p}} \subset \mathfrak{S}_{\ell_{\mathfrak{p}}}$.

Proof. The proof is by contradiction. Suppose there exists a sequence of positive numbers $\epsilon_{0,a} \to 0$ and $(u'_{(j),a}, (\mathfrak{y}^{(j),a}, \vec{T}^{(j),a}, \vec{\theta}^{(j),a}), (\vec{w}_c'^{(j),a}; c \in \mathfrak{B}))$ for $j = 1, 2$ and $a = 1, 2, \dots$ such that:

(1) The object $(u'_{(1),a}, (\mathfrak{y}^{(1),a}, \vec{T}^{(1),a}, \vec{\theta}^{(1),a}), (\vec{w}_c'^{(1),a}; c \in \mathfrak{B}))$ is weakly equivalent to the object $(u'_{(2),a}, (\mathfrak{y}^{(2),a}, \vec{T}^{(2),a}, \vec{\theta}^{(2),a}), (\vec{w}_c'^{(2),a}; c \in \mathfrak{B}))$.

(2) $\mathfrak{Y}^{(j),a} = \overline{\Phi}(\mathfrak{y}^{(j),a}, \vec{T}^{(j),a}, \vec{\theta}^{(j),a}) \in \mathcal{M}_{k+1,\ell+\ell_{\mathfrak{p}}}$.

(3) The objects $(u'_{(j),a}, (\mathfrak{y}^{(j),a}, \vec{T}^{(j),a}, \vec{\theta}^{(j),a}), (\vec{w}_c'^{(j),a}; c \in \mathfrak{B}))$ are representatives of elements of $\mathfrak{U}_{k+1,(\ell;\ell_{\mathfrak{p}},(\ell_c))}(\beta, \mathfrak{p}; \mathfrak{B})_{\epsilon_{0,a},\vec{T}_0}$.

(4) There is no $v \in \Gamma_{\mathfrak p}$ satisfying $(\mathfrak y^{(2),a}, \vec T^{(2),a}, \vec\theta^{(2),a}) = v_*(\mathfrak y^{(1),a}, \vec T^{(1),a}, \vec\theta^{(1),a})$.

We will deduce contradiction. By assumption there exist $\vec R_a \to \infty$ and biholomorphic maps $v_a : \mathfrak Y^{(1),a} \to \mathfrak Y^{(2),a}$ such that

(I) $|u'_{(2),a} \circ v_a - u'_{(1),a}|_{C^{10}(K_{\rm v}^{+\vec R_a})} < \epsilon_{0,a}$.

(II) The diameter of $u'_{(j),a}$ image of each connected component of the complement of the union of the extended cores $K_{\rm v}^{+\vec R_a}$ is smaller than $\epsilon_{0,a}$.

(III) $v_a(w'^{(1),a}_{c,i}) = w'^{(2),a}_{c,\sigma_c(i)}$, where $\sigma_c \in \mathfrak S_{\ell_c}$.

(IV) v_a sends the i-th boundary marked point of $\mathfrak Y^{(1),a}$ to the i-th boundary marked point of $\mathfrak Y^{(2),a}$. v_a sends 1-st,…,ℓ-th interior marked points of $\mathfrak Y^{(1),a}$ to the corresponding interior marked points of $\mathfrak Y^{(2),a}$. v_a sends $\ell+1,\dots,\ell+k,\dots\ell+\ell_{\mathfrak p}$-th interior marked points of $\mathfrak Y^{(1),a}$ to the $\ell+\sigma_a(1),\dots,\ell+\sigma_a(k),\dots \ell+\sigma_a(\ell_{\mathfrak p})$-th interior marked points of $\mathfrak Y^{(2),a}$, where $\sigma_a \in \mathfrak S_{\ell_{\mathfrak p}}$.

By (I) and (II) we may take a subsequence (still denoted by the same symbol) such that v_a converges to a biholomorphic map $v : \Sigma_{\mathfrak p} \to \Sigma_{\mathfrak p}$ such that $u_{\mathfrak p} \circ v = u_{\mathfrak p}$. Then (III) and (IV) imply that $v \in \Gamma_{\mathfrak p}$.

So changing $\mathfrak Y^{(2),a}$ by v we may assume $v =$ identity. Therefore v_a converges to identity. The stability then implies that v_a is identity. This contradicts to (4). □

Definition 2.60. Let $\mathfrak q^+ = (\mathfrak Y, u', (\vec w'_c; c \in \mathfrak B)) \in \mathfrak U_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak B)_{\epsilon_0,\vec T_0}$. We define

$$E_c(\mathfrak q^+) \subset \bigoplus_{{\rm v}\in C^0(\mathcal G(\mathfrak Y))} \Gamma_0({\rm Int}\, K_{\rm v}^{\rm obst}; (u')^*TX \otimes \Lambda^{01})$$

as follows, where $\mathcal G(\mathfrak Y)$ is the combinatorial type of $(\mathfrak Y, u')$. We regard $K_{\rm v}^{\rm obst}$ as a subset of $\mathfrak Y$. We note that $\mathfrak p \cup \vec w_c^{\mathfrak p}$ is $\epsilon_{\mathfrak p_c}$-close to $\mathfrak p_c \cup \vec w_{\mathfrak p_c}$ and $(\mathfrak Y \cup \vec w'_c, u')$ is ϵ_0-close to $\mathfrak p \cup \vec w_c^{\mathfrak p}$ in the sense of Definition 2.38. Therefore we have

$$I^{{\rm v},\mathfrak p_c}_{(\mathfrak y_c,u_c),(\mathfrak Y\cup\vec w'_c,u')} : E_{\mathfrak p_c,{\rm v}}(\mathfrak y_c,u_c) \to \Gamma_0({\rm Int}\, K_{\rm v}^{\rm obst}; (u')^*TX \otimes \Lambda^{01}). \tag{2.207}$$

Here $\mathfrak p_c = (\mathfrak x_c, u_c)$ and $\mathfrak Y \cup \vec w'_c = \overline\Phi(\mathfrak y_c, \vec T, \vec\theta)$. We regard $K_c^{\rm obst}$ as a subset of $\mathfrak x_c$ also. (Note that the core of $\mathfrak Y$ is canonically identified with the core of $\mathfrak y_c$.) Then we define

$$E_c(\mathfrak q^+) = \sum_{{\rm v}\in C^0(\mathcal G(\mathfrak Y))} I^{{\rm v},\mathfrak p_c}_{(\mathfrak y_c,u_c),(\mathfrak Y\cup\vec w'_c,u')}(E_{\mathfrak p_c,{\rm v}}(\mathfrak y_c,u_c)) \tag{2.208}$$

and put

$$\mathcal E_{\mathfrak B}(\mathfrak q^+) = \sum_{c\in\mathfrak B} E_c(\mathfrak q^+). \tag{2.209}$$

For $\mathfrak A \subset \mathfrak B$ we put

$$\mathcal E_{\mathfrak A}(\mathfrak q^+) = \sum_{c\in\mathfrak A} E_c(\mathfrak q^+). \tag{2.210}$$

Remark 2.61. When we define $E_c(\mathfrak q^+)$, we use the additional marked points $\vec w'_c$ and $\vec w_{\mathfrak p_c}$ that are assigned to $\mathfrak p_c$. So this subspace is taken in a way independent of $\mathfrak p$. This is important to prove that the coordinate change satisfies the cocycle condition later. We explained this point in [Fu2, the last three lines in the answer to question 4].

The next lemma is a consequence of Lemmas 2.59 and 2.42.

Lemma 2.62. *Suppose that* $(\mathfrak{Y}^{(1)}, u'^{(1)}, (\vec{w}_c'^{(1)}; c \in \mathfrak{B}))$ *is weakly equivalent to* $(\mathfrak{Y}^{(2)}, u'^{(2)}, (\vec{w}_c'^{(2)}; c \in \mathfrak{B}))$ *and* v *is as in Lemma 2.59. We put* $\mathfrak{q}^{+(j)} = (\mathfrak{Y}^{(j)}, u'^{(j)}, (\vec{w}_c'^{(j)}; c \in \mathfrak{B}))$. *Then*

$$E_c(\mathfrak{q}^{+(2)}) = v_* E_c(\mathfrak{q}^{+(1)}).$$

Now we define:

Definition 2.63. The *thickened moduli space* $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B})_{\epsilon_0,\vec{T}_0}$ is the subset of $\mathfrak{U}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{B})_{\epsilon_0,\vec{T}_0}$ consisting of the equivalence classes of elements $\mathfrak{q}^+ = (\mathfrak{Y}, u', (\vec{w}_c'; c \in \mathfrak{B})) \in \mathfrak{U}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{B})_{\epsilon_0,\vec{T}_0}$ that satsify

$$\overline{\partial} u' \equiv 0 \mod \mathcal{E}_{\mathfrak{A}}(\mathfrak{q}^+). \tag{2.211}$$

In case $\mathfrak{A} = \mathfrak{B}$ we write $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A})_{\epsilon_0,\vec{T}_0}$.

Lemma 2.64. *Assume* $\mathfrak{A} \neq \emptyset$. *We can choose* ϵ_0, $\epsilon_{\mathfrak{p}_c}$ *sufficiently small and* $\vec{T}_0$ *sufficiently large such that the following holds after extending the core of* $\mathfrak{p} \cup \vec{w}_{\mathfrak{p}}$.

(1) *If* $\mathfrak{q}^+ = (\mathfrak{Y}, u', (\vec{w}_c'; c \in \mathfrak{B}))$ *is in* $\mathfrak{U}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{B})_{\epsilon_0,\vec{T}_0}$ *then the equation (2.211) is Fredholm regular.*
(2) *If* $\mathfrak{q}^+ = (\mathfrak{Y}, u', (\vec{w}_c'; c \in \mathfrak{B}))$ *is in* $\mathfrak{U}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{B})_{\epsilon_0,\vec{T}_0}$ *then* $\mathfrak{q}^+$ *is evaluation map transversal.*
(3) $\mathfrak{p} \in \mathfrak{U}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{B})_{\epsilon_0,\vec{T}_0}$.

Here the definition of Fredholm regularity is the same as Definition 2.44 and the definition of evaluation map transversality is the same as Definition 2.45. The proof of Lemma 2.64 is the same as that of Proposition 2.48.

Remark 2.65. More precisely we first choose $\epsilon_{\mathfrak{p}_c}$ so that Lemma 2.64 holds for $\mathfrak{q}^+ = \mathfrak{p} \cup \vec{w}_{\mathfrak{p}}$. (The choice of $\epsilon_{\mathfrak{p}_c}$ is done at the stage when we take $\mathfrak{M}^+(\mathfrak{p}_c)$ in Definition 2.49.) Then we take ϵ_0 small so that the Lemma 2.64 holds for any element $\mathfrak{q}^+$ of $\mathfrak{U}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{B})_{\epsilon_0,\vec{T}_0}$.

Corollary 2.66. *If* ϵ_0, $\epsilon_{\mathfrak{p}_c}$ *small then* $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B})_{\epsilon_0,\vec{T}_0}$ *has a structure of smooth manifold stratawise. The dimension of the top stratum is*

$$\dim \mathcal{M}_{k+1,\ell}(\beta) + 2\sum_{c\in\mathfrak{B}} \ell_c + 2\ell_{\mathfrak{p}} + \sum_{c\in\mathfrak{A}} \dim_{\mathbb{R}} E_c.$$

Here $\dim \mathcal{M}_{k+1,\ell}(\beta)$ *is a virtual dimension that is given by*

$$\dim \mathcal{M}_{k+1,\ell}(\beta) = k + 1 + 2\ell - 3 + 2\mu(\beta).$$

($\mu(\beta)$ *is the Maslov index.) The dimension of the stratum* $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B};\mathcal{G})_{\epsilon_0,\vec{T}_0}$ *is*

$$\dim \mathcal{M}_{k+1,\ell}(\beta) + 2\sum_{c\in\mathfrak{B}} \ell_c + 2\ell_{\mathfrak{p}} + \sum_{c\in\mathfrak{A}} \dim_{\mathbb{R}} E_c - 2\#C^1_{\rm c}(\mathcal{G}) - \#C^1_{\rm o}(\mathcal{G}).$$

$\Gamma_{\mathfrak{p}}$ *acts effectively on* $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B})_{\epsilon_0,\vec{T}_0}$.

Corollary 2.66 is an immediate consequence of Lemma 2.64, implicit function theorem and index calculation.

Remark 2.67. We can define the topology of $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B})_{\epsilon_0,\vec{T}_0}$ in the same way as the topology of $\mathcal{M}_{k+1,\ell}(\beta)$. We omit it here and will define the topology of $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B})_{\epsilon_0,\vec{T}_0}$ in the next subsection. (Definiton 2.71.)

So far we have described the case of $\mathcal{M}_{k+1,\ell}(\beta)$. The case of $\mathcal{M}^{\rm cl}_{\ell}(\alpha)$ is similar with obvious modification.

2.5. **Gluing analysis in the general case.** The purpose of this subsection is to generalize Theorems 1.10 and 1.34 to the case of the thickened moduli space $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)_{\epsilon_0,\vec T_0}$ we defined in the last subsection. Actually this generalization is straightforward.

We first state the result. Let $\mathcal{G}_{\mathfrak p}$ be the combinatorial type of $\mathfrak p$. We first consider the stratum $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathcal{G}_{\mathfrak p})_{\epsilon_0}$. We did not include $\vec T_0$ in the notation since this parameter does not play a role in our stratum. (Note $T_{{\rm e},0}$ is the gluing parameter. We do not perform gluing to obtain an element in the same startum as $\mathfrak p$.) We write

$$V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B;\epsilon_0)=\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B;\mathcal{G}_{\mathfrak p})_{\epsilon_0}. \tag{2.212}$$

This space in this subsection plays the role of $V_1\times_L V_2$ in Theorem 1.10. In case $\mathfrak B=\mathfrak A$, we put

$$V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\epsilon_0):=V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak A;\epsilon_0).$$

Lemma 2.68. *$V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B;\epsilon_0)$ has a structure of smooth manifold.*

Proof. This is a special case of Corollary 2.66 and is a consequence of Lemma 2.50 (2) and (3). We give a proof for completeness.

Let $c\in\mathfrak C(\mathfrak p)$. Since $\mathfrak p\prec\mathfrak p_c$, there exists a map $\pi:\mathcal{G}_{\mathfrak p_c}\to\mathcal{G}_{\mathfrak p}$. For each ${\rm v}'\in C^0(\mathcal{G}_{\mathfrak p_c})$ we obtain an element $\mathfrak p_{c,{\rm v}'}\in\mathcal{M}_{k_{{\rm v}'}+1,\ell'_{\rm v}}(\beta_{{\rm v}'})$ and $\mathfrak p_{c,{\rm v}'}\cup\vec w_{c,{\rm v}'}\in\mathcal{M}_{k_{{\rm v}'}+1,\ell_{\rm v}+\ell_{c,{\rm v}'}}(\beta_{{\rm v}'})$. For ${\rm v}\in C^0_{\rm d}(\mathcal{G}_{\mathfrak p})$ the union of $\mathfrak p_{c,{\rm v}'}$ for all ${\rm v}'$ with $\pi({\rm v}')={\rm v}$ is an element $\mathfrak p_{c,{\rm v}}\in\mathcal{M}_{k_{{\rm v}+1},\ell_{\rm v}}(\beta_{\rm v})$. Together with the union of $\vec w_{c,{\rm v}'}$'s it gives $\mathfrak p_{c,{\rm v}}\cup\vec w_{c,{\rm v}}\in\mathcal{M}_{k_{{\rm v}+1},\ell_{\rm v}+\ell_{c,{\rm v}}}(\beta_{\rm v})$. The obstruction bundle data centered at $\mathfrak p_c$ induces one centered at $\mathfrak p_{c,{\rm v}}$ in an obvious way.

Let $\mathfrak p_{\rm v}\in\mathcal{M}_{k_{{\rm v}+1},\ell_{\rm v}}(\beta_{\rm v})$ be an element obtained by restricting various data of $\mathfrak p$ to the irreducible component of $\mathfrak x_{\mathfrak p}$ corresponding to the vertex v in an obvious way. We have additional marked points $\vec w_c^{\mathfrak p_{\rm v}}$ by restricting $\vec w_c^{\mathfrak p}$. Then $\mathfrak p_{\rm v}\cup\vec w_c^{\mathfrak p_{\rm v}}$ is ϵ_c close to $\mathfrak p_{c,{\rm v}}\cup\vec w_{c,{\rm v}}$.

We have taken the additional marked points $\vec w_{\mathfrak p}$ on $\mathfrak p$. Let $\vec w_{\mathfrak p,{\rm v}}$ be a part of it that lies on the irreducible component $\mathfrak p_{\rm v}$ Then $\mathfrak p_{\rm v}\cup\vec w_{\mathfrak p,{\rm v}}\in\mathcal{M}_{k_{{\rm v}+1},\ell_{\rm v}+\ell_{\mathfrak p,{\rm v}}}(\beta_{\rm v})$.

Using $\mathfrak p_{c,{\rm v}},\vec w_{c,{\rm v}},\mathfrak p_{\rm v},\vec w_{\mathfrak p,{\rm v}},\vec w_c^{\mathfrak p_{\rm v}}$ etc., we define $\mathcal{M}_{k_{\rm v}+1,(\ell_{\rm v},\ell_{\mathfrak p,{\rm v}},(\ell_{c,{\rm v}}))}(\beta_{\rm v};\mathfrak p_{\rm v};\mathfrak A;\mathfrak B;{\rm point})_{\epsilon_0}$. (Note that $\mathfrak p_{\rm v}$ is irreducible. So the corresponding graph is trivial, that is the graph without edge.) We note again that $\mathfrak p_{\rm v}$ is irreducible and is source stable. So the thickened moduli space $\mathcal{M}_{k_{\rm v}+1,(\ell_{\rm v},\ell_{\mathfrak p,{\rm v}},(\ell_{c,{\rm v}}))}(\beta_{\rm v};\mathfrak p_{\rm v};\mathfrak A;\mathfrak B;{\rm point})_{\epsilon_0}$ is the set parametrized by the solutions of the equations

$$\overline\partial u'\equiv 0\mod\mathcal{E}_{\mathfrak B}(u')$$

together with the complex structure of the source. By Lemma 2.50 (2) the linearized operator of this equiation is surjective. Therefore $\mathcal{M}_{k_{\rm v}+1,(\ell_{\rm v},\ell_{\mathfrak p,{\rm v}},(\ell_{c,{\rm v}}))}(\beta_{\rm v};\mathfrak p_{\rm v};\mathfrak A;\mathfrak B;{\rm point})_{\epsilon_0}$ is a smooth manifold on a neighborhood of $(\mathfrak p_{\rm v},\vec w_{\mathfrak p,{\rm v}},(\vec w_c^{\mathfrak p_{\rm v}}))$ for each ${\rm v}\in C^0_{\rm d}(\mathcal{G}_{\mathfrak p})$. (Note that we add marked points so that there is no automorphism of elements of $\mathcal{M}_{k_{\rm v}+1,\ell_{\rm v}}(\beta_{\rm v})$. So it is not only an orbifold but is also a manifold.) The case ${\rm v}\in{\rm C}^0_{\rm s}(\mathcal{G}_{\mathfrak p})$ can be discussed in the same way and obtain $\mathcal{M}^{\rm cl}_{(\ell_{\rm v},\ell_{\mathfrak p,{\rm v}},(\ell_{c,{\rm v}}))}(\beta_{\rm v};\mathfrak p_{\rm v};\mathfrak A;\mathfrak B;{\rm point})_{\epsilon_0}$, that is also a smooth manifold.

We take the product of them for all $\mathrm{v} \in \mathrm{C}^0(\mathcal{G}_{\mathfrak{p}})$. By taking evaluation maps we have

$$\begin{aligned}
&\prod_{\mathrm{v}\in \mathrm{C}^0_{\mathrm{d}}(\mathcal{G}_{\mathfrak{p}})} \mathcal{M}_{k_{\mathrm{v}}+1,(\ell_{\mathrm{v}},\ell_{\mathfrak{p},\mathrm{v}},(\ell_{c,\mathrm{v}}))}(\beta_{\mathrm{v}};\mathfrak{p}_{\mathrm{v}};\mathfrak{A};\mathfrak{B};\mathrm{point})_{\epsilon_0} \\
&\times \prod_{\mathrm{v}\in \mathrm{C}^0_{\mathrm{s}}(\mathcal{G}_{\mathfrak{p}})} \mathcal{M}^{\mathrm{cl}}_{(\ell_{\mathrm{v}},\ell_{\mathfrak{p},\mathrm{v}},(\ell_{c,\mathrm{v}}))}(\beta_{\mathrm{v}};\mathfrak{p}_{\mathrm{v}};\mathfrak{A};\mathfrak{B};\mathrm{point})_{\epsilon_0} \\
&\to \left(\prod_{\mathrm{e}\in \mathrm{C}^1_{\mathrm{o}}(\mathcal{G}_{\mathfrak{p}})} L \times \prod_{\mathrm{e}\in \mathrm{C}^1_{\mathrm{c}}(\mathcal{G}_{\mathfrak{p}})} X\right)^2 .
\end{aligned}$$

Lemma 2.50 (3) implies that this map is transversal to the diagonal set $\prod_{\mathrm{e}\in \mathrm{C}^1_{\mathrm{o}}(\mathcal{G}_{\mathfrak{p}})} L\times \prod_{\mathrm{e}\in \mathrm{C}^1_{\mathrm{c}}(\mathcal{G}_{\mathfrak{p}})} X = L^{\#C^1_{\mathrm{o}}(\mathcal{G}_{\mathfrak{p}})} \times X^{\#C^1_{\mathrm{c}}(\mathcal{G}_{\mathfrak{p}})}$. The inverse image of the diagonal set is $V_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\epsilon_0)$. □

The gluing we will perform below defines a map

$$\begin{aligned}
\mathrm{Glu} : V_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B};\epsilon_1) \times (\vec{T}^{\mathrm{o}}_0,\infty] \times ((\vec{T}^{\mathrm{c}}_0,\infty] \times \vec{S}^1) \\
\to \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B})_{\epsilon_0,\vec{T}_0}.
\end{aligned} \tag{2.213}$$

For a fixed $(\vec{T},\vec{\theta})$ we denote the restriction of Glu to $V_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B};\epsilon_1)\times \{(\vec{T},\vec{\theta})\}$ by $\mathrm{Glu}_{(\vec{T},\vec{\theta})}$.

Definition 2.69. $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B};(\vec{T},\vec{\theta}))_{\epsilon_0,\vec{T}_0}$ is a subset of the space $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B})_{\epsilon_0,\vec{T}_0}$ consisting of the equivalence classes of $(\mathfrak{Y},u')$ such that $\mathfrak{Y} = \overline{\Phi}(\mathfrak{y},\vec{T},\vec{\theta})$ where the combinatorial type of $\mathfrak{y}$ is $\mathcal{G}_{\mathfrak{p}}$. In case $\mathfrak{A}=\mathfrak{B}$, we put

$$\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A})_{\epsilon_0,\vec{T}_0} = \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{A})_{\epsilon_0,\vec{T}_0}.$$

Theorem 2.70. *For each sufficiently small ϵ_3, and sufficiently large $\vec{T}$, there exist ϵ_2, ϵ_4 and a $\Gamma^+_{\mathfrak{p}}$ equivariant map*

$$\begin{aligned}
\mathrm{Glu}_{(\vec{T},\vec{\theta})} : &V_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B};\epsilon_4) \\
&\to \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B};(\vec{T},\vec{\theta}))_{\epsilon_2}
\end{aligned} \tag{2.214}$$

which is a diffeomorphism onto its image. The image of $\mathrm{Glu}_{(\vec{T},\vec{\theta})}$ *contains the space* $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B};(\vec{T},\vec{\theta}))_{\epsilon_3}$.

Here $\vec{T}$ beging sufficiently large means that each of its component is sufficiently large. Theorem 2.70 is a generalization of Theorem 1.10.

Definition 2.71. We define a topology on $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B};(\vec{T},\vec{\theta}))_{\epsilon}$ for $\epsilon < \epsilon_3$ and $\vec{T}_0$ large so that Glu is a homeomorphism to the image.

It is easy to see that this topology coincides with the topology that is defined in the same way as the topology of $\mathcal{M}_{k+1,\ell}(\beta)$.

To state a generalization of Theorem 1.34, that is the exponential decay estimate of T derivatives, we take $\vec{R}$ and the extended core $K^{+\vec{R}}_{\mathrm{v}}$ as in (2.201). By restriction we define a map

$$\begin{aligned}
&\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B};(\vec{T},\vec{\theta}))_{\epsilon_2,\vec{T}_0} \\
&\qquad \to C^{\infty}((K^{+\vec{R}}_{\mathrm{v}}, K^{+\vec{R}}_{\mathrm{v}} \cap \partial\Sigma_{\mathfrak{p},\mathrm{v}}),(X,L)).
\end{aligned} \tag{2.215}$$

We compose it with $\mathrm{Glu}_{(\vec T,\vec\theta)}$ and obtain $\mathrm{Glures}_{(\vec T,\vec\theta),\mathrm{v},\vec R}$.

Theorem 2.72. *For each m and $\vec R$ there exist $T(m)$, $C_{6,m,\vec R}$ and δ such that the following holds for $T^{\rm o}_{\rm e} > T(m)$, $T^{\rm c}_{\rm e} > T(m)$ and $n + |\vec k_T| + |\vec k_\theta| \le m - 10$ and $|\vec k_T| + |\vec k_\theta| > 0$.*

$$\left\Vert \nabla_\rho^n \frac{\partial^{|\vec k_T|}}{\partial T^{\vec k_T}} \frac{\partial^{|\vec k_\theta|}}{\partial \theta^{\vec k_\theta}} \mathrm{Glures}_{(\vec T,\vec\theta),\mathrm{v},\vec R} \right\Vert_{L^2_{m+1-|\vec k_T|-|\vec k_\theta|}} < C_{6,m,\vec R}e^{-\delta'(\vec k_T\cdot \vec T + \vec k_\theta\cdot \vec T^{\rm c})}. \tag{2.216}$$

Here ∇_ρ^n is the n-th derivative in $\rho \in V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B;\epsilon_2)$ direction and $\delta' > 0$ depends only on δ and m.

The proofs of Theorems 2.70 and 2.72 occupy the rest of this subsection. We begin with introducing some notations. Suppose that $(\mathfrak x^{\rho,+}, u^\rho, (\vec w^\rho_c))$ is a representative of an element ρ of $V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\epsilon_0)$. We put $\Sigma_{\mathfrak x^{\rho,+}} = \Sigma^\rho$. Its marked points are denoted by $\vec z^\rho$, $\vec z^{\rm int,\rho}$ and $\vec w^\rho_{\mathfrak p}$, $\vec w^\rho_c$. Here w's are additional marked points. We divide each of the irreducible components $\Sigma^\rho_{\rm v}$ of Σ^ρ as

$$\begin{aligned} K^\rho_{\rm v} \cup &\bigcup_{\substack{{\rm e}\in C^1_{\rm o}(\mathcal G)\\ \text{e is an outgoing edge of v}}} (0,\infty)\times[0,1] \\ \cup &\bigcup_{\substack{{\rm e}\in C^1_{\rm o}(\mathcal G)\\ \text{e is an incoming edge of v}}} (-\infty,0)\times[0,1] \\ \cup &\bigcup_{\substack{{\rm e}\in C^1_{\rm c}(\mathcal G)\\ \text{e is an outgoing edge of v}}} (0,\infty)\times S^1 \\ \cup &\bigcup_{\substack{{\rm e}\in C^1_{\rm c}(\mathcal G)\\ \text{e is an incoming edge of v}}} (-\infty,0)\times S^1, \end{aligned} \tag{2.217}$$

where the coordinates of the 2-nd, 3-rd, 4-th, and 5-th summands are $(\tau'_{\rm e}, t_{\rm e})$, $(\tau''_{\rm e}, t_{\rm e})$, $(\tau'_{\rm e}, t'_{\rm e})$, and $(\tau''_{\rm e}, t''_{\rm e})$, respectively. Here $\tau'_{\rm e} \in (0,\infty)$, $\tau''_{\rm e} \in (-\infty,0)$.

We call the end corresponding to e the e-*th end.*

We recall

$$\tau_e = \tau'_{\rm e} - 5T_{\rm e} = \tau''_{\rm e} + 5T_{\rm e}, \tag{2.218}$$

$$t_{\rm e} = t'_{\rm e} = t''_{\rm e} - \theta_{\rm e}. \tag{2.219}$$

We put

$$u^\rho_{\rm v} = u^\rho|_{K_{\rm v}}, \qquad u^\rho_{\rm e} = u^\rho|_{\text{e-th neck region}}.$$

We denote by $\Sigma_{\mathfrak Y} = \Sigma^\rho_{\vec T,\vec\theta}$ a representative of $\mathfrak Y = \overline\Phi(\mathfrak y,\vec T,\vec\theta)$. The curve $\Sigma^\rho_{\vec T,\vec\theta}$ is a union

$$\begin{aligned} \bigcup_{{\rm v}\in C^0(\mathcal G_{\mathfrak p})} K^\rho_{\rm v} \cup \bigcup_{{\rm e}\in C^1_{\rm o}(\mathcal G)} [-5T_{\rm e},5T_{\rm e}]\times[0,1] \\ \cup \bigcup_{{\rm e}\in C^1_{\rm c}(\mathcal G)} [-5T_{\rm e},5T_{\rm e}]\times S^1. \end{aligned} \tag{2.220}$$

The coordinates of the 2nd and 3rd terms are $\tau_{\rm e}$ and $t_{\rm e}$.

We call $[-5T_{\rm e},5T_{\rm e}]\times[0,1]$ or $[-5T_{\rm e},5T_{\rm e}]\times S^1$ the e-*th neck.*

In case $T_{\rm e} = \infty$, the curve $\Sigma^{\rho}_{\vec T,\vec\theta}$ contains $([0,\infty) \cup (-\infty,0]) \times [0,1]$ or $([0,\infty) \cup (-\infty,0]) \times S^1$ corresponding to the e-th edge. We call $([0,\infty) \times [0,1]$ (or $\times S^1$) the *outgoing* e-*th end* and $(-\infty,0] \times [0,1]$ (or S^1) the *incoming* e-*th end*.

We call $K_{\rm v}$ the v-*th core*.

The restriction of u^ρ to $K_{\rm v}$ is written as $u^\rho_{\rm v}$. The restriction of u^ρ to the e-neck is written as $u^\rho_{\rm e}$.

For each e, let $\rm v_1$ and $\rm v_2$ be its incoming and outgoing vertices. We have

$$\lim_{\tau_{\rm e}\to -\infty} u^\rho_{\rm v_2}(\tau_{\rm e}, t_{\rm e}) = \lim_{\tau_{\rm e}\to \infty} u^\rho_{\rm v_1}(\tau_{\rm e}, t_{\rm e}), \tag{2.221}$$

and (2.221) is independent of $t_{\rm e}$. We write this limit as $p^\rho_{\rm e}$. We take a Darboux coordinate in a neighborhood of each $p^\rho_{\rm e}$ such that L is flat in this coordinate. We choose the map E such that (1.31) holds in this neighborhood of $p^\rho_{\rm e}$.

For $\rm e \in C^1_o(\mathcal G_{\mathfrak p})$ with $T_{\rm e} \ne \infty$, we define

$$\begin{aligned}
\mathcal A_{{\rm e},T} &= [-T_{\rm e}-1, -T_{\rm e}+1] \times [0,1] \subset [-5T_{\rm e}, 5T_{\rm e}] \times [0,1],\\
\mathcal B_{{\rm e},T} &= [T_{\rm e}-1, T_{\rm e}+1] \times [0,1] \subset [-5T_{\rm e}, 5T_{\rm e}] \times [0,1],\\
\mathcal X_{{\rm e},T} &= [-1, +1] \times [0,1] \subset [-5T_{\rm e}, 5T_{\rm e}] \times [0,1].
\end{aligned} \tag{2.222}$$

In case $\rm e \in C^1_c(\mathcal G_{\mathfrak p})$, the sets $\mathcal A_{{\rm e},T}$, $\mathcal B_{{\rm e},T}$, $\mathcal X_{{\rm e},T}$ are defined in the same way as above replacing $[0,1]$ by S^1.

If v is a vertex of e then $\mathcal A_{{\rm e},T}$, $\mathcal B_{{\rm e},T}$, $\mathcal X_{{\rm e},T}$ may be regarded as a subset of $\Sigma^\rho_{\rm v}$ also.

Let $\chi^{\leftarrow}_{{\rm e},\mathcal A}$, $\chi^{\rightarrow}_{{\rm e},\mathcal A}$ be smooth functions on $[-5T_{\rm e}, 5T_{\rm e}] \times [0,1]$ or $[-5T_{\rm e}, 5T_{\rm e}] \times S^1$ such that

$$\chi^{\leftarrow}_{{\rm e},\mathcal A}(\tau_{\rm e}, t_{\rm e}) = \begin{cases} 1 & \tau_{\rm e} < -T_{\rm e} - 1 \\ 0 & \tau_{\rm e} > -T_{\rm e} + 1. \end{cases} \tag{2.223}$$

$$\chi^{\rightarrow}_{{\rm e},\mathcal A} = 1 - \chi^{\leftarrow}_{{\rm e},\mathcal A}.$$

We define

$$\chi^{\leftarrow}_{{\rm e},\mathcal B}(\tau_{\rm e}, t_{\rm e}) = \begin{cases} 1 & \tau_{\rm e} < T_{\rm e} - 1 \\ 0 & \tau_{\rm e} > T_{\rm e} + 1. \end{cases} \tag{2.224}$$

$$\chi^{\rightarrow}_{{\rm e},\mathcal B} = 1 - \chi^{\leftarrow}_{{\rm e},\mathcal B}.$$

We define

$$\chi^{\leftarrow}_{{\rm e},\mathcal X}(\tau_{\rm e}, t_{\rm e}) = \begin{cases} 1 & \tau_{\rm e} < -1 \\ 0 & \tau_{\rm e} > 1. \end{cases} \tag{2.225}$$

$$\chi^{\rightarrow}_{{\rm e},\mathcal X} = 1 - \chi^{\leftarrow}_{{\rm e},\mathcal X}.$$

We extend these functions to $\Sigma^\rho_{\vec T,\vec\theta}$ and $\Sigma^\rho_{\rm v}$ so that they are locally constant on its core. We denote them by the same symbol.

We next introduce weighted Sobolev norms and their local versions for sections on $\Sigma^\rho_{\rm v}$ as follows. We define a smooth function $e_{{\rm v},\delta} : \Sigma^\rho_{\rm v} \to [1,\infty)$ by

$$e_{{\rm v},\delta}(\tau_{\rm e}, t_{\rm e}) \begin{cases} = 1 & \text{on } K_{\rm v}, \\ = e^{\delta|\tau_{\rm e} + 5T_{\rm e}|} & \text{if } \tau_{\rm e} > 1 - 5T_{\rm e}, \text{ and e is an outgoing edge of v}, \\ \in [1,10] & \text{if } \tau_{\rm e} < 1 - 5T_{\rm e}, \text{ and e is an outgoing edge of v}, \\ = e^{\delta|\tau - 5T_{\rm e}|} & \text{if } \tau_{\rm e} < 5T_{\rm e} - 1, \text{ and e is an incoming edge of v}, \\ \in [1,10] & \text{if } \tau_{\rm e} > 5T_{\rm e} - 1, \text{ and e is an incoming edge of v}. \end{cases} \tag{2.226}$$

We also define a weight function $e_{\vec T,\delta} : \Sigma^{\rho}_{\vec T,\vec\theta} \to [1,\infty)$ as follows:

$$e_{\vec T,\delta}(\tau_{\rm e},t_{\rm e}) \begin{cases} = e^{\delta|\tau_{\rm e}-5T_{\rm e}|} & \text{if } 1 < \tau_{\rm e} < 5T_{\rm e}-1, \\ = e^{\delta|\tau+5T_{\rm e}|} & \text{if } -1 > \tau > 1-5T_{\rm e}, \\ = 1 & \text{on } K_{\rm v}, \\ \in [1,10] & \text{if } |\tau_{\rm e}-5T_{\rm e}| < 1 \text{ or } |\tau_{\rm e}+5T_{\rm e}| < 1, \\ \in [e^{5T_{\rm e}\delta}/10, e^{5T_{\rm e}\delta}] & \text{if } |\tau_{\rm e}| < 1. \end{cases} \tag{2.227}$$

The weighted Sobolev norm we use for $L^2_{m,\delta}(\Sigma^{\rho}_{\rm v}; (u^{\rho}_{\rm v})^*TX \otimes \Lambda^{01})$ is given by

$$\|s\|^2_{L^2_{m,\delta}} = \sum_{k=0}^{m} \int_{\Sigma^{\rho}_{\rm v}} e_{{\rm v},\delta} |\nabla^k s|^2 {\rm vol}_{\Sigma^{\rho}_{\rm v}}. \tag{2.228}$$

Definition 2.73. The Sobolev space $L^2_{m+1,\delta}((\Sigma^{\rho}_{\rm v},\partial\Sigma^{\rho}_{\rm v}); (u^{\rho}_{\rm v})^*TX, (u^{\rho}_{\rm v})^*TL)$ consists of elements $(s,\vec v)$ with the following properties.

1. $\vec v = (v_{\rm e})$ where e runs on the set of edges of v and $v_{\rm e} \in T_{p^{\rho}_{\rm e}}(X)$ (in case ${\rm e} \in {\rm C}^1_{\rm c}(\mathcal G)$) or $v_{\rm e} \in T_{p^{\rho}_{\rm e}}(L)$ (in case ${\rm e} \in {\rm C}^1_{\rm o}(\mathcal G)$).
2. The following norm is finite.

$$\begin{aligned} \|(s,\vec v)\|^2_{L^2_{m+1,\delta}} = &\sum_{k=0}^{m+1} \int_{K_{\rm v}} |\nabla^k s|^2 {\rm vol}_{\Sigma_i} + \sum_{\text{e: edges of v}} \|v_{\rm e}\|^2 \\ &+ \sum_{k=0}^{m+1} \sum_{\text{e: edges of v}} \int_{\text{e-th end}} e_{{\rm v},\delta} |\nabla^k (s - {\rm Pal}(v_{\rm e}))|^2 {\rm vol}_{\Sigma^{\rho}_{\rm v}}. \end{aligned} \tag{2.229}$$

Definition 2.74. We define

$$\begin{aligned} D{\rm ev}_{\mathcal G_{\mathfrak p}} : &\bigoplus_{{\rm v} \in C^0_{\rm o}(\mathcal G_{\mathfrak p})} L^2_{m+1,\delta}((\Sigma^{\rho}_{\rm v},\partial\Sigma^{\rho}_{\rm v}); (u^{\rho}_{\rm v})^*TX, (u^{\rho}_{\rm v})^*TL) \\ &\oplus \bigoplus_{{\rm v} \in C^0_{\rm c}(\mathcal G_{\mathfrak p})} L^2_{m+1,\delta}(\Sigma^{\rho}_{\rm v}; (u^{\rho}_{\rm v})^*TX) \\ &\to \bigoplus_{{\rm e} \in {\rm C}^1_{\rm o}(\mathcal G_{\mathfrak p})} T_{p^{\rho}_{\rm e}} L \oplus \bigoplus_{{\rm e} \in {\rm C}^1_{\rm c}(\mathcal G_{\mathfrak p})} T_{p^{\rho}_{\rm e}} X \end{aligned} \tag{2.230}$$

as in (2.200).

Definition 2.75. We denote the kernel of (2.230) by

$$L^2_{m+1,\delta}((\Sigma^{\rho},\partial\Sigma^{\rho}); (u^{\rho})^*TX, (u^{\rho})^*TL).$$

We next define weighted Sobolev norms for the sections of various bundles on $\Sigma^{\rho}_{\vec T,\vec\theta}$. Let

$$u' : (\Sigma^{\rho}_{\vec T,\vec\theta}, \partial\Sigma^{\rho}_{\vec T,\vec\theta}) \to (X,L)$$

be a smooth map of homology class β that is pseudo-holomorphic in the neck region and has finite energy. (We include the case when u' is not pseudo-holomorphic in the neck region but satisfies the same exponential decay estimate as the pseudo-holomorphic curve.) We first consider the case when all $T_{\rm e} \ne \infty$. In this case $\Sigma^{\rho}_{\vec T,\vec\theta}$ is compact. We consider an element

$$s \in L^2_{m+1}((\Sigma^{\rho}_{\vec T,\vec\theta}, \partial\Sigma^{\rho}_{\vec T,\vec\theta}); (u')^*TX, (u')^*TL).$$

Since we take m large, the section s is continuous. We take a point $(0,1/2)_{\rm e}$ in the e-th neck. Since $s \in L^2_{m+1}$ its value $s((0,1/2)_{\rm e}) \in T_{u'((0,1/2)_{\rm e})}X$ is well-defined.

We take a coordinate around $p^{\rho}_{\rm e}$ such that in case ${\rm e} \in {\rm C}^1_{\rm o}(\mathcal G)$ our Lagrangian submanifold L is linear in this coordinate around $p^{\rho}_{\rm e}$. We use this trivialization to find a canonical trivialization of TX in a neighbofhood of $p^{\rho}_{\rm e}$. We use this trivialization to define Pal below. We put

$$\begin{aligned}\Vert s\Vert^2_{L^2_{m+1,\delta}} = &\sum_{k=0}^{m+1}\sum_{\rm v}\int_{K_{\rm v}} \vert\nabla^k s\vert^2 {\rm vol}_{\Sigma^{\rho}_{\rm v}} \\ &+ \sum_{k=0}^{m+1}\sum_{\rm e}\int_{\text{e-th neck}} e_{\vec T,\delta}\vert\nabla^k(s-{\rm Pal}(s(0,1/2)_{\rm e}))\vert^2 dt_{\rm e}d\tau_{\rm e} \\ &+ \sum_{\rm e}\Vert s((0,1/2)_{\rm e}))\Vert^2.\end{aligned} \tag{2.231}$$

For a section $s \in L^2_m(\Sigma^{\rho}_{\vec T,\vec\theta}; u^*TX\otimes\Lambda^{01})$ we define

$$\Vert s\Vert^2_{L^2_{m,\delta}} = \sum_{k=0}^{m}\int_{\Sigma^{\rho}_{\vec T,\vec\theta}} e_{T,\delta}\vert\nabla^k s\vert^2{\rm vol}_{\Sigma^{\rho}_{\vec T,\vec\theta}}. \tag{2.232}$$

We next consider the case when some of the edges e have infinite length, namely $T_{\rm e} = \infty$. Let $C^{1,\rm inf}_{\rm o}(\mathcal G_{\mathfrak p},\vec T)$ (resp. $C^{1,\rm inf}_{\rm c}(\mathcal G_{\mathfrak p},\vec T)$) be the set of elements e in $C^1_{\rm o}(\mathcal G_{\mathfrak p})$ (resp. $C^1_{\rm c}(\mathcal G_{\mathfrak p})$) with $T_{\rm e} = \infty$ and let $C^{1,\rm fin}_{\rm o}(\mathcal G_{\mathfrak p},\vec T)$ (resp. $C^{1,\rm fin}_{\rm c}(\mathcal G_{\mathfrak p},\vec T)$) be the set of elements ${\rm e} \in C^1_{\rm o}(\mathcal G_{\mathfrak p})$ (resp. $C^1_{\rm c}(\mathcal G_{\mathfrak p})$) with $T_{\rm e} \ne \infty$. Note the ends of $\Sigma^{\rho}_{\vec T,\vec\theta}$ correspond two to one to $C^{1,\rm inf}_{\rm o}(\mathcal G_{\mathfrak p},\vec T)\cup C^{1,\rm inf}_{\rm c}(\mathcal G_{\mathfrak p},\vec T)$. The ends that correspond to an element e of $C^{1,\rm inf}_{\rm o}(\mathcal G_{\mathfrak p},\vec T)$ is $([-5T_{\rm e},\infty)\times[0,1])\cup(-\infty,5T_{\rm e}]\times[0,1])$ and the ends that correspond to ${\rm e} \in C^{1,\rm inf}_{\rm c}(\mathcal G_{\mathfrak p},\vec T)$ is $([-5T_{\rm e},\infty)\times S^1)\cup(-\infty,5T_{\rm e}]\times S^1)$. We have a weight function $e_{{\rm v},\delta}(\tau_{\rm e},t_{\rm e})$ on it.

Definition 2.76. An element of

$$L^2_{m+1,\delta}((\Sigma^{\rho}_{\vec T,\vec\theta},\partial\Sigma^{\rho}_{\vec T,\vec\theta});(u')^*TX,(u')^*TL)$$

is a pair $(s,\vec v)$ such that

1. s is a section of $(u')^*TX$ on $\Sigma^{\rho}_{\vec T,\vec\theta}$ minus singular points $z_{\rm e}$ with $T_{\rm e} = \infty$.
2. s is locally of L^2_{m+1} class.
3. On $\partial\Sigma^{\rho}_{\vec T,\vec\theta}$ the restriction of s is in $(u')^*TL$.
4. $\vec v = (v_{\rm e})$ where e runs in $C^{1,\rm inf}(\mathcal G_{\mathfrak p},\vec T)$ and $v_{\rm e}$ is as in Definition 2.73 (1).
5. For each e with $T_{\rm e} = \infty$, the integral

$$\begin{aligned}&\sum_{k=0}^{m+1}\int_0^{\infty}\int_{t_{\rm e}} e_{{\rm v},\delta}(\tau_{\rm e},t_{\rm e})\vert\nabla^k(s(\tau_{\rm e},t_{\rm e})-{\rm Pal}(v_{\rm e}))\vert^2 d\tau_{\rm e}dt_{\rm e} \\ &+\sum_{k=0}^{m+1}\int_{-\infty}^{0}\int_{t_{\rm e}} e_{{\rm v},\delta}(\tau_{\rm e},t_{\rm e})\vert\nabla^k(s(\tau_{\rm e},t_{\rm e})-{\rm Pal}(v_{\rm e}))\vert^2 d\tau_{\rm e}dt_{\rm e}\end{aligned} \tag{2.233}$$

is finite. (Here we integrate over $t_{\rm e} \in [0,1]$ (resp. $t_{\rm e} \in S^1$) if ${\rm e} \in C^{1,\rm inf}_{\rm o}(\mathcal G_{\mathfrak p},\vec T)$ (resp. ${\rm e} \in C^{1,\rm inf}_{\rm c}(\mathcal G_{\mathfrak p},\vec T)$).

We define

$$\|(s,\vec{v})\|^2_{L^2_{m+1,\delta}} = (2.231) + \sum_{\mathrm{e}\in C^{1,\mathrm{inf}}(\mathcal{G}_{\mathfrak{p}},\vec{T})} (2.233) + \sum_{\mathrm{e}\in C^{1,\mathrm{inf}}(\mathcal{G}_{\mathfrak{p}},\vec{T})} \|v_{\mathrm{e}}\|^2. \tag{2.234}$$

An element of

$$L^2_{m,\delta}(\Sigma^{\rho}_{\vec{T},\vec{\theta}};(u')^*TX\otimes\Lambda^{01})$$

is a section s of the bundle $(u')^*TX\otimes\Lambda^{01}$ such that it is locally of L^2_m-class and

$$\begin{aligned}&\sum_{k=0}^m\int_0^\infty\int_{t_{\mathrm{e}}} e_{\mathrm{v},\delta}|\nabla^k s(\tau_{\mathrm{e}},t_{\mathrm{e}})|^2 d\tau_{\mathrm{e}}dt_{\mathrm{e}}\\ &+\sum_{k=0}^m\int_{-\infty}^0\int_{t_{\mathrm{e}}} e_{\mathrm{v},\delta}|\nabla^k(s(\tau_{\mathrm{e}},t_{\mathrm{e}})|^2 d\tau_{\mathrm{e}}dt_{\mathrm{e}}\end{aligned} \tag{2.235}$$

is finite. We define

$$\|s\|^2_{L^2_{m,\delta}} = (2.232) + \sum_{\mathrm{e}\in C^{1,\mathrm{inf}}(\mathcal{G}_{\mathfrak{p}},\vec{T})} (2.235). \tag{2.236}$$

For a subset W of $\Sigma^{\rho}_{\mathrm{v}}$ or $\Sigma^{\rho}_{\vec{T},\vec{\theta}}$ we define $\|s\|_{L^2_{m,\delta}(W\subset\Sigma^{\rho}_{\mathrm{v}})}$, $\|s\|_{L^2_{m,\delta}(W\subset\Sigma^{\rho}_{\vec{T},\vec{\theta}})}$ by restricting the domain of the integration (2.232), (2.231), (2.234) or (2.236) to W.

Let $(s_j,\vec{v}_j)\in L^2_{m+1,\delta}((\Sigma^{\rho}_{\mathrm{v}},\partial\Sigma^{\rho}_{\mathrm{v}});(u^{\rho}_{\mathrm{v}})^*TX,(u^{\rho}_{\mathrm{v}})^*TL)$ for $j=1,2$. We define an inner product among them by:

$$\begin{aligned}&\langle\!\langle(s_1,\vec{v}_1),(s_2,\vec{v}_2)\rangle\!\rangle_{L^2_\delta}\\ =&\sum_{\mathrm{e}\in C^1(\mathcal{G}_{\mathfrak{p}})}\int_{\text{e-th neck}} e_{\vec{T},\delta}(s_1-\mathrm{Pal}(v_{1,\mathrm{e}}),s_2-\mathrm{Pal}(v_{2,\mathrm{e}}))\\ &+\sum_{\mathrm{v}\in C^0(\mathcal{G}_{\mathfrak{p}})}\int_{K_{\mathrm{v}}}(s_1,s_2)+\sum_{\mathrm{e}\in C^1(\mathcal{G}_{\mathfrak{p}})}(v_{1,\mathrm{e}},v_{2,\mathrm{e}}).\end{aligned} \tag{2.237}$$

Now we start the gluing process. Let us start with the maps

$$u^{\rho}_{\mathrm{v}}:(\Sigma^{\rho}_{\mathrm{v}},\partial\Sigma^{\rho}_{\mathrm{v}})\to(X,L)$$

for each v so that $(u^{\rho}_{\mathrm{v}};\mathrm{v}\in C^0(\mathcal{G}_{\mathfrak{p}}))$ consists an element of $V_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\epsilon_2)$. Let $(\vec{T},\vec{\theta})\in(\vec{T}^{\mathrm{o}}_0,\infty]\times((\vec{T}^{\mathrm{c}}_0,\infty]\times\vec{S}^1)$. For $\kappa=0,1,2,\dots$, we will define a series of maps

$$u^{\rho}_{\vec{T},\vec{\theta},(\kappa)} : (\Sigma^{\rho}_{\vec{T},\vec{\theta}},\partial\Sigma^{\rho}_{\vec{T},\vec{\theta}})\to(X,L) \tag{2.238}$$

$$\hat{u}^{\rho}_{\mathrm{v},\vec{T},\vec{\theta},(\kappa)} : (\Sigma^{\rho}_{\mathrm{v}},\partial\Sigma^{\rho}_{\mathrm{v}})\to(X,L) \tag{2.239}$$

and elements

$$\mathfrak{e}^{\rho}_{c,\vec{T},\vec{\theta},(\kappa)} \in E_c=\bigoplus_{\mathrm{v}\in C^0(\mathcal{G}_{\mathfrak{p}_c})}E_{c,\mathrm{v}} \tag{2.240}$$

$$\mathrm{Err}^{\rho}_{\mathrm{v},\vec{T},\vec{\theta},(\kappa)} \in L^2_{m,\delta}(\Sigma^{\rho}_{\mathrm{v}};(\hat{u}^{\rho}_{\mathrm{v},\vec{T},\vec{\theta},(\kappa)})^*TX\otimes\Lambda^{01}). \tag{2.241}$$

Note $E_{c,\mathrm{v}}\subset\Gamma(K_{\mathrm{v}};u^*_{\mathfrak{p}_c}TX\otimes\Lambda^{01})$ is a finite dimensional space which we take as a part of the obstruction bundle data centered at $\mathfrak{p}_c$.

Moreover we will define $V^{\rho}_{\vec T,\vec\theta,\mathrm{v},(\kappa)}$ for $\mathrm{v} \in C^0(\mathcal{G}_{\mathfrak p})$ and $\Delta p^{\rho}_{\mathrm{e},\vec T,\vec\theta,(\kappa)}$ for $\mathrm{e} \in C^1(\mathcal{G}_{\mathfrak p})$. The pair $((V^{\rho}_{\vec T,\vec\theta,\mathrm{v},(\kappa)}),(\Delta p^{\rho}_{\mathrm{e},\vec T,\vec\theta,(\kappa)}))$ is an element of the weighted Sobolev space $L^2_{m+1,\delta}((\Sigma^{\rho}_{\mathrm{v}},\partial\Sigma^{\rho}_{\mathrm{v}});(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(\kappa-1)})^*TX,(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(\kappa-1)})^*TL)$.

The construction of these objects is a straightforward generalization of the construction given by Subsection 1.3 and proceed by induction on κ as follows.

Pregluing: We first define an approximate solution $u^{\rho}_{\vec T,\vec\theta,(0)}$. For $\mathrm{e} \in C^1(\mathcal{G}_{\mathfrak p})$ we denote by $\mathrm{v}_{\leftarrow}(\mathrm{e})$ and $\mathrm{v}_{\rightarrow}(\mathrm{e})$ its two vertices. Here e is an outgoing edge of $\mathrm{v}_{\leftarrow}(\mathrm{e})$ and is an incoming edge of $\mathrm{v}_{\rightarrow}(\mathrm{e})$. We put:

$$u^{\rho}_{\vec T,\vec\theta,(0)} = \begin{cases} \chi^{\leftarrow}_{\mathrm{e},\mathcal{B}}(u^{\rho}_{\mathrm{v}_{\leftarrow}(\mathrm{e})} - p^{\rho}_{\mathrm{e}}) + \chi^{\rightarrow}_{\mathrm{e},\mathcal{A}}(u^{\rho}_{\mathrm{v}_{\rightarrow}(\mathrm{e})} - p^{\rho}_{\mathrm{e}}) + p^{\rho}_{\mathrm{e}} & \text{on the e-th neck} \\ u^{\rho}_{\mathrm{v}} & \text{on } K_{\mathrm{v}}. \end{cases} \tag{2.242}$$

Step 0-3: We next define

$$\sum_{c\in\mathfrak{A}} \mathfrak{e}^{\rho}_{c,\vec T,\vec\theta,(0)} = \overline\partial u^{\rho}_{\mathrm{v}}, \qquad \text{on } K_{\mathrm{v}}. \tag{2.243}$$

Here we identify $E_c \cong E_c(u^{\rho}_{\mathrm{v}})$ on K_{v} by the parallel transport as we did in Definition 2.60. See also Definition 2.41. Note that $\overline\partial u^{\rho}_{\mathrm{v}}$ is contained in $\oplus E_c$ since $(u^{\rho}_{\mathrm{v}};\mathrm{v} \in C^0(\mathcal{G}_{\mathfrak p}))$ is an element of $V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\epsilon_0)$.

We put

$$\mathfrak{se}^{\rho}_{\vec T,\vec\theta,(0)} := \sum_{c\in\mathfrak{A}} \mathfrak{e}^{\rho}_{c,\vec T,\vec\theta,(0)}. \tag{2.244}$$

Step 0-4: We next define

$$\mathrm{Err}^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)} = \begin{cases} \chi^{\leftarrow}_{\mathrm{e},\mathcal{X}}\overline\partial u^{\rho}_{\vec T,\vec\theta,(0)} & \text{on the e-th neck if e is outgoing} \\ \chi^{\rightarrow}_{\mathrm{e},\mathcal{X}}\overline\partial u^{\rho}_{\vec T,\vec\theta,(0)} & \text{on the e-th neck if e is incoming} \\ \overline\partial u^{\rho}_{\vec T,\vec\theta,(0)} - \mathfrak{se}^{\rho}_{\vec T,\vec\theta,(0)} & \text{on } K_{\mathrm{v}}. \end{cases} \tag{2.245}$$

See Remark 2.80.

Step 1-1: We put

$$\begin{aligned} &\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}(z) \\ &= \begin{cases} \chi^{\leftarrow}_{\mathrm{e},\mathcal{B}}(\tau_{\mathrm{e}} - T_{\mathrm{e}}, t_{\mathrm{e}})\, u^{\rho}_{\vec T,\vec\theta,(0)}(\tau_{\mathrm{e}}, t_{\mathrm{e}}) + \chi^{\rightarrow}_{\mathrm{e},\mathcal{B}}(\tau_{\mathrm{e}} - T_{\mathrm{e}}, t_{\mathrm{e}}) p^{\rho}_{\mathrm{e}} & \\ \qquad \text{if } z = (\tau_{\mathrm{e}}, t_{\mathrm{e}}) \text{ is on the e-th neck that is outgoing} & \\ \chi^{\rightarrow}_{\mathrm{e},\mathcal{A}}(\tau_{\mathrm{e}} - T_{\mathrm{e}}, t_{\mathrm{e}}) u^{\rho}_{\vec T,\vec\theta,(0)}(\tau, t) + \chi^{\leftarrow}_{\mathrm{e},\mathcal{A}}(\tau_{\mathrm{e}} - T_{\mathrm{e}}, t_{\mathrm{e}}) p^{\rho}_{\mathrm{e}} & \\ \qquad \text{if } z = (\tau_{\mathrm{e}}, t_{\mathrm{e}}) \text{ is on the e-th neck that is incoming} & \\ u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}(z) \qquad \text{if } z \in K_{\mathrm{v}}. & \end{cases} \end{aligned} \tag{2.246}$$

We denote the (covariant) linearization of the Cauchy-Riemann equation at this map $\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}$ by

$$\begin{aligned} D_{\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}}\overline\partial : L^2_{m+1,\delta}((\Sigma_{\mathrm{v}},\partial\Sigma_{\mathrm{v}});(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)})^*TX,(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)})^*TL) \\ \to L^2_{m,\delta}(\Sigma_{\mathrm{v}};(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)})^*TX \otimes \Lambda^{01}). \end{aligned} \tag{2.247}$$

We next study the obstruction bundle E_c. We recall that at $u^{\rho}_{\vec T,\vec\theta,(0)}$ the obstruction bundle $E_c(u^{\rho}_{\vec T,\vec\theta,(0)})$ was defined as follows. (See Definition 2.60.) We use the added marked points $\vec w^{\rho}_c$ and consider $\Sigma^{\rho}_{\vec T,\vec\theta} \cup \vec w^{\rho}_c$. Here, by abuse of notation, we include the $k+1$ boudary and ℓ interior marked points in the notation $\Sigma^{\rho}_{\vec T,\vec\theta}$. (The additional marked points $\vec w^{\rho}_{\mathfrak p}$ and $\vec w^{\rho}_c$ are *not* included.) By assumption $\Sigma^{\rho}_{\vec T,\vec\theta} \cup \vec w^{\rho}_c$ is $(\epsilon_c + o(\epsilon_0))$-close to $\mathfrak p_c$. Therefore the diffeomorphism between cores of $\Sigma_{\mathfrak p_c}$ and of $\Sigma^{\rho}_{\vec T,\vec\theta}$ is determined, by the obstruction bundle data $\mathfrak E_{\mathfrak p_c}$. Using this diffeomorphism and the parallel transport we have

$$I^{\mathrm v,\mathfrak p_c}_{(\mathfrak y_c,u_c),(\Sigma^{\rho}_{\vec T,\vec\theta}\cup\vec w^{\rho}_c,u^{\rho}_{\vec T,\vec\theta,(0)})} : E_{c,\mathrm v}(\mathfrak y_c,u_c) \to \Gamma(K_{\mathrm v};(u^{\rho}_{\vec T,\vec\theta,(0)})^*TX\otimes\Lambda^{01}). \tag{2.248}$$

The notation in (2.248) is as follows. There is a map $\pi : \mathcal G_{\mathfrak p_c} \to \mathcal G_{\mathfrak p}$ shrinking several edges. For $\mathrm v \in C^0(\mathcal G_{\mathfrak p})$ we put

$$E_{c,\mathrm v} = \bigoplus_{\substack{\mathrm v'\in C^0(\mathcal G_{\mathfrak p_c})\\ \pi(\mathrm v')=\mathrm v}} E_{c,\mathrm v'}$$

where $E_{c,\mathrm v'}$ is the obstruction bundle that is included in the obstruction bundle data $\mathfrak E_{\mathfrak p_c}$ at $\mathfrak p_c$. It determines $E_{c,\mathrm v}(\mathfrak y_c,u_c) = \bigoplus_{\substack{\mathrm v'\in C^0(\mathcal G_{\mathfrak p_c})\\ \pi(\mathrm v')=\mathrm v}} E_{c,\mathrm v'}(\mathfrak y_c,u_c)$. Then (2.248) is defined by Definition 2.41.

Remark 2.77. In Definiton 2.57 (6) we assumed that the image of $K^{\mathrm{obst}}_{\mathrm v,c}$ by the diffeomorphism mentioned above is always contained in the core of $\Sigma^{\rho}_{\vec T,\vec\theta}$. (Here $K^{\mathrm{obst}}_{\mathrm v,c}$ is the support of $E_{c,\mathrm v}$.) Note by the core we mean the core with respect to the coordinate at infinity that is included as a part of the stabilization data at $\mathfrak p$ here.

The vector space $E_c(u^{\rho}_{\vec T,\vec\theta,(0)})$ is the sum over $\mathrm v\in C^0(\mathcal G_{\mathfrak p})$ of the images of (2.248).

We next consider the obstruction bundle at $\hat u^{\rho}_{\mathrm v,\vec T,\vec\theta,(0)}$. A technical point we need to take care of here is that the obstruction bundle we use is *not* $E_c(\coprod_{\mathrm v\in C^0(\mathcal G_{\mathfrak p})}\hat u^{\rho}_{\mathrm v,\vec T,\vec\theta,(0)})$ but is slightly different from it. Let $K^{\mathrm{obst}}_{\mathrm v,c} \subset K_{\mathrm v} \subset \Sigma^{\rho}_{\vec T,\vec\theta}$ be the image of the set $K^{\mathrm{obst}}_{\mathrm v,c}$ by the above mentioned diffeomorphism that is induced by the stabilization data at $\mathfrak p$. We remark that we may regard $K_{\mathrm v}$ as a subset of $\Sigma^{\rho}_{\mathrm v}$ also by using the stabilization data at $\mathfrak p$. Moreover on $K_{\mathrm v}$ we have $\hat u^{\rho}_{\mathrm v,\vec T,\vec\theta,(0)} = u^{\rho}_{\vec T,\vec\theta,(0)}$. So we have

$$\begin{aligned}\text{Image of (2.248) } &\subset \Gamma(K_{\mathrm v};(u^{\rho}_{\vec T,\vec\theta,(0)})^*TX\otimes\Lambda^{01})\\ &= \bigoplus_{\mathrm v\in C^0(\mathcal G_{\mathfrak p})}\Gamma(K_{\mathrm v};(\hat u^{\rho}_{\mathrm v,\vec T,\vec\theta,(0)})^*TX\otimes\Lambda^{01}).\end{aligned} \tag{2.249}$$

Definition 2.78. We regard the left hand side of (2.249) as a subspace of

$$\bigoplus_{\mathrm v\in C^0(\mathcal G_{\mathfrak p})}\Gamma(K_{\mathrm v};(\hat u^{\rho}_{\mathrm v,\vec T,\vec\theta,(0)})^*TX\otimes\Lambda^{01})$$

and denote it by

$$\bigoplus_{\mathrm v\in C^0(\mathcal G_{\mathfrak p})} E'_c(\hat u^{\rho}_{\mathrm v,\vec T,\vec\theta,(0)}) \subset \bigoplus_{\mathrm v\in C^0(\mathcal G_{\mathfrak p})} L^2_{m,\delta}(\Sigma_{\mathrm v};(\hat u^{\rho}_{\mathrm v,\vec T,\vec\theta,(0)})^*TX\otimes\Lambda^{01}).$$

We also define

$$\mathcal{E}'_{\mathfrak{p},\mathrm{v},\mathfrak{A}}(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}) = \bigoplus_{c\in\mathfrak{A}} E'_c(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}), \quad \mathcal{E}'_{\mathfrak{p},\mathfrak{A}}(\hat u^{\rho}_{\vec T,\vec\theta,(0)}) = \bigoplus_{\mathrm{v}\in C^0(\mathcal{G}_{\mathfrak{p}})} \mathcal{E}'_{\mathfrak{p},\mathrm{v},\mathfrak{A}}(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}).$$

Remark 2.79. The reason why $E'_c(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}) \ne E_c(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)})$ is as follows. The union of the domains of $\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}$ over v is $\Sigma_{\mathfrak{p}}$. When we identify the core of $\Sigma_{\mathfrak{p}}$ with the core of $\Sigma^{\rho}_{\vec T,\vec\theta}$, we use the additional marked points $\vec w_{\mathfrak{p}}$ included in the stabilization data at $\mathfrak{p}$. We now consider the two diffeomorphisms:

$$K^{\mathrm{obst}}_{\mathrm{v},c} \longrightarrow \text{Core of } \Sigma^{\rho}_{\vec T,\vec\theta} \longrightarrow \text{Core of } \Sigma_{\mathfrak{p}} \tag{2.250}$$

$$K^{\mathrm{obst}}_{\mathrm{v},c} \longrightarrow \text{Core of } \Sigma_{\mathfrak{p}}. \tag{2.251}$$

We note that the diffeomorphism of the second arrow of (2.250) is defined by using the additional marked points $\vec w_{\mathfrak{p}}$. The other arrows are defined by using the additional marked points $\vec w_{\mathfrak{p}_c}$. Therefore in general (2.250) $\ne$ (2.251). The definition of $E'_c(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)})$ uses (2.250) and the definition of $E_c(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)})$ uses (2.251). This phenomenon does not occur in the situation of Section 1. This is because we took $\mathfrak{p} = \mathfrak{p}_c$ in Section 1.

Remark 2.80. In the situation of Section 1 we have $\mathrm{Err}^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)} = 0$ on the core K_{v}. However this is not the case in the current situation . In fact, by definition we have

$$\sum_{\mathrm{v}\in C^0(\mathcal{G}_{\mathfrak{p}})} \mathrm{Err}^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)} = \overline\partial u^{\rho}_{\vec T,\vec\theta,(0)} - \mathfrak{se}^{\rho}_{\vec T,\vec\theta,(0)} \tag{2.252}$$

and

$$\mathfrak{se}^{\rho}_{\vec T,\vec\theta,(0)} = \sum_{c\in\mathfrak{A}} \mathfrak{e}^{\rho}_{c,\vec T,\vec\theta,(0)} = \overline\partial u^{\rho}_{\mathrm{v}} \tag{2.253}$$

on K_{v}. Moreover $u^{\rho}_{\mathrm{v}} = u^{\rho}_{\vec T,\vec\theta,(0)}$ on K_{v}. However (2.252) is nonzero because the way how we identify an element $\mathfrak{e}^{\rho}_{c,\vec T,\vec\theta,(0)} \in E_c$ as a section on K_{v} are different between the case of u^{ρ}_{v} and of $u^{\rho}_{\vec T,\vec\theta,(0)}$. Namely, in (2.252) we regard $\mathfrak{e}^{\rho}_{c,\vec T,\vec\theta,(0)}$ (that is a part of $\mathfrak{se}^{\rho}_{\vec T,\vec\theta,(0)}$) as an element of $E_c(u^{\rho}_{\vec T,\vec\theta,(0)})$. In (2.253) we regard $\mathfrak{e}^{\rho}_{c,\vec T,\vec\theta,(0)}$ as an element of $E_c(u^{\rho}_{\mathrm{v}})$.

We identify $K_{\mathrm{v}} \subset \Sigma^{\rho}_{\vec T,\vec\theta}$ with $K_{\mathrm{v}} \subset \Sigma^{\rho}_{\mathrm{v}}$ by using the stabilization data at $\mathfrak{p}$. Thus $\mathfrak{e}^{\rho}_{c,\vec T,\vec\theta,(0)}$ in (2.252) is also regarded as an element of $E'_c(u^{\rho}_{\mathrm{v}})$. So $\mathrm{Err}^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}$ is nonzero on K_{v} because of $E'_c(u^{\rho}_{\mathrm{v}}) \ne E_c(u^{\rho}_{\mathrm{v}})$. But this difference is of exponentially small. Namely we have the next lemma.

Lemma 2.81. *Put $T_{\min} = \min\{T_{\mathrm{e}} \mid \mathrm{e} \in C^1(\mathcal{G}_{\mathfrak{p}})\}$. Then there exists T_m such that the following inequality holds*

$$\left\Vert \nabla^n_{\rho} \frac{\partial^{|\vec k_T|}}{\partial T^{\vec k_T}} \frac{\partial^{|\vec k_\theta|}}{\partial \theta^{\vec k_\theta}} \mathrm{Err}^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)} \right\Vert_{L^2_{m-|\vec k_T|-|\vec k_\theta|-1,\delta}(\Sigma^{\rho}_{\mathrm{v}})} < C_{7,m} e^{-\delta T_{\min}} \tag{2.254}$$

for $|\vec k_T| + |\vec k_\theta| \le m - 10$ and $T_{\min} > T_m$.

The proof is given later right after the proof of Lemma 2.90.

In Definition 2.78 we defined $\mathcal{E}'_{\mathfrak{p},\mathrm{v},\mathfrak{A}}(\cdot)$ for $\cdot = \hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}$. We next extend it to nearby maps. Let $u'_{\mathrm{v}} : (\Sigma^{\rho}_{\mathrm{v}}, \partial\Sigma^{\rho}_{\mathrm{v}}) \to (X, L)$ be a smooth map which is sufficiently close to $\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}$ in C^{10} sense on K_{v}. We define $\mathcal E'_{\mathfrak p,\mathrm{v},\mathfrak A}(u'_{\mathrm{v}})$ as follows. We identify K_{v} with a subset of $\Sigma^{\rho}_{\vec T,\vec\theta}$ by using the additional marked points $\vec w^{\rho}_{\mathfrak p}$. Take any $u'' : (\Sigma^{\rho}_{\vec T,\vec\theta},\partial\Sigma^{\rho}_{\vec T,\vec\theta}) \to (X,L)$ that coincides with u'_{v} on K_{v} and is enough close to $u_{\mathfrak p}$ so that $E_c(u'') = \bigoplus_{\mathrm{v}'\in C^0(\mathcal G_{\mathfrak p_c})} E_{c,\mathrm{v}'}(u'')$ is defined. We put

$$E_{c,\mathrm{v}}(u'') = \bigoplus_{\substack{\mathrm{v}'\in C^0(\mathcal G_{\mathfrak p_c})\\ \pi(\mathrm{v}')=\mathrm{v}}} E_{c,\mathrm{v}'}(u'').$$

By definition, $E_{c,\mathrm{v}}(u'')$ is independent of u'' but depends only on u'_{v} and is in $\Gamma(K_{\mathrm{v}};(u'_{\mathrm{v}})^*TX\otimes\Lambda^{01})$. Again using the diffeomorphism which is defined by the marked points $\vec w^{\rho}_{\mathfrak p}$ we identify this space as a subspace of $\Gamma(\Sigma^{\rho}_{\mathrm{v}};(u'_{\mathrm{v}})^*TX\otimes\Lambda^{01})$. That is by definition $E'_{c,\mathrm{v}}(u'_{\mathrm{v}})$. (This is the case $\mathrm{v}\in C^0_{\mathrm{d}}(\mathcal G_{\mathfrak p})$. The case of $\mathrm{v}\in C^0_{\mathrm{s}}(\mathcal G_{\mathfrak p})$ is similar.) We put

$$\mathcal E'_{\mathfrak p,\mathrm{v},\mathfrak A}(u'_{\mathrm{v}}) = \sum_{c\in\mathfrak A} E'_{c,\mathrm{v}}(u'_{\mathrm{v}}), \qquad \mathcal E'_{\mathfrak p,\mathfrak A}(u') = \sum_{\mathrm{v}\in C^0(\mathcal G_{\mathfrak p})} \mathcal E'_{\mathfrak p,\mathrm{v},\mathfrak A}(u'_{\mathrm{v}}). \tag{2.255}$$

Let

$$\Pi_{\mathcal E'_{\mathfrak p,\mathfrak A}(u')} : \bigoplus_{\mathrm{v}\in C^0(\mathcal G_{\mathfrak p})} L^2_{m,\delta}(\Sigma^{\rho}_{\mathrm{v}};(u'_{\mathrm{v}})^*TX\otimes\Lambda^{01}) \to \mathcal E'_{\mathfrak p,\mathfrak A}(u')$$

be the L^2-orthogonal projection. We next define its derivation by an element

$$v=(v_{\mathrm{v}}) \in \bigoplus_{\mathrm{v}\in C^0_{\mathrm{o}}(\mathcal G_{\mathfrak p})} \Gamma((\Sigma^{\rho}_{\mathrm{v}},\partial\Sigma^{\rho}_{\mathrm{v}});(u'_{\mathrm{v}})^*TX,(u'_{\mathrm{v}})^*TL) \oplus \bigoplus_{\mathrm{v}\in C^0_{\mathrm{o}}(\mathcal G_{\mathfrak p})} \Gamma(\Sigma^{\rho}_{\mathrm{v}};(u'_{\mathrm{v}})^*TX)$$

by

$$(D_{u'_{\mathrm{v}}}\mathcal E'_{\mathfrak p,\mathfrak A})((A_{\mathrm{v}}),(v_{\mathrm{v}})) = \frac{d}{ds}(\Pi_{\mathcal E'_{\mathfrak p,\mathfrak A}(\mathrm{E}(u'_{\mathrm{v}},sv_{\mathrm{v}}))}(A_{\mathrm{v}}))|_{s=0} \tag{2.256}$$

as in (1.40), where

$$A_{\mathrm{v}} \in L^2_{m,\delta}(\Sigma^{\rho}_{\mathrm{v}};(u'_{\mathrm{v}})^*TX\otimes\Lambda^{01}).$$

We use the operator

$$V \mapsto D_{\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}}\overline\partial(V) - (D_{\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}}\mathcal E'_{\mathfrak p,\mathfrak A})(\mathfrak{se}^{\rho}_{\vec T,\vec\theta,(0)},V) \tag{2.257}$$

as the linearization of the Cauchy-Riemann equiation modulo $\mathcal E'_{\mathfrak A}$. [15]

We recall that

$$L^2_{m+1,\delta}((\Sigma^{\rho},\partial\Sigma^{\rho});(\hat u^{\rho}_{\vec T,\vec\theta,(0)})^*TX,(\hat u^{\rho}_{\vec T,\vec\theta,(0)})^*TL)$$

is the kernel of (2.230) for $\hat u^{\rho}_{\vec T,\vec\theta,(0)} = (\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)})_{\mathrm{v}\in C^0(\mathcal G_{\mathfrak p})}$. The direct sum of (2.257) induces an operator on $L^2_{m+1,\delta}((\Sigma^{\rho},\partial\Sigma^{\rho});(\hat u^{\rho}_{\vec T,\vec\theta,(0)})^*TX,(\hat u^{\rho}_{\vec T,\vec\theta,(0)})^*TL)$ by restriction.

Lemma 2.82. *The sum of the image of the direct sum of the operators (2.257) on*

$$L^2_{m+1,\delta}((\Sigma^{\rho},\partial\Sigma^{\rho});(\hat u^{\rho}_{\vec T,\vec\theta,(0)})^*TX,(\hat u^{\rho}_{\vec T,\vec\theta,(0)})^*TL)$$

[15] Here we consider $\mathcal E_{\mathfrak A}$ and not $\mathcal E'_{\mathfrak A}$. Note we are studying the Cauchy-Riemann equation for $u^{\rho}_{\vec T,\vec\theta,(0)}$. The obsutruction space $\mathcal E'_{\mathfrak A}(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)})$ is sent to $\mathcal E_{\mathfrak A}(u^{\rho}_{\vec T,\vec\theta,(0)})$ by the identification using the stablization data at $\mathfrak p$.

and the subspace $\mathcal{E}'_{\mathfrak{p},\mathfrak{A}}(\hat u^{\rho}_{\vec T,\vec\theta,(0)})$ *is*

$$\bigoplus_{\mathrm{v}\in C^0(\mathcal{G}_{\mathfrak{p}})} L^2_{m,\delta}(\Sigma^{\rho}_{\mathrm{v}};(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)})^*TX\otimes\Lambda^{01})$$

if $\vec T$ *is sufficiently large.*

Proof. This is a consequence of Lemma 2.64. □

Lemma 2.82 is a generalization of Lemma 1.16.

Definition 2.83. The L^2 orthogonal complement of

$$\left(D_{\hat u^{\rho}_{\vec T,\vec\theta,(0)}}\overline{\partial}-(D_{\hat u^{\rho}_{\vec T,\vec\theta,(0)}}\mathcal{E}'_{\mathfrak{p},\mathfrak{A}}))(\mathfrak{se}^{\rho}_{\vec T,\vec\theta,(0)},\cdot)\right)^{-1}(\mathcal{E}'_{\mathfrak{p},\mathfrak{A}}(\hat u^{\rho}_{\vec T,\vec\theta,(0)}))$$

in

$$L^2_{m+1,\delta}((\Sigma^{\rho},\partial\Sigma^{\rho});(\hat u^{\rho}_{\vec T,\vec\theta,(0)})^*TX,(\hat u^{\rho}_{\vec T,\vec\theta,(0)})^*TL)$$

is denoted by $\mathfrak{H}(\rho,\vec T,\vec\theta)$.

We take $\vec T=\vec\infty=(\infty,\dots,\infty)$ and write $\mathfrak{H}(\rho)=\mathfrak{H}(\rho,\vec\infty,\vec\theta_0)$. Then the restriction of (2.257) to $\mathfrak{H}(\rho)$ induces an isomorphism to

$$\bigoplus_{\mathrm{v}\in C^0(\mathcal{G}_{\mathfrak{p}})} L^2_{m,\delta}(\Sigma^{\rho}_{\mathrm{v}};(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)})^*TX\otimes\Lambda^{01})/\mathcal{E}'_{\mathfrak{p},\mathfrak{A}}(\hat u^{\rho}_{\vec T,\vec\theta,(0)})$$

for sufficiently large $\vec T$.

Definition 2.84. We define $V^{\rho}_{\vec T,\vec\theta,\mathrm{v},(1)}$ for $\mathrm{v}\in C^0(\mathcal{G}_{\mathfrak{p}})$ and $\Delta p^{\rho}_{\mathrm{e},\vec T,\vec\theta,(1)}$ for $\mathrm{e}\in C^1(\mathcal{G}_{\mathfrak{p}})$ so that $((V^{\rho}_{\vec T,\vec\theta,\mathrm{v},(1)})_{\mathrm{v}},(\Delta p^{\rho}_{\mathrm{e},\vec T,\vec\theta,(1)})_{\mathrm{e}})\in\mathfrak{H}(\rho)$ is the unique element such that

$$\begin{aligned}D_{\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}}\overline{\partial}(V^{\rho}_{\vec T,\vec\theta,\mathrm{v},(1)})-(D_{\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}}\mathcal{E}'_{\mathfrak{p},\mathfrak{A}})(\mathfrak{se}^{\rho}_{\vec T,\vec\theta,(0)},V^{\rho}_{\vec T,\vec\theta,\mathrm{v},(1)})\\ +\mathrm{Err}^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}\in\mathcal{E}'_{\mathfrak{p},\mathfrak{A}}(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)})\end{aligned} \tag{2.258}$$

and

$$\lim_{\tau_{\mathrm{e}}\to\pm\infty}V^{\rho}_{\vec T,\vec\theta,\mathrm{v},(1)}(\tau_{\mathrm{e}},t_{\mathrm{e}})=\Delta p^{\rho}_{\mathrm{e},\vec T,\vec\theta,(1)}, \tag{2.259}$$

where $\pm\infty=+\infty$ if e is outgoing and $=-\infty$ if e is incoming.

Step 1-2:

Definition 2.85. We define $u^{\rho}_{\vec T,\vec\theta,(1)}(z)$ as follows. (Here E is the map as in (1.30).)

(1) If $z\in K_{\mathrm{v}}$, we put

$$u^{\rho}_{\vec T,\vec\theta,(1)}(z)=\mathrm{E}(u^{\rho}_{\vec T,\vec\theta,(0)}(z),V^{\rho}_{\vec T,\vec\theta,\mathrm{v},(1)}(z)). \tag{2.260}$$

(2) If $z=(\tau_{\mathrm{e}},t_{\mathrm{e}})\in[-5T_{\mathrm{e}},5T_{\mathrm{e}}]\times[0,1]$ or S^1, we put

$$\begin{aligned}u^{\rho}_{\vec T,\vec\theta,(1)}(\tau_{\mathrm{e}},t_{\mathrm{e}})=&\chi^{\leftarrow}_{\mathrm{v}_{\leftarrow}(\mathrm{e}),\mathcal{B}}(\tau_{\mathrm{e}},t_{\mathrm{e}})(V^{\rho}_{\vec T,\vec\theta,\mathrm{v}_{\leftarrow}(\mathrm{e}),(1)}(\tau_{\mathrm{e}},t_{\mathrm{e}})-\Delta p^{\rho}_{\mathrm{e},\vec T,\vec\theta,(1)})\\&+\chi^{\rightarrow}_{\mathrm{v}_{\rightarrow}(\mathrm{e}),\mathcal{A}}(\tau_{\mathrm{e}},t_{\mathrm{e}})(V^{\rho}_{\vec T,\vec\theta,\mathrm{v}_{\rightarrow}(\mathrm{e}),(1)}(\tau_{\mathrm{e}},t_{\mathrm{e}})-\Delta p^{\rho}_{\mathrm{e},\vec T,\vec\theta,(1)})\\&+u^{\rho}_{\vec T,\vec\theta,(0)}(\tau_{\mathrm{e}},t_{\mathrm{e}})+\Delta p^{\rho}_{\mathrm{e},\vec T,\vec\theta,(1)}.\end{aligned} \tag{2.261}$$

Step 1-3: We define:

$$\mathfrak{e}^{\rho}_{1,T,(1)} = \Pi_{\mathcal{E}_{\mathfrak{p},\mathfrak{A}}(\mathrm{E}(u^{\rho}_{\vec T,\vec\theta,(0)},V^{\rho}_{\vec T,\vec\theta,\mathrm{v},(1)})}(\overline{\partial}\mathrm{E}(u^{\rho}_{\vec T,\vec\theta,(0)},V^{\rho}_{\vec T,\vec\theta,\mathrm{v},(1)})) \tag{2.262}$$

and

$$\mathfrak{se}^{\rho}_{\vec T,\vec\theta,(1)} = \mathfrak{e}^{\rho}_{\vec T,\vec\theta,(0)} + \mathfrak{e}^{\rho}_{\vec T,\vec\theta,(1)}. \tag{2.263}$$

Step 1-4: We take $0<\mu<1$ and fix it throughout the proof of this subsection.

Definition 2.86. We put

$$\mathrm{Err}^{\rho}_{\mathrm{v},\vec T,\vec\theta,(1)} = \begin{cases} \chi^{\leftarrow}_{\mathrm{e},\mathcal{X}}\overline{\partial}u^{\rho}_{\vec T,\vec\theta,(1)} & \text{on the e-th neck if e is outgoing} \\ \chi^{\rightarrow}_{\mathrm{e},\mathcal{X}}\overline{\partial}u^{\rho}_{\vec T,\vec\theta,(1)} & \text{on the e-th neck if e is incoming} \\ \overline{\partial}u^{\rho}_{\vec T,\vec\theta,(0)} - \mathfrak{se}^{\rho}_{\vec T,\vec\theta,(1)} & \text{on } K_{\mathrm{v}}. \end{cases} \tag{2.264}$$

We extend them by 0 outside a compact set and will regard them as elements of the function space $L^2_{m,\delta}(\Sigma^{\rho}_{\mathrm{v}};(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(1)})^*TX\otimes\Lambda^{01})$, where $\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(1)}$ will be defined in the next step.

We put $p^{\rho}_{\mathrm{e},\vec T,\vec\theta,(1)} = p^{\rho}_{\mathrm{e},\vec T,\vec\theta,(0)} + \Delta p^{\rho}_{\mathrm{e},\vec T,\vec\theta,(1)}$.

We now come back to Step 2-1 and continue inductively on κ.

The main estimate of those objects are the next lemma. We put $R_{(\mathrm{v},\mathrm{e})} = 5T_{\mathrm{e}}+1$ and $\vec R = (R_{(\mathrm{v},\mathrm{e})})$.

Proposition 2.87. *There exist $T_m, C_{8,m}, C_{9,m}, C_{10,m}, \epsilon_{5,m} > 0$ and $0<\mu<1$ such that the following inequalities hold if $T_{\mathrm{e}} > T_m$ for all e. We put $\vec T = (T_e; e\in C^1(\mathcal{G}_{\mathfrak{p}}))$ and $T_{\min} = \min\{T_{\mathrm{e}} \mid \mathrm{e}\in C^1(\mathcal{G}_{\mathfrak{p}})\}$.*

$$\left\|\left((V^{\rho}_{\vec T,\vec\theta,\mathrm{v},(\kappa)}),(\Delta p^{\rho}_{\mathrm{e},\vec T,\vec\theta,(\kappa)})\right)\right\|_{L^2_{m+1,\delta}(\Sigma^{\rho}_{\mathrm{v}})} < C_{8,m}\mu^{\kappa-1}e^{-\delta T_{\min}}, \tag{2.265}$$

$$\left\|(\Delta p^{\rho}_{\mathrm{e},\vec T,\vec\theta,(\kappa)})\right\| < C_{8,m}\mu^{\kappa-1}e^{-\delta T_{\min}}, \tag{2.266}$$

$$\left\|u^{\rho}_{\vec T,\vec\theta,(\kappa)} - u^{\rho}_{\vec T,\vec\theta,(0)}\right\|_{L^2_{m+1,\delta}(K_{\mathrm{v}}^{+\vec R})} < C_{9,m}e^{-\delta T_{\min}}, \tag{2.267}$$

$$\left\|\mathrm{Err}^{\rho}_{\mathrm{v},\vec T,\vec\theta,(\kappa)}\right\|_{L^2_{m,\delta}(\Sigma^{\rho}_{\mathrm{v}})} < C_{10,m}\epsilon_{5,m}\mu^{\kappa}e^{-\delta T_{\min}}, \tag{2.268}$$

$$\left\|\mathfrak{e}^{\rho}_{\vec T,\vec\theta,(\kappa)}\right\|_{L^2_{m}(K_{\mathrm{v}}^{\mathrm{obst}})} < C_{10,m}\mu^{\kappa-1}e^{-\delta T_{\min}}, \tag{2.269}$$

where we assume $\kappa\ge 1$ in (2.269).

Proof. The proof is the same as the discussion in Subsection 1.3 and so is omitted.[16] □

(2.265) implies that the limit of $u^{\rho}_{\vec T,\vec\theta,(\kappa)}$ converges as κ goes to ∞ after C^k topology for each k if $T_{\mathrm{e}} > T_{k+10}$ for all e. We define

$$\mathrm{Glu}_{\vec T,\vec\theta}(\rho) = \lim_{\kappa\to\infty} u^{\rho}_{\vec T,\vec\theta,(\kappa)} = u^{\rho}_{\vec T,\vec\theta}. \tag{2.270}$$

[16] Actually we need some new argument for the case $\kappa = 0$ of (2.268). We will discuss it later during the proof of Lemma 2.89.

(2.268) and (2.269) imply

$$\overline{\partial} u^{\rho}_{\vec T,\vec\theta} = \sum_{\kappa=0}^{\infty} \mathfrak{e}^{\rho}_{\vec T,\vec\theta,(\kappa)} \in \mathcal{E}_{\mathfrak{A}}(\overline{\partial} u^{\rho}_{\vec T,\vec\theta}).$$

Therefore

$$u^{\rho}_{\vec T,\vec\theta} \in \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};(\vec T^{\rm o},\vec T^{\rm c},\vec\theta))_{\epsilon_2,\vec T_0}.$$

We thus have defined $\mathrm{Glu}_{\vec T,\vec\theta}$.

We next prove Theorem 2.72. The main part of the proof is the next lemma.

Proposition 2.88. *There exist $T_m, C_{11,m}, C_{12,m}, C_{13,m}, C_{14,m}, \epsilon_{2,m} > 0$ and $0 < \mu < 1$ such that the following inequalities hold if $T_{\rm e} > T_m$ for all* e.

Let $\mathrm{e}_0 \in C^1_{\rm o}(\mathcal{G}_{\mathfrak{p}})$. Then for each $\vec k_T$, $\vec k_\theta$ we have

$$\left\Vert \nabla_{\rho}^n \frac{\partial^{|\vec k_T|}}{\partial T^{\vec k_T}}\frac{\partial^{|\vec k_\theta|}}{\partial \theta^{\vec k_\theta}}\frac{\partial}{\partial T_{\mathrm{e}_0}}((V^{\rho}_{\vec T,\vec\theta,{\rm v},(\kappa)}),(\Delta p^{\rho}_{{\rm e},\vec T,\vec\theta,(\kappa)}))\right\Vert_{L^2_{m+1-|\vec k_T|-|\vec k_\theta|-1,\delta}(\Sigma_{\rm v}^{\rho})} \tag{2.271}$$
$$< C_{11,m}\mu^{\kappa-1}e^{-\delta T_{\mathrm{e}_0}},$$

$$\left\Vert \nabla_{\rho}^n \frac{\partial^{|\vec k_T|}}{\partial T^{\vec k_T}}\frac{\partial^{|\vec k_\theta|}}{\partial \theta^{\vec k_\theta}}\frac{\partial}{\partial T_{\mathrm{e}_0}}(\Delta p^{\rho}_{{\rm e},\vec T,\vec\theta,(\kappa)})\right\Vert < C_{11,m}\mu^{\kappa-1}e^{-\delta T_{\mathrm{e}_0}}, \tag{2.272}$$

$$\left\Vert \nabla_{\rho}^n \frac{\partial^{|\vec k_T|}}{\partial T^{\vec k_T}}\frac{\partial^{|\vec k_\theta|}}{\partial \theta^{\vec k_\theta}}\frac{\partial}{\partial T_{\mathrm{e}_0}}u^{\rho}_{\vec T,\vec\theta,(\kappa)}\right\Vert_{L^2_{m+1-|\vec k_T|-|\vec k_\theta|-1,\delta}(K_{\rm v}^{+\vec R})} < C_{12,m}e^{-\delta T_{\mathrm{e}_0}}, \tag{2.273}$$

$$\left\Vert \nabla_{\rho}^n \frac{\partial^{|\vec k_T|}}{\partial T^{\vec k_T}}\frac{\partial^{|\vec k_\theta|}}{\partial \theta^{\vec k_\theta}}\frac{\partial}{\partial T_{\mathrm{e}_0}}\mathrm{Err}^{\rho}_{{\rm v},\vec T,\vec\theta,(\kappa)}\right\Vert_{L^2_{m-|\vec k_T|-|\vec k_\theta|-1,\delta}(\Sigma_{\rm v}^{\rho})} \tag{2.274}$$
$$< C_{13,m}\epsilon_{6,m}\mu^{\kappa}e^{-\delta T_{\mathrm{e}_0}},$$

$$\left\Vert \nabla_{\rho}^n \frac{\partial^{|\vec k_T|}}{\partial T^{\vec k_T}}\frac{\partial^{|\vec k_\theta|}}{\partial \theta^{\vec k_\theta}}\frac{\partial}{\partial T_{\mathrm{e}_0}}\mathfrak{e}^{\rho}_{\vec T,\vec\theta,(\kappa)}\right\Vert_{L^2_{m-|\vec k_T|-|\vec k_\theta|-1}(K_{\rm v}^{\rm obst})} < C_{14,m}\mu^{\kappa-1}e^{-\delta T_{\mathrm{e}_0}}, \tag{2.275}$$

for $|\vec k_T| + |\vec k_\theta| + n < m - 11$.

Let $\mathrm{e}_0 \in C^1_{\rm c}(\mathcal{G}_{\mathfrak{p}})$. Then the same inequalities as above hold if we replace $\frac{\partial}{\partial T_{\mathrm{e}_0}}$ by $\frac{\partial}{\partial \theta_{\mathrm{e}_0}}$.

Proposition 2.88 ⇒ Theorem 2.72. Note if $k_{\mathrm{e}_0} \ne 0$ or $\theta_{\mathrm{e}_0} \ne 0$ then

$$\vec k_T \cdot \vec T + \vec k_\theta \cdot \vec T^{\rm c} \le 2k \max\{T_{\rm e} \mid k_{T,{\rm e}} \ne 0, \text{ or } k_{\theta,{\rm e}} \ne 0\}.$$

It is then easy to see that Proposition 2.88 implies Theorem 2.72 by putting $\delta' = \delta/2k$. □

Proof of Proposition 2.88. The proof is mostly the same as the argument of Subsection 1.4. The new part is the proof of the next lemma.

Lemma 2.89. *Let $\mathrm{e}_0 \in C^1_{\rm c}(\mathcal{G}_{\mathfrak{p}})$. We have*

$$\left\Vert \nabla_{\rho}^n \frac{\partial^{|\vec k_T|}}{\partial T^{\vec k_T}}\frac{\partial^{|\vec k_\theta|}}{\partial \theta^{\vec k_\theta}}\frac{\partial}{\partial T_{\mathrm{e}_0}}\mathrm{Err}^{\rho}_{{\rm v},\vec T,\vec\theta,(0)}\right\Vert_{L^2_{m-|\vec k_T|-|\vec k_\theta|-1,\delta}(\Sigma_{\rm v}^{\rho})} < C_{15,m}e^{-\delta T_{\mathrm{e}_0}} \tag{2.276}$$

and

$$\left\| \nabla_\rho^n \frac{\partial^{|\vec k_T|}}{\partial T^{\vec k_T}} \frac{\partial^{|\vec k_\theta|}}{\partial \theta^{\vec k_\theta}} \frac{\partial}{\partial \theta_{\mathrm{e}_0}} \mathrm{Err}^\rho_{\mathrm{v},\vec T,\vec\theta,(0)} \right\|_{L^2_{m-|\vec k_T|-|\vec k_\theta|-1,\delta}(\Sigma^\rho_{\mathrm{v}})} < C_{15,m} e^{-\delta T_{\mathrm{e}_0}}. \tag{2.277}$$

Proof. We recall (2.245),

$$\mathrm{Err}^\rho_{\mathrm{v},\vec T,\vec\theta,(0)} = \begin{cases} \chi^{\leftarrow}_{\mathrm{e},\mathcal X}\overline\partial u^\rho_{\vec T,\vec\theta,(0)} & \text{on the e-th neck if e is outgoing} \\ \chi^{\rightarrow}_{\mathrm{e},\mathcal X}\overline\partial u^\rho_{\vec T,\vec\theta,(0)} & \text{on the e-th neck if e is incoming} \\ \overline\partial u^\rho_{\vec T,\vec\theta,(0)} - \mathfrak{se}^\rho_{\vec T,\vec\theta,(0)} & \text{on } K_{\mathrm{v}}. \end{cases} \tag{2.278}$$

We first estimate $\mathrm{Err}^\rho_{\mathrm{v},\vec T,\vec\theta,(0)}$ on the neck region. Let $\mathrm{e} \in C^1_{\mathrm{c}}(\mathcal G_{\mathfrak p})$ is an outgoing edge of v. Let v′ be the other vertex of e. We have

$$\begin{aligned} &\mathrm{Err}^\rho_{\mathrm{v},\vec T,\vec\theta,(0)}(\tau'_{\mathrm{e}},t'_{\mathrm{e}}) \\ &= (1-\chi(\tau'_{\mathrm{e}}-5T_{\mathrm{e}}))\overline\partial\Big(p^\rho_{\mathrm{e}} + (1-\chi(\tau'_{\mathrm{e}}-6T_{\mathrm{e}}))(u^\rho_{\mathrm{v}}(\tau'_{\mathrm{e}},t'_{\mathrm{e}}) - p^\rho_{\mathrm{e}}) \\ &\qquad + \chi(\tau'_{\mathrm{e}}-4T_{\mathrm{e}})(u^\rho_{\mathrm{v}'}(\tau'_{\mathrm{e}}-10T_{\mathrm{e}},t'_{\mathrm{e}}+\theta_{\mathrm{e}}) - p^\rho_{\mathrm{e}})\Big). \end{aligned} \tag{2.279}$$

Note that we use the coordinates $(\tau'_{\mathrm{e}},t'_{\mathrm{e}})$ for u^ρ_{v} and $(\tau''_{\mathrm{e}},t''_{\mathrm{e}})$ for $u^\rho_{\mathrm{v}'}$. (See (2.218), (2.219).) The function χ is as in (1.115).

If $\mathrm{e}_0 \ne \mathrm{e}$, then $\partial/\partial T_{\mathrm{e}_0}$ or $\partial/\partial\theta_{\mathrm{e}_0}$ of (2.279) is zero.

Let us study $\partial/\partial T_{\mathrm{e}}$ or $\partial/\partial\theta_{\mathrm{e}}$ of (2.279) in case $\mathrm{e}_0 = \mathrm{e}$. We apply $\partial/\partial\theta_{\mathrm{e}}$ to the third line of (2.279) to obtain

$$\begin{aligned} &(1-\chi(\tau'_{\mathrm{e}}-5T_{\mathrm{e}}))\frac{\partial}{\partial\theta_{\mathrm{e}}}\overline\partial\left(\chi(\tau'_{\mathrm{e}}-4T_{\mathrm{e}})u^\rho_{\mathrm{v}'}(\tau'_{\mathrm{e}}-10T_{\mathrm{e}},t'_{\mathrm{e}}+\theta_{\mathrm{e}})\right) \\ &= (1-\chi(\tau'_{\mathrm{e}}-5T_{\mathrm{e}}))\chi(\tau'_{\mathrm{e}}-4T_{\mathrm{e}})\overline\partial\left(\frac{\partial}{\partial t'_{\mathrm{e}}}u^\rho_{\mathrm{v}'}(\tau'_{\mathrm{e}}-10T_{\mathrm{e}},t'_{\mathrm{e}}+\theta_{\mathrm{e}})\right). \end{aligned} \tag{2.280}$$

Support of (2.280) is in the domain $4T_{\mathrm{e}}-1 \le \tau'_{\mathrm{e}} \le 5T_{\mathrm{e}}+1$ that is $-6T_{\mathrm{e}}-1 \le \tau''_{\mathrm{e}} \le -5T_{\mathrm{e}}+1$. There the C^m norm of $u^\rho_{\mathrm{v}'}$ is estimated as

$$\|u^\rho_{\mathrm{v}'}\|_{C^m([-6T_{\mathrm{e}}-1,-5T_{\mathrm{e}}+1))} \le C_{11,m}\, e^{-5T_{\mathrm{e}}\delta_1}.$$

On the other hand, the weight function $e_{\mathrm{v},\delta}$ given in (2.226) is estimated by $e^{5T_{\mathrm{e}}\delta}$ on the support. (See (2.226).) Therefore this term has the required estimated. (Note $\delta < \delta_1/10$.) The other term or other case of the estimate on the neck region is similar.

We next estimate $\mathrm{Err}^\rho_{\mathrm{v},\vec T,\vec\theta,(0)}$ on the core. As we explained in Remark 2.80 this is nonzero because of the difference of the parametrization of the core. So to study it, we need to discuss the dependence of the parametrization of the core on the coordinate at infinity. Proposition 2.23, Coroolary 2.24 and Lemma 2.26 give the estimate we need to study.

We consider $\mathfrak p_c$ and the obstruction bundle data $\mathfrak E_{\mathfrak p_c}$ there. Let $\mathcal G_c$ be the combinatorial type of $\mathfrak p_c$. Note $\mathfrak p \in \mathfrak W_{\mathfrak p_c}$ and $(\mathfrak x_{\mathfrak p} \cup \vec w^{\mathfrak p}_c, u_{\mathfrak p})$ is $\epsilon_{\mathfrak p_c}$-close to $\mathfrak p_c$. Let $\mathcal G(\mathfrak p,c)$ be the combinatorial type of $(\mathfrak x_{\mathfrak p} \cup \vec w^{\mathfrak p}_c, u_{\mathfrak p})$. By Definition 2.38 (1) we have $\mathcal G_c \succ \mathcal G(\mathfrak p,c)$. Let

$$\mathfrak x_{\mathfrak p} \cup \vec w^{\mathfrak p}_c = \overline\Phi(\mathfrak y_1,\vec T_1,\vec\theta_1).$$

Note that the singular point of $\mathfrak{p}$ corresponds one to one to the edges e of $\mathfrak{y}_1$ such that $T_{1,\mathrm{e}} = \infty$.

For each $\mathrm{v}' \in C^0(\mathcal{G}_{\mathfrak{p}_c})$, we denote the corresponding core of $\Sigma_{\mathfrak{p}_c}$ by $K^c_{\mathrm{v}'}$. We may also regard

$$K^c_{\mathrm{v}'} \subset \Sigma_{\mathfrak{p}}.$$

Let $\pi : \mathcal{G}_{\mathfrak{p}_c} \to \mathcal{G}_{\mathfrak{p}}$ be a map shrinking the edges e with $T_{\mathrm{e}} \neq \infty$. We put $\mathrm{v} = \pi(\mathrm{v}')$. Then there exists $\vec{R}$ such that

$$K^c_{\mathrm{v}'} \subset K^{+\vec{R}}_{\mathrm{v}}. \tag{2.281}$$

Here the right hand side is the core of the coordinate at infinity of $\mathfrak{p}$, that is included in the stabilization data of $\mathfrak{p}$. The inclusion (2.281) is obtained from the map $\mathfrak{v}_{\xi,\mathfrak{y},\vec{T},\vec{\theta}}$ appearing in Lemma 2.26 as follows.

We put

$$\{\mathrm{v}(i) \mid i = 1, \dots, n_{c,\mathrm{v}}\} = \{\mathrm{v}' \in C^0(\mathcal{G}_{\mathfrak{p}_c}) \mid \pi(\mathrm{v}') = \mathrm{v}\}.$$

We consider the union

$$K^c_{\mathrm{v},0} = \bigcup_{i=1}^{n_{c,\mathrm{v}}} K^{\mathrm{obst}}_{\mathrm{v}(i)} \subset \Sigma_{\mathfrak{p}_c}.$$

We consider $\Sigma^{\rho}_{\vec{T},\vec{\theta}}$ that is a domain of $u^{\rho}_{\vec{T},\vec{\theta},(0)}$. The parameter ρ includes both the marked points $\vec{w}^{\rho}_c$ and $\vec{w}^{\rho}_{\mathfrak{p}}$. By forgetting $\vec{w}^{\rho}_{\mathfrak{p}}$ we have an embedding

$$\mathfrak{v}_{c,\mathrm{v}(i),\rho,\vec{T},\vec{\theta}} : K^c_{\mathrm{v}(i)} \to \Sigma^{\rho}_{\vec{T},\vec{\theta}}.$$

(Here the parameter $\vec{w}^{\rho}_{\mathfrak{p}}$ (that is a part of ρ) plays the role of the parameter $\xi \in Q$ in Lemma 2.26.)

By forgetting $\vec{w}^{\rho}_c$ we have an embedding

$$\mathfrak{v}_{\mathfrak{p},\mathrm{v},\rho,\vec{T},\vec{\theta}} : K_{\mathrm{v}} \to \Sigma^{\rho}_{\vec{T},\vec{\theta}}.$$

We consider $K^{\mathrm{obst}}_{\mathrm{v}(i)} \subset K^c_{\mathrm{v}(i)}$ that is a compact set we fixed as a part of the obstruction bundle data centered at $\mathfrak{p}_c$. By Remark 2.77, we may assume

$$\mathfrak{v}_{c,\mathrm{v}(i),\rho,\vec{T},\vec{\theta}}(K^{\mathrm{obst}}_{\mathrm{v}(i)}) \subset \mathfrak{v}_{\mathfrak{p},\mathrm{v},\rho,\vec{T},\vec{\theta}}(K_{\mathrm{v}}).$$

Therefore taking union over $i = 1, \dots, n_{c,\mathrm{v}}$ we obtain

$$\mathfrak{v}_{(\mathfrak{p},c),\mathrm{v},\rho,\vec{T},\vec{\theta}} := \mathfrak{v}^{-1}_{\mathfrak{p},\mathrm{v},\rho,\vec{T},\vec{\theta}} \circ \left(\coprod_{i=1}^{n_{c,\mathrm{v}}} \mathfrak{v}_{c,\mathrm{v}(i),\rho,\vec{T},\vec{\theta}} \right) : K^c_{\mathrm{v},0} \to K_{\mathrm{v}}. \tag{2.282}$$

We denote this map by

$$\mathrm{Res}(\mathfrak{v}_{(\mathfrak{p},c),\mathrm{v},\rho,\vec{T},\vec{\theta}}) \in C^m(K^c_{\mathrm{v},0}, K_{\mathrm{v}}).$$

We can estimate it by using Lemma 2.26 that is a family version of Proposition 2.23 and Corollary 2.24. (See Lemma 2.90 below.)

We next describe the way how $\mathfrak{v}_{(\mathfrak{p},c),\mathrm{v},\rho,\vec{T},\vec{\theta}}$ and its estimate are related to the estimate of $\mathrm{Err}^{\rho}_{\mathrm{v},\vec{T},\vec{\theta},(0)}$. We first recall that

$$\overline{\partial} u^{\rho}_{\mathrm{v}} \in \bigoplus_{c \in \mathfrak{A}} E_{c,\mathrm{v}}$$

by assumption. We denote by $\mathfrak{e}^{\rho}_{c,\vec{T},\vec{\theta},(0)}$ the sum of its $E_{c,\mathrm{v}}$ components over v. It is actually independent of $\vec{T},\vec{\theta}$. So we write it $\mathfrak{e}^{\rho}_{c,(0)}$ here. We remark that we identify

$$E_{c,\mathrm{v}} \subset \Gamma_0(K_{\mathrm{v}}; (u^{\rho}_{\mathrm{v}})^* TX \otimes \Lambda^{01})$$

using the obstruction bundle data centered at $\mathfrak{p}_c$. Here $K_{\rm v} \subset \Sigma_{\mathfrak{y}}$. (Note that the combinatorial type of $\mathfrak{y}$ is the same as $\mathfrak{p}$.)

In (2.242), we used $u^{\rho}_{\rm v}$ to obtain a map

$$u^{\rho}_{\vec T,\vec\theta,(0)} : (\Sigma^{\rho}_{\vec T,\vec\theta}, \partial\Sigma^{\rho}_{\vec T,\vec\theta}) \to (X,L).$$

Moreover $u^{\rho}_{\rm v} = u^{\rho}_{\vec T,\vec\theta,(0)}$ on $K_{\rm v}$. However

$$E_{c,{\rm v}}(u^{\rho}_{\rm v}) \ne E_{c,{\rm v}}(u^{\rho}_{\vec T,\vec\theta,(0)}),$$

as subsets of

$$\Gamma(K_{\rm v}; (u^{\rho}_{\rm v})^*TX \otimes \Lambda^{01}) = \Gamma(K_{\rm v}; (u^{\rho}_{\vec T,\vec\theta,(0)})^*TX \otimes \Lambda^{01}).$$

In fact, $E_{c,{\rm v}}(u^{\rho}_{\vec T,\vec\theta,(0)})$ is defined by the diffeomorphism $\mathfrak{v}_{(\mathfrak{p},c),{\rm v},\rho,\vec T,\vec\theta}$ and $E_{c,{\rm v}}(u^{\rho}_{\rm v})$ is defined by the diffeomorphism $\mathfrak{v}_{(\mathfrak{p},c),{\rm v},\rho,\vec\infty}$.

Therefore, by definition, ${\rm Err}^{\rho}_{{\rm v},\vec T,\vec\theta,(0)}$ on $K_{\rm v}$ is

$$\overline\partial u^{\rho}_{\vec T,\vec\theta,(0)} - \sum_c \mathfrak{e}^{\rho}_{c,(0)} = \sum_c \left(\mathfrak{e}^{\rho,1}_{c,(0)} - \mathfrak{e}^{\rho,2}_{c,(0)}\right), \tag{2.283}$$

where $\mathfrak{e}^{\rho,1}_{c,(0)} \in \bigoplus_{{\rm v}\in C^0(\mathcal{G}_{\mathfrak{p}})} E_{c,{\rm v}}(u^{\rho}_{\vec T,\vec\theta,(0)})$ and $\mathfrak{e}^{\rho,2}_{c,(0)} \in \bigoplus_{{\rm v}\in C^0(\mathcal{G}_{\mathfrak{p}})} E_{c,{\rm v}}(u^{\rho}_{\rm v})$ are defined as follows:

$$\begin{aligned}\mathfrak{e}^{\rho,1}_{c,(0)}(\mathfrak{v}_{(\mathfrak{p},c),{\rm v},\rho,\vec T,\vec\theta}(z)) &= {\rm Pal}_{u_{\mathfrak{p}_c,{\rm v}}(z),u^{\rho}_{\rm v}(\mathfrak{v}_{(\mathfrak{p},c),{\rm v},\rho,\vec T,\vec\theta}(z))}(\mathfrak{e}^{\rho}_{c,(0)}),\\ \mathfrak{e}^{\rho,2}_{c,(0)}(\mathfrak{v}_{(\mathfrak{p},c),{\rm v},\rho,\vec\infty}(z)) &= {\rm Pal}_{u_{\mathfrak{p}_c,{\rm v}}(z),u^{\rho}_{\rm v}(\mathfrak{v}_{(\mathfrak{p},c),{\rm v},\rho,\vec\infty}(z))}(\mathfrak{e}^{\rho}_{c,(0)}).\end{aligned} \tag{2.284}$$

Thus Lemma 2.90 below implies

$$\left\Vert {\rm Err}^{\rho}_{{\rm v},\vec T,\vec\theta,(0)}\right\Vert_{L^2_{m,\delta}(K_{\rm v})} < C_{8,m}\epsilon_{1,m}e^{-\delta T_{\min}}.$$

This is the case $\kappa = 0$ of (2.268) on $K_{\rm v}$.

Proposition 2.23 implies the estimate (2.276) and (2.277) on $K_{\rm v}$. The proof of Lemma 2.89 is complete assuming Lemma 2.90. □

Lemma 2.90. *There exist $C_{15,k}$, T_k such that for each ${\rm e} \in C^1_{\rm c}(\mathcal{G}_{\mathfrak{p}})$ we have:*

$$\begin{aligned}\left\Vert \nabla^n_{\mathfrak{y}_2}\frac{\partial^{|\vec k_T|}}{\partial T_2^{\vec k_T}}\frac{\partial^{|\vec k_\theta|}}{\partial \theta_2^{\vec k_\theta}}\frac{\partial}{\partial T_{2,{\rm e}_0}}(\mathfrak{v}_{(\mathfrak{p},c),{\rm v},\rho,\vec T,\vec\theta})\right\Vert_{C^k} &< C_{15,k}e^{-\delta_2 T_{2,{\rm e}_0}},\\ \left\Vert \nabla^n_{\mathfrak{y}_2}\frac{\partial^{|\vec k_T|}}{\partial T_2^{\vec k_T}}\frac{\partial^{|\vec k_\theta|}}{\partial \theta_2^{\vec k_\theta}}\frac{\partial}{\partial \theta_{2,{\rm e}_0}}(\mathfrak{v}_{(\mathfrak{p},c),{\rm v},\rho,\vec T,\vec\theta})\right\Vert_{C^k} &< C_{15,k}e^{-\delta_2 T_{2,{\rm e}_0}},\end{aligned} \tag{2.285}$$

whenever $T_{2,{\rm e}}$ is greater than T_k and $|\vec k_T| + |\vec k_\theta| + n \le k$.

The first inequality holds for ${\rm e} \in C^1_{\rm o}(\mathcal{G}_{\mathfrak{p}})$ also.

Proof. It suffices to prove the same estimate for $\mathfrak{v}_{\mathfrak{p},{\rm v},\rho,\vec T,\vec\theta}$ and $\mathfrak{v}_{c,{\rm v}(i),\rho,\vec T,\vec\theta}$. Note $\rho \in V_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\epsilon_0)$ contains various data. We use only a part of such a data. We blow recall the parametre space which contains only the data we use below.

Let $\mathfrak{V}(\mathfrak{x}_{\mathfrak{p}} \cup \vec w^{\mathfrak{p}}_c)$ be a neighborhood of $\mathfrak{x}_{\mathfrak{p}} \cup \vec w^{\mathfrak{p}}_c$ in the stratum of the Deligne-Mumford moduli space that consists of elements of the same combinatorial type as $\mathfrak{x}_{\mathfrak{p}} \cup \vec w^{\mathfrak{p}}_c$. We also take $\mathfrak{V}(\mathfrak{x}_{\mathfrak{p}} \cup \vec w_{\mathfrak{p}})$ and $\mathfrak{V}(\mathfrak{x}_{\mathfrak{p}} \cup \vec w^{\mathfrak{p}}_c \cup \vec w_{\mathfrak{p}})$ that are neighborhoods in the stratum of the Deligne-Mumford moduli space of $\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}_{\mathfrak{p}}$ and $\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}^{\mathfrak{p}}_c \cup \vec{w}_{\mathfrak{p}}$, respectively.

We can take those three neighborhoods so that there exist Q_1 and Q_2 such that

$$Q_1 \times \mathfrak{V}(\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}^{\mathfrak{p}}_c) \cong \mathfrak{V}(\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}^{\mathfrak{p}}_c \cup \vec{w}_{\mathfrak{p}}) \cong Q_2 \times \mathfrak{V}(\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}_{\mathfrak{p}}) \tag{2.286}$$

and that the isomorphisms in (2.286) is compatible with the forgetful maps

$$\mathfrak{V}(\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}^{\mathfrak{p}}_c \cup \vec{w}_{\mathfrak{p}}) \to \mathfrak{V}(\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}^{\mathfrak{p}}_c)$$

and

$$\mathfrak{V}(\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}^{\mathfrak{p}}_c \cup \vec{w}_{\mathfrak{p}}) \to \mathfrak{V}(\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}_{\mathfrak{p}}).$$

We consider the univeral family

$$\mathfrak{M}(\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}^{\mathfrak{p}}_c \cup \vec{w}_{\mathfrak{p}}) \to \mathfrak{V}(\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}^{\mathfrak{p}}_c \cup \vec{w}_{\mathfrak{p}}).$$

Together with other data it gives a coordinate at infinity. We take any of them.

Using (2.286), this coordinate at infinity of $\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}^{\mathfrak{p}}_c \cup \vec{w}_{\mathfrak{p}}$ induces a Q_1-parametrized family of coordinates at infinity of $\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}^{\mathfrak{p}}_c$ and a Q_2-parametrized family of coordinates at infinity of $\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}_{\mathfrak{p}}$. (See Definition 2.25 for the definition of a Q-parametrized family of coordinates at infinity.)

Compared with the given coordinate at infinities of $\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}^{\mathfrak{p}}_c$ and of $\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}_{\mathfrak{p}}$ we obtain the maps $\mathfrak{v}_{\mathfrak{p},\mathrm{v},\rho,\vec{T},\vec{\theta}}$ and $\mathfrak{v}_{c,\mathrm{v}(i),\rho,\vec{T},\vec{\theta}}$. Therefore Lemma 2.90 follows from Lemma 2.26. □

We thus have completed the first step of the induction to prove Proposition 2.88. The other step is similar to the proof of Theorem 1.34.

When we study T_{e} and θ_{e} derivatives and prove Lemma 2.88, we again need to estimate the T_{e} and θ_{e} derivatives of the map

$$E_c \to \Gamma_0(K_{\mathrm{v}}; (u^{\rho}_{\vec{T},\vec{\theta},(\kappa)})^*TX \otimes \Lambda^{01}).$$

This map is defined by using the diffeomorphism $\mathfrak{v}_{(\mathfrak{p},c),\mathrm{v},\rho,\vec{T},\vec{\theta}}$. Therefore we can use Lemma 2.90 in the same way as above to obtain the required estimate.[17]

The proof of Proposition 2.88 is complete. □

Proof of Lemma 2.81. We can prove Lemma 2.81 by integrating the inequality in Lemma 2.90. □

Thus we have proved Theorem 2.72.

We can use it in the same way as in Subsection 1.5 to prove surjectivity and injectivity of the map $\mathrm{Glu}_{\vec{T},\vec{\theta}}$.

To show that $\mathrm{Glu}_{\vec{T},\vec{\theta}}$ is $\Gamma^+_{\mathfrak{p}}$-equivariant, we only need to remark that if $\mathfrak{p}_c \in \mathfrak{C}(\mathfrak{p})$ then $\Gamma^+_{\mathfrak{p}} \subseteq \Gamma^+_{\mathfrak{p}_c}$. (In fact all the constructions are equivariant.)

The proof of Theorem 2.70 is complete. □

Remark 2.91. We close this subsection with another technical remark. Theorems 2.70 and 2.72 imply that

$$\begin{aligned}\mathrm{Glu} : V_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B};\epsilon_1) \times (\vec{T}^{\mathrm{o}}_0,\infty] \times ((\vec{T}^{\mathrm{c}}_0,\infty] \times \vec{S}^1) \\ \to \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B})_{\epsilon_0,\vec{T}_0}\end{aligned}$$

[17]We remark that E_c is a fnite dimensional vector space consisting of smooth sections with compact support. So estimating the effect of change of variables of its element by $\mathfrak{v}_{(\mathfrak{p},c),\mathrm{v},\rho,\vec{T},\vec{\theta}}$ is easy using Lemma 2.90.

is a strata-wise C^m diffeomorphism if $T_{\mathrm{e},0}$ for all e is larger than a number *depending* on m. Using Theorem 2.72 we can define smooth structures on both sides so that the map becomes a C^m diffeomorphism. (See Subsection 2.7. We will use $s_{\mathrm{e}} = T_{\mathrm{e}}^{-1}$ as a coordinate.)

Note that the domain and the target of Glu have strata-wise C^∞ structure.[18] However, the construction we gave does not show that Glu is of C^∞-class. This is not really an issue for our purpose of defining virtual fundamental chain or cycle. Indeed, Kuranishi structure of C^k class with sufficiently large k is enough for such a purpose. (C^1-structure is enough.)

On the other hand, as we will explain in Subsection 3.2, Theorems 2.70 and 2.72 are enough to prove the existence of Kuranishi structure of C^∞ class. Except in Subsection 3.2, we fix m and will construct a Kuranishi structure of C^m class. For this purpose we choose $T_{\mathrm{e},0}$ so that it is larger than T_{10m}. Therefore our construction of Glu works on $L^2_{10m+1,\delta}$.

2.6. **Cutting down the solution space by transversals.** In Subsection 2.5, we described the thickened moduli space $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0,\vec T_0}$ by a gluing construction. Its dimension is given by

$$\begin{aligned}
&\dim \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0,\vec T_0}\\
&= \operatorname{virdim} \mathcal{M}_{k+1,\ell}(\beta) + \dim_{\mathbb R} \mathcal{E}_{\mathfrak A} + (2\ell_{\mathfrak p} + 2\sum_{c\in\mathfrak B} \ell_c)\\
&= k+1+2\ell+\mu(\beta)-3+\dim_{\mathbb R} \mathcal{E}_{\mathfrak A} + (2\ell_{\mathfrak p} + 2\sum_{c\in\mathfrak B} \ell_c).
\end{aligned}$$

Note that the dimension of the Kuranishi neighborhood of $\mathfrak p$ in $\mathcal{M}_{k+1,\ell}(\beta)$ must be $\operatorname{virdim}\mathcal{M}_{k+1,\ell}(\beta) + \dim_{\mathbb R}\mathcal{E}_{\mathfrak A}$. Therefore we need to cut down this moduli space $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0,\vec T_0}$ to obtain a Kuranishi neighborhood. We do so by requiring the transversal constraint as in Definition 2.49. We will define it below in a slightly generalized form. (For example, we define it for $(\mathfrak x,u)$ such that u is not necessarily pseudo-holomorphic but satisfies the equation $\overline{\partial} u \equiv 0 \mod \mathcal{E}_{\mathfrak A}(u)$ only.)

Let $\mathfrak p \in \mathcal{M}_{k+1,\ell}(\beta)$ and $\emptyset \ne \mathfrak A \subseteq \mathfrak B \subseteq \mathfrak C(\mathfrak p)$. We consider a subset $\mathfrak B^- \subseteq \mathfrak B$ with $\mathfrak A \subseteq \mathfrak B^-$. Let $\vec w_{\mathfrak p} = (w_{\mathfrak p,1},\dots,w_{\mathfrak p,\ell_{\mathfrak p}})$ be a symmetric stabilization of $\mathfrak x_{\mathfrak p}$ that is a part of the stabilization data at $\mathfrak p$. Let $I \subset \{1,\dots,\ell_{\mathfrak p}\}$ and we consider $\vec w_{\mathfrak p}^- = (w_{\mathfrak p,i}; i\in I)$. For simplicity of notation we put $I = \{1,\dots,\ell_{\mathfrak p}^-\}$. We assume that $\vec w_{\mathfrak p}^-$ is already a symmetric stabilization of $\mathfrak x_{\mathfrak p}$. It induces a stabilization data at $\mathfrak p$ in an obvious way. We thus obtain $\mathcal{M}_{k+1,(\ell,\ell^-_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B^-)_{\epsilon_0,\vec T_0}$.

Definition 2.92. An element $(\mathfrak Y, u', (\vec w'_c; c\in\mathfrak B))$ of $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0,\vec T_0}$ is said to satisfy the (partial) *transversal constraint* for $\vec w_{\mathfrak p}\setminus \vec w_{\mathfrak p}^-$ and $\mathfrak B\setminus\mathfrak B^-$ if the following holds.

(1) If $i > \ell^-_{\mathfrak p}$ then $u'(w'_{\mathfrak p,i}) \in \mathcal{D}_{\mathfrak p,i}$. Here $w'_{\mathfrak p,i}$, $i=1,\dots,\ell_{\mathfrak p}$ denote the $(\ell+1)$-th, $\dots$, $(\ell+\ell_{\mathfrak p})$-th interior marked points of $\mathfrak Y$.
(2) If $c\in\mathfrak B\backslash\mathfrak B^-$ and $i=1,\dots,\ell_c$ then $u'(w'_{c,i}) \in \mathcal{D}_{c,i}$. Here $\vec w'_c = (w'_{c,1},\dots,w'_{c,\ell_c})$.

[18] This is an easy consequence of implicit function theorem.

We denote by

$$\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)^{\vec w^-_{\mathfrak p},\mathfrak B^-}_{\epsilon_0,\vec T_0}$$

the set of all elements of the thickened moduli space $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0,\vec T_0}$ satisfying transversal constraint for $\vec w_{\mathfrak p}\setminus\vec w^-_{\mathfrak p}$ and $\mathfrak B\setminus\mathfrak B^-$.

Our next goal is to show that $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)^{\vec w^-_{\mathfrak p},\mathfrak A^-}_{\epsilon_0,\vec T_0}$ is homeomorphic to $\mathcal{M}_{k+1,(\ell,\ell^-_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B^-)_{\epsilon_0,\vec T_0}$. (Proposition 2.95.) To prove this we first define an appropriate forgetful map.

Definition 2.93. Let $(\mathfrak Y,u',(\vec w'_c;c\in\mathfrak B))\in\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0,\vec T_0}$. Note $\mathfrak Y=\mathfrak Y_0\cup\vec w_{\mathfrak p}$ and $\vec w_{\mathfrak p}$ consists of $\ell_{\mathfrak p}$ interior marked points. We take only $\ell^-_{\mathfrak p}$ of them and put $\vec w^-_{\mathfrak p}$ and put $\mathfrak Y^-=\mathfrak Y_0\cup\vec w^-_{\mathfrak p}$. We assume that $\mathfrak Y^-$ is stable and $\mathfrak x_{\mathfrak p}\cup\vec w^-_{\mathfrak p}$ is also stable. We also assume that $\Gamma_{\mathfrak p}$ preserves $\vec w_{\mathfrak p}$ as a set. We define the forgetful map by:

$$\mathfrak{forget}_{\mathfrak B,\mathfrak B^-;\vec w_{\mathfrak p},\vec w^-_{\mathfrak p}}(\mathfrak Y,u',(\vec w'_c;c\in\mathfrak B))=(\mathfrak Y^-,u',(\vec w'_c;c\in\mathfrak B^-)).\tag{2.287}$$

Lemma 2.94. *The map $\mathfrak{forget}_{\mathfrak B,\mathfrak B^-;\vec w_{\mathfrak p},\vec w^-_{\mathfrak p}}$ defines*

$$\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0,\vec T_0}\to\mathcal{M}_{k+1,(\ell,\ell^-_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B^-)_{\epsilon_0,\vec T_0}.$$

This map is a continuous and strata-wise smooth submersion. The fiber is $2(\ell_{\mathfrak p}-\ell^-_{\mathfrak p})+2\sum_{c\in\mathfrak B\setminus\mathfrak B^-}\ell_c$ dimensional.

Proof. We note that $\mathfrak Y^-$ is still stable. (This is because $\mathfrak x_{\mathfrak p}\cup\vec w^-_{\mathfrak p}$ is stable.) Therefore $\mathfrak{forget}_{\mathfrak B,\mathfrak B^-;\vec w_{\mathfrak p},\vec w^-_{\mathfrak p}}$ preserves stratification. Note we forget the position of the $\ell_{\mathfrak p}-\ell^-_{\mathfrak p}+\sum_{c\in\mathfrak B\setminus\mathfrak B^-}\ell_c$ marked points. There is no constraint for those marked points other than those coming from the condition that $(\mathfrak Y,u')$ is ϵ_0-close to $(\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p},u_{\mathfrak p})$ and $(\mathfrak Y_0\cup\vec w'_c,u')$ are ϵ_0-close to $\mathfrak p\cup\vec w^{\mathfrak p}_c$ for all $c\in\mathcal A$. These are open conditions. Therefore this map is a strata-wise smooth submersion and the fiber is $2(\ell_{\mathfrak p}-\ell^-_{\mathfrak p})+2\sum_{c\in\mathfrak B\setminus\mathfrak B^-}\ell_c$ dimensional. ☐

Proposition 2.95. *The following holds if $\epsilon_0,\epsilon_{\mathfrak p_c}$ are sufficiently small.*

1. *The space $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)^{\vec w^-_{\mathfrak p},\mathfrak B^-}_{\epsilon_0,\vec T_0}$ is a strata-wise smooth submanifold of our thickened moduli space $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0,\vec T_0}$ of codimension $2(\ell_{\mathfrak p}-\ell^-_{\mathfrak p})+2\sum_{c\in\mathfrak B\setminus\mathfrak B^-}\ell_c$.*
2. *The restriction of $\mathfrak{forget}_{\mathfrak B,\mathfrak B^-;\vec w_{\mathfrak p},\vec w^-_{\mathfrak p}}$ induces a homeomorphism*

$$\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)^{\vec w^-_{\mathfrak p},\mathfrak B^-}_{\epsilon_0,\vec T_0}\to\mathcal{M}_{k+1,(\ell,\ell^-_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B^-)_{\epsilon_0,\vec T_0}$$

that is a strata-wise diffeomorphism.

Remark 2.96. Note that if $c\in\mathcal B$ then $\mathfrak p\in\mathfrak M_{\mathfrak p_c}$ and ϵ_c is used to define $\mathfrak M_{\mathfrak p_c}$. (See Definition 2.51.)

Proof. We consider the evaluation maps at the $(\ell_{\mathfrak p}-\ell^-_{\mathfrak p})+\sum_{c\in\mathfrak B\setminus\mathfrak B^-}\ell_c$ marked points that we forget by the map $\mathfrak{forget}_{\mathfrak B,\mathfrak B^-;\vec w_{\mathfrak p},\vec w^-_{\mathfrak p}}$. It defines a continuous and strata-wise smooth map

$$\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0,\vec T_0}\to X^{(\ell_{\mathfrak p}-\ell^-_{\mathfrak p})+\sum_{c\in\mathfrak B\setminus\mathfrak B^-}\ell_c}.\tag{2.288}$$

We consider the submanifold

$$\prod_{i=\ell_{\mathfrak p}^-+1}^{\ell_{\mathfrak p}} \mathcal D_{\mathfrak p,i} \times \prod_{c\in\mathfrak B\setminus\mathfrak B^-}\prod_{i=1}^{\ell_c}\mathcal D_{c,i} \tag{2.289}$$

of the right hand side of (2.288). By Proposition 2.48 (2), the map (2.288) is transversal to (2.289) at $\mathfrak p$ if $\epsilon_{\mathfrak p_c}$ is sufficiently small. Therefore we may assume (2.288) is transversal to (2.289) everywhere. Since $\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)^{\vec w_{\mathfrak p}^-,\mathfrak B^-}_{\epsilon_0,\vec T_0}$ is the inverse image of (2.289) by the map (2.288), the statement (1) follows.

By choosing ϵ_0 sufficiently small we can ensure that the image under the map (2.288) of each fiber of the map $\mathfrak{forget}_{\mathfrak B,\mathfrak B^-;\vec w_{\mathfrak p},\vec w_{\mathfrak p}^-}$ intersects with the submanifold (2.289) at one point. Moreover by stability the elements of $\mathcal M_{k+1,(\ell,\ell_{\mathfrak p}^-,(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B^-)_{\epsilon_0,\vec T_0}$ have no automorphism. The statement (2) follows. □

We next consider a similar but a slightly different case of transversal constraint. Namely:

Definition 2.97. An element $(\mathfrak Y,u',(\vec w'_c;c\in\mathfrak B))$ of $\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0,\vec T_0}$ is said to satisfy the *transversal constraint at all additional marked points* if the following holds. Let $w'_{\mathfrak p,i}$, $i=1,\dots,\ell_{\mathfrak p}$ denote the $(\ell+1)$-th, ..., $(\ell+\ell_{\mathfrak p})$-th interior marked points of $\mathfrak Y$. We put $\vec w'_c=(w'_{c,1},\dots,w'_{c,\ell_c})$.

(1) For all $i=1,\dots,\ell_{\mathfrak p}$ we have $u'(w'_{\mathfrak p,i})\in\mathcal D_{\mathfrak p,i}$.
(2) For all $c\in\mathfrak B$ and $i=1,\dots,\ell_{\mathfrak c}$ we have $u'(w'_{c,i})\in\mathcal D_{c,i}$.

We denote by $\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)^{\rm trans}_{\epsilon_0,\vec T_0}$ the set of all elements of the thickened moduli space $\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0,\vec T_0}$ satisfying transversal constraint at all additional marked points.

Lemma 2.98. *The set $\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)^{\rm trans}_{\epsilon_0,\vec T_0}$ is a closed subset of our space $\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0,\vec T_0}$ and is a strata-wise smooth submanifold of codimension $2\ell_{\mathfrak p}+2\sum_{c\in\mathfrak B}\ell_c$.*

Remark 2.99. We note that the map Glu is a homeomorphism onto its image of the thickened moduli space $\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)^{\rm trans}_{\epsilon_0,\vec T_0}$.

Proof. By Proposition 2.95 it suffices to consider the case $\mathfrak A=\mathfrak B$. By the way similar to the proof of Proposition 2.95 we define

$$\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)_{\epsilon_0,\vec T_0}\to X^{\ell_{\mathfrak p}+\sum_{c\in\mathfrak A}\ell_c} \tag{2.290}$$

that is an evaluation map at all the added marked points. If ϵ_0 is small, then (2.290) is transversal to

$$\prod_{i=1}^{\ell_{\mathfrak p}}\mathcal D_{\mathfrak p,i}\times\prod_{c\in\mathfrak A}\prod_{i=1}^{\ell_c}\mathcal D_{c,i}. \tag{2.291}$$

Since $\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec T_0}$ is the inverse image of (2.291) by the map (2.290), the lemma follows. □

Definition 2.100. We denote by $\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec T_0}\cap\mathfrak s^{-1}(0)$ the set of all $(\mathfrak Y,u',(\vec w'_c;c\in\mathfrak A))\in\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec T_0}$ such that u' is pseudo-holomorphic.

Our space $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec T_0} \cap \mathfrak s^{-1}(0)$ is a closed subset of the moduli space $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec T_0}$.

By forgetting all the additional marked points we obtain a map

$$\mathfrak{forget} : \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec T_0} \cap \mathfrak s^{-1}(0) \to \mathcal{M}_{k+1,\ell}(\beta). \tag{2.292}$$

We recall that we have injective homomorphisms

$$\Gamma_{\mathfrak p} \to \mathfrak S_{\ell_{\mathfrak p}} \times \prod_{c\in\mathfrak A}\mathfrak S_{\ell_c},$$
$$\Gamma_{\mathfrak p}^+ \to \mathfrak S_{\ell} \times \mathfrak S_{\ell_{\mathfrak p}} \times \prod_{c\in\mathfrak A}\mathfrak S_{\ell_c}.$$

The group $\Gamma_{\mathfrak p}^+$ acts on $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)_{\epsilon_0,\vec T_0}$ as follows. We regard $\Gamma_{\mathfrak p}^+ \subset \mathfrak S_{\ell}\times\mathfrak S_{\ell_{\mathfrak p}}\times\prod_{c\in\mathfrak A}\mathfrak S_{\ell_c}$. Then the action of $\Gamma_{\mathfrak p}^+$ on $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)_{\epsilon_0,\vec T_0}$ is by exchanging the interior marked points. It is easy to see that $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec T_0}$ is invariant under this action. Therefore (2.292) induces a map

$$\overline{\mathfrak{forget}} : \left(\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec T_0} \cap \mathfrak s^{-1}(0)\right)/\Gamma_{\mathfrak p} \to \mathcal{M}_{k+1,\ell}(\beta). \tag{2.293}$$

Remark 2.101. The map (2.293) induces a map

$$\left(\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec T_0} \cap \mathfrak s^{-1}(0)\right)/\Gamma_{\mathfrak p}^+ \to \mathcal{M}_{k+1,\ell}(\beta)/\mathfrak S_{\ell}.$$

See Remark 2.16. We can use this remark to construct an $\mathfrak S_\ell$ invariant Kuranishi structure on $\mathcal{M}_{k+1,\ell}(\beta)$.

Proposition 2.102. *The map (2.293) is a homeomorphism onto an open neighborhood of* $\mathfrak p$.

Proof. The geometric intuition behind this proposition is clear. We will give a detailed proof below for completeness sake. We first review the definition of the topology of $\mathcal{M}_{k+1,\ell}(\beta)$ given in [FOn1, Definition 10.2, 10.3], [FOOO1, Definition 7.1.39, 7.1.42].

Definition 2.103. Let $\mathfrak p_a = ((\Sigma_a,\vec z_a,\vec z_a^{\rm int}),u_a), \mathfrak p_\infty = ((\Sigma,\vec z,\vec z^{\rm int}),u) \in \mathcal{M}_{k+1,\ell}(\beta)$. We *assume* $(\Sigma_a,\vec z_a,\vec z_a^{\rm int})$ and $(\Sigma,\vec z,\vec z^{\rm int})$ are stable. We say that a sequence $((\Sigma_a,\vec z_a,\vec z_a^{\rm int}),u_a)$ *stably converges* to $((\Sigma,\vec z,\vec z^{\rm int}),u)$ and write

$$\lim_{a\to\infty}{\rm s}\ \mathfrak p_a = \mathfrak p_\infty$$

if the following holds.

(1) We assume
$$\lim_{a\to\infty}(\Sigma_a,\vec z_a,\vec z_a^{\rm int}) = (\Sigma,\vec z,\vec z^{\rm int})$$
in the Deligne-Mumford moduli space $\mathcal{M}_{k+1,\ell}$. We take a coordinate at infinity of $(\Sigma,\vec z,\vec z^{\rm int})$. It determines a diffeomorphism between cores of Σ_a and of Σ for large a.

(2) For each ϵ we can extend the core appropriately so that there exists a_0 such that (2),(3) hold for $a > a_0$.
$$|u_a - u|_{C^1({\rm Core})} < \epsilon.$$
Here we regard u_a and u as maps from the core of Σ_a and Σ by the above mentioned diffeomorphism.

(3) The diameter of the image of each of the connected component of the neck region by u_a is smaller than ϵ.

Definition 2.104. Let $\mathfrak{p}_a = ((\Sigma_a, \vec z_a, \vec z_a^{\rm int}), u_a)$, $\mathfrak{p}_\infty = ((\Sigma, \vec z, \vec z^{\rm int}), u) \in \mathcal{M}_{k+1,\ell}(\beta)$. We say that $\mathfrak{p}_a$ converges to $\mathfrak{p}_\infty$ and write

$$\lim_{a\to\infty} \mathfrak{p}_a = \mathfrak{p}_\infty$$

if there exist $\ell' \ge 0$ and $\mathfrak{q}_a = ((\Sigma_a, \vec z_a, \vec z_a^{\rm int} \cup \vec z_a^{+,{\rm int}}), u_a)$, $\mathfrak{q}_\infty = ((\Sigma, \vec z, \vec z^{\rm int} \cup \vec z_\infty^{+,{\rm int}}), u) \in \mathcal{M}_{k+1,\ell+\ell'}(\beta)$ such that

$$\lims_{a\to\infty} \mathfrak{q}_a = \mathfrak{q}_\infty \tag{2.294}$$

and

$$\mathfrak{forget}_{(k+1;\ell+\ell'),(k+1;\ell)}(\mathfrak{q}_a) = \mathfrak{p}_a, \quad \mathfrak{forget}_{(k+1;\ell+\ell'),(k+1;\ell)}(\mathfrak{q}_\infty) = \mathfrak{p}_\infty. \tag{2.295}$$

Here

$$\mathfrak{forget}_{(k+1;\ell+\ell'),(k+1;\ell)} : \mathcal{M}_{k+1,\ell+\ell'}(\beta) \to \mathcal{M}_{k+1,\ell}(\beta)$$

is a map forgetting $(\ell+1)$-st,...,$(\ell+\ell')$-st (interior) marked points (and shrinking the irreducible components that become unstable. See [FOOO1, p 419].)

Now we prove the following:

Lemma 2.105. *If ϵ_0, $\epsilon_{\mathfrak{p}_c}$ are sufficiently small, then the image of (2.293) is an open subset of $\mathcal{M}_{k+1,\ell}(\beta)$.*

Proof. Let

$$\mathfrak{p}' \in \overline{\mathfrak{forget}}\Big(\big(\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A})^{\rm trans}_{\epsilon_0,\vec T_0} \cap \mathfrak{s}^{-1}(0)\big)/\Gamma_{\mathfrak{p}}\Big)$$

and $\mathfrak{p}_a \in \mathcal{M}_{k+1,\ell}(\beta)$ such that $\lim_{a\to\infty}\mathfrak{p}_a = \mathfrak{p}'$. We will prove

$$\mathfrak{p}_a \in \overline{\mathfrak{forget}}\Big(\big(\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A})^{\rm trans}_{\epsilon_0,\vec T_0} \cap \mathfrak{s}^{-1}(0)\big)/\Gamma_{\mathfrak{p}}\Big)$$

for all sufficiently large a.

We put $\mathfrak{p}' = (\mathfrak{Y}_0, u')$ and

$$(\mathfrak{Y}_0 \cup \vec w'_{\mathfrak{p}}, u', (\vec w'_c; c \in \mathfrak{A})) \in \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A})^{\rm trans}_{\epsilon_0,\vec T_0} \cap \mathfrak{s}^{-1}(0).$$

We also put $\mathfrak{p}_a = (\mathfrak{x}_{\mathfrak{p}_a}, u_{\mathfrak{p}_a})$. By Definition 2.103, there exists $\mathfrak{q}_a, \mathfrak{q}_\infty \in \mathcal{M}_{k+1,\ell+\ell'}(\beta)$ such that (2.294) holds and

$$\mathfrak{forget}_{(k+1;\ell+\ell'),(k+1;\ell)}(\mathfrak{q}_a) = \mathfrak{p}_a, \quad \mathfrak{forget}_{(k+1;\ell+\ell'),(k+1;\ell)}(\mathfrak{q}_\infty) = \mathfrak{p}'. \tag{2.296}$$

Let $\vec z_a^{+,{\rm int}} \subset \mathfrak{x}_{\mathfrak{q}_a}, \vec z_\infty^{+,{\rm int}} \subset \mathfrak{x}_{\mathfrak{q}_\infty}$ be the interior marked points that are not the marked points of $\mathfrak{p}_a$ or of $\mathfrak{p}'$. By perturbing $\mathfrak{q}_a$ and $\mathfrak{q}_\infty$ a bit we may assume

$$\begin{aligned} u_{\mathfrak{q}_a}(z_{a,i}^{+,{\rm int}}) &\notin \bigcup_{i=1}^{\ell_{\mathfrak{p}}} \mathcal{D}_{\mathfrak{p},i} \cup \bigcup_{c\in\mathfrak{A}}\bigcup_{i=1}^{\ell_c} \mathcal{D}_{c,i}, \\ u_{\mathfrak{q}_\infty}(z_{\infty,i}^{+,{\rm int}}) &\notin \bigcup_{i=1}^{\ell_{\mathfrak{p}}} \mathcal{D}_{\mathfrak{p},i} \cup \bigcup_{c\in\mathfrak{A}}\bigcup_{i=1}^{\ell_c} \mathcal{D}_{c,i}. \end{aligned} \tag{2.297}$$

We consider the map $\Sigma_{\mathfrak{q}_\infty} \to \Sigma_{\mathfrak{p}'}$ that shrinks the irreducible components which become unstable after forgetting $(\ell+1)$-th,..., $(\ell+\ell')$-th marked points $\vec z_\infty^{+,{\rm int}}$ of $\mathfrak{x}_{\mathfrak{q}_\infty}$. By (2.297) none of the points $\vec w'_{\mathfrak{p}}$, $\vec w'_c$ are contained in the image of the irreducible components of $\Sigma_{\mathfrak q_\infty}$ that we shrink. Therefore $\vec w'_{\mathfrak p}, \vec w'_c \subset \Sigma_{\mathfrak p'}$ may be regarded as points of $\Sigma_{\mathfrak q_\infty}$.

Then by extending the core if necessary we may assume that $\vec w'_{\mathfrak p}, \vec w'_c$ are in the core of $\Sigma_{\mathfrak q_\infty}$. Here we use the coordinate at infinity that appears in the definition of $\underset{a\to\infty}{\text{lims}}\ \mathfrak q_a = \mathfrak q_\infty$.

We note that

$$u_{\mathfrak q_\infty}(w'_{\mathfrak p,i}) \in \mathcal D_{\mathfrak p,i}, \quad u_{\mathfrak q_\infty}(w'_{c,i}) \in \mathcal D_{c,i}.$$

We also note that $u_{\mathfrak q_a}$ converges to $u_{\mathfrak q_\infty}$ in C^1-topology on the core. Moreover $u_{\mathfrak q_\infty}$ is transversal to $\mathcal D_{\mathfrak p,i}$ (resp. $\mathcal D_{c,i}$) at $u_{\mathfrak q_\infty}(w'_{\mathfrak p,i})$ (resp. $u_{\mathfrak q_\infty}(w'_{c,i})$). Therefore, for sufficiently large a there exist $w'_{a,\mathfrak p,i}, w'_{a,c,i} \in \Sigma_{\mathfrak q_a}$ with the following properties.

(1) $u_{\mathfrak q_a}(w'_{a,\mathfrak p,i}) \in \mathcal D_{\mathfrak p,i}$.
(2) $u_{\mathfrak q_a}(w'_{a,c,i}) \in \mathcal D_{c,i}$.
(3) $\lim_{a\to\infty} w'_{a,\mathfrak p,i} = w'_{\mathfrak p,i}$.
(4) $\lim_{a\to\infty} w'_{a,c,i} = w'_{c,i}$.

Here in the statements (3) and (4) we use the identification of the core of $\Sigma_{\mathfrak q_a}$ and of $\Sigma_{\mathfrak q_\infty}$ induced by the coordinate at infinity that appears in the definition of $\underset{a\to\infty}{\text{lims}}\ \mathfrak q_a = \mathfrak q_\infty$. We send $w'_{a,\mathfrak p,i}$ by the map $\Sigma_{\mathfrak q_a} \to \Sigma_{\mathfrak p_a}$ and denote it by the same symbol. We thus obtain $\vec w'_{a,\mathfrak p} \subset \Sigma_{\mathfrak p_a}$. The additional marked points $w'_{a,c,i}$ induce $\vec w'_{a,c} \subset \Sigma_{\mathfrak p_a}$ in the same way.

Using (1)-(4) above and the fact that $u_{\mathfrak q_a}$ converges to $u_{\mathfrak q_\infty}$ in C^1-topology we can easily show that

$$(\mathfrak x_{\mathfrak p_a} \cup \vec w'_{a,\mathfrak p}, u_{\mathfrak p_a}, (\vec w'_{a,c}; c \in \mathfrak A)) \in \mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\text{trans}}_{\epsilon_0,\vec T_0} \cap \mathfrak s^{-1}(0)$$

for sufficiently large a. Thus we have

$$\begin{aligned}\mathfrak p_a &= \overline{\mathfrak{forget}}((\mathfrak x_{\mathfrak p_a} \cup \vec w'_{a,\mathfrak p}, u_{\mathfrak p_a}, (\vec w'_{a,c}; c \in \mathfrak A)))\\ &\in \overline{\mathfrak{forget}}\left(\big(\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\text{trans}}_{\epsilon_0,\vec T_0} \cap \mathfrak s^{-1}(0)\big)/\Gamma_{\mathfrak p}\right)\end{aligned}$$

for sufficiently large a. The proof of Lemma 2.105 is complete. □

Lemma 2.106. *If ϵ_0 is sufficiently small, then the map (2.293) is injective.*

Proof. The proof is by contradiction. We assume that there exists $\epsilon_0^{(n)}$ with $\epsilon_0^{(n)} \to 0$ as $n \to \infty$, and

$$\begin{aligned}&(\mathfrak Y_{j;(n),0} \cup \vec w'_{j;(n),\mathfrak p}, u'_{j;(n)}, (\vec w'_{j;(n),c}; c \in \mathfrak A))\\ &\in \mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\text{trans}}_{\epsilon_0^{(n)},\vec T_0} \cap \mathfrak s^{-1}(0)\end{aligned} \tag{2.298}$$

for $j = 1, 2$. Here we extend the core of the coordinate at infinity of $\mathfrak p$ by $\vec R_{(n)} \to \infty$ to define the right hand side of (2.298). We assume

$$(\mathfrak Y_{1;(n),0}, u'_{1;(n)}) \sim (\mathfrak Y_{2;(n),0}, u'_{2;(n)}) \tag{2.299}$$

in $\mathcal M_{k+1,\ell}(\beta)$ but

$$\begin{aligned}&[(\mathfrak Y_{1;(n),0} \cup \vec w'_{1;(n),\mathfrak p}, u'_{1;(n)}, (\vec w'_{1;(n),c}; c \in \mathfrak A))]\\ &\ne [(\mathfrak Y_{2;(n),0} \cup \vec w'_{2;(n),\mathfrak p}, u'_{2;(n)}, (\vec w'_{2;(n),c}; c \in \mathfrak A))]\end{aligned} \tag{2.300}$$

in $\big(\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\text{trans}}_{\epsilon_0^{(n)},\vec T_0} \cap \mathfrak s^{-1}(0)\big)/\Gamma_{\mathfrak p}$. We will deduce contradiction.

The condition (2.299) implies that there exists $v_{(n)} : \Sigma_{\mathfrak Y_{1;(n),0}} \to \Sigma_{\mathfrak Y_{2;(n),0}}$ with the following properties.

(1) $v_{(n)}$ is a biholomorphic map.
(2) $u'_{2;(n)} \circ v_{(n)} = u'_{1;(n)}$.
(3) $v_{(n)}$ sends $k+1$ boundary marked points and ℓ interior marked points of $\mathfrak{Y}_{1;(n),0}$ to the corresponding marked points of $\mathfrak{Y}_{2;(n),0}$.

We take a coordinate at infinity associated to the stabilization data at $\mathfrak{p}$. Then (2.298) implies that the core of $\mathfrak{Y}_{j;(n),0}$ $(j = 1, 2)$ is identified with the extended core $(K^{\mathfrak{p}}_{\rm v})^{+\vec{R}_{(n)}}$ of $\mathfrak{p}$. This identification may not preserve complex structures but preserves the $k+1$ boundary and $\ell + \ell'$ interior marked points. Therefore $v_{(n)}$ induces

$$v_{(n)} : K^{\mathfrak{p}}_{0,{\rm v}} \to (K^{\mathfrak{p}}_{\rm v})^{+\vec{R}_{(n)}}$$

where $K^{\mathfrak{p}}_{0,{\rm v}}$ is a compact set such that $w'_{1;(n),\mathfrak{p},i}, w'_{1;(n),c,i} \in K^{\mathfrak{p}}_{0,{\rm v}}$. (We may extend the core so that we can find such $K^{\mathfrak{p}}_{0,{\rm v}}$.)

We may take $\vec{R}_{(n)} \to \infty$ so that the $u'_{j;(n)}$ image of each of the connected components of the complement of $(K^{\mathfrak{p}}_{\rm v})^{+\vec{R}_{(n)}}$ has diameter $< \epsilon^{(n)}_0$.

We consider the complex structure of $\Sigma_{\mathfrak{p}}$ on $(K^{\mathfrak{p}}_{\rm v})^{+\vec{R}_{(n)}}$ and denote it by $j_{\mathfrak{p}}$. Then we have

$$\lim_{n\to\infty} \|(v_{(n)})_* j_{\mathfrak{p}} - j_{\mathfrak{p}}\|_{C^1\left((K^{\mathfrak{p}}_{\rm v})^{+\vec{R}^-_{(n)}}\right)} = 0 \tag{2.301}$$

where $\vec{R}^-_{(n)} \to \infty$ is chosen so that $v_{(n)}((K^{\mathfrak{p}}_{\rm v})^{+\vec{R}^-_{(n)}}) \subset (K^{\mathfrak{p}}_{\rm v})^{+\vec{R}_{(n)}}$.

On the other hand by Property (4) above we have

$$\lim_{n\to\infty} \|u \circ v_{(n)} - u\|_{C^1\left((K^{\mathfrak{p}}_{\rm v})^{+\vec{R}^-_{(n)}}\right)} = 0. \tag{2.302}$$

We use (2.301) and (2.302) to prove the following.

Sublemma 2.107. *After taking a subsequence if necessary, there exists $v' \in \Gamma_{\mathfrak{p}}$ such that*

$$\lim_{n\to\infty} \|v_{(n)} - v'\|_{C^1((K^{\mathfrak{p}}_{\rm v})^{+\vec{R}})} = 0$$

for any $\vec{R}$.

Proof. Since $v_{(n)}$ is biholomorphic with respect to a pair of complex structures converging to $(j_{\mathfrak{p}}, j_{\mathfrak{p}})$, we can use Gromov compactness to show that it converges in compact C^∞ topology outside finitely many points after taking a subsequence if necessary. Let v' be the limit. By the Property (2) above we have $u \circ v' = u$.

On the irreducible component of $\mathfrak{x}_{\mathfrak{p}}$ where u is not constant, we use $u \circ v' = u$ together with the fact that $v_{(n)}$ is biholomorphic to show that there is no bubble on this component. Namely $v_{(n)}$ converges everywhere on this component.

The irreducible component of $\mathfrak{x}_{\mathfrak{p}}$ where u is trivial is stable since $\mathfrak{p}$ is stable. We note that v' preserves the marked points of $\mathfrak{p}$. It implies that v' is not a constant map on this component. Then using the fact that $v_{(n)}$ is biholomorphic we can again show that there is no bubble on this component.

We thus proved that $v_{(n)}$ converges to v' everywhere. It is then easy to see that $v' \in \Gamma_{\mathfrak{p}}$. □

By replacing $(\mathfrak{Y}_{2;(n),0} \cup \vec{w}'_{2;(n),\mathfrak{p}}, u'_2, (\vec{w}'_{2;(n),c}; c \in \mathfrak{A}))$ using the action of $v' \in \Gamma_{\mathfrak{p}}$, we may assume that

$$\lim_{n\to\infty} \|v_{(n)} - \text{identity}\|_{C^1(K^{\mathfrak{p}}_{0,{\rm v}})} = 0. \tag{2.303}$$

Then, $u'_{1;(n)}(w'_{1;(n),\mathfrak p,i}), u'_{2;(n)}(w'_{2;(n),\mathfrak p,i}) \in \mathcal D_{\mathfrak p,i}$ imply

$$v_{(n)}(w'_{1;(n),\mathfrak p,i},) = w'_{2;(n),\mathfrak p,i}. \tag{2.304}$$

We next take coordinate at infinity associated to the obstruction bundle data centered at $\mathfrak p_c$. Then we can think of the restriction $v_{(n)} : K^{\mathfrak p_c}_{0,\mathrm v} \to K^{\mathfrak p_c}_{\mathrm v}$, which satisfies

$$\lim_{n\to\infty} \|v_{(n)} - \text{identity}\|_{C^1(K^{\mathfrak p_c}_{0,\mathrm v})} = 0. \tag{2.305}$$

(In fact, we may take $\vec R$ so that for each $\mathrm v \in C^0(\mathcal G_{\mathfrak p_c})$ we have $\mathrm v' \in C^0(\mathcal G_{\mathfrak p})$ such that $K^{\mathfrak p_c}_{\mathrm v} \subset (K^{\mathfrak p}_{\mathrm v'})^{+\vec R}$.)

Then, $u'_{1;(n)}(w'_{1;(n),c,i}), u'_{2;(n)}(w'_{2;(n),c,i}) \in \mathcal D_{c,i}$ imply

$$v_{(n)}(w'_{1;(n),c,i},) = w'_{2;(n),c,i}. \tag{2.306}$$

Property (1),(2) and (2.305), (2.306) contradict to (2.300). The proof of Lemma 2.106 is complete. □

Lemma 2.108. *If ϵ_0, $\epsilon_{\mathfrak p_c}$ are sufficiently small, then (2.293) is a homeomorphism onto its image.*

Proof. It is easy to see that the map (2.293) is continuous. It is injective by Lemma 2.106. It suffices to show that the converse is continuous. The proof of the continuity of the converse is similar to the proof of Lemma 2.105. We however repeat the detail of the proof for completeness sake. Let

$$(\mathfrak x_{\mathfrak p_a} \cup \vec w'_{a,\mathfrak p}, u_{\mathfrak p_a}, (\vec w'_{a,c}; c \in \mathfrak A)) \in \mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec T_0} \cap \mathfrak s^{-1}(0)$$

and

$$(\mathfrak x_{\mathfrak p_\infty} \cup \vec w'_{\infty,\mathfrak p}, u_{\mathfrak p_\infty}, (\vec w'_{\infty,c}; c \in \mathfrak A)) \in \mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec T_0} \cap \mathfrak s^{-1}(0).$$

We put $\mathfrak p_\infty = (\mathfrak x_{\mathfrak p_\infty}, u_{\mathfrak p_\infty})$, $\mathfrak p_a = (\mathfrak x_{\mathfrak p_a}, u_{\mathfrak p_a})$ and assume

$$\lim_{a\to\infty} \mathfrak p_a = \mathfrak p_\infty \tag{2.307}$$

in $\mathcal M_{k+1,\ell}(\beta)$.

By Definition 2.103, there exist $\mathfrak q_a, \mathfrak q_\infty \in \mathcal M_{k+1,\ell+\ell'}(\beta)$ such that (2.294) and

$$\mathfrak{forget}_{(k+1;\ell+\ell'),(k+1;\ell)}(\mathfrak q_a) = \mathfrak p_a, \quad \mathfrak{forget}_{(k+1;\ell+\ell'),(k+1;\ell)}(\mathfrak q_\infty) = \mathfrak p_\infty. \tag{2.308}$$

Let $\vec z^{+,\rm int}_a \subset \mathfrak x^{+,\rm int}_{\mathfrak q_a}, \vec z^{+,\rm int}_\infty \subset \mathfrak x_{\mathfrak q_\infty}$ be the marked points of $\mathfrak q_a, \mathfrak q_\infty$ that are not marked points of $\mathfrak p_a$ or of $\mathfrak p_\infty$. By perturbing $\mathfrak q_a$ and $\mathfrak q_\infty$ a bit we may assume (2.297).

We consider the map $\Sigma_{\mathfrak q_\infty} \to \Sigma_{\mathfrak p_\infty}$ that shrinks the components which become unstable after forgetting $(\ell+1)$-th,…, $(\ell+\ell')$-th marked points $\vec z^{+,\rm int}_\infty$ of $\mathfrak x_{\mathfrak q_\infty}$. By (2.297) none of the points $\vec w'_{\infty,\mathfrak p}$, $\vec w'_{\infty,c}$ are contained in the image of the components of $\Sigma_{\mathfrak q_\infty}$ that we shrink. So $\vec w'_{\infty,\mathfrak p}, \vec w'_{\infty,c} \subset \Sigma_{\mathfrak p'}$ may be regarded as points of $\Sigma_{\mathfrak q_\infty}$.

Then by extending the core if necessary we may regard that $\vec w'_{\infty,\mathfrak p}, \vec w'_{\infty,c}$ are in the core of $\Sigma_{\mathfrak q_\infty}$. Here we use the coordinate at infinity that appears in the definition of $\lims_{a\to\infty} \mathfrak q_a = \mathfrak q_\infty$.

We remark that $u_{\mathfrak q_\infty}(w'_{\infty,c,i}) \in \mathcal D_{c,i}$. We also remark that $u_{\mathfrak q_a}$ converges to $u_{\mathfrak q_\infty}$ in C^1-topology on the core. Moreover $u_{\mathfrak q_\infty}$ is transversal to $\mathcal D_{\mathfrak p,i}$ (resp. $\mathcal D_{c,i}$) at $u_{\mathfrak q_\infty}(w'_{\infty,\mathfrak p,i})$ (resp. $u_{\mathfrak q_\infty}(w'_{\infty,c,i})$). Therefore, for sufficiently large a there exist $w''_{a,\mathfrak p,i}, w''_{a,c,i} \in \Sigma_{\mathfrak q_a}$ with the following properties.

(1) $u_{\mathfrak q_a}(w''_{a,\mathfrak p,i}) \in \mathcal D_{\mathfrak p,i}$.

(2) $u_{\mathfrak q_a}(w''_{a,c,i}) \in \mathcal D_{c,i}$.
(3) $\lim_{a\to\infty} w''_{a,\mathfrak p,i} = w'_{\infty,\mathfrak p,i}$.
(4) $\lim_{a\to\infty} w''_{a,c,i} = w'_{\infty,c,i}$.

Here in (3)(4) we use the identification of the core of $\Sigma_{\mathfrak q_a}$ and of $\Sigma_{\mathfrak q_\infty}$ induced by the coordinate at infinity that appears in the definition of $\lim\mathrm{s}_{a\to\infty}\ \mathfrak q_a = \mathfrak q_\infty$. We send $w''_{a,\mathfrak p,i}$ by the map $\Sigma_{\mathfrak q_a} \to \Sigma_{\mathfrak p_a}$ and denote it by the same symbol. We thus obtain $\vec w''_{a,\mathfrak p} \subset \Sigma_{\mathfrak p_a}$. The additional marked points $w''_{a,c,i}$ induce $\vec w''_{a,c} \subset \Sigma_{\mathfrak p_a}$ in the same way.

Sublemma 2.109. *$w''_{a,\mathfrak p,i} = w'_{a,\mathfrak p,i}$ and $w''_{a,c,i} = w'_{a,c,i}$ if ϵ_0 and $\epsilon_{\mathfrak p_c}$ are small and a is large.*

Proof. Note $(\mathfrak x_{\mathfrak p_a} \cup \vec w_{a,\mathfrak p_a}, u_{\mathfrak p_a})$ and $(\mathfrak x_{\mathfrak p_\infty} \cup \vec w_{\infty,\mathfrak p_\infty}, u_{\mathfrak p_\infty})$ are both ϵ_0-close to $(\mathfrak x_{\mathfrak p}, \vec w_{\mathfrak p}, u_{\mathfrak p})$. Then we can choose ϵ_0 small so that (3) above implies

$$d(w'_{a,\mathfrak p,i}, w''_{a,\mathfrak p,i}) \le 3\epsilon_0$$

for sufficiently large a. We can also show that

$$d(w'_{a,c,i}, w''_{a,c,i}) \le 3(o(\epsilon_0) + \epsilon_{\mathfrak p_c})$$

in the same way. (Here $\lim_{\epsilon_0\to 0} o(\epsilon_0) = 0$.) On the other hand we have $u_{\mathfrak q_a}(w'_{a,\mathfrak p,i}) \in \mathcal D_{\mathfrak p,i}$, $u_{\mathfrak q_a}(w'_{a,c,i}) \in \mathcal D_{c,i}$. They imply the sublemma. □

Remark 2.110. In the last step we need to assume $\epsilon_{\mathfrak p_c}$ small. More precisely, when we take $\epsilon_{\mathfrak p_c}$ at the stage of Definition 2.51 we require the following.

If $d(w'_{c,i}, w''_{c,i}) \le 4\epsilon_{\mathfrak p_c}$, $w'_{c,i}, w''_{c,i} \in \Sigma_{\mathfrak p}$ and $u_{\mathfrak p}(w'_{c,i}) \in \mathcal D_{c,i}$, $u_{\mathfrak p}(w''_{c,i}) \in \mathcal D_{c,i}$, then $w'_{c,i} = w''_{c,i}$.

We next choose ϵ_0 so small that the same statement holds for $\mathfrak p_a$, with $4\epsilon_{\mathfrak p}$ replaced by $3\epsilon_{\mathfrak p_c}$.

Now (3)(4) above imply

$$\lim_{a\to\infty} (\mathfrak x_{\mathfrak p_a} \cup \vec w'_{a,\mathfrak p}, u_{\mathfrak p_a}, (\vec w'_{a,c}; c \in \mathfrak A)) = (\mathfrak x_{\mathfrak p_\infty} \cup \vec w'_{\infty,\mathfrak p}, u_{\mathfrak p_\infty}, (\vec w'_{\infty,c}; c \in \mathfrak A))$$

in $\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta; \mathfrak p; \mathfrak A)^{\mathrm{trans}}_{\epsilon_0,\vec T_0} \cap \mathfrak s^{-1}(0)$ as required. □

The proof of Proposition 2.102 is complete. □

Proof of Lemma 2.50. Lemma 2.50 is actually the same as Lemma 2.106 except the following point. We remark that at the stage when we state Lemma 2.50 we did not prove Theorems 2.70 and 2.72. In fact, to fix the obstruction bundle E_c we used Lemma 2.50. However the argument here is not circular by the following reason.

When we prove Lemma 2.50, we take an obstruction bundle data centered at $\mathfrak p$ only, the same point as the one we start the gluing construction. We use the obstruction bundle induced by this obsruction bundle data to go throught the gluing argument (proof of Theorems 2.70 and 2.72.) We do not need the conclusion of Lemma 2.50 for the glueing argument. Then we obtain Glu. We use this map to go through the proof of Lemma 2.106 and prove Lemma 2.50. □

Remark 2.111. In Definition 2.51 we mentioned that we prove open-ness of the set $\mathfrak W^+(\mathfrak p)$ in Subsection 2.6. Indeed it follows from Lemma 2.105. We remark that open-ness of $\mathfrak W^+(\mathfrak p)$ was used to define the set $\mathfrak C(\mathfrak p)$ and so was used in the proof of Theorems 2.70 and 2.72. However the argument is not circular by the same reason as we explained in the proof of Lemma 2.50 above.

2.7. **Construction of Kuranishi chart.** In Lemma 2.94, Propositin 2.95, Lemma 2.98, *strata-wise* differentiable structures of the spaces $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)^{\vec w_{\mathfrak p}^-,\mathfrak B^-}_{\epsilon_0,\vec T_0}$ and $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec T_0}$ or maps among them are discussed. These spaces are actually differentiable manifolds with corners and the maps are differentiable maps between them. As we mentioned in [FOOO1, page 771-773] this is a consequence of the exponential decay estimate (Theorems 1.34 and 2.72). We first discuss this point in detail here.

Let $V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B;\epsilon_1)$ be as in (2.212). We put

$$\begin{aligned}&V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B;\epsilon_1)^{\vec w_{\mathfrak p}^-,\mathfrak B^-}\\ &= \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\vec w_{\mathfrak p}^-,\mathfrak B^-}_{\epsilon_2,\vec T_0}\cap V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\epsilon_1)\end{aligned}\tag{2.309}$$

$$\begin{aligned}&V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\epsilon_1)^{\rm trans}\\ &= \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_2,\vec T_0}\cap V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\epsilon_1).\end{aligned}\tag{2.310}$$

(See Definitions 2.92, 2.97.) We note that the right hand side is independent of ϵ_2 and $\vec T_0$ if ϵ_1 is sufficiently small. By Proposition 2.95 (1) and Lemma 2.98, $V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B;\epsilon_1)^{\vec w_{\mathfrak p}^-,\mathfrak B^-}$ and $V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\epsilon_1)^{\rm trans}$ are C^m-submanifolds.

Proposition 2.112. *There exist strata-wise C^m-maps*

$$\begin{aligned}{\rm End}_{\vec w_{\mathfrak p}^-,\mathfrak B^-}:&V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B;\epsilon_1)^{\vec w_{\mathfrak p}^-,\mathfrak B^-}\times(\vec T_0^{\rm o},\infty]\times((\vec T_0^{\rm c},\infty]\times\vec S^1)\\ &\to V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B;\epsilon_1)\end{aligned}$$

and

$$\begin{aligned}{\rm End}_{\rm trans}: V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\epsilon_1)^{\rm trans}&\times(\vec T_0^{\rm o},\infty]\times((\vec T_0^{\rm c},\infty]\times\vec S^1)\\ &\to V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\epsilon_0)\end{aligned}$$

with the following properties.

(1) $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)^{\vec w_{\mathfrak p}^-,\mathfrak B^-}_{\epsilon_0,\vec T_0}$ *is described by the map* ${\rm End}_{\vec w_{\mathfrak p}^-,\mathfrak B^-}$ *as follows:*

$$\begin{aligned}&\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)^{\vec w_{\mathfrak p}^-,\mathfrak B^-}_{\epsilon_0,\vec T_0}\\ &=\Big\{{\rm Glu}({\rm End}_{\vec w_{\mathfrak p}^-,\mathfrak B^-}(\mathfrak q,(\vec T,\vec\theta)),\vec T,\vec\theta)\\ &\qquad\mid(\mathfrak q,(\vec T,\vec\theta))\in V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B;\epsilon_1)^{\vec w_{\mathfrak p}^-,\mathfrak B^-}\times(\vec T_0^{\rm o},\infty]\times((\vec T_0^{\rm c},\infty]\times\vec S^1)\Big\}.\end{aligned}$$

We also have

$$\begin{aligned}&\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec T_0}\\ &=\Big\{{\rm Glu}({\rm End}_{\rm trans}(\mathfrak q,(\vec T,\vec\theta)),\vec T,\vec\theta)\\ &\qquad\mid(\mathfrak q,(\vec T,\vec\theta))\in V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\epsilon_1)^{\rm trans}\times(\vec T_0^{\rm o},\infty]\times((\vec T_0^{\rm c},\infty]\times\vec S^1)\Big\}.\end{aligned}$$

(2) *The maps* $\mathrm{End}_{\vec w_{\mathfrak p}^-,\mathfrak B^-}$ *and* $\mathrm{End}_{\rm trans}$ *enjoy the following exponential decay estimate.*

$$\left\Vert \nabla_{\mathfrak q}^n \frac{\partial^{|\vec k_T|}}{\partial T^{\vec k_T}}\frac{\partial^{|\vec k_\theta|}}{\partial \theta^{\vec k_\theta}} \mathrm{End}_{\vec w_{\mathfrak p}^-,\mathfrak B^-}\right\Vert_{C^0} < C_{16,m,\vec R}e^{-\delta'(\vec k_T\cdot\vec T+\vec k_\theta\cdot\vec T^{\rm c})} \tag{2.311}$$

$$\left\Vert \nabla_{\mathfrak q}^n \frac{\partial^{|\vec k_T|}}{\partial T^{\vec k_T}}\frac{\partial^{|\vec k_\theta|}}{\partial \theta^{\vec k_\theta}} \mathrm{End}_{\rm trans}\right\Vert_{C^0} < C_{16,m,\vec R}e^{-\delta'(\vec k_T\cdot\vec T+\vec k_\theta\cdot\vec T^{\rm c})} \tag{2.312}$$

if $n+|\vec k_T|+|\vec k_\theta|\le m$. *Here* $\nabla_{\mathfrak q}^n$ *is a derivation of the direction of the parameter space* $V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B;\epsilon_1)^{\vec w_{\mathfrak p}^-,\mathfrak B^-}$ *or of the parameter space* $V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\epsilon_1)^{\rm trans}$.

Proof. We prove the estimate for the case of $\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)^{\vec w_{\mathfrak p}^-,\mathfrak B^-}_{\epsilon_0,\vec T_0}$. The other case is entirely similar.

We consider the evaluation map (2.288)

$$\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0,\vec T_0} \to X^{(\ell_{\mathfrak p}-\ell_{\mathfrak p}^-)+\sum_{c\in\mathfrak B\setminus\mathfrak B^-}\ell_c} \tag{2.313}$$

and compose it with (2.212)

$$\begin{aligned}\mathrm{Glu}: V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B;\epsilon_1)\times(\vec T_0^{\rm o},\infty]\times((\vec T_0^{\rm c},\infty]\times\vec S^1)\\ \to \mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0,\vec T_0}\end{aligned}$$

to obtain

$$\begin{aligned}\mathrm{ev}_{\vec w_{\mathfrak p}^-,\mathfrak B^-}: V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B;\epsilon_1)\times(\vec T_0^{\rm o},\infty]\times((\vec T_0^{\rm c},\infty]\times\vec S^1)\\ \to X^{(\ell_{\mathfrak p}-\ell_{\mathfrak p}^-)+\sum_{c\in\mathfrak B\setminus\mathfrak B^-}\ell_c}.\end{aligned} \tag{2.314}$$

Lemma 2.113. *The map* $\mathrm{ev}_{\vec w_{\mathfrak p}^-,\mathfrak B^-}$ *enjoys the following exponential decay estimate.*

$$\left\Vert \nabla_{\rho}^n \frac{\partial^{|\vec k_T|}}{\partial T^{\vec k_T}}\frac{\partial^{|\vec k_\theta|}}{\partial \theta^{\vec k_\theta}} \mathrm{ev}_{\vec w_{\mathfrak p}^-,\mathfrak B^-}\right\Vert_{C^0} < C_{17,m,\vec R}e^{-\delta'(\vec k_T\cdot\vec T+\vec k_\theta\cdot\vec T^{\rm c})}, \tag{2.315}$$

if $n+|\vec k_T|+|\vec k_\theta|\le m$. *Here* ∇_{ρ}^n *is a derivation of the direction of the parameter space* $V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B;\epsilon_1)^{\vec w_{\mathfrak p}^-,\mathfrak B^-}$.

Proof. We remark that (2.314) factors through

$$\begin{aligned}\mathrm{Glures}: &V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B;\epsilon_1)\times(\vec T_0^{\rm o},\infty]\times((\vec T_0^{\rm c},\infty]\times\vec S^1)\\ &\to \prod_{{\rm v}\in C^0(\mathcal G_{\mathfrak p})} L^2_m((K_{\rm v}^{+\vec R},K_{\rm v}^{+\vec R}\cap\partial\Sigma_{\mathfrak p,{\rm v}}),(X,L)).\end{aligned} \tag{2.316}$$

In fact we may take $\vec R$ so that all the marked points are in the extended core $\bigcup_{{\rm v}\in C^0(\mathcal G_{\mathfrak p})}K_{\rm v}^{+\vec R}$. Therefore the lemma is an immediate consequence of Theorem 2.72. ∎

By definition, we have:

$$V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B;\epsilon_0)^{\vec w_{\mathfrak p}^-,\mathfrak B^-} = \mathrm{ev}^{-1}_{\vec w_{\mathfrak p}^-,\mathfrak B^-}\left(\prod_{i=\ell_{\mathfrak p}^-+1}^{\ell_{\mathfrak p}}\mathcal D_{\mathfrak p,i}\times\prod_{c\in\mathfrak B\setminus\mathfrak B^-}\prod_{i=1}^{\ell_c}\mathcal D_{c,i}\right).$$

(See the proof of Proposition 2.95.) Proposition 2.112 is then a consequence of Lemma 2.113 and the implicit function theorem. □

We next change the coordinate of $(\vec{T}_0^{\rm o},\infty] \times ((\vec{T}_0^{\rm c},\infty] \times \vec{S}^1)$. The original coordinates are $((T_{\rm e}),(\theta_{\rm e})) \in (\vec{T}_0^{\rm o},\infty] \times ((\vec{T}_0^{\rm c},\infty] \times \vec{S}^1)$.

Definition 2.114. We define

$$\begin{aligned} s_{\rm e} = \frac{1}{T_{\rm e}} &\in \left[0,\frac{1}{T_{{\rm e},0}}\right), \qquad \text{if } {\rm e} \in C^1_{\rm o}(\mathcal{G}_{\mathfrak p}),\\ \mathfrak{z}_{\rm e} = \frac{1}{T_{\rm e}}\exp(2\pi\sqrt{-1}\theta_{\rm e}) &\in D^2\left(\frac{1}{T_{{\rm e},0}}\right), \qquad \text{if } {\rm e} \in C^1_{\rm c}(\mathcal{G}_{\mathfrak p}). \end{aligned} \tag{2.317}$$

We also put $s_{\rm e} = 0$ (resp. $\mathfrak{z}_{\rm e} = 0$) if $T_{\rm e} = \infty$. Here we put $D^2(r) = \{z \in \mathbb{C} \mid |z| < r\}$.

By this change of coordinates, $(\vec{T}_0^{\rm o},\infty] \times ((\vec{T}_0^{\rm c},\infty] \times \vec{S}^1)$ is identified with

$$\prod_{{\rm e}\in C^1_{\rm o}(\mathcal{G}_{\mathfrak p})}\left[0,\frac{1}{T_{{\rm e},0}}\right) \times \prod_{{\rm e}\in C^1_{\rm o}(\mathcal{G}_{\mathfrak p})} D^2\left(\frac{1}{T_{{\rm e},0}}\right). \tag{2.318}$$

Definition 2.115. We denote the right hand side of (2.318) as $[0,(\vec{T}_0^{\rm o})^{-1}) \times D^2((\vec{T}_0^{\rm c})^{-1})$.

Remark 2.116. The space $[0,(\vec{T}_0^{\rm o})^{-1}) \times D^2((\vec{T}_0^{\rm c})^{-1})$ has a stratification that is induced by the stratification

$$[0,1/T_{{\rm e},0}) = \{0\} \cup (0,1/T_{{\rm e},0})$$

and

$$D^2(1/T_{{\rm e},0}) = \{0\} \cup (D^2(1/T_{{\rm e},0}) \setminus \{0\}).$$

This stratification corresponds to the stratification of $(\vec{T}_0^{\rm o},\infty] \times ((\vec{T}_0^{\rm c},\infty] \times \vec{S}^1)$ that we defined before, by the homeomorphism (2.317).

We note that $[0,(\vec{T}_0^{\rm o})^{-1}) \times D^2((\vec{T}_0^{\rm c})^{-1})$ is a smooth manifold with corner. The above stratification is finer than its stratification associated to the structure of manifold with corner.

We then regard Glu as a map

$$\begin{aligned} {\rm Glu}' : V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B;\epsilon_1) &\times [0,(\vec{T}_0^{\rm o})^{-1}) \times D^2((\vec{T}_0^{\rm c})^{-1})\\ &\to \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0,\vec{T}_0}. \end{aligned} \tag{2.319}$$

Corollary 2.117. *The inverse image*

$$({\rm Glu}')^{-1}\left(\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)^{\vec{w}^-_{\mathfrak p},\mathfrak B^-}_{\epsilon_0,\vec{T}_0}\right)$$

is a C^m-submanifold of $V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B;\epsilon_1) \times [0,(\vec{T}_0^{\rm o})^{-1}) \times D^2((\vec{T}_0^{\rm c})^{-1})$. It is transversal to the strata of the stratification mentioned in Remark 2.116.

The same holds for $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec{T}_0}$.

This is an immediate consequence of Proposition 2.112.

Remark 2.118. We can actually promote this C^m structure to a C^∞-structure as we will explain in Subsection 3.2. The same remark applies to all the constructions of Subsections 2.7-2.10.

Definition 2.119. We put

$$V_{k+1,\ell}((\beta;\mathfrak p;\mathfrak A);\epsilon_0,\vec T_0) = \mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec T_0}$$

and regard it as a C^m-manifold with corner so that Glu$'$ is a C^m-diffeomorphism.

Lemma 2.120. *The action of* $\Gamma_{\mathfrak p}$ *on* $V_{k+1,\ell}((\beta;\mathfrak p;\mathfrak A);\epsilon_0,\vec T_0)$ *is of* C^m*-class.*

Proof. Note the $\Gamma_{\mathfrak p}$-action on $(\vec T^{\rm o}_0,\infty]\times((\vec T^{\rm c}_0,\infty]\times\vec S^1)$ is by exchanging the factors associated to the edges e and by the rotation of the S^1 factors. Therefore it becomes a smooth action on $[0,(\vec T^{\rm o}_0)^{-1})\times D^2((\vec T^{\rm c}_0)^{-1})$. By construction Glu$'$ is $\Gamma_{\mathfrak p}$-equivariant. The lemma follows. □

The orbifold $V_{k+1,\ell}((\beta;\mathfrak p;\mathfrak A);\epsilon_0,\vec T_0)/\Gamma_{\mathfrak p}$ is a chart of the Kuranishi neighborhood of $\mathfrak p$ which we define in this subsection. Note we may assume that the action of $\Gamma_{\mathfrak p}$ to $V_{k+1,\ell}((\beta;\mathfrak p;\mathfrak A);\epsilon_0,\vec T_0)$ is effective, by increasing the obstruction bundle if necessary.

We next define an obstruction bundle. Recall that we fixed a complex vector space E_c for each $c\in\mathfrak A$. ($E_c=\bigoplus_{{\rm v}\in C^0(\mathcal G_{\mathfrak p_c})}E_{c,{\rm v}}$ and $E_{c,{\rm v}}$ is a subspace of $\Gamma_0({\rm Int}\,K^{\rm obst}_{\rm v};u^*_{\mathfrak p_c}TX\otimes\Lambda^{01})$.) By Definition 2.33 (5), E_c carries a $\Gamma_{\mathfrak p_c}$ action. It follows that $\Gamma_{\mathfrak p}\subset\Gamma_{\mathfrak p_c}$, because $\mathfrak p\cup\vec w^{\mathfrak p}_c$ is ϵ_c-close to $\mathfrak p_c\cup\vec w_{\mathfrak p_c}$. Therefore we have a $\Gamma_{\mathfrak p}$-action on

$$E_{\mathfrak A}=\bigoplus_{c\in\mathfrak A}E_c.$$

Definition 2.121. The obstruction bundle of our Kuranishi chart is the bundle

$$\frac{\Big(V_{k+1,\ell}((\beta;\mathfrak p;\mathfrak A);\epsilon_0,\vec T_0)\times E_{\mathfrak A}\Big)}{\Gamma_{\mathfrak p}}\to\frac{\Big(V_{k+1,\ell}((\beta;\mathfrak p;\mathfrak A);\epsilon_0,\vec T_0)\Big)}{\Gamma_{\mathfrak p}}. \tag{2.320}$$

We next define the Kuranishi map, that is a section of the obstruction bundle. Let $\mathfrak q^+=(\mathfrak x_{\mathfrak q},u_{\mathfrak q};(\vec w^{\mathfrak q}_c;c\in\mathfrak A))\in\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec T_0}$. By definition we have

$$\overline\partial u_{\mathfrak q}\in\mathcal E_{\mathfrak A}(\mathfrak q^+).$$

By Definition 2.60 we have an isomorphism (2.207)

$$I^{{\rm v},\mathfrak p_c}_{(\mathfrak y_c,u_c),(\mathfrak x_{\mathfrak q}\cup\vec w^{\mathfrak q}_c,u_{\mathfrak q})}:E_{\mathfrak p_c,{\rm v}}(\mathfrak y_c,u_c)\to\Gamma_0({\rm Int}\,K^{\rm obst}_{\rm v};(u_{\mathfrak q})^*TX\otimes\Lambda^{01}). \tag{2.321}$$

The direct sum of the right hand side over $c\in\mathfrak A$ and ${\rm v}\in C^0(\mathcal G_{\mathfrak p_c})$ is by definition $\mathcal E_{\mathfrak A}(\mathfrak q^+)$. Sending the element $\overline\partial u_{\mathfrak q}$ by the inverse of $I^{{\rm v},\mathfrak p_c}_{(\mathfrak y_c,u_c),(\mathfrak x_{\mathfrak q}\cup\vec w^{\mathfrak q}_c,u_{\mathfrak q})}$ we obtain an element

$$\bigoplus_{\substack{c\in\mathfrak A\\ {\rm v}\in C^0(\mathcal G_{\mathfrak p_c})}}{I^{{\rm v},\mathfrak p_c}_{(\mathfrak y_c,u_c),(\mathfrak x_{\mathfrak q}\cup\vec w^{\mathfrak q}_c,u_{\mathfrak q})}}^{-1}(\overline\partial u_{\mathfrak q})\in E_{\mathfrak A}. \tag{2.322}$$

Definition 2.122. We denote the element (2.322) by $\mathfrak s(\mathfrak q^+)$. The section $\mathfrak s$ is called the *Kuranishi map.*

Lemma 2.123. *The section* $\mathfrak s$ *defined above is a section of* C^m*-class of the obstruction bundle in Definition 2.121 and is* $\Gamma_{\mathfrak p}$*-equivariant.*

Proof. The $\Gamma_{\mathfrak p}$-equivariance is immediate from its construction.

To prove that $\mathfrak s$ is of C^m-class, we first remark that $\mathfrak s$ is extended to the thickened moduli space $\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)_{\epsilon_0,\vec T_0}$ by the same formula. We consider the composition of $\mathfrak q^+ \mapsto \mathfrak s(\mathfrak q^+)$ with the map Glu' (2.319). Since $K_{\rm v}^{\rm obst}$ lies in the core this composition factors through Glures (2.317). (Here we identify $(\vec T_0^{\rm o},\infty] \times ((\vec T_0^{\rm c},\infty]\times \vec S^1)$ with $[0,(\vec T_0^{\rm o})^{-1})\times D^2((\vec T_0^{\rm c})^{-1})$.) Therefore by Theorem 2.72 we have

$$\left\Vert \nabla_{\rho}^n \frac{\partial^{|\vec k_T|}}{\partial T^{\vec k_T}}\frac{\partial^{|\vec k_\theta|}}{\partial \theta^{\vec k_\theta}}(\mathfrak s\circ \mathrm{Glu})\right\Vert_{C^0} < C_{18,m,\vec R}e^{-\delta'(\vec k_T\cdot \vec T+\vec k_\theta\cdot\vec T^{\rm c})}, \tag{2.323}$$

if $n+|\vec k_T|+|\vec k_\theta|\le m$. Therefore $\mathfrak s$ is of C^m-class. □

We note that the zero set of the section $\mathfrak s$ coincides with the set

$$\mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec T_0}\cap \mathfrak s^{-1}(0)$$

which we defined in Definition 2.100.

Definition 2.124. We define a local parametrization map

$$\psi : \frac{\mathfrak s^{-1}(0)}{\Gamma_{\mathfrak p}} \to \mathcal M_{k+1,\ell}(\beta)$$

to be the map (2.293).

Proposition 2.102 implies that ψ is a homeomorphism to an open neighborhood of $\mathfrak p$.

In summary we have proved the following:

Proposition 2.125. *Let $\mathfrak p\in \mathcal M_{k+1,\ell}(\beta)$. We take a stabilization data at $\mathfrak p$ and $\mathfrak A\subset \mathfrak C(\mathfrak p)$. ($\mathfrak A\ne\emptyset$.) Then there exists a Kuranishi neighborhood of $\mathcal M_{k+1,\ell}(\beta)$ at $\mathfrak p$. Namely :*

1. *An (effective) orbifold $V_{k+1,\ell}((\beta;\mathfrak p;\mathfrak A);\epsilon_0,\vec T_0)/\Gamma_{\mathfrak p}$.*
2. *A vector bundle*
$$\frac{\Big(V_{k+1,\ell}((\beta;\mathfrak p;\mathfrak A);\epsilon_0,\vec T_0)\times E_{\mathfrak A}\Big)}{\Gamma_{\mathfrak p}} \to \frac{\Big(V_{k+1,\ell}((\beta;\mathfrak p;\mathfrak A);\epsilon_0,\vec T_0)\Big)}{\Gamma_{\mathfrak p}}$$
on it.
3. *Its section $\mathfrak s$ of C^m-class.*
4. *A homeomorphism*
$$\psi : \frac{\mathfrak s^{-1}(0)}{\Gamma_{\mathfrak p}} \to \mathcal M_{k+1,\ell}(\beta)$$
onto an open neighborhood of $\mathfrak p$ in $\mathcal M_{k+1,\ell}(\beta)$.

Before closing this subsection, we prove that the evaluation maps on $\mathcal M_{k+1,\ell}(\beta)$ are extended to our Kuranishi neighborhood as C^m-maps.

We consider the map

$$\mathrm{ev} : V_{k+1,\ell}((\beta;\mathfrak p;\mathfrak A);\epsilon_0,\vec T_0) = \mathcal M_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec T_0} \to L^{k+1}\times X^{\ell} \tag{2.324}$$

that is the evaluation map at the 0-th,…,k-th boundary marked points and 1st - ℓ-th interior marked points.

Lemma 2.126. *The map (2.324) is a C^m-map and is $\Gamma_{\mathfrak p}$-equivariant.*

Proof. We first remark that (2.324) extends to $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A})_{\epsilon_0,\vec{T}_0}$. Its composition with Glu factors through Glures (2.317). Therefore by Theorem 2.72 we have

$$\left\Vert \nabla_\rho^n \frac{\partial^{|\vec{k}_T|}}{\partial T^{\vec{k}_T}} \frac{\partial^{|\vec{k}_\theta|}}{\partial \theta^{\vec{k}_\theta}} (\mathrm{ev}\circ\mathrm{Glu}) \right\Vert_{C^0} < C_{19,m,\vec{R}} e^{-\delta'(\vec{k}_T\cdot\vec{T}+\vec{k}_\theta\cdot\vec{T}^{\rm c})}, \tag{2.325}$$

if $n+|\vec{k}_T|+|\vec{k}_\theta|\le m$. Therefore ev is of C^m-class. $\Gamma_{\mathfrak{p}}$ equivariance is immediate from definition. □

Remark 2.127. Proposition 2.125 holds and can be proved when we replace $\mathcal{M}_{k+1,\ell}(\beta)$ by $\mathcal{M}^{\rm cl}_{\ell}(\alpha)$. The proof is the same.

2.8. **Coordinate change - I: Change of the stabilization and of the coordinate at infinity.** In this subsection and the next, we define coordinate change between Kuranishi charts we constructed in the last subsection and prove a version of compatibility of the coordinate changes. In Subsection 2.10 we will adjust the sizes of the Kuranishi charts and of the domains of the coordinate changes so that they literally satsifiy the definition of the Kuranishi structure.

We begin with recalling the facts we have proved so far. We take a finite set $\{\mathfrak{p}_c \mid c\in\mathfrak{C}\}\subset\mathcal{M}_{k+1,\ell}(\beta)$ and fix an obstruction bundle data $\mathfrak{E}_{\mathfrak{p}_c}$ centered at each $\mathfrak{p}_c$.

Let $\mathfrak{w}_{\mathfrak{p}}$ be a stabilization data at $\mathfrak{p}\in\mathcal{M}_{k+1,\ell}(\beta)$. The stabilization data $\mathfrak{w}_{\mathfrak{p}}$ consists of the following:

1. The additional marked points $\vec{w}_{\mathfrak{p}}$ of $\mathfrak{x}_{\mathfrak{p}}$.
2. The codimension 2 submanifolds $\mathcal{D}_{\mathfrak{p},i}$.
3. A coordinate at infinity of $\mathfrak{x}_{\mathfrak{p}}\cup\vec{w}_{\mathfrak{p}}$.

By an abuse of notation we denote the coordinate at infinity also by $\mathfrak{w}_{\mathfrak{p}}$ from now on. Let $\ell_{\mathfrak{p}}=\#\vec{w}_{\mathfrak{p}}$ and $\mathfrak{A}\subset\mathfrak{C}(\mathfrak{p})$. We always assume that $\mathfrak{A}\neq\emptyset$.

By taking a sufficiently small ϵ_0 and sufficiently large $\vec{T}_0$, we obtained a Kuranishi chart at $\mathfrak{p}$ by Proposition 2.125. The Kuranishi neighborhood is $V_{k+1,\ell}((\beta;\mathfrak{p};\mathfrak{A});\epsilon_0,\vec{T}_0)/\Gamma_{\mathfrak{p}}$. This Kuranish chart depends on $\epsilon_0,\vec{T}_0$ as well as $\mathfrak{w}_{\mathfrak{p}}$. During the construction of the coordinate change, we need to shrink this chart several times. We use a pair of positive numbers $(\mathfrak{o},\mathcal{T})$ to specify the size as follows. We consider

$$\begin{aligned}\mathrm{Glu}: V_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\epsilon_1)\times(\vec{T}^{\rm o}_0,\infty]\times((\vec{T}^{\rm c}_0,\infty]\times\vec{S}^1)\\ \to\mathcal{M}^{\mathfrak{w}_{\mathfrak{p}}}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A})_{\epsilon_0,\vec{T}_0}.\end{aligned}\tag{2.326}$$

Remark 2.128. Here and hereafter we include the symbol $\mathfrak{w}_{\mathfrak{p}}$ in the notation of the thickened moduli space, to show the stabilization data at $\mathfrak{p}$ that we use to define it. In fact the dependence of the thickened moduli space on the stabilization data is an important point to study in this subsection.

$V_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\epsilon_1)$ is a smooth manifold. We fix a metric on it. Let

$$B^{\mathfrak{w}_{\mathfrak{p}}}_{\mathfrak{o}}(\mathfrak{p};V_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\epsilon_1))\tag{2.327}$$

be the $\mathfrak{o}$ neighborhood of $\mathfrak{p}$ in this space. We put $T_{{\rm e},0}=\mathcal{T}$ for all e and denote it by $\vec{\mathcal{T}}$. Since this space is independent of ϵ_1 if $\mathfrak{o}$ is sufficiently small compared to ϵ_1 we omit ϵ_1 from the notation. We consider

$$B^{\mathfrak{w}_{\mathfrak{p}}}_{\mathfrak{o}}(\mathfrak{p};V_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A}))\times(\vec{\mathcal{T}},\infty]\times((\vec{\mathcal{T}},\infty]\times\vec{S}^1).\tag{2.328}$$

Definition 2.129. We say that $(\mathfrak{o},\mathcal{T})$ is $\mathfrak{w}_{\mathfrak{p}}$ *admissible* if the domain of the map (2.326) includes (2.328). We say it is admissible if it is clear which stabilization data we take.

We say $(\mathfrak{o},\mathcal{T}) > (\mathfrak{o}',\mathcal{T}')$ if $\mathfrak{o} > \mathfrak{o}'$ and $1/\mathcal{T} > 1/\mathcal{T}'$.

Definition 2.130. We denote by $V(\mathfrak{p},\mathfrak{w}_{\mathfrak{p}};(\mathfrak{o},\mathcal{T});\mathfrak{A})$ the intersection of the image of the set (2.328) by the map (2.326) and $\mathcal{M}^{\mathfrak{w}_{\mathfrak{p}}}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A})^{\rm trans}_{\epsilon_0,\vec{T}_0}$.

The restrictions of the obstruction bundle, Kuranishi map, and the map ψ to $V(\mathfrak{p},\mathfrak{w}_{\mathfrak{p}};(\mathfrak{o},\mathcal{T});\mathfrak{A})$ are written as $\mathcal{E}_{\mathfrak{p},\mathfrak{w}_{\mathfrak{p}};(\mathfrak{o},\mathcal{T});\mathfrak{A}}$. and $\mathfrak{s}_{\mathfrak{p},\mathfrak{w}_{\mathfrak{p}};(\mathfrak{o},\mathcal{T});\mathfrak{A}}$, $\psi_{\mathfrak{p},\mathfrak{w}_{\mathfrak{p}};(\mathfrak{o},\mathcal{T});\mathfrak{A}}$, respectively.

They define a Kuranishi chart. Sometimes we denote by $V(\mathfrak{p},\mathfrak{w}_{\mathfrak{p}};(\mathfrak{o},\mathcal{T});\mathfrak{A})$ this Kuranishi chart, by an abuse of notation.

The main result of this subsection is the following.

Proposition 2.131. *Let $\mathfrak{w}^{(j)}_{\mathfrak{p}}$, $(j=1,2)$ be stabilization data at $\mathfrak{p}$ and $\mathfrak{A} \supseteq \mathfrak{A}^{(1)} \supseteq \mathfrak{A}^{(2)} \ne \emptyset$. Suppose $(\mathfrak{o}^{(1)},\mathcal{T}^{(1)})$ is $\mathfrak{w}^{(1)}_{\mathfrak{p}}$ admissible.*

Then there exists $(\mathfrak{o}^{(2)}_0,\mathcal{T}^{(2)}_0)$ such that if $(\mathfrak{o}^{(2)},\mathcal{T}^{(2)}) < (\mathfrak{o}^{(2)}_0,\mathcal{T}^{(2)}_0)$ then $(\mathfrak{o}^{(2)},\mathcal{T}^{(2)})$ is $\mathfrak{w}^{(2)}_{\mathfrak{p}}$ admissible and we have a coordinate change from $V(\mathfrak{p},\mathfrak{w}^{(2)}_{\mathfrak{p}};(\mathfrak{o}^{(2)},\mathcal{T}^{(2)});\mathfrak{A}^{(2)})$ to $V(\mathfrak{p},\mathfrak{w}^{(1)}_{\mathfrak{p}};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(1)})$. Namely there exists $(\varphi_{12},\widehat{\varphi}_{12})$ with the following properties.

(1)
$$\varphi_{12} : V(\mathfrak{p},\mathfrak{w}^{(2)}_{\mathfrak{p}};(\mathfrak{o}^{(2)},\mathcal{T}^{(2)});\mathfrak{A}^{(2)}) \to V(\mathfrak{p},\mathfrak{w}^{(1)}_{\mathfrak{p}};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(1)})$$
is a $\Gamma_{\mathfrak{p}}$-equivariant C^m embedding.

(2)
$$\widehat{\varphi}_{12} : \mathcal{E}_{\mathfrak{p},\mathfrak{w}^{(2)}_{\mathfrak{p}};(\mathfrak{o}^{(2)},\mathcal{T}^{(2)});\mathfrak{A}^{(2)}} \to \mathcal{E}_{\mathfrak{p},\mathfrak{w}^{(1)}_{\mathfrak{p}};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(1)}}$$
is a $\Gamma_{\mathfrak{p}}$-equivariant embedding of vector bundles of C^m-class that covers φ_{12}.

(3) *The next equality holds.*
$$\mathfrak{s}_{\mathfrak{p},\mathfrak{w}^{(1)}_{\mathfrak{p}};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(1)}} \circ \varphi_{12} = \widehat{\varphi}_{12} \circ \mathfrak{s}_{\mathfrak{p},\mathfrak{w}^{(2)}_{\mathfrak{p}};(\mathfrak{o}^{(2)},\mathcal{T}^{(2)});\mathfrak{A}^{(2)}}.$$

(4) *The next equality holds on $\mathfrak{s}^{-1}_{\mathfrak{p},\mathfrak{w}^{(2)}_{\mathfrak{p}};(\mathfrak{o}^{(2)},\mathcal{T}^{(2)});\mathfrak{A}^{(2)}}(0)$.*
$$\tilde{\psi}_{\mathfrak{p},\mathfrak{w}^{(1)}_{\mathfrak{p}};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(1)}} \circ \varphi_{12} = \tilde{\psi}_{\mathfrak{p},\mathfrak{w}^{(2)}_{\mathfrak{p}};(\mathfrak{o}^{(2)},\mathcal{T}^{(2)});\mathfrak{A}^{(2)}}.$$
Here $\tilde{\psi}_{\mathfrak{p},\mathfrak{w}^{(1)}_{\mathfrak{p}};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(1)}}$ is the composition of $\psi_{\mathfrak{p},\mathfrak{w}^{(1)}_{\mathfrak{p}};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(1)}}$ and the projection map
$$V(\mathfrak{p},\mathfrak{w}^{(1)}_{\mathfrak{p}};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(1)}) \to V(\mathfrak{p},\mathfrak{w}^{(1)}_{\mathfrak{p}};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(1)})/\Gamma_{\mathfrak{p}}.$$
The definition of $\tilde{\psi}_{\mathfrak{p},\mathfrak{w}^{(2)}_{\mathfrak{p}};(\mathfrak{o}^{(2)},\mathcal{T}^{(2)});\mathfrak{A}^{(2)}}$ is similar.

(5) *Let $\mathfrak{q}^{(2)} \in V(\mathfrak{p},\mathfrak{w}^{(2)}_{\mathfrak{p}};(\mathfrak{o}^{(2)},\mathcal{T}^{(1)});\mathfrak{A}^{(2)})$ and $\mathfrak{q}^{(1)} = \varphi_{12}(\mathfrak{p})$. Then the derivative of $\mathfrak{s}_{\mathfrak{p},\mathfrak{w}^{(2)}_{\mathfrak{p}};(\mathfrak{o}^{(2)},\mathcal{T}^{(2)});\mathfrak{A}^{(2)}}$ induces an isomorphism*
$$\frac{T_{\mathfrak{q}^{(1)}}V(\mathfrak{p},\mathfrak{w}^{(1)}_{\mathfrak{p}};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(1)})}{T_{\mathfrak{q}^{(2)}}V(\mathfrak{p},\mathfrak{w}^{(2)}_{\mathfrak{p}};(\mathfrak{o}^{(2)},\mathcal{T}^{(2)});\mathfrak{A}^{(2)})} \cong \frac{\left(\mathcal{E}_{\mathfrak{p},\mathfrak{w}^{(1)}_{\mathfrak{p}};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(1)}}\right)_{\mathfrak{q}^{(1)}}}{\left(\mathcal{E}_{\mathfrak{p},\mathfrak{w}^{(2)}_{\mathfrak{p}};(\mathfrak{o}^{(2)},\mathcal{T}^{(2)});\mathfrak{A}^{(2}}\right)_{\mathfrak{q}^{(2)}}}.$$

Proof. We divide the proof into several cases.

Case 1: The case $\vec w_{\mathfrak p}^{(1)} = \vec w_{\mathfrak p}^{(2)}$, $\mathcal D_{\mathfrak p,i}^{(1)} = \mathcal D_{\mathfrak p,i}^{(2)}$ and $\mathfrak A^{(1)} = \mathfrak A^{(2)}$.

This is the case when only the coordinate at infinity $\mathfrak w_{\mathfrak p}^{(1)}$ is different from $\mathfrak w_{\mathfrak p}^{(2)}$. A part of the data of the coordinate at infinity is a fiber bundle (2.154) that is:

$$\pi : \mathfrak M^{(j)}_{(\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p})_{\rm v}} \to \mathfrak V^{(j)}((\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p})_{\rm v}) \tag{2.329}$$

where $\mathfrak V^{(j)}((\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p})_{\rm v})$ is a neighborhood of $(\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p})_{\rm v}$ in the Deligne-Mumford moduli space $\mathcal M_{k_{\rm v}+1,\ell_{\rm v}}$ or $\mathcal M^{\rm cl}_{\ell_{\rm v}}$. (${\rm v}\in C^0(\mathcal G_{\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p}})$.) We choose $\mathfrak V^{(2)-}((\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p})_{\rm v})\subset \mathfrak V^{(j)}((\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p})_{\rm v})$ an open neighborhood of $(\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p})_{\rm v}$ so that

$$\mathfrak V^{(2)-}((\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p})_{\rm v})\subset \mathfrak V^{(1)}((\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p})_{\rm v}). \tag{2.330}$$

We put $\mathfrak M^{(2)-}_{(\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p})_{\rm v}} = \pi^{-1}(\mathfrak V^{(2)-}((\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p})_{\rm v}))$. Then there exists a unique bundle map

$$\Phi_{12} : \mathfrak M^{(2)-}_{(\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p})_{\rm v}} \to \mathfrak M^{(1)}_{(\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p})_{\rm v}}$$

that preserves the marked points and is a fiberwise biholomorphic map. This is because of the stability. By extending the core of $\mathfrak w_{\mathfrak p}^{(2)}$ we may assume

$$\Phi_{12}(\mathfrak K^{(2)-}_{(\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p})_{\rm v}}) \supset (\mathfrak K^{(1)}_{(\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p})_{\rm v}}) \cap \pi^{-1}(\mathfrak V^{(2)-}((\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p})_{\rm v})). \tag{2.331}$$

Lemma 2.132. *Let ϵ_0 and $\mathcal T^{(1)}$ be given, then there exist ϵ_0', $\mathcal T^{(2)}$ such that*

$$\mathcal M^{\mathfrak w_{\mathfrak p}^{(2)-}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)_{\epsilon_0',\vec{\mathcal T}^{(2)}} \subset \mathcal M^{\mathfrak w_{\mathfrak p}^{(1)}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)_{\epsilon_0,\vec{\mathcal T}^{(1)}}. \tag{2.332}$$

Here we define $\mathfrak w_{\mathfrak p}^{(2)-}$ from $\mathfrak w_{\mathfrak p}^{(2)}$ by shrinking $\mathfrak V^{(2)}((\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p})_{\rm v})$ to $\mathfrak V^{(2)-}((\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p})_{\rm v})$ and extending the core so that (2.331) is satisfied and use it to define the left hand side.

Proof. Since the equation (2.211) is independent of the stabilization data at $\mathfrak p$, it suffices to show

$$\mathfrak U^{\mathfrak w_{\mathfrak p}^{(2)-}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p)_{\epsilon_0',\vec{\mathcal T}^{(2)}} \subseteq \mathfrak U^{\mathfrak w_{\mathfrak p}^{(1)}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p)_{\epsilon_0,\vec{\mathcal T}^{(1)}}.$$

Here the meaning of the symbol '(2)−' and '(1)' is similar to (2.332).

An element of $\mathfrak U^{\mathfrak w_{\mathfrak p}^{(2)-}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p)_{\epsilon_0',\vec{\mathcal T}^{(2)}}$ is $(\mathfrak Y_0\cup\vec w_{\mathfrak p}', u', (\vec w_c'))$. Let us check that it satisfies (1)-(4) of Definition 2.58 applied to $\mathfrak U^{\mathfrak w_{\mathfrak p}^{(1)}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p)_{\epsilon_0,\vec{\mathcal T}^{(1)}}$.

(1) is obvious. (2) follows from (2.331). (4) is also obvious.

We will prove (3). We note that $\mathfrak p$ is ϵ_0 close to $\mathfrak p$ itself by our choice. So the diameter of the $u_{\mathfrak p}$ image of each connected component of the neck region (with respect to $\mathfrak w^{(1)}$) is smaller than ϵ_0. We take ϵ_0' so that the diameter of the $u_{\mathfrak p}$ image of each connected component of the neck region (with respect to $\mathfrak w^{(1)}$) is smaller than $\epsilon_0 - 2\epsilon_0'$. Now since the C^0 distance between u' and $u_{\mathfrak p}$ on the core of $\mathfrak w^{(2)}$ is small than ϵ_0',

$$\begin{aligned}&u'\big(\text{e-th neck with respect to } \mathfrak w_{\mathfrak p}^{(1)}\big)\\ &\subset \epsilon_0' \text{ neighborhood of } u_{\mathfrak p}\big(\text{e-th neck with respect to } \mathfrak w_{\mathfrak p}^{(2)}\big).\end{aligned}$$

(3) follows. ∎

Using the fact that $\mathcal{D}^{(1)}_{\mathfrak p,i} = \mathcal{D}^{(2)}_{\mathfrak p,i}$, Lemma 2.132 implies

$$\mathcal{M}^{\mathfrak{w}_{\mathfrak p}^{(2)-}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon'_0,\vec{\mathcal T}^{(2)}} \subset \mathcal{M}^{\mathfrak{w}_{\mathfrak p}^{(1)}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)^{\rm trans}_{\epsilon_0,\vec{\mathcal T}^{(1)}}. \tag{2.333}$$

Let

$$\begin{aligned}{\rm Glu}^{(1)} :&B^{\mathfrak{w}_{\mathfrak p}^{(1)}}_{\mathfrak o^{(1)}}(\mathfrak p; V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)) \\ &\times (\vec{\mathcal T}^{(1)},\infty] \times ((\vec{\mathcal T}^{(1)},\infty]\times \vec S^1) \to \mathcal{M}^{\mathfrak{w}_{\mathfrak p}^{(1)}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)_{\epsilon_0,\vec{\mathcal T}^{(1)}}\end{aligned} \tag{2.334}$$

and

$$\begin{aligned}{\rm Glu}^{(2)-} :&B^{\mathfrak{w}_{\mathfrak p}^{(2)-}}_{\mathfrak o^{(2)}}(\mathfrak p; V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)) \\ &\times (\vec{\mathcal T}^{(2)},\infty] \times ((\vec{\mathcal T}^{(2)},\infty]\times \vec S^1) \to \mathcal{M}^{\mathfrak{w}_{\mathfrak p}^{(2)-}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)_{\epsilon_0,\vec{\mathcal T}^{(2)}}\end{aligned}$$

be appropriate restrictions of (2.326). Its image is an open neighborhood of $\mathfrak p \cup \vec w_{\mathfrak p}$. Therefore there exists $(\mathfrak o^{(2)}_0, \mathcal T^{(2)}_0)$ such that for any $(\mathfrak o^{(2)}, \mathcal T^{(2)}) < (\mathfrak o^{(2)}_0, \mathcal T^{(2)}_0)$ we have

$$\begin{aligned}&{\rm Glu}^{(2)-}\big(B^{\mathfrak{w}_{\mathfrak p}^{(2)-}}_{\mathfrak o^{(2)}}(\mathfrak p; V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)) \times (\vec{\mathcal T}^{(2)},\infty] \times ((\vec{\mathcal T}^{(2)},\infty]\times \vec S^1)\big) \\ &\subset {\rm Glu}^{(1)}\big(B^{\mathfrak{w}_{\mathfrak p}^{(1)}}_{\mathfrak o^{(1)}}(\mathfrak p; V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)) \times (\vec{\mathcal T}^{(1)},\infty] \times ((\vec{\mathcal T}^{(1)},\infty]\times \vec S^1)\big).\end{aligned} \tag{2.335}$$

This in turn implies

$$V(\mathfrak p, \mathfrak w_{\mathfrak p}^{(2)}; (\mathfrak o^{(2)}, \mathcal T^{(2)}); \mathfrak A) \subset V(\mathfrak p, \mathfrak w_{\mathfrak p}^{(1)}; (\mathfrak o^{(1)}, \mathcal T^{(1)}); \mathfrak A).$$

Let φ_{12} be this natural inclusion.

Lemma 2.133. *φ_{12} is a C^m-map.*

Proof. Let

$$\begin{aligned}&\hat V(\mathfrak p, \mathfrak w_{\mathfrak p}^{(2)}; (\mathfrak o^{(2)}, \mathcal T^{(2)}); \mathfrak A) \\ &\subset B^{\mathfrak{w}_{\mathfrak p}^{(2)-}}_{\mathfrak o^{(2)}}(\mathfrak p; V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)) \times (\vec{\mathcal T}^{(2)},\infty] \times ((\vec{\mathcal T}^{(2)},\infty]\times \vec S^1)\end{aligned}$$

be the inverse image of $V(\mathfrak p, \mathfrak w_{\mathfrak p}^{(2)-}; (\mathfrak o^{(2)}, \mathcal T^{(2)}); \mathfrak A)$ by ${\rm Glu}^{(2)-}$ and let

$$\begin{aligned}&\hat V(\mathfrak p, \mathfrak w_{\mathfrak p}^{(1)}; (\mathfrak o^{(1)}, \mathcal T^{(1)}); \mathfrak A) \\ &\subset B^{\mathfrak{w}_{\mathfrak p}^{(1)}}_{\mathfrak o^{(1)}}(\mathfrak p; V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)) \times (\vec{\mathcal T}^{(1)},\infty] \times ((\vec{\mathcal T}^{(1)},\infty]\times \vec S^1)\end{aligned}$$

be the inverse image of $V(\mathfrak p, \mathfrak w_{\mathfrak p}^{(1)}; (\mathfrak o^{(1)}, \mathcal T^{(1)}); \mathfrak A)$ by ${\rm Glu}^{(1)}$.

We consider the maps

$$B^{\mathfrak{w}_{\mathfrak p}^{(1)}}_{\mathfrak o^{(1)}}(\mathfrak p; V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)) \to \prod_{{\rm v}\in C^0(\mathcal G_{\mathfrak p})} \mathfrak V^{(1)}((\mathfrak x_{\mathfrak p}\cup \vec w_{\mathfrak p})_{\rm v})$$

$$B^{\mathfrak{w}_{\mathfrak p}^{(2)-}}_{\mathfrak o^{(2)}}(\mathfrak p; V_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A)) \to \prod_{{\rm v}\in C^0(\mathcal G_{\mathfrak p})} \mathfrak V^{(2)-}((\mathfrak x_{\mathfrak p}\cup \vec w_{\mathfrak p})_{\rm v})$$

that forget the maps. (Namely it sends $(\mathfrak y, u')$ to $\mathfrak y$.)

We then define a map

$$
\begin{aligned}
&\mathfrak{F}^{(1)} : \hat{V}(\mathfrak{p}, \mathfrak{w}_{\mathfrak{p}}^{(1)}; (\mathfrak{o}^{(1)}, \mathcal{T}^{(1)}); \mathfrak{A}) \\
\to &\prod_{\mathrm{v}\in C^0(\mathcal{G}_{\mathfrak{p}})} C^m((K_{\mathrm{v},(1)}^{+\vec{R}_{(1)}}, K_{\mathrm{v},(1)}^{+\vec{R}_{(1)}} \cap \partial\Sigma_{\mathrm{v},(1)}), (X, L)) \\
&\times \prod_{\mathrm{v}\in C^0(\mathcal{G}_{\mathfrak{p}})} \mathfrak{V}^{(1)}((\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}_{\mathfrak{p}})_{\mathrm{v}}) \\
&\times (\vec{\mathcal{T}}^{(1)}, \infty] \times ((\vec{\mathcal{T}}^{(1)}, \infty] \times \vec{S}^1).
\end{aligned}
\tag{2.336}
$$

Here the first factor is induced by the map

$$
\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}^{\mathfrak{w}_{\mathfrak{p}}^{(1)}}(\beta; \mathfrak{p}; \mathfrak{A})_{\epsilon_0,\vec{T}_0} \to L^2_{m+10}((K_{\mathrm{v},(1)}^{+\vec{R}_{(1)}}, K_{\mathrm{v},(1)}^{+\vec{R}_{(1)}} \cap \partial\Sigma_{\mathrm{v},(1)}), (X, L))
$$

that is the map $\mathrm{Glu}^{(1)}$ followed by the restriction of the domain to the core $K_{\mathrm{v},(1)}^{+\vec{R}_{(1)}}$. (See (2.215).) (We put the symbol (1) in $K_{\mathrm{v},(1)}^{+\vec{R}_{(1)}}$) to clarify that this core is induced by $\mathfrak{w}_{\mathfrak{p}}^{(1)}$.) We chose $T_{\mathrm{e},0}$ so that the gluing construction works for L^2_{10m+1}. (See the end of Subsection 2.5.) The second and the third factors are the obvious projections. The map $\mathfrak{F}^{(1)}$ is a C^m embedding of the C^m manifold $\hat{V}(\mathfrak{p}, \mathfrak{w}_{\mathfrak{p}}^{(2)}; (\mathfrak{o}^{(2)}, \mathcal{T}^{(2)}); \mathfrak{A})$, with corners.

We also consider a similar embedding

$$
\begin{aligned}
&\mathfrak{F}^{(2)} : \hat{V}(\mathfrak{p}, \mathfrak{w}_{\mathfrak{p}}^{(2)}; (\mathfrak{o}^{(2)}, \mathcal{T}^{(2)}); \mathfrak{A}) \\
\to &\prod_{\mathrm{v}\in C^0(\mathcal{G}_{\mathfrak{p}})} C^{2m}((K_{\mathrm{v},(2)}^{+\vec{R}_{(2)}}), K_{\mathrm{v},(2)}^{+\vec{R}_{(2)}} \cap \partial\Sigma_{\mathrm{v},(2)}), (X, L)) \\
&\times \prod_{\mathrm{v}\in C^0(\mathcal{G}_{\mathfrak{p}})} \mathfrak{V}^{(2)}((\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}_{\mathfrak{p}})_{\mathrm{v}}) \\
&\times (\vec{\mathcal{T}}^{(2)}, \infty] \times ((\vec{\mathcal{T}}^{(2)}, \infty] \times \vec{S}^1).
\end{aligned}
\tag{2.337}
$$

We denote by $\mathfrak{X}(1, m)$ the right hand side of (2.336) and by $\mathfrak{X}(2, 2m)$ the right hand side of (2.337).

We next study the change of parametrization of the core. Let us use the notation in Proposition 2.23. For $(\rho, \vec{T}, \vec{\theta}) \in \prod_{\mathrm{v}\in C^0(\mathcal{G}_{\mathfrak{p}})} \mathfrak{V}^{(2)}((\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}_{\mathfrak{p}})_{\mathrm{v}}) \times (\vec{\mathcal{T}}^{(2)}, \infty] \times ((\vec{\mathcal{T}}^{(2)}, \infty] \times \vec{S}^1)$ we have a map

$$
\mathfrak{v}_{\rho,\vec{T},\vec{\theta}} : \Sigma^{(2)}_{\vec{T},\vec{\theta}} \to \Sigma^{(1)}_{\Phi_{12}(\rho,\vec{T},\vec{\theta})}.
$$

The source $\Sigma^{(2)}_{\vec{T},\vec{\theta}}$ is obtained using the coordinate at infinity $\mathfrak{w}_{\mathfrak{p}}^{(2)}$ and the target $\Sigma^{(1)}_{\overline{\Phi}_{12}(\rho,\vec{T},\vec{\theta})}$ is obtained using the coordinate at infinity $\mathfrak{w}_{\mathfrak{p}}^{(1)}$. We may assume that

$$
\mathfrak{v}_{\rho,\vec{T},\vec{\theta}}(K_{\mathrm{v},(2)}^{+\vec{R}_{(2)}}) \subset K_{\mathrm{v},(1)}^{+\vec{R}_{(1)}}.
$$

We then define a map

$$
\mathfrak{H}_{12} : \mathfrak{X}(2, 2m) \to \mathfrak{X}(1, m)
$$

by the formula

$$
\mathfrak{H}_{12}(u, (\rho, \vec{T}, \vec{\theta})) = (u \circ \mathfrak{v}_{(\rho,\vec{T},\vec{\theta})}, \overline{\Phi}_{12}(\rho, \vec{T}, \vec{\theta})).
\tag{2.338}
$$

Sublemma 2.134. *$\mathfrak{H}_{12}$ is a C^m-map.*

Proof. By Proposition 2.19, the map $\overline{\Phi}_{12}$ is a C^m diffeomorphism. Therefore the second and the third factors of $\mathfrak{H}_{12}$ is a C^m-map. The first factor is of C^m-class because of Proposition 2.23 and a well-known fact that the map $C^m(M_1, M_2) \times C^{2m}(M_2, M_3) \to C^m(M_1, M_3)$ given by $(v, u) \mapsto u \circ v$ is a C^m map. □

On the other hand we have:

Sublemma 2.135.

$$\mathfrak{H}_{12} \circ \mathfrak{F}^{(2)} = \mathfrak{F}^{(1)} \circ \varphi_{12}.$$

This is immedate from the construction.

Since $\mathfrak{F}^{(2)}$ and $\mathfrak{F}^{(1)}$ are both C^m embeddings, Sublemmas 2.134 and 2.135 imply Lemma 2.133. □

The map φ_{12} is obviously $\Gamma_{\mathfrak{p}}$ equivariant. We then define

$$\begin{aligned}\widehat{\varphi}_{12} = \varphi_{12} \times \text{identity} :& V(\mathfrak{p}, \mathfrak{w}_{\mathfrak{p}}^{(2)}; (\mathfrak{o}^{(2)}, \mathcal{T}^{(2)}); \mathfrak{A}) \times \bigoplus_{c \in \mathfrak{A}} E_c \\ &\subset V(\mathfrak{p}, \mathfrak{w}_{\mathfrak{p}}^{(1)}; (\mathfrak{o}^{(1)}, \mathcal{T}^{(1)}); \mathfrak{A}) \times \bigoplus_{c \in \mathfrak{A}} E_c.\end{aligned}$$

Conditions (2)-(5) are trivial to verify. It also follows that the maps obtained are $\Gamma_{\mathfrak{p}}$-equivariant. (In the situation of this subsection, $\Gamma_{\mathfrak{p}}$-equivariance is always trivial to prove. So we do not mention it any more.)

Case 2: The case $\mathfrak{w}_{\mathfrak{p}}^{(1)} = \mathfrak{w}_{\mathfrak{p}}^{(2)}$ and $\mathfrak{A}^{(1)} \neq \mathfrak{A}^{(2)}$.

Assume that $\mathfrak{B} \supseteq \mathfrak{A}^{(1)} \supset \mathfrak{A}^{(2)}$ ($\mathfrak{B} \subseteq \mathfrak{C}(\mathfrak{p})$). If we regard

$$V(\mathfrak{p}, \mathfrak{w}_{\mathfrak{p}}; (\mathfrak{o}^{(1)}, \mathcal{T}^{(1)}); \mathfrak{A}^{(1)}) \subset \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}^{\mathfrak{w}_{\mathfrak{p}}^{(1)}}(\beta; \mathfrak{p}; \mathfrak{A}^{(1)}; \mathfrak{B})^{\text{trans}}_{\epsilon_0, \vec{\mathcal{T}}^{(1)}},$$

then we may also regard

$$V(\mathfrak{p}, \mathfrak{w}_{\mathfrak{p}}; (\mathfrak{o}^{(1)}, \mathcal{T}^{(1)}); \mathfrak{A}^{(2)}) \subset \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}^{\mathfrak{w}_{\mathfrak{p}}^{(1)}}(\beta; \mathfrak{p}; \mathfrak{A}^{(1)}; \mathfrak{B})^{\text{trans}}_{\epsilon_0, \vec{\mathcal{T}}^{(1)}}.$$

Moreover

$$V(\mathfrak{p}, \mathfrak{w}_{\mathfrak{p}}; (\mathfrak{o}^{(1)}, \mathcal{T}^{(1)}); \mathfrak{A}^{(2)}) \subset V(\mathfrak{p}, \mathfrak{w}_{\mathfrak{p}}; (\mathfrak{o}^{(1)}, \mathcal{T}^{(1)}); \mathfrak{A}^{(1)}). \tag{2.339}$$

We can show that (2.339) is a C^m-map in the same way as the proof of Lemma 2.133. (Actually the proof is easier since there is no coordinate change of the source and so $\mathfrak{H}_{12}$ is the identity map in the situation of Case 2.)

Furthermore an element $(\mathfrak{Y}, u', (\vec{w}'_{a,c}; c \in \mathfrak{B}))$ of $V(\mathfrak{p}, \mathfrak{w}_{\mathfrak{p}}; (\mathfrak{o}^{(1)}, \mathcal{T}^{(1)}); \mathfrak{A}^{(1)})$ is in $V(\mathfrak{p}, \mathfrak{w}_{\mathfrak{p}}; (\mathfrak{o}^{(1)}, \mathcal{T}^{(1)}); \mathfrak{A}^{(2)})$ if and only if

$$\mathfrak{s}_{\mathfrak{p},\mathfrak{w}_{\mathfrak{p}};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(1)}}(\mathfrak{Y}, u', (\vec{w}'_{a,c}; c \in \mathfrak{B})) = \overline{\partial} u' \in \mathcal{E}_{\mathfrak{p},\mathfrak{w}_{\mathfrak{p}};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(2)}}. \tag{2.340}$$

We put $\mathfrak{q}^+ = (\mathfrak{Y}, u', (\vec{w}'_{a,c}; c \in \mathfrak{B}))$. By Lemmas 2.64 and 2.98, $d_{\mathfrak{q}^+}\mathfrak{s}$ induces an isomorphism:

$$\frac{T_{\mathfrak{q}^+} V(\mathfrak{p}, \mathfrak{w}_{\mathfrak{p}}; (\mathfrak{o}^{(1)}, \mathcal{T}^{(1)}); \mathfrak{A}^{(1)})}{T_{\mathfrak{q}^+} V(\mathfrak{p}, \mathfrak{w}_{\mathfrak{p}}; (\mathfrak{o}^{(1)}, \mathcal{T}^{(1)}); \mathfrak{A}^{(2)})} \cong \frac{\left(\mathcal{E}_{\mathfrak{p},\mathfrak{w}_{\mathfrak{p}};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(1)}}\right)_{\mathfrak{q}^+}}{\left(\mathcal{E}_{\mathfrak{p},\mathfrak{w}_{\mathfrak{p}};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(2)}}\right)_{\mathfrak{q}^+}}.$$

We have thus obtained a coordinate change in this case.

The other two cases are as follows.

Case 3: The case $\vec{w}_{\mathfrak p}^{(1)} \subset \vec{w}_{\mathfrak p}^{(2)}$ and $\mathfrak A^{(1)} = \mathfrak A^{(2)}$. The stabilization data $\mathfrak w_{\mathfrak p}^{(1)}$ is induced from $\mathfrak w_{\mathfrak p}^{(2)}$.

Case 4: The case $\vec{w}_{\mathfrak p}^{(1)} \supset \vec{w}_{\mathfrak p}^{(2)}$ and $\mathfrak A^{(1)} = \mathfrak A^{(2)}$. The stabilization data $\mathfrak w_{\mathfrak p}^{(2)}$ is induced from $\mathfrak w_{\mathfrak p}^{(1)}$.

Let us explain the notion that 'stabilization data $\mathfrak w_{\mathfrak p}^{(1)}$ is induced from $\mathfrak w_{\mathfrak p}^{(2)}$.' Suppose $\vec{w}_{\mathfrak p}^{(1)} \subset \vec{w}_{\mathfrak p}^{(2)}$. Let

$$\pi : \bigodot_{\mathrm v \in C^0(\mathcal G_{\mathfrak p})} \mathfrak M^{(2)}_{(\mathfrak x_{\mathfrak p}\cup \vec{w}_{\mathfrak p}^{(2)})_{\mathrm v}} \to \prod_{\mathrm v \in C^0(\mathcal G_{\mathfrak p})} \mathfrak V^{(2)}((\mathfrak x_{\mathfrak p} \cup \vec{w}_{\mathfrak p}^{(2)})_{\mathrm v}) \tag{2.341}$$

be the fiber bundle (2.156) that is a part of the data included in $\mathfrak w_{\mathfrak p}^{(2)}$. Here $\mathfrak V^{(2)}((\mathfrak x_{\mathfrak p} \cup \vec{w}_{\mathfrak p}^{(2)})_{\mathrm v})$ is an open neighborhood of $(\mathfrak x_{\mathfrak p} \cup w_{\mathfrak p}^{(2)})_{\mathrm v}$ in $\mathcal M_{k_{\mathrm v}+1,\ell_{\mathrm v}+\ell_{\mathrm v}^{(2)}}$ or in $\mathcal M^{\rm cl}_{\ell_{\mathrm v}+\ell_{\mathrm v}^{(2)}}$. (They are contained in the top stratum of the Deligne-Mumford moduli spaces.)

Forgetful map of the marked points in $\vec{w}_{\mathfrak p}^{(2)} \setminus \vec{w}_{\mathfrak p}^{(1)}$ induces a map

$$\mathfrak{forget}_{\mathrm v} : \mathcal M_{k_{\mathrm v}+1,\ell_{\mathrm v}+\ell_{\mathrm v}^{(2)}} \to \mathcal M_{k_{\mathrm v}+1,\ell_{\mathrm v}+\ell_{\mathrm v}^{(1)}}$$

etc. We put

$$\mathfrak{forget}_{\mathrm v}(\mathfrak V^{(2)}((\mathfrak x_{\mathfrak p} \cup \vec{w}_{\mathfrak p}^{(2)})_{\mathrm v})) = \mathfrak V^{(1),+}((\mathfrak x_{\mathfrak p} \cup \vec{w}_{\mathfrak p}^{(1)})_{\mathrm v}).$$

We take $\mathfrak V^{(1)}((\mathfrak x_{\mathfrak p} \cup \vec{w}_{\mathfrak p}^{(1)})_{\mathrm v}) \subset \mathfrak V^{(1),+}((\mathfrak x_{\mathfrak p} \cup \vec{w}_{\mathfrak p}^{(1)})_{\mathrm v})$ that is a neighborhood of $(\mathfrak x_{\mathfrak p} \cup \vec{w}_{\mathfrak p}^{(1)})_{\mathrm v}$ such that there exists a section

$$\mathfrak{sect}_{\mathrm v} : \mathfrak V^{(1)}((\mathfrak x_{\mathfrak p} \cup \vec{w}_{\mathfrak p}^{(1)})_{\mathrm v}) \to \mathfrak{forget}(\mathfrak V^{(2)}((\mathfrak x_{\mathfrak p} \cup \vec{w}_{\mathfrak p}^{(2)})_{\mathrm v})). \tag{2.342}$$

Then we can pull back (2.341) by $\mathfrak{sect} = (\mathfrak{sect}_{\mathrm v})$ to obtain a fiber bundle

$$\pi : \bigodot_{\mathrm v \in C^0(\mathcal G_{\mathfrak p})} \mathfrak M^{(1)}_{(\mathfrak x_{\mathfrak p}\cup \vec{w}_{\mathfrak p}^{(1)})_{\mathrm v}} \to \prod_{\mathrm v \in C^0(\mathcal G_{\mathfrak p})} \mathfrak V^{(1)}((\mathfrak x_{\mathfrak p} \cup \vec{w}_{\mathfrak p}^{(1)})_{\mathrm v}). \tag{2.343}$$

Moreover we can pull back a trivialization of the fiber bundle (2.341) to one of the fiber bundle (2.343). Thus we obtain a coordinate at infinity of $(\mathfrak x_{\mathfrak p} \cup w_{\mathfrak p}^{(1)})_{\mathrm v}$.

Definition 2.136. We call the coordinate at infinity obtained as above the *coordinate at infinity induced from* $\mathfrak w_{\mathfrak p}^{(2)}$.

We also take codimension 2 submanifolds $\mathcal D_{\mathfrak p,i}^{(1)}$ that are included as a part of the stabilizaton data $\mathfrak w_{\mathfrak p}^{(1)}$, so that $\mathcal D_{\mathfrak p,i}^{(1)} = \mathcal D_{\mathfrak p,i}^{(2)}$ for $i = 1,\dots,\#\vec{w}_{\mathfrak p}^{(1)}$. We thus have obtained a stablization date $\mathfrak w_{\mathfrak p}^{(1)}$. We call it *the stablization data induced from* $\mathfrak w_{\mathfrak p}^{(2)}$.

We now construct a coordinate change of the Kuranishi structures in Case 3. In Definition 2.93 we defined a forgetful map

$$\begin{aligned}\mathfrak{forget}_{\mathfrak B,\mathfrak B^-;\vec w_{\mathfrak p},\vec w_{\mathfrak p}^-} &: \mathcal M^{\mathfrak w_{\mathfrak p}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0',\vec{\mathcal T}^{(2)}} \\ &\to \mathcal M^{\mathfrak w_{\mathfrak p}^-}_{k+1,(\ell,\ell_{\mathfrak p}^-,(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B^-)_{\epsilon_0,\vec{\mathcal T}^{(1)}}.\end{aligned}$$

Here we shrink the base space of (2.341) so that this map is well-defined. We need to extend the core of the domain and replace ϵ_0 by ϵ_0' in the same way as in Lemma 2.133. We then obtain a stabilization data, which we denote by $\mathfrak m_{\mathfrak p}^{(2)-}$.

Taking $\vec w_{\mathfrak p} = \vec w_{\mathfrak p}^{(2)-}$ and $\vec w_{\mathfrak p}^{-} = \vec w_{\mathfrak p}^{(1)}$ and $\mathfrak B^- = \mathfrak B$ we have

$$\begin{aligned}\mathfrak{forget}_{\mathfrak B,\mathfrak B;\vec w_{\mathfrak p}^{(2)-},\vec w_{\mathfrak p}^{(1)}} : \mathcal M^{\mathfrak w_{\mathfrak p}^{(2)-}}_{k+1,(\ell,\ell_{\mathfrak p}^{(2)},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon'_0,\vec{\mathcal T}^{(2)}} \\ \to \mathcal M^{\mathfrak w_{\mathfrak p}^{(1)}}_{k+1,(\ell,\ell_{\mathfrak p}^{(1)},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0,\vec{\mathcal T}^{(1)}}.\end{aligned}$$

It induces a map

$$\mathcal M^{\mathfrak w_{\mathfrak p}^{(2)}}_{k+1,(\ell,\ell_{\mathfrak p}^{(2)},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)^{\vec w_{\mathfrak p}^{(2)}\setminus \vec w_{\mathfrak p}^{(1)}}_{\epsilon'_0,\vec{\mathcal T}^{(2)}} \to \mathcal M^{\mathfrak w_{\mathfrak p}^{(1)}}_{k+1,(\ell,\ell_{\mathfrak p}^{(1)},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0,\vec{\mathcal T}^{(1)}} \tag{2.344}$$

which is a strata-wise differentiable open embedding by Proposition 2.95. We denote the map (2.344) by $\tilde\varphi_{12}$.

Lemma 2.137. *$\tilde\varphi_{12}$ is of C^m-class in a neighborhood of $\mathfrak p \cup \vec w_{\mathfrak p}^{(2)}$.*

Proof. The proof is similar to the proof of Lemma 2.133. We use Lemma 2.26 which is a parametrized version of Propositions 2.19 and 2.23. Let

$$\mathfrak x_{\mathfrak p} \cup \vec w_{\mathfrak p}^{(2)} = \tilde{\mathfrak x} \in \prod_{\mathrm v\in C^0(\mathcal G_{\mathfrak p})} \mathfrak V^{(2)}((\mathfrak x_{\mathfrak p}\cup \vec w_{\mathfrak p}^{(2)})_{\mathrm v})$$

and $\mathfrak{forget}(\tilde{\mathfrak x}) = \mathfrak x = \mathfrak x_{\mathfrak p}\cup \vec w_{\mathfrak p}^{(1)}$. Let $\mathfrak V^{(1)-}((\mathfrak x_{\mathfrak p}\cup \vec w_{\mathfrak p}^{(1)})_{\mathrm v})$ be a neighborhood of $\mathfrak p$.

Let $\mathfrak{sect}_{(1),\mathrm v}$ be the section we chose in (2.342). It gives a stabilization data $\mathfrak w_{\mathfrak p}^{(1)}$. We take

$$\mathfrak{sect}_{(2),\mathrm v} : Q_{\mathrm v} \times \mathfrak V^{(1)-}((\mathfrak x_{\mathfrak p}\cup \vec w_{\mathfrak p}^{(1)})_{\mathrm v}) \to \mathfrak V^{(2)}((\mathfrak x_{\mathfrak p}\cup \vec w_{\mathfrak p}^{(2)})_{\mathrm v})$$

such that the following condition is satisfied.

Condition 2.138. (1) $\mathfrak{forget}(\mathfrak{sect}_{(2),\mathrm v}(\xi,\mathfrak y_{\mathrm v})) = \mathfrak y_{\mathrm v}$.
(2) $\mathfrak{sect}_{(2),\mathrm v}$ is a diffeomorphism onto an open neighborhood of $\tilde{\mathfrak x}_{\mathrm v}$.

Pulling back $\mathfrak w_{\mathfrak p}^{(2)}$ by $\mathfrak{sect}_{(2)}$ we have a $Q=\prod Q_{\mathrm v}$-parametrized family of stabilization data, which we call $\tilde{\mathfrak w}_{\mathfrak p}^{(2)}$. We denote the image of $\mathfrak{sect}_{(2),\mathrm v}$ by $\mathfrak V^{(2)-}((\mathfrak x_{\mathfrak p}\cup\vec w_{\mathfrak p}^{(2)})_{\mathrm v})$.

We use $\mathfrak w_{\mathfrak p}^{(1)}$ in the same was as in the proof of Lemma 2.133 to obtain

$$\begin{aligned}&\mathfrak F^{(1)} : \hat V^-(\mathfrak p, \mathfrak w_{\mathfrak p}^{(1)}; (\mathfrak o^{(1)},\mathcal T^{(1)});\mathfrak A)\\ \to &\prod_{\mathrm v\in C^0(\mathcal G_{\mathfrak p})} C^m((K^{+\vec R_{(1)}}_{\mathrm v,(1)}, K^{+\vec R_{(1)}}_{\mathrm v,(1)}\cap \partial\Sigma_{\mathrm v,(1)}),(X,L))\\ &\times \prod_{\mathrm v\in C^0(\mathcal G_{\mathfrak p})} \mathfrak V^{(1)-}((\mathfrak x_{\mathfrak p}\cup \vec w_{\mathfrak p})_{\mathrm v})\\ &\times((\vec{\mathcal T}^{(1)},\infty]\times((\vec{\mathcal T}^{(1)},\infty]\times \vec S^1).\end{aligned} \tag{2.345}$$

(Here we put $-$ in $\hat V^-(\mathfrak p,\mathfrak w_{\mathfrak p}^{(1)};(\mathfrak o^{(1)},\mathcal T^{(1)});\mathfrak A)$ to clarify that this space uses $\mathfrak V^{(1)-}((\mathfrak x_{\mathfrak p}\cup \vec w_{\mathfrak p})_{\mathrm v})$.)

We use $\mathfrak{w}_{\mathfrak p}^{(2)}$ to obtain

$$\begin{aligned}
&\mathfrak{F}^{(2)} : \hat V^-(\mathfrak p, \mathfrak{w}_{\mathfrak p}^{(2)}; (\mathfrak o^{(2)}, \mathcal T^{(2)}); \mathfrak A) \\
\to &\prod_{\mathrm v \in C^0(\mathcal G_{\mathfrak p})} C^{2m}((K_{\mathrm v,(2)}^{+\vec R_{(2)}}), K_{\mathrm v,(2)}^{+\vec R_{(2)}} \cap \partial \Sigma_{\mathrm v,(2)}), (X, L)) \\
&\times \prod_{\mathrm v \in C^0(\mathcal G_{\mathfrak p})} \mathfrak V^{(2)-}((\mathfrak x_{\mathfrak p} \cup \vec w_{\mathfrak p})_{\mathrm v}) \\
&\times ((\vec{\mathcal T}^{(2)}, \infty] \times ((\vec{\mathcal T}^{(2)}, \infty] \times \vec S^1).
\end{aligned} \tag{2.346}$$

Let $\mathfrak X(1,m)$, $\mathfrak X(2,2m)$ be the spaces in the right hand side of (2.345), (2.346) respectively.

We apply Lemma 2.26 to the family of coordinates at infinity $\tilde{\mathfrak w}_{\mathfrak p}^{(2)}$ and the coordinate at infinity $\mathfrak w_{\mathfrak p}^{(1)}$. It gives estimates of the map $\overline\Phi_{12}$ defined in (2.177) and $\mathfrak v_{(\xi,\rho,\vec T,\vec\theta)}$ as in (2.178).

We define $\mathfrak H_{12} : \mathfrak X(2,2m) \to \mathfrak X(1,m)$ by

$$\mathfrak H_{12}(u, \mathfrak{sect}_{(2)}(\xi,\rho), (\vec T, \vec \theta)) = (u \circ \mathfrak v_{(\xi,\rho,\vec T,\vec\theta)}, \overline\Phi_{12}(\xi,\rho,\vec T,\vec\theta)). \tag{2.347}$$

By construction we have

$$\mathfrak H_{12} \circ \mathfrak F^{(2)} = \mathfrak F^{(1)} \circ \tilde\varphi_{12}. \tag{2.348}$$

Lemma 2.26 implies that $\mathfrak H_{12}$ is a C^m-map. Moreover $\mathfrak F^{(1)}$ and $\mathfrak F^{(2)}$ are C^m-embeddings. Therefore $\tilde\varphi_{12}$ is a C^m-map on $\hat V^-(\mathfrak p, \mathfrak w_{\mathfrak p}^{(2)}; (\mathfrak o^{(2)}, \mathcal T^{(2)}); \mathfrak A)$. The proof of Lemma 2.137 is complete. □

We go back to the construction of coordinate change in Case 3. By requiring the transversal constraint at all the marked points, $\tilde\varphi_{12}$ induces a required coordinate change φ_{12}. Since $\mathfrak A^{(1)} = \mathfrak A^{(2)}$, it is easy to find the bundle map $\hat\varphi_{12}$ that has the required properties.

Remark 2.139. Note that the map (2.344) and the coordinate change φ_{12} we obtain are independent of the choice of the section of (2.342). But φ_{12} depends on the codimension 2 submanifolds we take, since the process to take trans depends on them. We use the coordinate at infinity (or the map $\mathfrak{sect}_{\mathrm v}$ of (2.342)) only to *prove* that φ_{12} is of C^m-class.

Using the fact that the map (2.344) is a local diffeomorphism the construction of the coordinate change in Case 4 is an inverse of one in Case 3.

We have thus constructed the coordinate change in the 4 cases above. The general case can be constructed by a composition of them.

Let us be given $(\mathfrak w_{\mathfrak p}^{(1)}, \mathfrak A^{(1)})$ and $(\mathfrak w_{\mathfrak p}^{(2)}, \mathfrak A^{(2)})$. We say that the pair $((\mathfrak w_{\mathfrak p}^{(1)}, \mathfrak A^{(1)}), (\mathfrak w_{\mathfrak p}^{(2)}, \mathfrak A^{(2)}))$ is of Type 1,2,3,4, if we can apply Case 1,2,3,4, respectively. We say the coordinate change obtained *the coordiate change of Type* 1,2,3,4, respectively.

Lemma 2.140. *For given* $(\mathfrak w_{\mathfrak p}^{(1)}, \mathfrak A^{(1)})$ *and* $(\mathfrak w_{\mathfrak p}^{(6)}, \mathfrak A^{(6)})$ *with* $\vec w^{(1)} \cap \vec w^{(6)} = \emptyset$, *there exist* $(\mathfrak w_{\mathfrak p}^{(j)}, \mathfrak A^{(j)})$ *for* $j = 2, \dots, 5$ *such that:*

The pair $((\mathfrak w_{\mathfrak p}^{(1)}, \mathfrak A^{(1)}), (\mathfrak w_{\mathfrak p}^{(2)}, \mathfrak A^{(2)}))$ *is of type* 2,
The pair $((\mathfrak w_{\mathfrak p}^{(2)}, \mathfrak A^{(2)}), (\mathfrak w_{\mathfrak p}^{(3)}, \mathfrak A^{(3)}))$ *is of type* 1,
The pair $((\mathfrak w_{\mathfrak p}^{(3)}, \mathfrak A^{(3)}), (\mathfrak w_{\mathfrak p}^{(4)}, \mathfrak A^{(4)}))$ *is of type* 3,

The pair $((\mathfrak{w}_{\mathfrak{p}}^{(4)},\mathfrak{A}^{(4)}),(\mathfrak{w}_{\mathfrak{p}}^{(5)},\mathfrak{A}^{(5)}))$ *is of type* 4*,*

The pair $((\mathfrak{w}_{\mathfrak{p}}^{(5)},\mathfrak{A}^{(5)}),(\mathfrak{w}_{\mathfrak{p}}^{(6)},\mathfrak{A}^{(6)}))$ *is of type* 1*.*

Proof. We put $(\mathfrak{w}_{\mathfrak{p}}^{(2)},\mathfrak{A}^{(2)}) = (\mathfrak{w}_{\mathfrak{p}}^{(1)},\mathfrak{A}^{(6)})$ and $\mathfrak{A}^{(j)} = \mathfrak{A}^{(6)}$ for all $j = 2,\dots,6$.

Let $\vec{w}_{\mathfrak{p}}^{(4)} = \vec{w}_{\mathfrak{p}}^{(1)} \cup \vec{w}_{\mathfrak{p}}^{(6)}$. (Note this is a disjoint union by assumption.) We take (any) coordinate at infinity for $\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}_{\mathfrak{p}}^{(4)}$. The codimension 2 submanifolds are determined from the data given in $\mathfrak{w}_{\mathfrak{p}}^{(1)}$ and $\mathfrak{w}_{\mathfrak{p}}^{(6)}$. We thus defined $(\mathfrak{w}_{\mathfrak{p}}^{(4)},\mathfrak{A}^{(4)})$.

We take the coordiates at infinity that is induced from $\mathfrak{w}_{\mathfrak{p}}^{(4)}$ so that the set of additional marked points are $\vec{w}_{\mathfrak{p}}^{(1)}$ and $\vec{w}_{\mathfrak{p}}^{(6)}$. We thus obtain $(\mathfrak{w}_{\mathfrak{p}}^{(3)},\mathfrak{A}^{(3)})$ and $(\mathfrak{w}_{\mathfrak{p}}^{(5)},\mathfrak{A}^{(5)})$, respectively. It is easy to see that they have required properties. □

Remark 2.141. We need the hypothesis $\vec{w}_{\mathfrak{p}}^{(1)} \cap \vec{w}_{\mathfrak{p}}^{(6)} = \emptyset$ in Lemma 2.140. Otherwise it might happen that $w_{\mathfrak{p},i}^{(1)} = w_{\mathfrak{p},j}^{(6)}$ but $\mathcal{D}_{\mathfrak{p},i}^{(1)} \ne \mathcal{D}_{\mathfrak{p},j}^{(6)}$.

By Lemma 2.140 we can define a coordinate change for the pairs $(\mathfrak{w}_{\mathfrak{p}}^{(1)},\mathfrak{A}^{(1)})$ and $(\mathfrak{w}_{\mathfrak{p}}^{(2)},\mathfrak{A}^{(2)})$ as the composition of 5 coordinate changes. We have thus constructed the required coordinate change

$$\varphi_{12} : V(\mathfrak{p},\mathfrak{w}_{\mathfrak{p}}^{(2)};(\mathfrak{o}^{(2)},\mathcal{T}^{(2)});\mathfrak{A}^{(2)}) \to V(\mathfrak{p},\mathfrak{w}_{\mathfrak{p}}^{(1)};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(1)})$$

in case $\vec{w}_{\mathfrak{p}}^{(1)} \cap \vec{w}_{\mathfrak{p}}^{(2)} = \emptyset$.

In general cases we take $\mathfrak{w}_{\mathfrak{p}}^{(0)}$ such that $\vec{w}_{\mathfrak{p}}^{(1)} \cap \vec{w}_{\mathfrak{p}}^{(0)} = \vec{w}_{\mathfrak{p}}^{(2)} \cap \vec{w}_{\mathfrak{p}}^{(0)} = \emptyset$ and put

$$\varphi_{12} = \varphi_{10} \circ \varphi_{02}.$$

The proof of Proposition 2.131 is complete. □

We remark that in the proof of Lemma 2.140 we made a choice of coordinate at infinity of $\mathfrak{x}_{\mathfrak{p}} \cup \vec{w}_{\mathfrak{p}}^{(4)}$. We also take $\mathfrak{w}_{\mathfrak{p}}^{(0)}$ at the last step of the proof of Proposition 2.131. However the resulting coordinate change is independent of these choices if we shrink the domain. Namely we have:

Lemma 2.142. *We use the notation in Proposition 2.131. If two different choices of* $(\mathfrak{o}_0^{(2),j},\mathcal{T}_0^{(2),j})$ $(j = 1,2)$ *and* $(\varphi_{12}^{j},\widehat{\varphi}_{12}^{j})$ $(j = 1,2)$ *are made, then there exists* $(\mathfrak{o}^{(3)},\mathcal{T}^{(3)})$ *such that* $(\mathfrak{o}^{(3)},\mathcal{T}^{(3)}) < (\mathfrak{o}_0^{(2),j},\mathcal{T}_0^{(2),j})$ $(j = 1,2)$ *and*

$$(\varphi_{12}^{1},\widehat{\varphi}_{12}^{1}) = (\varphi_{12}^{2},\widehat{\varphi}_{12}^{2})$$

on $V(\mathfrak{p},\mathfrak{w}_{\mathfrak{p}}^{(2)};(\mathfrak{o}^{(3)},\mathcal{T}^{(3)});\mathfrak{A}^{(2)})$.

Proof. We first prove the next lemma.

Lemma 2.143. *Let* $\vec{w}_{\mathfrak{p}}^{(1)} \subset \vec{w}_{\mathfrak{p}}^{(2)}$*. Let* $\mathfrak{w}_{\mathfrak{p}}^{(i,j)}$ $i = 1,2$, $j = 1,2$ *be the stabilization data at* $\mathfrak{p}$ *such that the additional marked points associated to* $\mathfrak{w}_{\mathfrak{p}}^{(i,j)}$ *is* $\vec{w}_{\mathfrak{p}}^{(j)}$*.*

We assume that $((\mathfrak{w}_{\mathfrak{p}}^{(i,1)},\mathfrak{A}),(\mathfrak{w}_{\mathfrak{p}}^{(i,2)},\mathfrak{A}))$ *is type* 3*.*[19]

Let $\varphi_{(i,j);(i',j')}$ *be the coordinate change from the coordinate associated with* $\mathfrak{w}_{\mathfrak{p}}^{(i',j')}$ *to one associated with* $\mathfrak{w}_{\mathfrak{p}}^{(i,j)}$*. Then we have*

$$\varphi_{(1,1);(1,2)} \circ \varphi_{(1,2);(2,2)} = \varphi_{(1,1);(2,1)} \circ \varphi_{(2,1);(2,2)} \tag{2.349}$$

[19]Namely we assume that the coordinate at infinity of $\mathfrak{w}_{\mathfrak{p}}^{(i,1)}$ is induced by that of $\mathfrak{w}_{\mathfrak{p}}^{(i,2)}$ and the submanifolds we assigned in Definition 2.57 (2) coincide each other when they are assigned to the same marked points.

on a small neighborhood of $\mathfrak{p}$ *in the Kuranishi chart assciated with* $\mathfrak{w}_{\mathfrak{p}}^{(2,2)}$. *The same equality holds for* $\widehat{\varphi}_{(i,j);(i',j')}$.

The same conclusion holds when $\vec{w}_{\mathfrak{p}}^{(2)} \subset \vec{w}_{\mathfrak{p}}^{(1)}$ *and replace 'type* 3*' by 'type* 4*'.*

Proof. This lemma as well as several other lemmas that appear later, is a consequence of the following general observation.

We consider an open subset $\mathcal{U} \subset \mathcal{M}_{k+1,\ell+\ell'}$ of the Deligne-Mumford moduli space. Let

$$\pi : \mathfrak{M}(\mathcal{U}) \to \mathcal{U}$$

be the restriction of the universal family to $\mathcal{U}$. Suppose we have a *topological space* Ξ consisting of pairs $(\mathfrak{x}, u')$ where $\mathfrak{x} \in \mathcal{U}$ and $u' : \pi^{-1}(\mathfrak{x}) \to X$ is a smooth map. Here we emphasis that we regard Ξ as a topological space and do not need to use any other structure such as smooth structure.

Suppose $(V_i, E_i, \mathfrak{s}_i, \psi_i)$ is a Kuranishi charts at $\mathfrak{p}$. We assume that the coordinate change φ_{ji} is defined as follows: Suppose that there exists a homeomorphism $\Phi_i : V_i \to \Xi$ onto an open neighborhood of $\mathfrak{x}$ with $\mathfrak{x} = \Phi_i(\mathfrak{p})$ for all of i and

$$\varphi_{ji} = \Phi_j^{-1} \circ \Phi_i$$

holds on a neighborhood of $\mathfrak{p}$. Then we have

$$\varphi_{12} \circ \varphi_{23} = \varphi_{13}$$

on a neighborhood of $\mathfrak{p}$. This observation is obvious.

Remark 2.144. Later we will use a slightly more general case. Namely we consider the case when there are $V_{i,j}$ and $\Phi_{i,j} : V_{i,j} \to \Xi$ for $(i,j) = (1,1), \dots, (1,m)$ and $(i,j) = (2,1), \dots, (2,n)$. We assume $V_{1,1} = V_{2,1}$ and $V_{1,m} = V_{2,n}$. Supoose $\mathfrak{x} = \Phi_{i,j}(\mathfrak{p})$ is independent of i, j and $\Phi_{i,j}$ is a homeomorphism onto a neighborhood of $\mathfrak{x}$. We put: $\varphi_{(i,j)(i,j+1)} = \Phi_{i,j}^{-1} \circ \Phi_{i,j+1}$. Then we have

$$\varphi_{(1,1)(1,2)} \circ \cdots \circ \varphi_{(1,m-1)(1,m)} = \varphi_{(2,1)(2,2)} \circ \cdots \circ \varphi_{(2,n-1)(2,n)}$$

on a neighborhood of $\mathfrak{p}$. This is again obvious.

Now we apply the observation above to the situation of Lemma 2.143. The role of Ξ is taken by

$$\mathcal{M}^{\mathfrak{w}_{\mathfrak{p}}^{(2,2)}}_{k+1,(\ell,\ell_{\mathfrak{p}}^{(2)},(\ell_c))}(\beta; \mathfrak{p}; \mathfrak{A}; \mathfrak{B})^{\rm trans}_{\epsilon_0,\vec{\mathcal{T}}^{(2)}}.$$

We note that this set depends on the coordinate at infinity. However Lemma 2.132 implies that it is independent of the coordinate at infinity on a neighborhood of $\mathfrak{p}$. We have thus proved (2.349).

Note the bundle maps $\widehat{\varphi}_{(i,j);(i',j')}$ are nothing but the identity maps on the fiber in our situation. The proof of Lemma 2.143 is complete. □

Lemma 2.142 for the case $\vec{w}_{\mathfrak{p}}^{(1)} \cap \vec{w}_{\mathfrak{p}}^{(2)} = \emptyset$ is immediate from Lemma 2.143.

Let us prove the general case. We need to prove the independence of the coordinate change of the choice of $\mathfrak{w}_{\mathfrak{p}}^{(0)}$. Let $\mathfrak{w}_{\mathfrak{p}}^{(0,1)}$, $\mathfrak{w}_{\mathfrak{p}}^{(0,2)}$ be two such choices. Namely we assume $\vec{w}_{\mathfrak{p}}^{(1)} \cap \vec{w}_{\mathfrak{p}}^{(0,i)} = \vec{w}_{\mathfrak{p}}^{(2)} \cap \vec{w}_{\mathfrak{p}}^{(0,i)} = \emptyset$ for $i = 1, 2$. We first assume $\vec{w}_{\mathfrak{p}}^{(0,1)} \cap \vec{w}_{\mathfrak{p}}^{(0,2)} = \emptyset$ in addition. We put $\vec{w}_{\mathfrak{p}}^{(0)} = \vec{w}_{\mathfrak{p}}^{(0,1)} \cup \vec{w}_{\mathfrak{p}}^{(0,2)}$. We take a stabilization data $\mathfrak{w}_{\mathfrak{p}}^{(0)}$ so that the codimension 2 submanifolds are induced by $\mathfrak{w}_{\mathfrak{p}}^{(0,i)}$. Then, $\varphi_{(0,i),0}$ are composition of coordinate change of type 3 and of type 1 and $\varphi_{0,(0,i)}$ are composition of coordinate change of type 4 and of type 1. Therefore from the first part of the proof we have

$$\begin{aligned}\varphi_{1(0,1)} \circ \varphi_{(0,1)2} &= \varphi_{1(0,1)} \circ \varphi_{(0,1)0} \circ \varphi_{0(0,1)} \circ \varphi_{(0,1)2} \\ &= \varphi_{10} \circ \varphi_{02} = \varphi_{1(0,2)} \circ \varphi_{(0,2)2}\end{aligned}$$

as required.[20]

To remove the condition $\vec{w}_{\mathfrak p}^{(0,1)} \cap \vec{w}_{\mathfrak p}^{(0,2)} = \emptyset$ it suffices to remark that there exists $\vec{w}_{\mathfrak p}^{(0,3)}$ such that $\vec{w}_{\mathfrak p}^{(1)} \cap \vec{w}_{\mathfrak p}^{(0,3)} = \vec{w}_{\mathfrak p}^{(2)} \cap \vec{w}_{\mathfrak p}^{(0,3)} = \emptyset$ and $\vec{w}_{\mathfrak p}^{(0,1)} \cap \vec{w}_{\mathfrak p}^{(0,3)} = \vec{w}_{\mathfrak p}^{(0,2)} \cap \vec{w}_{\mathfrak p}^{(0,3)} = \emptyset$. The proof of Lemma 2.142 is complete. □

Now we prove the compatibility of the coordinate transformations stated in Proposition 2.131.

Lemma 2.145. *Let $(\mathfrak w_{\mathfrak p}^{(j)}, \mathfrak A^{(j)})$ be a pair of stabilization data at $\mathfrak p$ and $\mathfrak A^{(j)} \subset \mathfrak C(\mathfrak p)$, for $j = 1, 2, 3$. Suppose $\mathfrak A^{(1)} \supseteq \mathfrak A^{(2)} \supseteq \mathfrak A^{(3)} \neq \emptyset$ and let $(\mathfrak o^{(1)}, \mathcal T^{(1)})$ be admissible for $(\mathfrak w_{\mathfrak p}^{(1)}, \mathfrak A^{(1)})$.*

By Proposition 2.131 we have admissible $(\mathfrak o^{(2)}, \mathcal T^{(2)})$ and $(\mathfrak o^{(3)}, \mathcal T^{(3)})$ such that the coordinate change

$$(\varphi_{1j}, \hat\varphi_{1j}) : V(\mathfrak p, \mathfrak w_{\mathfrak p}^{(j)}; (\mathfrak o, \mathcal T); \mathfrak A^{(j)}) \to V(\mathfrak p, \mathfrak w_{\mathfrak p}^{(1)}; (\mathfrak o^{(1)}, \mathcal T^{(1)}); \mathfrak A^{(2)})$$

exists if $(\mathfrak o^{(j)}, \mathcal T^{(j)}) > (\mathfrak o, \mathcal T)$. (Here $j = 2, 3$).

By Proposition 2.131 there exists admissible $(\mathfrak o^{(4)}, \mathcal T^{(4)})$ such that a coordinate change

$$(\varphi_{23}, \hat\varphi_{23}) : V(\mathfrak p, \mathfrak w_{\mathfrak p}^{(3)}; (\mathfrak o, \mathcal T); \mathfrak A^{(3)}) \to V(\mathfrak p, \mathfrak w_{\mathfrak p}^{(2)}; (\mathfrak o^{(2)}, \mathcal T^{(2)}); \mathfrak A^{(2)})$$

exists if $(\mathfrak o^{(4)}, \mathcal T^{(4)}) > (\mathfrak o, \mathcal T)$.

Now there exists $(\mathfrak o^{(5)}, \mathcal T^{(5)})$ with $(\mathfrak o^{(5)}, \mathcal T^{(5)}) < (\mathfrak o^{(j)}, \mathcal T^{(j)})$ $(j = 3, 4)$ such that we have

$$(\varphi_{13}, \hat\varphi_{13}) = (\varphi_{12}, \hat\varphi_{12}) \circ (\varphi_{23}, \hat\varphi_{23}) \tag{2.350}$$

on $V(\mathfrak p, \mathfrak w_{\mathfrak p}^{(3)}; (\mathfrak o^{(5)}, \mathcal T^{(5)}); \mathfrak A^{(3)})$.

Proof. We first prove the case when $\vec{w}_{\mathfrak p}^{(1)}$, $\vec{w}_{\mathfrak p}^{(2)}$, $\vec{w}_{\mathfrak p}^{(3)}$ are mutually disjoint.

We note that we may assume $\mathfrak A^{(1)} = \mathfrak A^{(2)} = \mathfrak A^{(3)}$. In fact the coordinate change of type 2 (that is the coordinate change which replaces $\mathfrak A$ by its subset $\mathfrak A^-$), is defined by inclusion of the domains so that $\mathfrak A^-$ is obtained from $\mathfrak A$ by the equation (2.340). This process commutes with other types of coordinate changes. So we assume $\mathfrak A^{(1)} = \mathfrak A^{(2)} = \mathfrak A^{(3)} = \mathfrak A$.

We also note that the composition of two coordinate changes of type j (for $j = 1, \dots, 4$) is again a coordinate change of type j.

Now using Lemma 2.140, we can find $\mathfrak w_{\mathfrak p}^{(i,j)}$ $i = 1, 2, 3$, $j = 2, \dots, 6$ such that $(\mathfrak w_{\mathfrak p}^{(i,j)}, \mathfrak w_{\mathfrak p}^{(i,j+1)})$ is as in the conclusion of Lemma 2.140 and

$$\mathfrak w_{\mathfrak p}^{(1,2)} = \mathfrak w_{\mathfrak p}^{(3,2)} = \mathfrak w_{\mathfrak p}^{(1)}, \quad \mathfrak w_{\mathfrak p}^{(1,6)} = \mathfrak w_{\mathfrak p}^{(2,2)} = \mathfrak w_{\mathfrak p}^{(2)}, \quad \mathfrak w_{\mathfrak p}^{(2,6)} = \mathfrak w_{\mathfrak p}^{(3,6)} = \mathfrak w_{\mathfrak p}^{(3)}.$$

[20] Here $\varphi_{1(0,1)}$ is the coordinate change from the Kuranishi chart associated with $\vec{w}_{\mathfrak p}^{(0,1)}$ to the one associated with $\vec{w}_{\mathfrak p}^{1}$. The notation of other coordinate changes are similar.

Then

$$\begin{aligned}\varphi_{12} &= \varphi_{(1,2)(1,3)} \circ \varphi_{(1,3)(1,4)} \circ \varphi_{(1,4)(1,5)} \circ \varphi_{(1,5)(1,6)},\\ \varphi_{23} &= \varphi_{(2,2),(2,3)} \circ \varphi_{(2,3)(2,4)} \circ \varphi_{(2,4)(2,5)} \circ \varphi_{(2,5)(1,6)},\\ \varphi_{13} &= \varphi_{(3,2)(3,3)} \circ \varphi_{(3,3)(1,4)} \circ \varphi_{(3,4)(3,5)} \circ \varphi_{(3,5)(3,6)}.\end{aligned}$$

Therefore we can apply the general observation mentioned in the course of the proof of Lemma 2.143 in the form of Remark 2.144 to prove Lemma 2.145 in our case.

In fact we can take Ξ as follows. We consider $\vec w_{\mathfrak p}^{(i,4)}$ for $i = 1,2,3$ and put $\vec w_{\mathfrak p} = \vec w_{\mathfrak p}^{(1,4)} \cup \vec w_{\mathfrak p}^{(2,4)} \cup \vec w_{\mathfrak p}^{(3,4)}$. We take (any) coordinate at infinity of $\mathfrak x_{\mathfrak p} \cup \vec w_{\mathfrak p}$. We take the codimension 2 submanifolds $\mathcal D_{\mathfrak p,i}$ (that is a part of the data $\mathfrak w_{\mathfrak p}$) so that they coincide with those taken for $\mathfrak w_{\mathfrak p}^{(i)}$, $i = 1,2,3$. (Note we use the assumption that $\vec w_{\mathfrak p}^{(1)}$, $\vec w_{\mathfrak p}^{(2)}$, $\vec w_{\mathfrak p}^{(3)}$ are mutually disjoint here.) We have thus defined the stabilization data $\mathfrak w_{\mathfrak p}$. Then

$$\Xi = \mathcal M^{\mathfrak w_{\mathfrak p}}_{k+1,(\ell,\ell_{\mathfrak p}^{(+)},(\ell_c))}(\beta;\mathfrak p;\mathfrak A;\mathfrak B)^{\rm trans}_{\epsilon_0,\vec{\mathcal T}_1},$$

where $\ell_{\mathfrak p}^{(+)} = \#\vec w_{\mathfrak p}$.

We finally remove the condition that $\vec w_{\mathfrak p}^{(1)}$, $\vec w_{\mathfrak p}^{(2)}$, $\vec w_{\mathfrak p}^{(3)}$ are mutually disjoint. We take $\vec w_{\mathfrak p}^{(4)}, \vec w_{\mathfrak p}^{(5)}$ such that

$$\vec w_{\mathfrak p}^{(i)} \cap \vec w_{\mathfrak p}^{(4)} = \emptyset = \vec w_{\mathfrak p}^{(i)} \cap \vec w_{\mathfrak p}^{(5)}$$

for $i = 1,2,3$ and $\vec w_{\mathfrak p}^{(4)} \cap \vec w_{\mathfrak p}^{(5)} = \emptyset$. We also take codimention two transversal submanifolds $\mathcal D_i$ for each of those additional marked points. We have thus obtained the stabilization data $\mathfrak w_{\mathfrak p}^{(4)}$, $\mathfrak w_{\mathfrak p}^{(5)}$. Then we have

$$\varphi_{12} \circ \varphi_{23} = \varphi_{15} \circ \varphi_{52} \circ \varphi_{24} \circ \varphi_{43} = \varphi_{15} \circ \varphi_{54} \circ \varphi_{43} = \varphi_{15} \circ \varphi_{53} = \varphi_{13}.$$

Here the first and the last equalities are the definitions. The second and the third equalities follow from the case of Lemma 2.145 which we already proved. The proof of Lemma 2.145 is complete. □

2.9. **Coordinate change - II: Coordinate change among different strata.** In this subsection we construct coordinate changes between the Kuranishi charts we constructed in Proposition 2.125 for the general case. Let $\mathfrak p(1) \in \mathcal M_{k+1,\ell}(\beta)$. We take a stabilization data $\mathfrak w_{\mathfrak p(1)}$ at $\mathfrak p(1)$ and $\mathfrak A^{(1)} \subseteq \mathfrak C(\mathfrak p(1))$. We use them to define Kuranishi neighborhood $V(\mathfrak p(1), \mathfrak w_{\mathfrak p(1)}; (\mathfrak o^{(1)}, \mathcal T^{(1)}); \mathfrak A^{(1)})$ given in Definition 2.124. Let

$$\psi_{\mathfrak p(1),\mathfrak w_{\mathfrak p(1)};(\mathfrak o^{(1)},\mathcal T^{(1)});\mathfrak A^{(1)}} : \mathfrak s^{-1}_{\mathfrak p(1),\mathfrak w_{\mathfrak p(1)};(\mathfrak o^{(1)},\mathcal T^{(1)});\mathfrak A^{(1)}}(0)/\Gamma_{\mathfrak p(1)} \to \mathcal M_{k+1,\ell}(\beta) \tag{2.351}$$

be the map in Proposition 2.125. We assume that $\mathfrak p(2)$ is contained in its image.

We will define the notion of *induced stabilization data* at $\mathfrak p(2)$. We recall that the stabilization data $\mathfrak w_{\mathfrak p(1)}$ includes the fiber bundle (2.156)

$$\pi : \bigodot_{{\rm v}\in C^0(\mathcal G_{\mathfrak p(1)})} \mathfrak M^{(1)}((\mathfrak x_{\mathfrak p(1)} \cup \vec w_{\mathfrak p(1)})_{\rm v}) \to \prod_{{\rm v}\in C^0(\mathcal G_{\mathfrak p(1)})} \mathfrak V^{(1)}((\mathfrak x_{\mathfrak p(1)} \cup \vec w_{\mathfrak p(1)})_{\rm v}). \tag{2.352}$$

Here $\mathfrak V^{(1)}((\mathfrak x_{\mathfrak p(1)} \cup \vec w_{\mathfrak p(1)})_{\rm v})$ is a neighborhood of $(\mathfrak x_{\mathfrak p(1)} \cup \vec w_{\mathfrak p(1)})_{\rm v}$ in the Deligne-Mumford moduli space $\mathcal M_{k_{\rm v}+1,\ell_{\rm v}+\ell_{\mathfrak p(1),{\rm v}}}$. The product in the right hand side of (2.352) is identified with a neighborhood of $\mathfrak x_{\mathfrak p(1)} \cup \vec w_{\mathfrak p(1)}$ in the stratum $\mathcal M_{k+1,\ell+\ell_{\mathfrak p(1)}}(\mathcal G_{\mathfrak p(1)\cup\vec w_{\mathfrak p(1)}})$ of the Deligne-Mumford moduli space $\mathcal M_{k+1,\ell+\ell_{\mathfrak p(1)}}$. We denote this neighborhood by $\mathfrak V(\mathfrak x_{\mathfrak p(1)} \cup \vec w_{\mathfrak p(1)})$.

Condition 2.146. We consider a symmetric stabilization $\vec{w}_{\mathfrak{p}(2)}$ on $\mathfrak{x}_{\mathfrak{p}(2)}$, an element $\sigma_0 \in \mathfrak{V}(\mathfrak{x}_{\mathfrak{p}(1)} \cup \vec{w}_{\mathfrak{p}(1)})$ and $(\vec{S}_0^{\rm o}, (\vec{S}_0^{\rm c}, \vec{\theta}_0))) \in (\vec{\mathcal{T}}^{(1)}, \infty] \times ((\vec{\mathcal{T}}^{(1)}, \infty] \times \vec{S}^1)$ that satisfy the following two conditions.

(1) $\mathfrak{x}_{\mathfrak{p}(2)} \cup \vec{w}_{\mathfrak{p}(2)} = \overline{\Phi}(\sigma_0; \vec{S}_0^{\rm o}, (\vec{S}_0^{\rm c}, \vec{\theta}_0))$.
(2) $\mathfrak{p}(2) \cup \vec{w}_{\mathfrak{p}(2)}$ satisfies the treansversal constraint at all marked points. Namely for each $i = 1, \dots, \ell_{\mathfrak{p}(1)}$ we have

$$u_{\mathfrak{p}(2)}(w_{\mathfrak{p}(2),i}) \in \mathcal{D}_{\mathfrak{p}(1),i}.$$

Here $\mathcal{D}_{\mathfrak{p}(1),i}$ is a codimension 2 submanifold included in the stabilization data $\mathfrak{w}_{\mathfrak{p}(1)}$. (We remark $\#\vec{w}_{\mathfrak{p}(2)} = \#\vec{w}_{\mathfrak{p}(1)} = \ell_{\mathfrak{p}(1)}$.)

An element of $\Gamma_{\mathfrak{p}(1)}$ is regarded as an element of the permutation group $\mathfrak{S}_{\ell_{\mathfrak{p}(1)}}$. So it transforms $\vec{w}_{\mathfrak{p}(2)}$ by permutation. The group $\Gamma_{\mathfrak{p}(1)}$ acts also on the set of pairs $(\sigma_0; \vec{S}_0^{\rm o}, (\vec{S}_0^{\rm c}, \vec{\theta}_0))$. We then have the following:

Lemma 2.147. *The set of triples $(\vec{w}_{\mathfrak{p}(2)}, \sigma_0; \vec{S}_0^{\rm o}, (\vec{S}_0^{\rm c}, \vec{\theta}_0))$ satisfying Condition 2.146 consists of a single $\Gamma_{\mathfrak{p}(1)}$-orbit.*

Proof. This is an immediate consequence of Proposition 2.102. □

We continue the construction of the induced stabilization data at $\mathfrak{p}(2)$. Let $\mathcal{G}_{\mathfrak{p}(2)\cup\vec{w}_{\mathfrak{p}(2)}}$ be the combinatorial type of $\mathfrak{p}(2) \cup \vec{w}_{\mathfrak{p}(2)}$. In general it is different from the combinatorial type $\mathcal{G}_{\mathfrak{p}(1)\cup\vec{w}_{\mathfrak{p}(1)}}$ of $\mathfrak{p}(1) \cup \vec{w}_{\mathfrak{p}(1)}$. In fact the graph $\mathcal{G}_{\mathfrak{p}(2)\cup\vec{w}_{\mathfrak{p}(2)}}$ is obtained from the graph $\mathcal{G}_{\mathfrak{p}(1)\cup\vec{w}_{\mathfrak{p}(1)}}$ by shrinking all the edges e such that $S_{0,\rm e} \ne \infty$. We denote by $C^{1,\rm fin}(\mathcal{G}_{\mathfrak{p}(1)\cup\vec{w}_{\mathfrak{p}(1)}})$ the set of edges e with $S_{0,\rm e} \ne \infty$. We have

$$C^1(\mathcal{G}_{\mathfrak{p}(1)\cup\vec{w}_{\mathfrak{p}(1)}}) = C^{1,\rm fin}(\mathcal{G}_{\mathfrak{p}(1)\cup\vec{w}_{\mathfrak{p}(1)}}) \sqcup C^1(\mathcal{G}_{\mathfrak{p}(2)\cup\vec{w}_{\mathfrak{p}(2)}}). \tag{2.353}$$

Here the right hand side is the *disjoint* union. Choose $\Delta S \in \mathbb{R}_{>0}$ that is sufficiently smaller than $S_{0,\rm e}$. (We may take for example $\Delta S = 1$.)

Let $\mathfrak{V}^{(2)}(\mathfrak{x}_{\mathfrak{p}(2)} \cup \vec{w}_{\mathfrak{p}(2)})$ be a neighborhood of $\mathfrak{x}_{\mathfrak{p}(2)} \cup \vec{w}_{\mathfrak{p}(2)}$ in the stratum $\mathcal{M}_{k+1,\ell+\ell_{\mathfrak{p}(1)}}(\mathcal{G}_{\mathfrak{p}(2)\cup\vec{w}_{\mathfrak{p}(2)}})$ of the Deligne-Mumford moduli space $\mathcal{M}_{k+1,\ell+\ell_{\mathfrak{p}(1)}}$. We can take them so that there exists an identification

$$\begin{aligned}\mathfrak{V}^{(2)}(\mathfrak{x}_{\mathfrak{p}(2)} \cup \vec{w}_{\mathfrak{p}(2)}) = &\mathfrak{V}^{(1)}(\mathfrak{x}_{\mathfrak{p}(1)} \cup \vec{w}_{\mathfrak{p}(1)}) \\ &\times \prod_{{\rm e} \in C^{1,\rm fin}_{\rm o}(\mathcal{G}_{\mathfrak{p}(1)\cup\vec{w}_{\mathfrak{p}(1)}})} ((S_{0,\rm e} - \Delta S, S_{0,\rm e} + \Delta S) \times [0,1]) \\ &\times \prod_{{\rm e} \in C^{1,\rm fin}_{\rm c}(\mathcal{G}_{\mathfrak{p}(1)\cup\vec{w}_{\mathfrak{p}(1)}})} ((S_{0,\rm e} - \Delta S, S_{0,\rm e} + \Delta S) \times S^1).\end{aligned} \tag{2.354}$$

Let $\overline{\rm v}$ be a vertex of $\mathcal{G}_{\mathfrak{p}(2)\cup\vec{w}_{\mathfrak{p}(2)}}$. We take the subgraph $\mathcal{G}_{\mathfrak{p}(1)\cup\vec{w}_{\mathfrak{p}(1)},\overline{\rm v}}$ of the graph $\mathcal{G}_{\mathfrak{p}(1)\cup\vec{w}_{\mathfrak{p}(1)}}$ as follows. There exists a map $\mathcal{G}_{\mathfrak{p}(1)\cup\vec{w}_{\mathfrak{p}(1)}} \to \mathcal{G}_{\mathfrak{p}(2)\cup\vec{w}_{\mathfrak{p}(2)}}$ that shrinks the edges e with $S_{0,\rm e} \ne \infty$. An edge ${\rm e} \in C^1(\mathcal{G}_{\mathfrak{p}(1)\cup\vec{w}_{\mathfrak{p}(1)}})$ is an edge of $\mathcal{G}_{\mathfrak{p}(1)\cup\vec{w}_{\mathfrak{p}(1)},\overline{\rm v}}$ if it goes to the point $\overline{\rm v}$ by this map, or it goes to the edge containing $\overline{\rm v}$ by this map.

Then we have

$$\begin{aligned}
&\mathfrak{V}^{(2)}((\mathfrak{x}_{\mathfrak{p}(2)} \cup \vec{w}_{\mathfrak{p}(2)})_{\overline{\mathrm{v}}}) \\
&= \prod_{\mathrm{v} \in C^0(\mathcal{G}_{\mathfrak{p}(1)\cup\vec{w}_{\mathfrak{p}(1)}},\overline{\mathrm{v}})} \mathfrak{V}^{(1)}((\mathfrak{x}_{\mathfrak{p}(1)} \cup \vec{w}_{\mathfrak{p}(1)})_{\mathrm{v}}) \\
&\times \prod_{\mathrm{e} \in C^{1,\mathrm{fin}}_{\mathrm{o}}(\mathcal{G}_{\mathfrak{p}(1)\cup\vec{w}_{\mathfrak{p}(1)}},\overline{\mathrm{v}})} ((S_{0,\mathrm{e}} - \Delta S, S_{0,\mathrm{e}} + \Delta S) \times [0,1]) \\
&\times \prod_{\mathrm{e} \in C^{1,\mathrm{fin}}_{\mathrm{c}}(\mathcal{G}_{\mathfrak{p}(1)\cup\vec{w}_{\mathfrak{p}(1)}},\overline{\mathrm{v}})} ((S_{0,\mathrm{e}} - \Delta S, S_{0,\mathrm{e}} + \Delta S) \times S^1).
\end{aligned} \tag{2.355}$$

The universal family over the Deligne-Mumford moduli space restricts to a fiber bundle

$$\pi : \mathfrak{M}^{(2)}((\mathfrak{x}_{\mathfrak{p}(2)} \cup \vec{w}_{\mathfrak{p}(2)})_{\overline{\mathrm{v}}}) \to \mathfrak{V}^{(2)}((\mathfrak{x}_{\mathfrak{p}(2)} \cup \vec{w}_{\mathfrak{p}(2)})_{\overline{\mathrm{v}}}). \tag{2.356}$$

The fiber at $(\sigma; \vec{S}^{\mathrm{o}}, (\vec{S}^{\mathrm{c}}, \vec{\theta}))$ of this bundle, which we denote by $\Sigma_{(\sigma;\vec{S}^{\mathrm{o}},(\vec{S}^{\mathrm{c}},\vec{\theta}))}$, is the union of the following three types of 2 dimensional manifolds.

(I) For each $\mathrm{v} \in C^0(\mathcal{G}_{\mathfrak{p}(2)\cup\vec{w}_{\mathfrak{p}(2)}})$ we consider the core $K_{\mathrm{v}}^{\sigma_{\mathrm{v}}}$ that is contained in $\Sigma_{\sigma_{\mathrm{v}}}$. (Here $\sigma_{\mathrm{v}} \in \mathfrak{V}^{(1)}((\mathfrak{x}_{\mathfrak{p}(1)} \cup \vec{w}_{\mathfrak{p}(1)})_{\mathrm{v}})$ is a compoent of σ and $\Sigma_{\sigma_{\mathrm{v}}}$ is a Riemann surface corresponding to this element σ_{v}.)

(II) If $\mathrm{e} \in C^1_{\mathrm{o}}(\mathcal{G}_{\mathfrak{p}(2)\cup\vec{w}_{\mathfrak{p}(2)}})$, $S_{0,\mathrm{e}} = \infty$ and e goes to an outgoing edges of $\overline{\mathrm{v}}$, we have $[0,\infty) \times [0,1]$.
If $\mathrm{e} \in C^1_{\mathrm{o}}(\mathcal{G}_{\mathfrak{p}(2)\cup\vec{w}_{\mathfrak{p}(2)}})$, $S_{0,\mathrm{e}} = \infty$ and e goes to an incoming edge of $\overline{\mathrm{v}}$, we have $(-\infty,0] \times [0,1]$.
If $\mathrm{e} \in C^1_{\mathrm{c}}(\mathcal{G}_{\mathfrak{p}(2)\cup\vec{w}_{\mathfrak{p}(2)}})$, $S_{0,\mathrm{e}} = \infty$ and e goes to an outgoing edge of $\overline{\mathrm{v}}$, we have $[0,\infty) \times S^1$.
If $\mathrm{e} \in C^1_{\mathrm{c}}(\mathcal{G}_{\mathfrak{p}(2)\cup\vec{w}_{\mathfrak{p}(2)}})$, $S_{0,\mathrm{e}} = \infty$ and e goes to an incoming edge of $\overline{\mathrm{v}}$, we have $(-\infty,0] \times S^1$.

(III) If $\mathrm{e} \in C^1_{\mathrm{o}}(\mathcal{G}_{\mathfrak{p}(2)\cup\vec{w}_{\mathfrak{p}(2)}})$, $S_{0,\mathrm{e}} \neq \infty$, we have $[-5S_{\mathrm{e}}, 5S_{\mathrm{e}}] \times [0,1]$. If $\mathrm{e} \in C^1_{\mathrm{c}}(\mathcal{G}_{\mathfrak{p}(2)\cup\vec{w}_{\mathfrak{p}(2)}})$, $S_{0,\mathrm{e}} \neq \infty$, we have $[-5S_{\mathrm{e}}, 5S_{\mathrm{e}}] \times S^1$.

Definition 2.148. The core $K_{\overline{\mathrm{v}}}$ of $\Sigma_{(\sigma;\vec{S}^{\mathrm{o}},(\vec{S}^{\mathrm{c}},\vec{\theta}))}$ is the union of the subsets of type I or type III.

On the complement of the core, the fiber bundle (2.356) has a trivialization, that is given by the identification of the subsets of type II with the standard set mentioned there. This trivialization preserves complex structures.

This trivialization extends to the subsets of type I. In fact, such an extension is a part of the data included in the coordinate at infinity of $\mathfrak{w}_{\mathfrak{p}(1)}$. Note that this extension of trivialization does not respect the fiberwise complex structure.

Note, however, that this trivialization does *not* extend to the trivialization of the fiber bundle (2.356) if there exists an edge $\mathrm{e} \in C^1_{\mathrm{c}}(\mathcal{G}_{\mathfrak{p}(2)\cup\vec{w}_{\mathfrak{p}(2)}})$ with $S_{0,\mathrm{e}} \neq \infty$. In fact, there exists an S^1 factor in (2.355) that corresponds to such an edge e and our fiber bundle has nontrivial monodromy around it, that is the Dehn twist at the domain $[-5S_{0,\mathrm{e}}, 5S_{0,\mathrm{e}}] \times S^1$.

Therefore to find a coordinate at infinity that satisfies Definition 2.10 (5) we need to restrict the domain. We take a sufficiently small $\Delta\theta$ (for example $\Delta\theta = 1/10$)

and put

$$\begin{aligned}
&\mathfrak{V}((\mathfrak{x}_{\mathfrak{p}(2)} \cup \vec{w}_{\mathfrak{p}(2)})_{\overline{\rm v}}) \\
=& \prod_{{\rm v} \in C^0(\mathcal{G}_{\mathfrak{p}(1)\cup\vec{w}_{\mathfrak{p}(1)}}, \overline{\rm v})} \mathfrak{V}((\mathfrak{x}_{\mathfrak{p}(1)} \cup \vec{w}_{\mathfrak{p}(1)})_{\rm v}) \\
&\times \prod_{{\rm e} \in C^{1,{\rm fin}}_{\rm o}(\mathcal{G}_{\mathfrak{p}(1)\cup\vec{w}_{\mathfrak{p}(1)}}, \overline{\rm v})} ((S_{0,{\rm e}} - \Delta S, S_{0,{\rm e}} + \Delta S) \times [0,1]) \\
&\times \prod_{{\rm e} \in C^{1,{\rm fin}}_{\rm c}(\mathcal{G}_{\mathfrak{p}(1)\cup\vec{w}_{\mathfrak{p}(1)}}, \overline{\rm v})} ((S_{0,{\rm e}} - \Delta S, S_{0,{\rm e}} + \Delta S) \times (\theta_{0,{\rm e}} - \Delta\theta, \theta_{0,{\rm e}} + \Delta\theta)).
\end{aligned} \tag{2.357}$$

(Note $\mathfrak{x}_{\mathfrak{p}(2)} \cup \vec{w}_{\mathfrak{p}(2)} = \overline{\Phi}(\sigma_0; \vec{S}^{\rm o}_0, (\vec{S}^{\rm c}_0, \vec{\theta}_0))$ and $\theta_{0,{\rm e}}$ is a component of $\vec{\theta}_0$).)

We consider the fiber bundle

$$\pi : \mathfrak{M}((\mathfrak{x}_{\mathfrak{p}(2)} \cup \vec{w}_{\mathfrak{p}(2)})_{\overline{\rm v}}) \to \mathfrak{V}((\mathfrak{x}_{\mathfrak{p}(2)} \cup \vec{w}_{\mathfrak{p}(2)})_{\overline{\rm v}}) \tag{2.358}$$

in place of (2.356).

Now we can extend the trivialization of the fiber bundle defined in the complement of the core, to the trivialization that is defined everywhere. (But it does not preserve the complex structures.) We have thus defined a coordinate at infinity of $\mathfrak{p}(2)$.

We take the codimension 2 submanifolds $\mathcal{D}_{\mathfrak{p}(1),i}$ that is a part of $\mathfrak{w}_{\mathfrak{p}(1)}$ and put

$$\mathcal{D}_{\mathfrak{p}(2),i} = \mathcal{D}_{\mathfrak{p}(1),i}.$$

Definition 2.149. The stabilization data at $\mathfrak{p}(2)$ that is obtained as above is called the *stabilization data induced by* $\mathfrak{w}_{\mathfrak{p}(1)}$.

Remark 2.150. There is more than one ways of extending the trivialization of the fiber bundle that is given on the part of type I and type II to the whole space. However the way to do so is determined if we take the following two families of diffeomorphisms.

(1) A family of diffeomorphisms from the rectangles $[-5S_{\rm e}, 5S_{\rm e}] \times [0,1]$ to $[0,1] \times [0,1]$ so that they are obvious isometries in a neighborhood of $\partial[-5S_{\rm e}, 5S_{\rm e}] \times [0,1]$. Here the parameter is $S_{\rm e} \in (S_{0,{\rm e}} - \Delta S, S_{0,{\rm e}} + \Delta S)$.
(2) A family of diffeomorphisms from the annuli $[-5S_{\rm e}, 5S_{\rm e}] \times S^1$ to $[0,1] \times S^1$ so that they are obvious isometries in a neighborhood of $\{-5S_{\rm e}\} \times S^1$ and is the rotation by $\theta_{\rm e}$ in a neighborhood $\{5S_{\rm e}\} \times S^1$. Here the parameter is $S_{\rm e} \in (S_{0,{\rm e}} - \Delta S, S_{0,{\rm e}} + \Delta S)$ and $\theta_{\rm e} \in (\theta_{0,{\rm e}} - \Delta\theta, \theta_{0,{\rm e}} + \Delta\theta)$.

Such families of diffeomorphisms obviously exist. We can take one and use it whenever we define the induced coordinate at infinity. In that sense the notion of induced coordinate at infinity and of induced stabilization data is well-defined. (Namely it can be taken independent of $\mathfrak{p}(1)$ for example.)

In Subsection 2.2, we discussed how the parametrization changes when we change the coordinate at infinity. There we defined a map Ψ_{12}. (See (2.168).) The following is obvious from definition. We use the notation in Propositions 2.19 and 2.23.

Lemma 2.151. *If we take the induced core on $\mathfrak{Y}_0$ then $\overline{\Phi}_{12} = \Psi_{12}$. Moreover $\mathfrak{v}_{\mathfrak{y}_2,\vec{T}_2,\vec{\theta}_2}$ is the identity map on the core $K_{\rm v}$.*

The first main result of this subsection is the following.

Proposition 2.152. *Let* $\mathfrak{p}(1) \in \mathcal{M}_{k+1,\ell}(\beta)$ *and take a stabilization data* $\mathfrak{w}_{\mathfrak{p}(1)}$ *at* $\mathfrak{p}(1)$ *and admissible* $(\mathfrak{o}^{(1)}, \mathcal{T}^{(1)})$. *Let* $\mathfrak{p}(2)$ *be in the image of (2.351). We take the induced stablization data* $\mathfrak{w}_{\mathfrak{p}(2)}$. *Let* $\mathfrak{A} \subseteq \mathfrak{C}(\mathfrak{p}(2)) \subseteq \mathfrak{C}(\mathfrak{p}(1))$.

Then there exists an admissible $(\mathfrak{o}_0^{(2)}, \mathcal{T}_0^{(2)})$ *such that if* $(\mathfrak{o}^{(2)}, \mathcal{T}^{(2)}) < (\mathfrak{o}_0^{(2)}, \mathcal{T}_0^{(2)})$ *there exists a coordinate change*

$$(\varphi_{12}, \hat\varphi_{12}) : V(\mathfrak{p}(2), \mathfrak{w}_{\mathfrak{p}(2)}; (\mathfrak{o}^{(2)}, \mathcal{T}^{(2)}); \mathfrak{A}) \to V(\mathfrak{p}(1), \mathfrak{w}_{\mathfrak{p}(1)}; (\mathfrak{o}^{(1)}, \mathcal{T}^{(1)}); \mathfrak{A}).$$

Proof. We have maps

$$\begin{aligned}\mathrm{Glu}^{(1)} :&B_{\mathfrak{o}^{(1)}}^{\mathfrak{w}_{\mathfrak{p}(1)}}(\mathfrak{p}; V_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p}(1);\mathfrak{A})) \times (\vec{\mathcal{T}}^{(1)},\infty] \times ((\vec{\mathcal{T}}^{(1)},\infty] \times \vec{S}^1) \\ &\to \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}(1)},(\ell_c))}^{\mathfrak{w}_{\mathfrak{p}(1)}}(\beta;\mathfrak{p}(1);\mathfrak{A})_{\epsilon_{0,1},\vec{\mathcal{T}}^{(1)}}\end{aligned} \tag{2.359}$$

and

$$\begin{aligned}\mathrm{Glu}^{(2)} :&B_{\mathfrak{o}^{(2)}}^{\mathfrak{w}_{\mathfrak{p}(2)}}(\mathfrak{p}; V_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p}(2);\mathfrak{A})) \times (\vec{\mathcal{T}}^{(2)},\infty] \times ((\vec{\mathcal{T}}^{(2)},\infty] \times \vec{S}^1) \\ &\to \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}(1)},(\ell_c))}^{\mathfrak{w}_{\mathfrak{p}(2)}}(\beta;\mathfrak{p}(2);\mathfrak{A})_{\epsilon_{0,2},\vec{\mathcal{T}}^{(2)}}\end{aligned} \tag{2.360}$$

by the gluing constructions at $\mathfrak{p}(1)$ and at $\mathfrak{p}(2)$ respectively. (More precisely for a given $\epsilon_{0,2}$, the map (2.360) is defined by choosing $\mathfrak{o}^{(2)}$ small and $\mathcal{T}^{(2)}$ large.)

By the assumption and Proposition 2.102, there exists $\vec{w}_{\mathfrak{p}(2),c}$ such that

$$(\mathfrak{p}(2) \cup \vec{w}_{\mathfrak{p}(2)}, (\vec{w}_{\mathfrak{p}(2),c})) \in \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}(1)},(\ell_c))}^{\mathfrak{w}_{\mathfrak{p}(1)}}(\beta;\mathfrak{p}(1);\mathfrak{A})^{\mathrm{trans}}_{\epsilon_{0,1},\vec{\mathcal{T}}^{(1)}}.$$

We observe

$$(\mathfrak{p}(2) \cup \vec{w}_{\mathfrak{p}(2)}, (\vec{w}_{\mathfrak{p}(2),c})) \in \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}(2)},(\ell_c))}^{\mathfrak{w}_{\mathfrak{p}(2)}}(\beta;\mathfrak{p}(1);\mathfrak{A})_{\epsilon_{0,2},\vec{\mathcal{T}}^{(2)}}$$

and the image of (2.360) defines a neighborhood basis when we move $\epsilon_{0,2}$. Therefore by taking $\epsilon_{0,2}$ small and $\mathcal{T}^{(2)}$ large, we may assume that

$$\begin{aligned}&\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}(1)},(\ell_c))}^{\mathfrak{w}_{\mathfrak{p}(2)}}(\beta;\mathfrak{p}(2);\mathfrak{A})_{\epsilon_{0,2},\vec{\mathcal{T}}^{(2)}} \\ &\subset \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}(1)},(\ell_c))}^{\mathfrak{w}_{\mathfrak{p}(1)}}(\beta;\mathfrak{p}(1);\mathfrak{A})_{\epsilon_{0,1},\vec{\mathcal{T}}^{(1)}}\end{aligned} \tag{2.361}$$

and this is an open embedding. By construction, the element of the thickened moduli space $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}(2)},(\ell_c))}^{\mathfrak{w}_{\mathfrak{p}(2)}}(\beta;\mathfrak{p}(2);\mathfrak{A})_{\epsilon_{0,2},\vec{\mathcal{T}}^{(2)}}$ satisfies the transversal constrain at all additional marked points with respect to $\mathfrak{w}_{\mathfrak{p}(1)}$ if and only if the transversal constraint at all additional marked points with respect to $\mathfrak{w}_{\mathfrak{p}(2)}$ is satisfied. Therefore

$$\begin{aligned}&\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}(1)},(\ell_c))}^{\mathfrak{w}_{\mathfrak{p}(2)}}(\beta;\mathfrak{p}(2);\mathfrak{A})^{\mathrm{trans}}_{\epsilon_{0,2},\vec{\mathcal{T}}^{(2)}} \\ &\subset \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}(1)},(\ell_c))}^{\mathfrak{w}_{\mathfrak{p}(1)}}(\beta;\mathfrak{p}(1);\mathfrak{A})^{\mathrm{trans}}_{\epsilon_{0,1},\vec{\mathcal{T}}^{(1)}}\end{aligned} \tag{2.362}$$

and this is an open embedding. We thus can define a continuous strata-wise C^m-map φ_{12} as the inclusion map. It is an open embedding of C^m-class strata-wise.

Lemma 2.153. φ_{12} *is of* C^m*-class.*

Proof. The proof is similar to that of Lemma 2.133. We repeat the detail for completeness. Let $\hat V(\mathfrak{p}(j), \mathfrak{w}_{\mathfrak{p}(j)}; (\mathfrak{o}^{(j)}, \mathcal{T}^{(j)}); \mathfrak{A})$ be the inverse image of $V(\mathfrak{p}(j), \mathfrak{w}_{\mathfrak{p}(j)}; (\mathfrak{o}^{(j)}, \mathcal{T}^{(j)}); \mathfrak{A})$ by $\mathrm{Glu}^{(j)}$. (Here $j = 1, 2$). It suffices to show that

$$\begin{aligned}\tilde\varphi_{12} = (\mathrm{Glu}^{(1)})^{-1} \circ \mathrm{Glu}^{(2)} :&\hat V(\mathfrak{p}(2), \mathfrak{w}_{\mathfrak{p}(2)}; (\mathfrak{o}^{(2)}, \mathcal{T}^{(2)}); \mathfrak{A}) \\ &\to \hat V(\mathfrak{p}(1), \mathfrak{w}_{\mathfrak{p}(1)}; (\mathfrak{o}^{(1)}, \mathcal{T}^{(1)}); \mathfrak{A})\end{aligned}$$

is of C^m-class. We obtain maps

$$\begin{aligned}\mathfrak{F}^{(j)} &: \hat{V}(\mathfrak{p}(j), \mathfrak{w}_{\mathfrak{p}(j)}; (\mathfrak{o}^{(j)}, \mathcal{T}^{(j)}); \mathfrak{A}) \\ &\to \prod_{\mathrm{v}\in C^0(\mathcal{G}_{\mathfrak{p}(1)})} C^m((K_{\mathrm{v}}^{+\vec{R}}, K_{\mathrm{v}}^{+\vec{R}} \cap \partial\Sigma_{\mathrm{v},(1)}), (X, L)) \\ &\quad\times \prod_{\mathrm{v}\in C^0(\mathcal{G}_{\mathfrak{p}(1)})} \mathfrak{V}((\mathfrak{x}_{\mathfrak{p}(1)} \cup \vec{w}_{\mathfrak{p}(1)})_{\mathrm{v}}) \times (\vec{\mathcal{T}}^{(j)}, \infty] \times (\vec{\mathcal{T}}^{(j)}, \infty] \times \vec{S}^1)\end{aligned} \tag{2.363}$$

in the same way as (2.336) for $j = 1, 2$. We remark here that we take the graph $\mathcal{G}_{\mathfrak{p}(1)}$ for the case $j = 2$ also. By applying Theorem 2.72 we find that (2.363) is an C^m-embedding for $j = 1$.

We will prove that (2.363) is a C^m-embedding for $j = 2$ also. It follows from Theorem 2.72 applied to the gluing at $\mathfrak{p}(2)$ that $\mathfrak{F}^{(2)}$ is of C^m-class. We put $\mathfrak{F}^{(2)} = (\mathfrak{F}_1^{(2)}, \mathfrak{F}_2^{(2)})$. Here $\mathfrak{F}_1^{(2)}$ (resp.$\mathfrak{F}_2^{(2)}$) is a map to the factor in the second line (resp. third line). It suffices to show that $\mathfrak{F}_1^{(2)}$ is a C^m-embedding on each of the fiber of $\mathfrak{F}_2^{(2)}$. Note that the factors of the third line parametrize the complex structure of the source. The fact that $\mathfrak{F}_1^{(2)}$ is an embedding on the fiber of $T_{\mathrm{e}} = \infty$ follows from Theorem 2.72 applied to the gluing at $\mathfrak{p}(1)$. Then we apply Theorem 2.72 to the gluing at $\mathfrak{p}(2)$ to show that $\mathfrak{F}_1^{(2)}$ is an embedding on the fiber of $\mathfrak{F}_2^{(2)}$ if $\mathcal{T}^{(2)}$ is sufficiently large.

Now using the obvious fact that $\mathfrak{F}^{(1)} \circ \tilde{\varphi}_{12} = \mathfrak{F}^{(2)}$, we conclude that $\tilde{\varphi}_{12}$ is a C^m-embedding. □

Remark 2.154. Contrary to the case of the proof of Lemma 2.133, we do have $\mathfrak{F}^{(1)} \circ \tilde{\varphi}_{12} = \mathfrak{F}^{(2)}$. This is because we are using the coordinate at infinity $\mathfrak{w}_{\mathfrak{p}(2)}$ that is induced from $\mathfrak{w}_{\mathfrak{p}(1)}$ and so the parametrization of the core is the same.

We thus have defined φ_{12}. We define $\hat{\varphi}_{12} = \varphi_{12} \times$ identity. It is easy to see that φ_{12} is $\Gamma_{\mathfrak{p}(2)}$-equivariant. Other properties are also easy to prove. The proof of Proposition 2.152 is now complete. □

Remark 2.155. In Lemma 2.147 we proved that the two choices of $\vec{w}_{(2)}$ are transformed each other under the $\Gamma_{\mathfrak{p}(1)}$ action. More precisely we have the following. The action of $\Gamma_{\mathfrak{p}(1)}$ is given by the permutation of the marked points $\vec{w}_{(2)}$. If $\gamma \in \Gamma_{\mathfrak{p}(2)}$ the permutation of $\vec{w}_{(2)}$ gives an equivalent element. Namely there exists a biholomorphic map $\mathfrak{x}_{\mathfrak{p}(2)} \cup \vec{w}_{(2)} \to \mathfrak{x}_{\mathfrak{p}(2)} \cup \gamma\vec{w}_{(2)}$.

In case $\gamma \notin \Gamma_{\mathfrak{p}(2)}$, $\mathfrak{x}_{\mathfrak{p}(2)} \cup \vec{w}_{(2)}$ is not biholomorphic to $\mathfrak{x}_{\mathfrak{p}(2)} \cup \gamma\vec{w}_{(2)}$. Each of the choice $\vec{w}_{(2)}$ and $\gamma\vec{w}_{(2)}$ induces a stabilization data at $\mathfrak{p}(2)$, which we write $\mathfrak{w}_{(2)}$ and $\gamma\mathfrak{w}_{(2)}$ respectively. They define the coordinate changes. We remark that there is a canonical diffeomorphism

$$\mathcal{M}^{\mathfrak{w}_{\mathfrak{p}(2)}}_{k+1,(\ell,\ell_{\mathfrak{p}(1)},(\ell_c))}(\beta; \mathfrak{p}(2); \mathfrak{A})^{\mathrm{trans}}_{\epsilon_{0,2},\vec{\mathcal{T}}^{(2)}} \cong \mathcal{M}^{\gamma\mathfrak{w}_{\mathfrak{p}(2)}}_{k+1,(\ell,\ell_{\mathfrak{p}(1)},(\ell_c))}(\beta; \mathfrak{p}(2); \mathfrak{A})^{\mathrm{trans}}_{\epsilon_{0,2},\vec{\mathcal{T}}^{(2)}}$$

by permutation of the marked points. Namely we have

$$\gamma : V(\mathfrak{p}(2), \mathfrak{w}_{\mathfrak{p}(2)}; (\mathfrak{o}^{(2)}, \mathcal{T}^{(2)}); \mathfrak{A}) \to V(\mathfrak{p}(2), \gamma\mathfrak{w}_{\mathfrak{p}(2)}; (\mathfrak{o}^{(2)}, \mathcal{T}^{(2)}); \mathfrak{A}).$$

On the other hand $\gamma \in \Gamma_{\mathfrak{p}(1)}$ acts on $V(\mathfrak{p}(1), \mathfrak{w}_{\mathfrak{p}(1)}; (\mathfrak{o}^{(1)}, \mathcal{T}^{(1)}); \mathfrak{A})$. Since our construction is $\Gamma_{\mathfrak{p}(1)}$ equivariant we have

$$\gamma \circ \varphi_{12} = \varphi_{12} \circ \gamma.$$

Here φ_{12} in the left hand side uses $\mathfrak{w}_{\mathfrak{p}(2)}$ and φ_{12} in the right hand side uses $\gamma\mathfrak{w}_{\mathfrak{p}(2)}$. This is the same as the case of coordinate change of the charts of orbifolds.

Combined with the result of the last subsection, Proposition 2.152 implies the following.

Corollary 2.156. *Let $\mathfrak{p}(1) \in \mathcal{M}_{k+1,\ell}(\beta)$. We take a stabilization data $\mathfrak{w}_{\mathfrak{p}(1)}$ at $\mathfrak{p}(1)$ and admissible $(\mathfrak{o}^{(1)},\mathcal{T}^{(1)})$.*

Let $\mathfrak{p}(2)$ be in the image of (2.351). We take a stablization data $\mathfrak{w}_{\mathfrak{p}(2)}$ at $\mathfrak{p}(2)$.

Then there exists an admissible $(\mathfrak{o}_0^{(2)},\mathcal{T}_0^{(2)})$ such that the following holds for $\mathfrak{A}^{(j)} \subseteq \mathfrak{C}(\mathfrak{p}(j))$ with $\mathfrak{A}^{(2)} \subseteq \mathfrak{A}^{(1)}$.

For any $(\mathfrak{o}^{(2)},\mathcal{T}^{(2)}) < (\mathfrak{o}_0^{(2)},\mathcal{T}_0^{(2)})$ then there exists a coordinate change $(\varphi_{12},\hat\varphi_{12})$ from the Kuranishi chart $V(\mathfrak{p}(2),\mathfrak{w}_{\mathfrak{p}(2)};(\mathfrak{o}^{(2)},\mathcal{T}^{(2)});\mathfrak{A}^{(2)})$ to the Kuranishi chart $V(\mathfrak{p}(1),\mathfrak{w}_{\mathfrak{p}(1)};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(1)})$.

Proof. Let $\mathfrak{w}'_{\mathfrak{p}(2)}$ be the stabilization data at $\mathfrak{p}(2)$ induced by $\mathfrak{w}_{\mathfrak{p}(1)}$. Then the required coordinate change is obtained by composing the three coordinate changes associated to the pairs, $((\mathfrak{w}_{\mathfrak{p}(1)},\mathfrak{A}^{(1)}),(\mathfrak{w}_{\mathfrak{p}(1)},\mathfrak{A}^{(2)}))$, $((\mathfrak{w}_{\mathfrak{p}(1)},\mathfrak{A}^{(2)}),(\mathfrak{w}'_{\mathfrak{p}(2)},\mathfrak{A}^{(2)}))$, $((\mathfrak{w}'_{\mathfrak{p}(2)},\mathfrak{A}^{(2)}),(\mathfrak{w}_{\mathfrak{p}(2)},\mathfrak{A}^{(2)}))$. They are obtained by Proposition 2.131, Proposition 2.152, Proposition 2.131, respectively. □

Remark 2.157. By construction the coordinate change given in Corollary 2.156 is independent of the choices involved in the definition, in a neighborhood of $\mathfrak{p}(2)$.

We next prove the compatibility of the coordinate changes in Corollary 2.156.

Proposition 2.158. *Let $\mathfrak{p}(1) \in \mathcal{M}_{k+1,\ell}(\beta)$. We take a stabilization data $\mathfrak{w}_{\mathfrak{p}(1)}$ at $\mathfrak{p}(1)$ and admissible $(\mathfrak{o}^{(1)},\mathcal{T}^{(1)})$.*

Let $\mathfrak{p}(2)$ be in the image of (2.351). We take a stablization data $\mathfrak{w}_{\mathfrak{p}(2)}$ at $\mathfrak{p}(2)$. Let $(\mathfrak{o}_0^{(2)},\mathcal{T}_0^{(2)})$ be as in Corollary 2.156.

Then there exists $\epsilon_7 = \epsilon_7(\mathfrak{p}(1),\mathfrak{w}_{\mathfrak{p}(1)},\mathfrak{p}(2),\mathfrak{w}_{\mathfrak{p}(2)})$ with the following properties for each $(\mathfrak{o}^{(2)},\mathcal{T}^{(2)}) < (\mathfrak{o}_0^{(2)},\mathcal{T}_0^{(2)})$.

Let $\mathfrak{p}(3) \in \mathcal{M}_{k+1,\ell}(\beta)$. We assume $d(\mathfrak{p}(2),\mathfrak{p}(3)) < \epsilon_7$.[21] *Then for any stabilization data $\mathfrak{m}_{\mathfrak{p}(3)}$ at $\mathfrak{p}(3)$, there exists admissible $(\mathfrak{o}_0^{(3)},\mathcal{T}_0^{(3)})$ such that if $(\mathfrak{o}^{(3)},\mathcal{T}^{(3)}) < (\mathfrak{o}_0^{(3)},\mathcal{T}_0^{(3)})$ and $\mathfrak{A}^{(j)} \subseteq \mathfrak{C}(\mathfrak{p}(j))$ ($j = 1,2,3$) with $\mathfrak{A}^{(1)} \supseteq \mathfrak{A}^{(2)} \supseteq \mathfrak{A}^{(3)}$, then we have the following.*

(1) *There exists a coordinate change*

$$(\varphi_{23},\hat\varphi_{23}) : V(\mathfrak{p}(3),\mathfrak{w}_{\mathfrak{p}(3)};(\mathfrak{o}^{(3)},\mathcal{T}^{(3)});\mathfrak{A}^{(3)}) \to V(\mathfrak{p}(2),\mathfrak{w}_{\mathfrak{p}(2)};(\mathfrak{o}^{(2)},\mathcal{T}^{(2)});\mathfrak{A}^{(2)})$$

as in Corollary 2.156.

(2) *There exists a coordinate change*

$$(\varphi_{13},\hat\varphi_{13}) : V(\mathfrak{p}(3),\mathfrak{w}_{\mathfrak{p}(3)};(\mathfrak{o}^{(3)},\mathcal{T}^{(3)});\mathfrak{A}^{(3)}) \to V(\mathfrak{p}(1),\mathfrak{w}_{\mathfrak{p}(1)};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(1)})$$

as in Corollary 2.156.

(3) *We have*

$$(\varphi_{13},\hat\varphi_{13}) = (\varphi_{12},\hat\varphi_{12}) \circ (\varphi_{23},\hat\varphi_{23}).$$

Here

$$(\varphi_{12},\hat\varphi_{12}) : V(\mathfrak{p}(2),\mathfrak{w}_{\mathfrak{p}(2)};(\mathfrak{o}^{(2)},\mathcal{T}^{(2)});\mathfrak{A}^{(2)}) \to V(\mathfrak{p}(1),\mathfrak{w}_{\mathfrak{p}(1)};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A}^{(1)})$$

[21] d here is any metric on $\mathcal{M}_{k+1,\ell}(\beta)$.

is the coordinate change in Corollary 2.156.

Proof. By the same reason as in the case of Proposition 2.152 we may assume $\mathfrak{A}^{(1)} = \mathfrak{A}^{(2)} = \mathfrak{A}^{(3)} = \mathfrak{A}$. So we will assume it throughout the proof.

We first prove the following.

Lemma 2.159. *Let $\mathfrak{w}^{(1)}_{\mathfrak{p}(2)}$ be the stabilization data at $\mathfrak{p}(2)$ induced by $\mathfrak{w}_{\mathfrak{p}(1)}$ and $\mathfrak{w}^{(1)}_{\mathfrak{p}(3)}$ the stabilization data at $\mathfrak{p}(3)$ induced by $\mathfrak{w}^{(1)}_{\mathfrak{p}(2)}$. Then $\mathfrak{w}^{(1)}_{\mathfrak{p}(3)}$ is the stabilization data induced by $\mathfrak{w}_{\mathfrak{p}(1)}$.*

The proof is obvious.

Lemma 2.160. *Let $\mathfrak{w}^{(1)}_{\mathfrak{p}(1)} = \mathfrak{w}_{\mathfrak{p}(1)}$ and $\mathfrak{w}^{(1)}_{\mathfrak{p}(2)}$, $\mathfrak{w}^{(1)}_{\mathfrak{p}(3)}$ be as in Lemma 2.159. We denote by $(\varphi_{ij},\hat\varphi_{ij})$ (for $1 \le i < j \le 3$) the coordinate changes induced by the pair $(\mathfrak{w}^{(1)}_{\mathfrak{p}(i)}, \mathfrak{w}^{(1)}_{\mathfrak{p}(j)})$. Then we have*

$$(\varphi_{12},\hat\varphi_{12}) \circ (\varphi_{23},\hat\varphi_{23}) = (\varphi_{13},\hat\varphi_{13}) \tag{2.364}$$

in a neighborhood of $\mathfrak{p}(3)$.

Proof. We can choose $\epsilon_{0,j}$ $(j = 1,2,3)$ such that

$$\begin{aligned}
&\mathcal{M}^{\mathfrak{w}^{(1)}_{\mathfrak{p}(3)}}_{k+1,(\ell,\ell_{\mathfrak{p}(1)},(\ell_c))}(\beta;\mathfrak{p}(3);\mathfrak{A})^{\rm trans}_{\epsilon_{0,3},\vec{\mathcal{T}}^{(3)}}\\
&\subset \mathcal{M}^{\mathfrak{w}^{(1)}_{\mathfrak{p}(2)}}_{k+1,(\ell,\ell_{\mathfrak{p}(1)},(\ell_c))}(\beta;\mathfrak{p}(2);\mathfrak{A})^{\rm trans}_{\epsilon_{0,2},\vec{\mathcal{T}}^{(2)}}\\
&\subset \mathcal{M}^{\mathfrak{w}^{(1)}_{\mathfrak{p}(1)}}_{k+1,(\ell,\ell_{\mathfrak{p}(1)},(\ell_c))}(\beta;\mathfrak{p}(1);\mathfrak{A})^{\rm trans}_{\epsilon_{0,1},\vec{\mathcal{T}}^{(1)}}.
\end{aligned}$$

The maps (2.364) are all induced by this inclusion in a neighborhood of $\mathfrak{p}(3)$. Hence the lemma. □

The proof of the next lemma is the main part of the proof of Proposition 2.158.

Lemma 2.161. *Let $\mathfrak{p}(2) \in \mathcal{M}_{k+1,\ell}(\beta)$ and let $\mathfrak{w}^{(1)}_{\mathfrak{p}(2)}$, $\mathfrak{w}^{(2)}_{\mathfrak{p}(2)}$ be two stabilization data at $\mathfrak{p}(2)$. Suppose a $\mathfrak{w}^{(1)}_{\mathfrak{p}(2)}$-admissible $(\mathfrak{o}^{(21)},\mathcal{T}^{(21)})$ is given. Take an $\mathfrak{w}^{(2)}_{\mathfrak{p}(2)}$-admissible $(\mathfrak{o}^{(22)}_0,\mathcal{T}^{(22)}_0)$ such that if $(\mathfrak{o}^{(22)},\mathcal{T}^{(22)}) < (\mathfrak{o}^{(22)}_0,\mathcal{T}^{(22)}_0)$ then there exists a coordinate change*

$$(\varphi_{(21)(22)},\hat\varphi_{(21)(22)}) : V(\mathfrak{p}(2),\mathfrak{w}^{(2)}_{\mathfrak{p}(2)};(\mathfrak{o}^{(22)},\mathcal{T}^{(22)});\mathfrak{A}) \to V(\mathfrak{p}(2),\mathfrak{w}^{(1)}_{\mathfrak{p}(2)};(\mathfrak{o}^{(21)},\mathcal{T}^{(21)});\mathfrak{A})$$

as in Proposition 2.131.

Then there exists $\epsilon_8 = \epsilon_8(\mathfrak{p}(2),\mathfrak{w}^{(1)}_{\mathfrak{p}(2)},\mathfrak{w}^{(2)}_{\mathfrak{p}(2)},(\mathfrak{o}^{(21)},\mathcal{T}^{(21)}),(\mathfrak{o}^{(22)},\mathcal{T}^{(22)}))$ such that if $\mathfrak{p}(3) \in \mathcal{M}_{k+1,\ell}(\beta)$, $d(\mathfrak{p}(2),\mathfrak{p}(3)) < \epsilon_8$ the following holds.

1. *There exists a stabilization data $\mathfrak{m}^{(1)}_{\mathfrak{p}(3)}$ at $\mathfrak{p}(3)$ induced from $\mathfrak{w}^{(1)}_{\mathfrak{p}(2)}$ and a stabilization data $\mathfrak{m}^{(2)}_{\mathfrak{p}(3)}$at $\mathfrak{p}(3)$ induced from $\mathfrak{w}^{(2)}_{\mathfrak{p}(2)}$.*
2. *There exists a $\mathfrak{w}^{(1)}_{\mathfrak{p}(3)}$-admissible $(\mathfrak{o}^{(31)}_0,\mathcal{T}^{(31)}_0)$ such that if $(\mathfrak{o}^{(31)},\mathcal{T}^{(31)}) <$ $(\mathfrak{o}^{(31)}_0,\mathcal{T}^{(31)}_0)$ then the cooridinate change*

$$(\varphi_{(21)(31)},\hat\varphi_{(21)(31)}) : V(\mathfrak{p}(3),\mathfrak{w}^{(1)}_{\mathfrak{p}(3)};(\mathfrak{o}^{(31)},\mathcal{T}^{(31)});\mathfrak{A}) \to V(\mathfrak{p}(2),\mathfrak{w}^{(1)}_{\mathfrak{p}(2)};(\mathfrak{o}^{(21)},\mathcal{T}^{(21)});\mathfrak{A})$$

as in Proposition 2.152 exists.

(3) *There exists a* $\mathfrak{w}^{(2)}_{\mathfrak{p}(3)}$*-admissible* $(\mathfrak{o}_0^{(32)}, \mathcal{T}_0^{(32)})$ *such that if* $(\mathfrak{o}^{(32)}, \mathcal{T}^{(32)}) < (\mathfrak{o}_0^{(32)}, \mathcal{T}_0^{(32)})$ *then the cooridinate change*

$$(\varphi_{(22)(32)}, \hat\varphi_{(22)(32)}) : V(\mathfrak{p}(3), \mathfrak{w}^{(2)}_{\mathfrak{p}(3)}; (\mathfrak{o}^{(32)}, \mathcal{T}^{(32)}); \mathfrak{A}) \to V(\mathfrak{p}(2), \mathfrak{w}^{(2)}_{\mathfrak{p}(2)}; (\mathfrak{o}^{(22)}, \mathcal{T}^{(22)}); \mathfrak{A})$$

as in Proposition 2.152 exists.

(4) *There exists a* $\mathfrak{w}^{(2)}_{\mathfrak{p}(3)}$*-admissible* $(\mathfrak{o}_0^{(32)\prime}, \mathcal{T}_0^{(32)\prime})$ *such that if* $(\mathfrak{o}^{(32)\prime}, \mathcal{T}^{(32)\prime}) < (\mathfrak{o}_0^{(32)\prime}, \mathcal{T}_0^{(32)\prime})$ *then the cooridinate change*

$$(\varphi_{(31)(32)}, \hat\varphi_{(31)(32)}) : V(\mathfrak{p}(3), \mathfrak{w}^{(2)}_{\mathfrak{p}(3)}; (\mathfrak{o}^{(32)\prime}, \mathcal{T}^{(32)\prime}); \mathfrak{A}) \to V(\mathfrak{p}(3), \mathfrak{w}^{(1)}_{\mathfrak{p}(3)}; (\mathfrak{o}^{(31)}, \mathcal{T}^{(31)}); \mathfrak{A})$$

as in Proposition 2.131 exists.

(5) *Suppose* $(\mathfrak{o}^{(32)\prime\prime}, \mathcal{T}^{(32)\prime\prime}) < (\mathfrak{o}_0^{(32)\prime}, \mathcal{T}_0^{(32)\prime})$ *and* $(\mathfrak{o}^{(32)\prime\prime}, \mathcal{T}^{(32)\prime\prime}) < (\mathfrak{o}_0^{(32)}, \mathcal{T}_0^{(32)})$. *Then we have*

$$\begin{aligned}&(\varphi_{(21)(22)}, \hat\varphi_{(21)(22)}) \circ (\varphi_{(22)(32)}, \hat\varphi_{(22)(32)}) \\ &= (\varphi_{(21)(31)}, \hat\varphi_{(21)(31)}) \circ (\varphi_{(31)(32)}, \hat\varphi_{(31)(32)})\end{aligned} \tag{2.365}$$

on $V(\mathfrak{p}(3), \mathfrak{w}^{(2)}_{\mathfrak{p}(3)}; (\mathfrak{o}^{(32)\prime\prime}, \mathcal{T}^{(32)\prime\prime}); \mathfrak{A})$.

Remark 2.162. The statement (1) above was proved at the begining of this subsection. The statements (2) and (3) above were proved by Proposition 2.152. The statement (4) above was proved by Proposition 2.131. So only the statement (5) is new in Lemma 2.161.

Lemma 2.161 $\Rightarrow$ *Proposition 2.158.* Let $\mathfrak{w}^{(1)}_{\mathfrak{p}(2)}$ be the stabilization data at $\mathfrak{p}(2)$ induced by $\mathfrak{w}_{\mathfrak{p}(1)}$.

We apply Lemma 2.161 to $\mathfrak{w}^{(1)}_{\mathfrak{p}(2)}$ and $\mathfrak{w}^{(2)}_{\mathfrak{p}(2)} = \mathfrak{w}_{\mathfrak{p}(2)}$. We then obtain ϵ_8. This ϵ_8 is ϵ_7 in Proposition 2.158. Suppose $\mathfrak{p}(3) \in \mathcal{M}_{k+1,\ell}(\beta)$, $d(\mathfrak{p}(2), \mathfrak{p}(3)) < \epsilon_8$. We obtain $\mathfrak{w}^{(1)}_{\mathfrak{p}(3)}, \mathfrak{w}^{(2)}_{\mathfrak{p}(3)}$ from Lemma 2.161 (1).

Using the pair of stabilization data $(\mathfrak{w}_{\mathfrak{p}(1)}, \mathfrak{w}^{(1)}_{\mathfrak{p}(2)})$ we obtain the coordinate change $(\varphi_{1(21)}, \hat\varphi_{1(21)})$ by Proposition 2.152.

Using the pair of stabilization data $(\mathfrak{w}^{(2)}_{\mathfrak{p}(3)}, \mathfrak{w}_{\mathfrak{p}(3)})$ we obtain the coordinate change $(\varphi_{(32)3}, \hat\varphi_{(32)3})$ by Proposition 2.131.

Now by using Lemma 2.161 (5) we have

$$\begin{aligned}&(\varphi_{1(21)}, \hat\varphi_{1(21)}) \circ (\varphi_{(21)(22)}, \hat\varphi_{(21)(22)}) \circ (\varphi_{(22)(32)}, \hat\varphi_{(22)(32)}) \circ (\varphi_{(32)3}, \hat\varphi_{(32)3}) \\ &= (\varphi_{1(21)}, \hat\varphi_{1(21)}) \circ (\varphi_{(21)(31)}, \hat\varphi_{(21)(31)}) \circ (\varphi_{(31)(32)}, \hat\varphi_{(31)(32)}) \circ (\varphi_{(32)3}, \hat\varphi_{(32)3})\end{aligned} \tag{2.366}$$

in a neighborhood of $\mathfrak{p}(3)$.

By definition of $(\varphi_{12}, \hat\varphi_{12})$, $(\varphi_{23}, \hat\varphi_{23})$ given in the proof of Corollary 2.156, we have

$$(\varphi_{1(21)}, \hat\varphi_{1(21)}) \circ (\varphi_{(21)(22)}, \hat\varphi_{(21)(22)}) = (\varphi_{12}, \hat\varphi_{12}) \tag{2.367}$$

and

$$(\varphi_{(22)(32)}, \hat\varphi_{(22)(32)}) \circ (\varphi_{(32)3}, \hat\varphi_{(32)3}) = (\varphi_{23}, \hat\varphi_{23}). \tag{2.368}$$

On the other hand, by Lemma 2.160 $(\varphi_{1(21)}, \hat\varphi_{1(21)}) \circ (\varphi_{(21)(31)}, \hat\varphi_{(21)(31)})$ is the coordinate change given by Proposition 2.152. By Lemma 2.145, $(\varphi_{(31)(32)}, \hat\varphi_{(31)(32)}) \circ$

$(\varphi_{(32)3},\hat\varphi_{(32)3})$ is the coordinate change given by Proposition 2.131. Therefore, by the definition given in the proof of Corollary 2.156,

$$\begin{aligned}(\varphi_{13},\hat\varphi_{13}) = (\varphi_{1(21)},\hat\varphi_{1(21)})\circ(\varphi_{(21)(31)},\hat\varphi_{(21)(31)})\\ \circ(\varphi_{(31)(32)},\hat\varphi_{(31)(32)})\circ(\varphi_{(32)3},\hat\varphi_{(32)3}).\end{aligned} \tag{2.369}$$

Proposition 2.158 follows from (2.366)-(2.369). □

Proof of Lemma 2.161. By definition, the coordinate change $(\varphi_{(21)(22)},\hat\varphi_{(21)(22)})$ is a composition of finitely many coordinate changes that are one of the types 1,3,4. (The notion of coordinate changes of type 1,3,4 is defined right before Lemma 2.140.) Therefore it suffices to prove the lemma in the case when $(\varphi_{(21)(22)},\hat\varphi_{(21)(22)})$ is one of types 1,3,4. We prove each of those cases below.

Case 1: $(\varphi_{(21)(22)},\hat\varphi_{(21)(22)})$ is of type 1.

We use the notation in the proof of Proposition 2.131 with $\mathfrak p$ being replaced by $\mathfrak p(2)$ or $\mathfrak p(3)$.

By Lemma 2.132 we have

$$\mathcal M^{\mathfrak w^{(2)-}_{\mathfrak p(2)}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p(2);\mathfrak A)_{\epsilon'_{0,2},\vec{\mathcal T}^{(2)\prime}} \subset \mathcal M^{\mathfrak w^{(1)}_{\mathfrak p(2)}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p(2);\mathfrak A)_{\epsilon_{0,2},\vec{\mathcal T}^{(1)}}. \tag{2.370}$$

(Here we replace ϵ_0,ϵ'_0 in (2.332) by $\epsilon_{0,2},\epsilon'_{0,2}$. We also put $\ell_{\mathfrak p}=\ell_{\mathfrak p(2)}=\ell_{\mathfrak p(3)}$.) Also by Lemma 2.132 we have

$$\mathcal M^{\mathfrak w^{(2)-}_{\mathfrak p(3)}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p(3);\mathfrak A)_{\epsilon'_{0,3},\vec{\mathcal T}^{(3)\prime}} \subset \mathcal M^{\mathfrak w^{(1)}_{\mathfrak p(3)}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p(3);\mathfrak A)_{\epsilon_{0,3},\vec{\mathcal T}^{(3)}}. \tag{2.371}$$

(Here we replace ϵ_0,ϵ'_0 in (2.332) by $\epsilon_{0,3},\epsilon'_{0,3}$.)

By the definition of type 1 we use the same codimension 2 submanifolds to put the transversal constraint. Therefore we have

$$\mathcal M^{\mathfrak w^{(2)-}_{\mathfrak p(2)}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p(2);\mathfrak A)^{\rm trans}_{\epsilon'_{0,2},\vec{\mathcal T}^{(2)\prime}} \subset \mathcal M^{\mathfrak w^{(1)}_{\mathfrak p(2)}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p(2);\mathfrak A)^{\rm trans}_{\epsilon_{0,2},\vec{\mathcal T}^{(2)}} \tag{2.372}$$

and

$$\mathcal M^{\mathfrak w^{(2)-}_{\mathfrak p(3)}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p(3);\mathfrak A)^{\rm trans}_{\epsilon'_{0,3},\vec{\mathcal T}^{(3)\prime}} \subset \mathcal M^{\mathfrak w^{(1)}_{\mathfrak p(3)}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p(3);\mathfrak A)^{\rm trans}_{\epsilon_{0,3},\vec{\mathcal T}^{(3)}}. \tag{2.373}$$

On the other hand, by (2.362) we have

$$\mathcal M^{\mathfrak w^{(1)}_{\mathfrak p(3)}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p(3);\mathfrak A)^{\rm trans}_{\epsilon_{0,3},\vec{\mathcal T}^{(3)}} \subset \mathcal M^{\mathfrak w^{(1)}_{\mathfrak p(2)}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p(2);\mathfrak A)^{\rm trans}_{\epsilon_{0,2},\vec{\mathcal T}^{(2)}}. \tag{2.374}$$

Note that the stabilization data $\mathfrak w^{(2)-}_{\mathfrak p(2)}$ and $\mathfrak w^{(2)-}_{\mathfrak p(3)}$ appearing in (2.372) and (2.373) are obtained by extending the core of the coordinate at infinity included in $\mathfrak w^{(2)}_{\mathfrak p(2)}$ and $\mathfrak w^{(2)}_{\mathfrak p(3)}$, respectively. Therefore by further extending the core we may assume that $\mathfrak w^{(2)-}_{\mathfrak p(3)}$ is induced from $\mathfrak w^{(2)-}_{\mathfrak p(2)}$. Therefore again by (2.362) we have

$$\mathcal M^{\mathfrak w^{(2)-}_{\mathfrak p(3)}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p(3);\mathfrak A)^{\rm trans}_{\epsilon''_{0,3},\vec{\mathcal T}^{(3)\prime\prime}} \subset \mathcal M^{\mathfrak w^{(2)-}_{\mathfrak p(2)}}_{k+1,(\ell,\ell_{\mathfrak p},(\ell_c))}(\beta;\mathfrak p(2);\mathfrak A)^{\rm trans}_{\epsilon'_{0,2},\vec{\mathcal T}^{(2)\prime}}. \tag{2.375}$$

By definition, the coordinate changes $\varphi_{(21)(22)}$, $\varphi_{(31)(32)}$, $\varphi_{(21)(31)}$, $\varphi_{(22)(32)}$ are the inclusion maps (2.372), (2.373), (2.374) and (2.375) in neighborhoods of $\mathfrak p(2)$, $\mathfrak p(3)$, $\mathfrak p(3)$, $\mathfrak p(3)$, respectively. The lemma is proved in this case.

Case 2: Void.

Case 4: $(\varphi_{(21)(22)},\hat\varphi_{(21)(22)})$ is of type 4.

We have $\vec{w}^{(1)}_{\mathfrak p(2)} \supset \vec{w}^{(2)}_{\mathfrak p(2)}$. Therefore $\vec{w}^{(1)}_{\mathfrak p(3)} \supset \vec{w}^{(2)}_{\mathfrak p(3)}$. It follows that $(\varphi_{(31)(32)}, \hat\varphi_{(31)(32)})$ is also of type 4. We have the following commutative diagram.

$$
\begin{array}{ccc}
\mathcal M^{\mathfrak w^{(1)}_{\mathfrak p(2)}}_{k+1,(\ell,\ell^{(1)}_{\mathfrak p(2)},(\ell_c))}(\beta;\mathfrak p(2);\mathfrak A)_{\epsilon_0,\vec{\mathcal T}^{(1)}} & \xrightarrow{\mathfrak{forget}_{\mathfrak A,\mathfrak A;\vec w^{(1)}_{\mathfrak p(2)},\vec w^{(2)}_{\mathfrak p(2)}}} & \mathcal M^{\mathfrak w^{(2)}_{\mathfrak p(2)}}_{k+1,(\ell,\ell^{(2)}_{\mathfrak p(2)},(\ell_c))}(\beta;\mathfrak p(2);\mathfrak A)_{\epsilon_0,\vec{\mathcal T}^{(2)}} \\
\uparrow \subset & & \uparrow \subset \\
\mathcal M^{\mathfrak w^{(1)}_{\mathfrak p(3)}}_{k+1,(\ell,\ell^{(1)}_{\mathfrak p(3)},(\ell_c))}(\beta;\mathfrak p(3);\mathfrak A)_{\epsilon'_0,\vec{\mathcal T}^{(1)\prime}} & \xrightarrow{\mathfrak{forget}_{\mathfrak A,\mathfrak A;\vec w^{(1)}_{\mathfrak p(3)},\vec w^{(2)}_{\mathfrak p(3)}}} & \mathcal M^{\mathfrak w^{(2)}_{\mathfrak p(3)}}_{k+1,(\ell,\ell^{(2)}_{\mathfrak p(3)},(\ell_c))}(\beta;\mathfrak p(3);\mathfrak A)_{\epsilon'_0,\vec{\mathcal T}^{(2)\prime}}
\end{array}
\tag{2.376}
$$

We note that we use the same codimension 2 submanifold to put transversal constraint. Therefore (2.376) induces:

$$
\begin{array}{ccc}
\mathcal M^{\mathfrak w^{(1)}_{\mathfrak p(2)}}_{k+1,(\ell,\ell^{(1)}_{\mathfrak p(2)},(\ell_c))}(\beta;\mathfrak p(2);\mathfrak A)^{\rm trans}_{\epsilon_0,\vec{\mathcal T}^{(1)}} & \xrightarrow{\mathfrak{forget}_{\mathfrak A,\mathfrak A;\vec w^{(1)}_{\mathfrak p(2)},\vec w^{(2)}_{\mathfrak p(2)}}} & \mathcal M^{\mathfrak w^{(2)}_{\mathfrak p(2)}}_{k+1,(\ell,\ell^{(2)}_{\mathfrak p(2)},(\ell_c))}(\beta;\mathfrak p(2);\mathfrak A)^{\rm trans}_{\epsilon_0,\vec{\mathcal T}^{(2)}} \\
\uparrow \subset & & \uparrow \subset \\
\mathcal M^{\mathfrak w^{(1)}_{\mathfrak p(3)}}_{k+1,(\ell,\ell^{(1)}_{\mathfrak p(3)},(\ell_c))}(\beta;\mathfrak p(3);\mathfrak A)^{\rm trans}_{\epsilon'_0,\vec{\mathcal T}^{(1)\prime}} & \xrightarrow{\mathfrak{forget}_{\mathfrak A,\mathfrak A;\vec w^{(1)}_{\mathfrak p(3)},\vec w^{(2)}_{\mathfrak p(3)}}} & \mathcal M^{\mathfrak w^{(2)}_{\mathfrak p(3)}}_{k+1,(\ell,\ell^{(2)}_{\mathfrak p(3)},(\ell_c))}(\beta;\mathfrak p(3);\mathfrak A)^{\rm trans}_{\epsilon'_0,\vec{\mathcal T}^{(2)\prime}}
\end{array}
\tag{2.377}
$$

The commutativity of (2.377) is Lemma 2.161 in this case.

Case 3: $(\varphi_{(21)(22)}, \hat\varphi_{(21)(22)})$ is of type 3.

We obtain the following commutative diagram in the same way.

$$
\begin{array}{ccc}
\mathcal M^{\mathfrak w^{(1)}_{\mathfrak p(2)}}_{k+1,(\ell,\ell^{(1)}_{\mathfrak p(2)},(\ell_c))}(\beta;\mathfrak p(2);\mathfrak A)^{\rm trans}_{\epsilon_0,\vec{\mathcal T}^{(1)}} & \xleftarrow{\mathfrak{forget}_{\mathfrak A,\mathfrak A;\vec w^{(2)}_{\mathfrak p(2)},\vec w^{(1)}_{\mathfrak p(2)}}} & \mathcal M^{\mathfrak w^{(2)}_{\mathfrak p(2)}}_{k+1,(\ell,\ell^{(2)}_{\mathfrak p(2)},(\ell_c))}(\beta;\mathfrak p(2);\mathfrak A)^{\rm trans}_{\epsilon_0,\vec{\mathcal T}^{(2)}} \\
\uparrow \subset & & \uparrow \subset \\
\mathcal M^{\mathfrak w^{(1)}_{\mathfrak p(3)}}_{k+1,(\ell,\ell^{(1)}_{\mathfrak p(3)},(\ell_c))}(\beta;\mathfrak p(3);\mathfrak A)^{\rm trans}_{\epsilon'_0,\vec{\mathcal T}^{(1)\prime}} & \xleftarrow{\mathfrak{forget}_{\mathfrak A,\mathfrak A;\vec w^{(2)}_{\mathfrak p(3)},\vec w^{(1)}_{\mathfrak p(3)}}} & \mathcal M^{\mathfrak w^{(2)}_{\mathfrak p(3)}}_{k+1,(\ell,\ell^{(2)}_{\mathfrak p(3)},(\ell_c))}(\beta;\mathfrak p(3);\mathfrak A)^{\rm trans}_{\epsilon'_0,\vec{\mathcal T}^{(2)\prime}}
\end{array}
\tag{2.378}
$$

All the above arrows are diffeomorphisms locally. This implies the lemma in this case. The proof of Lemma 2.161 is complete. □

The proof of Proosition 2.158 is complete. □

2.10. **Wrap-up of the construction of Kuranishi structure.** In this subsection we complete the proof of Theorem 2.3. We will prove the case of $\mathcal M_{k+1,\ell}(\beta)$. The case of $\mathcal M^{\rm cl}_{\ell}(\alpha)$ is the same.

In this subsection we fix a stabilization data $\mathfrak w_{\mathfrak p}$ at $\mathfrak p$ for each $\mathfrak p$ and always use it. We also take $\mathfrak A = \mathfrak C(\mathfrak p)$ unless otherwise specified. So we omit them from the notation of Kuranishi chart. We write $\mathfrak d = (\mathfrak o, \mathcal T)$. Thus we write

$$(V(\mathfrak p;\mathfrak d), \mathcal E_{(\mathfrak p;\mathfrak d)}, \mathfrak s_{(\mathfrak p;\mathfrak d)}, \psi_{(\mathfrak p;\mathfrak d)})$$

to denote our Kuranishi neighborhood.

For simplicity of notation we denote by $\tilde\psi_{(\mathfrak p;\mathfrak d)}$ the composition of $\psi_{(\mathfrak p;\mathfrak d)}$ and the projection $\mathfrak s^{-1}_{(\mathfrak p;\mathfrak d)}(0) \to \mathfrak s^{-1}_{(\mathfrak p;\mathfrak d)}(0)/\Gamma_{\mathfrak p}$.

The next lemma is the main technical lemma we use for the construction.

Lemma 2.163. *There exist finite subsets $\mathfrak{P}_j = \{\mathfrak{p}(j,i) \mid i = 1,\dots,N_j\} \subset \mathcal{M}_{k+1,\ell}(\beta)$ for $j = 1,2,3$ and admissible $\mathfrak{d}(j,1,i) > \mathfrak{d}(j,2,i)$ for $j = 1,2,3$, $i = 1,\dots,N_j$ such that they satisfy the following properties.*

(1) *If $j = 1,2,3$ then*
$$\bigcup_{i=1}^{N_j} \tilde\psi_{(\mathfrak{p}(j,i);\mathfrak{d}(j,2,i))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(j,i);\mathfrak{d}(j,2,i))}(0)) = \mathcal{M}_{k+1,\ell}(\beta).$$

(2) *The following holds for $j > j'$. If*
$$\mathfrak{p}(j,i) \in \tilde\psi_{(\mathfrak{p}(j',i');\mathfrak{d}(j',2,i'))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(j',i');\mathfrak{d}(j',2,i'))}(0)),$$
then there exists a coordinate change
$$\varphi_{(j',i'),(j,i)} : V(\mathfrak{p}(j,i);\mathfrak{d}(j,1,i)) \to V(\mathfrak{p}(j',i');\mathfrak{d}(j',1,i'))$$
as in Corollary 2.156.

(3) *Let $j = 1$ or 2, $i_1,\dots,i_m \in \{1,\dots,N_{j+1}\}$. Suppose*
$$\bigcap_{n=1}^{m} \tilde\psi_{(\mathfrak{p}(j+1,i_n);\mathfrak{d}(j+1,1,i_n))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(j+1,i_n);\mathfrak{d}(j+1,1,i_n))}(0)) \neq \emptyset,$$
then there exists i independent of n such that
$$\mathfrak{p}(j+1,i_n) \in \tilde\psi_{(\mathfrak{p}(j,i);\mathfrak{d}(j,2,i))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(j,i);\mathfrak{d}(j,2,i))}(0))$$
for any $n = 1,\dots,m$.

(4) *Let $i_j \in \{1,\dots,N_j\}$. If*
$$\mathfrak{p}(3,i_3) \in \tilde\psi_{(\mathfrak{p}(2,i_2);\mathfrak{d}(2,2,i_2))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(2,i_2);\mathfrak{d}(2,2,i_2))}(0))$$
and
$$\mathfrak{p}(2,i_2) \in \tilde\psi_{(\mathfrak{p}(1,i_1);\mathfrak{d}(1,2,i_1))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(1,i_1);\mathfrak{d}(1,2,i_1))}(0)),$$
then there exists a coordinate change
$$\varphi_{(1,i_1),(3,i_3)} : V(\mathfrak{p}(3,i_3);\mathfrak{d}(3,1,i_3)) \to V(\mathfrak{p}(1,i_1);\mathfrak{d}(1,1,i_1))$$
as in Corollary 2.156. Moreover we have
$$\varphi_{(1,i_1),(2,i_2)} \circ \varphi_{(2,i_2),(3,i_3)} = \varphi_{(1,i_1),(3,i_3)} \tag{2.379}$$
everywhere on $V(\mathfrak{p}(3,i_3);\mathfrak{d}(3,1,i_3))$.

Proof. For each $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(\beta)$, we take admissible $\mathfrak{d}(\mathfrak{p},1;1) > \mathfrak{d}(\mathfrak{p},1;2) > \mathfrak{d}(\mathfrak{p},1;3)$. Then we have $\mathfrak{P}_1 = \{\mathfrak{p}(1,i) \mid i = 1,\dots,N_1\}$ such that
$$\bigcup_{i=1}^{N_1} \tilde\psi_{(\mathfrak{p}(1,i);\mathfrak{d}(\mathfrak{p}(1,i),1;3))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(1,i);\mathfrak{d}(\mathfrak{p}(1,i),1;3))}(0)) = \mathcal{M}_{k+1,\ell}(\beta). \tag{2.380}$$
We put $\mathfrak{d}(1,1,i) = \mathfrak{d}(\mathfrak{p}(1,i),1;1)$, $\mathfrak{d}(1,2,i) = \mathfrak{d}(\mathfrak{p}(1,i),1;2)$. Then, since
$$\begin{aligned}&\tilde\psi_{(\mathfrak{p}(1,i);\mathfrak{d}(\mathfrak{p}(1,i),1;3))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(1,i);\mathfrak{d}(\mathfrak{p}(1,i),1;3)))}(0)) \\ &\subset \tilde\psi_{(\mathfrak{p}(1,i);\mathfrak{d}(1,2,i))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(1,i);\mathfrak{d}(1,2,i))}(0)),\end{aligned} \tag{2.381}$$
Lemma 2.163 (1) hods for $j = 1$.

For each $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(\beta)$ we take an admissible $\mathfrak{d}(\mathfrak{p},2;1)$ so that the following conditions hold.

Condition 2.164. (a) If $\mathfrak{p} \in \tilde\psi_{(\mathfrak{p}(1,i);\mathfrak{d}(1,2,i))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(1,i);\mathfrak{d}(1,2,i))}(0))$, then there exists a coordinate change

$$\varphi_{(1,i),(2,\mathfrak{p})} : V(\mathfrak{p};\mathfrak{d}(\mathfrak{p},2;1)) \to V(\mathfrak{p}(1,i);\mathfrak{d}(1,1,i))$$

as in Corollary 2.156.

(b) If

$$\tilde\psi_{(\mathfrak{p};\mathfrak{d}(\mathfrak{p},2;1))}(\mathfrak{s}^{-1}_{(\mathfrak{p};\mathfrak{d}(\mathfrak{p},2;1))}(0)) \cap \tilde\psi_{(\mathfrak{p}(1,i);\mathfrak{d}(\mathfrak{p}(1,i),1;3))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(1,i);\mathfrak{d}(\mathfrak{p}(1,i),1;3))}(0)) \ne \emptyset$$

then

$$\tilde\psi_{(\mathfrak{p};\mathfrak{d}(\mathfrak{p},2;1))}(\mathfrak{s}^{-1}_{(\mathfrak{p};\mathfrak{d}(\mathfrak{p},2;1))}(0)) \subseteq \tilde\psi_{(\mathfrak{p}(1,i);\mathfrak{d}(\mathfrak{p}(1,i),1;2))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(1,i);\mathfrak{d}(\mathfrak{p}(1,i),1;2))}(0)).$$

(c) Let $\epsilon_9(\mathfrak{p})$ be the positive number we define below. If an element $\mathfrak{q} \in \mathcal{M}_{k+1,\ell}(\beta)$ satisfies $\mathfrak{q} \in \tilde\psi_{(\mathfrak{p};\mathfrak{d}(\mathfrak{p},2;1))}(\mathfrak{s}^{-1}_{(\mathfrak{p};\mathfrak{d}(\mathfrak{p},2;1))}(0))$, then $d(\mathfrak{p},\mathfrak{q}) < \epsilon_9(\mathfrak{p})$.

Here $\epsilon_9(\mathfrak{p})$ is defined as follows. For each $i = 1,\dots,N_1$ we put $\mathfrak{p}(1) = \mathfrak{p}(1,i)$, $\mathfrak{p}(2) = \mathfrak{p}$ and apply Proposition 2.158. We then obtain $\epsilon_7(i,\mathfrak{p})$. We define

$$\epsilon_9(\mathfrak{p}) = \min\{\epsilon_7(i,\mathfrak{p}) \mid i = 1,\dots,N_1\}.$$

The existence of such $\mathfrak{d}(\mathfrak{p},2;1)$ is obvious. Furthermore for each $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(\beta)$, we take $\mathfrak{d}(\mathfrak{p},2;2), \mathfrak{d}(\mathfrak{p},2;3)$ such that $\mathfrak{d}(\mathfrak{p},2;1) > \mathfrak{d}(\mathfrak{p},2;2) > \mathfrak{d}(\mathfrak{p},2;3)$. Then we have $\mathfrak{P}_2 = \{\mathfrak{p}(2,i) \mid i = 1,\dots,N_2\}$ such that

$$\bigcup_{i=1}^{N_2} \tilde\psi_{(\mathfrak{p}(2,i);\mathfrak{d}(\mathfrak{p}(2,i),2;3))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(2,i);\mathfrak{d}(\mathfrak{p}(2,i),2;3))}(0)) = \mathcal{M}_{k+1,\ell}(\beta). \tag{2.382}$$

We put $\mathfrak{d}(2,1,i) = \mathfrak{d}(\mathfrak{p}(2,i),2;1)$, $\mathfrak{d}(2,2,i) = \mathfrak{d}(\mathfrak{p}(2,i),2;2)$. Then (2.382) and $\mathfrak{d}(\mathfrak{p},2;2) > \mathfrak{d}(\mathfrak{p},2;3)$ imply Lemma 2.163 (1) for $j = 2$. Lemma 2.163 (2) for $(j,j') = (2,1)$ follows immediately from Condition 2.164 (a).

Sublemma 2.165. *Lemma 2.163 (3) holds for $j = 1$.*

Proof. Suppose

$$\bigcap_{n=1}^{m} \tilde\psi_{(\mathfrak{p}(2,i_n);\mathfrak{d}(2,1,i_n))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(2,i_n);\mathfrak{d}(2,1,i_n))}(0)) \ne \emptyset.$$

Then (2.380) implies that there exists i such that

$$\bigcap_{n=1}^{m} \tilde\psi_{(\mathfrak{p}(2,i_n);\mathfrak{d}(2,1,i_n))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(2,i_n);\mathfrak{d}(2,1,i_n))}(0)) \cap \tilde\psi_{(\mathfrak{p}(1,i);\mathfrak{d}(\mathfrak{p}(1,i),1;3))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(1,i);\mathfrak{d}(\mathfrak{p}(1,i),1;3))}(0)) \ne \emptyset.$$

Therefore Condition 2.164 (b) and (2.381) imply

$$\tilde\psi_{(\mathfrak{p}(2,i_n);\mathfrak{d}(2,1,i_n))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(2,i_n);\mathfrak{d}(2,1,i_n))}(0)) \subset \tilde\psi_{(\mathfrak{p}(1,i);\mathfrak{d}(1,2,i))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(1,i);\mathfrak{d}(1,2,i))}(0))$$

for any n. In particular

$$\mathfrak{p}(2,i_n) \in \tilde\psi_{(\mathfrak{p}(1,i);\mathfrak{d}(1,2,i))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(1,i);\mathfrak{d}(1,2,i))}(0))$$

as required. □

For each $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(\beta)$ we take an admissible $\mathfrak{d}(\mathfrak{p},3;1)$ so that the following condisions hold.

Condition 2.166. (a) If $\mathfrak{p} \in \tilde\psi_{(\mathfrak{p}(2,i);\mathfrak{d}(2,2,i))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(2,i);\mathfrak{d}(2,2,i))}(0))$, then there exists a coordinate change

$$\varphi_{(2,i),(3,\mathfrak{p})} : V(\mathfrak{p};\mathfrak{d}(\mathfrak{p},3;1)) \to V(\mathfrak{p}(2,i);\mathfrak{d}(2,1,i))$$

as in Corollary 2.156.

(b) If

$$\tilde\psi_{(\mathfrak{p};\mathfrak{d}(\mathfrak{p},3;1))}(\mathfrak{s}^{-1}_{(\mathfrak{p};\mathfrak{d}(\mathfrak{p},3;1))}(0)) \cap \tilde\psi_{(\mathfrak{p}(2,i);\mathfrak{d}(\mathfrak{p}(2,i),2;3))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(2,i);\mathfrak{d}(\mathfrak{p}(2,i),2;3))}(0)) \ne \emptyset,$$

then

$$\tilde\psi_{(\mathfrak{p};\mathfrak{d}(\mathfrak{p},3;1))}(\mathfrak{s}^{-1}_{(\mathfrak{p};\mathfrak{d}(\mathfrak{p},3;1))}(0)) \subseteq \tilde\psi_{(\mathfrak{p}(2,i);\mathfrak{d}(\mathfrak{p}(2,i),2;2))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(2,i);\mathfrak{d}(\mathfrak{p}(2,i),2;2))}(0)).$$

(c) Void

(d) Let (i_1,i_2) be an arbitrary pair of integers such that

$$\mathfrak{p} \in \tilde\psi_{(\mathfrak{p}(2,i_2);\mathfrak{d}(2,2,i_2))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(2,i_2);\mathfrak{d}(2,2,i_2))}(0)),$$
$$\mathfrak{p}(2,i_2) \in \tilde\psi_{(\mathfrak{p}(1,i_1);\mathfrak{d}(1,2,i_1))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(1,i_1);\mathfrak{d}(1,2,i_1))}(0)).$$

Then

$$\mathfrak{d}(\mathfrak{p},3;1) < \mathfrak{d}(i_1,i_2).$$

Here the right hand side is defined below.

(e) Under the same assumption as in (d), there exists a coordinate change

$$\varphi_{(1,i_1),(3,\mathfrak{p})} : V(\mathfrak{p};\mathfrak{d}(\mathfrak{p},3;1)) \to V(\mathfrak{p}(1,i_1);\mathfrak{d}(1,1,i_1))$$

as in Corollary 2.156.

The definition of $\mathfrak{d}(i_1,i_2)$ is as follows. We put $\mathfrak{p}(1) = \mathfrak{p}(1,i_1)$ and $\mathfrak{p}(2) = \mathfrak{p}(2,i_2)$ and $\mathfrak{p}(3) = \mathfrak{p}$. We also put $(\mathfrak{o}^{(1)},\mathcal{T}^{(1)}) = \mathfrak{d}(1,1,i_1)$, $(\mathfrak{o}^{(2)},\mathcal{T}^{(2)}) = \mathfrak{d}(2,1,i_2)$. Using Condition 2.164 (c) we can apply Proposition 2.158 to obtain $(\mathfrak{o}^{(3)}_0,\mathcal{T}^{(3)}_0)$, which we put $\mathfrak{d}(i_1,i_2)$.

Existence of $\mathfrak{d}(\mathfrak{p},3;1)$ is obvious. Furthermore for each $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(\beta)$, we take $\mathfrak{d}(\mathfrak{p},3;2)$ with $\mathfrak{d}(\mathfrak{p},3;1) > \mathfrak{d}(\mathfrak{p},3;2)$. Then we have $\mathfrak{P}_3 = \{\mathfrak{p}(3,i) \mid i = 1,\dots,N_3\}$ such that

$$\bigcup_{i=1}^{N_3} \tilde\psi_{(\mathfrak{p}(3,i);\mathfrak{d}(\mathfrak{p}(3,i),3;2))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(3,i);\mathfrak{d}(\mathfrak{p}(3,i),3;2)))}(0)) = \mathcal{M}_{k+1,\ell}(\beta). \tag{2.383}$$

We put $\mathfrak{d}(3,1,i) = \mathfrak{d}(\mathfrak{p}(3,i),3;1)$, $\mathfrak{d}(3,2,i) = \mathfrak{d}(\mathfrak{p}(3,i),3;2)$.

Now Lemma 2.163 (1) for $j = 3$ follows from (2.383). Lemma 2.163 (2) for $(j,j') = (3,2),(3,1)$ follows from Condition 2.166 (a),(e). The proof of Lemma 2.163 (3) for $j = 2$ is the same as the proof of Sublemma 2.165.

Finally Lemma 2.163 (4) is a consequence of Condition 2.164 (c), Condition 2.166 (d)(e), and Proposition 2.158. The proof of Lemma 2.163 is complete. □

Proof of Theorem 2.3. We start the construction of a Kuranishi structure on $\mathcal{M}_{k+1,\ell}(\beta)$. Let $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(\beta)$. There exists $i(\mathfrak{p}) \in \{1,\dots,N_3\}$ such that

$$\mathfrak{p} \in \tilde\psi_{(\mathfrak{p}(3,i(\mathfrak{p}));\mathfrak{d}(\mathfrak{p}(3,i(\mathfrak{p})),3;2))}(\mathfrak{s}^{-1}_{(\mathfrak{p}(3,i(\mathfrak{p}));\mathfrak{d}(\mathfrak{p}(3,i(\mathfrak{p})),3;2))}(0)).$$

We take any such $i(\mathfrak{p})$ and fix it. Choose $\hat{\mathfrak{p}} \in V(\mathfrak{p}(3,i(\mathfrak{p}));\mathfrak{d}(3,1,i(\mathfrak{p})))$ such that

$$\tilde\psi_{(\mathfrak{p}(3,i(\mathfrak{p}));\mathfrak{d}(\mathfrak{p}(3,i(\mathfrak{p})),3;2))}(\hat{\mathfrak{p}}) = \mathfrak{p}.$$

We have an embedding

$$\bigoplus_{c\in\mathfrak C(\mathfrak p)} \mathcal E_c \subset \mathcal E_{(\mathfrak p(3,i(\mathfrak p));\mathfrak d(\mathfrak p(3,i(\mathfrak p)),3;2))}$$

of vector bundles. (This is because $\mathfrak C(\mathfrak p) \subseteq \mathfrak C(\mathfrak p(3,i(\mathfrak p)))$.) We take a neighborhood $V_{\mathfrak p}$ of $\hat{\mathfrak p}$ in the set

$$W^{(3)}_{\mathfrak p} = \left\{\mathfrak v \in V(\mathfrak p(3,i(\mathfrak p));\mathfrak d(3,1,i(\mathfrak p))) \mid \mathfrak s_{(\mathfrak p(3,i(\mathfrak p));\mathfrak d(\mathfrak p(3,i(\mathfrak p)),3;2)))}(\mathfrak v) \in \bigoplus_{c\in\mathfrak C(\mathfrak p)} \mathcal E_c\right\}$$

such that $V_{\mathfrak p}$ is $\Gamma_{\mathfrak p}$ invariant. The sum $\bigoplus_{c\in\mathfrak C(\mathfrak p)} \mathcal E_c$ defines a $\Gamma_{\mathfrak p}$ equivariant vector bundle on $V_{\mathfrak p}$ that we dente by $E_{\mathfrak p}$. The restriction to $V_{\mathfrak p}$ of the section $\mathfrak s_{(\mathfrak p(3,i(\mathfrak p));\mathfrak d(\mathfrak p(3,i(\mathfrak p)),3;2))}$ and the map $\tilde\psi_{(\mathfrak p(3,i(\mathfrak p));\mathfrak d(\mathfrak p(3,i(\mathfrak p)),3;2))}$ (divided by $\Gamma_{\mathfrak p(3,i(\mathfrak p))}$) is our $\mathfrak s_{\mathfrak p}$ and $\psi_{\mathfrak p}$. We can show easily that $(V_{\mathfrak p},\Gamma_{\mathfrak p},E_{\mathfrak p},\mathfrak s_{\mathfrak p},\psi_{\mathfrak p})$ is a Kuranishi chart of $\mathfrak p$.

We next define coordinate changes. Let $\mathfrak q \in \psi_{\mathfrak p}(\mathfrak s_{\mathfrak p}^{-1}(0))$. It implies $\mathfrak C(\mathfrak q) \subseteq \mathfrak C(\mathfrak p)$. We note that $i(\mathfrak p)$ may be different from $i(\mathfrak q)$. On the other hand, we have

$$\begin{aligned}&\tilde\psi_{(\mathfrak p(3,i(\mathfrak p));\mathfrak d(3,1,i(\mathfrak p)))}(\mathfrak s^{-1}_{(\mathfrak p(3,i(\mathfrak p));\mathfrak d(3,1,i(\mathfrak p)))}(0))\\ &\cap \tilde\psi_{(\mathfrak p(3,i(\mathfrak q));\mathfrak d(3,1,i(\mathfrak q)))}(\mathfrak s^{-1}_{(\mathfrak p(3,i_(\mathfrak q));\mathfrak d(3,1,i(\mathfrak q)))}(0)) \ne \emptyset.\end{aligned}$$

In fact, $\mathfrak q$ is contained in the intersection. Therefore by Lemma 2.163 (3), there exists $i(\mathfrak p,\mathfrak q)$ such that

$$\mathfrak p(3,i(\mathfrak p)),\mathfrak p(3,i(\mathfrak q)) \in \tilde\psi_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q));\mathfrak d(2,2,i(\mathfrak p,\mathfrak q)))}(\mathfrak s^{-1}_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q));\mathfrak d(2,2,i(\mathfrak p,\mathfrak q)))}(0)).$$

Therefore by Lemma 2.163 (2), we have coordinate changes:

$$\varphi_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q))(3,i(\mathfrak p))} : V(\mathfrak p(3,i(\mathfrak p));\mathfrak d(3,1,i(\mathfrak p))) \to V(\mathfrak p(2,i(\mathfrak p,\mathfrak q));\mathfrak d(2,1,i(\mathfrak p,\mathfrak q)))$$

and

$$\varphi_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q))(3,i(\mathfrak q))} : V(\mathfrak p(3,i(\mathfrak q));\mathfrak d(3,1,i(\mathfrak q))) \to V(\mathfrak p(2,i(\mathfrak p,\mathfrak q));\mathfrak d(2,1,i(\mathfrak p,\mathfrak q))).$$

We write them sometimes as $\varphi_{(\mathfrak p\mathfrak q)\mathfrak p}$, $\varphi_{(\mathfrak p\mathfrak q)\mathfrak q}$ for simplicity.

By the compatibility of ψ with coordinate changes,

$$\mathfrak q \in \varphi_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q))(3,i(\mathfrak p))}(V_{\mathfrak p}) \cap \varphi_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q))(3,i(\mathfrak q))}(V_{\mathfrak q}).$$

We consider

$$W^{(2)}_{\mathfrak q} = \left\{\mathfrak v \in V(\mathfrak p(2,i(\mathfrak p,\mathfrak q));\mathfrak d(2,1,i(\mathfrak p,\mathfrak q))) \;\middle|\; \mathfrak s_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q));\mathfrak d(2,1,i(\mathfrak p,\mathfrak q)))}(\mathfrak v) \in \bigoplus_{c\in\mathfrak C(\mathfrak q)} \mathcal E_c\right\}.$$

Both $\varphi_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q))(3,i(\mathfrak p))}(V_{\mathfrak p}) \cap W^{(2)}_{\mathfrak q}$ and $\varphi_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q))(3,i(\mathfrak q))}(V_{\mathfrak q})$ are open subsets of $W^{(2)}_{\mathfrak q}$. This fact is proved by dimension counting and by the fact that $\varphi_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q))(3,i(\mathfrak p))}$ and $\varphi_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q))(3,i(\mathfrak q))}$ are embeddings.

We put

$$V_{\mathfrak p\mathfrak q} = \varphi^{-1}_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q))(3,i(\mathfrak q))}(\varphi_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q))(3,i(\mathfrak p))}(V_{\mathfrak p}) \cap W_{\mathfrak q}) \tag{2.384}$$

and

$$\varphi_{\mathfrak p\mathfrak q} = \varphi^{-1}_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q))(3,i(\mathfrak p))} \circ \varphi_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q))(3,i(\mathfrak q))}. \tag{2.385}$$

We can define $\hat\varphi_{\mathfrak p\mathfrak q}$ by using $\hat\varphi_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q))(3,i(\mathfrak p))}$ and $\hat\varphi_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q))(3,i(\mathfrak q))}$. We have thus constructed a coordiate change.

We finally prove the compatibility of coordinate changes. Let $\mathfrak q \in \tilde\psi_{\mathfrak p}(\mathfrak s_{\mathfrak p}^{-1}(0))$, and $\mathfrak r \in \tilde\psi_{\mathfrak q}(\mathfrak s_{\mathfrak q}^{-1}(0))$. We then obtain $i(\mathfrak p,\mathfrak q)$, $i(\mathfrak p,\mathfrak r)$, $i(\mathfrak q,\mathfrak r)$ as above.

We note that

$$\begin{aligned}&\tilde\psi_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q));\mathfrak d(2,2,i(\mathfrak p,\mathfrak q)))}(\mathfrak s^{-1}_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q));\mathfrak d(2,2,i(\mathfrak p,\mathfrak q)))}(0))\\ &\supseteq \tilde\psi_{(\mathfrak p(3,i(\mathfrak q));\mathfrak d(3,1,i(\mathfrak q)))}(\mathfrak s^{-1}_{(\mathfrak p(3,i(\mathfrak q));\mathfrak d(3,1,i(\mathfrak q)))}(0))\\ &\supseteq \tilde\psi_{\mathfrak q}(\mathfrak s_{\mathfrak q}^{-1}(0)) \ni \mathfrak r.\end{aligned}$$

Therefore

$$\begin{aligned}&\tilde\psi_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q));\mathfrak d(2,2,i(\mathfrak p,\mathfrak q))}(\mathfrak s^{-1}_{(\mathfrak p(2,i(\mathfrak p,\mathfrak q));\mathfrak d(2,2,i(\mathfrak p,\mathfrak q)))}(0))\\ &\cap \tilde\psi_{(\mathfrak p(2,i(\mathfrak q,\mathfrak r));\mathfrak d(2,2,i(\mathfrak q,\mathfrak r)))}(\mathfrak s^{-1}_{(\mathfrak p(2,i(\mathfrak p,\mathfrak r));\mathfrak d(2,2,i(\mathfrak p,\mathfrak r)))}(0))\\ &\cap \tilde\psi_{(\mathfrak p(2,i(\mathfrak p,\mathfrak r));\mathfrak d(2,2,i(\mathfrak p,\mathfrak r)))}(\mathfrak s^{-1}_{(\mathfrak p(2,i(\mathfrak p,\mathfrak r));\mathfrak d(2,2,i(\mathfrak p,\mathfrak r)))}(0))\end{aligned}$$

is nonempty. Therefore Lemma 2.163 (2) and (3) imply that there exists $i(\mathfrak p,\mathfrak q,\mathfrak r)$ such that we have coordinate changes:

$$\begin{aligned}\varphi_{(\mathfrak p(1,i(\mathfrak p,\mathfrak q,\mathfrak r))(2,i(\mathfrak p,\mathfrak q))} :&V(\mathfrak p(2,i(\mathfrak p,\mathfrak q));\mathfrak d(2,1,i(\mathfrak p,\mathfrak q)))\\ &\to V(\mathfrak p(1,i(\mathfrak p,\mathfrak q,\mathfrak r));\mathfrak d(1,1,i(\mathfrak p,\mathfrak q,\mathfrak r)))\\ \varphi_{(\mathfrak p(1,i(\mathfrak p,\mathfrak q,\mathfrak r))(2,i(\mathfrak q,\mathfrak r))} :&V(\mathfrak p(2,i(\mathfrak q,\mathfrak r));\mathfrak d(2,1,i(\mathfrak q,\mathfrak r)))\\ &\to V(\mathfrak p(1,i(\mathfrak p,\mathfrak q,\mathfrak r));\mathfrak d(1,1,i(\mathfrak p,\mathfrak q,\mathfrak r)))\\ \varphi_{(\mathfrak p(1,i(\mathfrak p,\mathfrak q,\mathfrak r))(2,i(\mathfrak p,\mathfrak r))} :&V(\mathfrak p(2,i(\mathfrak p,\mathfrak r));\mathfrak d(2,1,i(\mathfrak p,\mathfrak r)))\\ &\to V(\mathfrak p(1,i(\mathfrak p,\mathfrak q,\mathfrak r));\mathfrak d(1,1,i(\mathfrak p,\mathfrak q,\mathfrak r))).\end{aligned}$$

We write them as $\varphi_{(\mathfrak p\mathfrak q\mathfrak r)(\mathfrak p\mathfrak q)}$, $\varphi_{(\mathfrak p\mathfrak q\mathfrak r)(\mathfrak q\mathfrak r)}$, $\varphi_{(\mathfrak p\mathfrak q\mathfrak r)(\mathfrak p\mathfrak r)}$. By Lemma 2.163 (4) we obtain

$$\begin{aligned}\varphi_{(\mathfrak p(1,i(\mathfrak p,\mathfrak q,\mathfrak r))(3,i(\mathfrak p))} :&V(\mathfrak p(3,i(\mathfrak p));\mathfrak d(3,1,i(\mathfrak p)))\\ &\to V(\mathfrak p(1,i(\mathfrak p,\mathfrak q,\mathfrak r));\mathfrak d(1,1,i(\mathfrak p,\mathfrak q,\mathfrak r)))\\ \varphi_{(\mathfrak p(1,i(\mathfrak p,\mathfrak q,\mathfrak r))(3,i(\mathfrak q))} :&V(\mathfrak p(3,i(\mathfrak q));\mathfrak d(3,1,i(\mathfrak q)))\\ &\to V(\mathfrak p(1,i(\mathfrak p,\mathfrak q,\mathfrak r));\mathfrak d(1,1,i(\mathfrak p,\mathfrak q,\mathfrak r)))\\ \varphi_{(\mathfrak p(1,i(\mathfrak p,\mathfrak q,\mathfrak r))(3,i(\mathfrak r))} :&V(\mathfrak p(3,i(\mathfrak r));\mathfrak d(3,1,i(\mathfrak r)))\\ &\to V(\mathfrak p(1,i(\mathfrak p,\mathfrak q,\mathfrak r));\mathfrak d(1,1,i(\mathfrak p,\mathfrak q,\mathfrak r))).\end{aligned}$$

We write them as $\varphi_{(\mathfrak p\mathfrak q\mathfrak r)\mathfrak p}$, $\varphi_{(\mathfrak p\mathfrak q\mathfrak r)\mathfrak q}$, $\varphi_{(\mathfrak p\mathfrak q\mathfrak r)\mathfrak r}$.

By Lemma 2.163 (4) we have

$$\begin{aligned}&\varphi_{(\mathfrak p\mathfrak q\mathfrak r)(\mathfrak p\mathfrak q)}\circ\varphi_{(\mathfrak p\mathfrak q)\mathfrak p} = \varphi_{(\mathfrak p\mathfrak q\mathfrak r)\mathfrak p}, \qquad \varphi_{(\mathfrak p\mathfrak q\mathfrak r)(\mathfrak p\mathfrak q)}\circ\varphi_{(\mathfrak p\mathfrak q)\mathfrak q} = \varphi_{(\mathfrak p\mathfrak q\mathfrak r)\mathfrak q},\\ &\varphi_{(\mathfrak p\mathfrak q\mathfrak r)(\mathfrak q\mathfrak r)}\circ\varphi_{(\mathfrak q\mathfrak r)\mathfrak q} = \varphi_{(\mathfrak p\mathfrak q\mathfrak r)\mathfrak q}, \qquad \varphi_{(\mathfrak p\mathfrak q\mathfrak r)(\mathfrak q\mathfrak r)}\circ\varphi_{(\mathfrak q\mathfrak r)\mathfrak r} = \varphi_{(\mathfrak p\mathfrak q\mathfrak r)\mathfrak r},\\ &\varphi_{(\mathfrak p\mathfrak q\mathfrak r)(\mathfrak p\mathfrak r)}\circ\varphi_{(\mathfrak p\mathfrak r)\mathfrak r} = \varphi_{(\mathfrak p\mathfrak q\mathfrak r)\mathfrak r}, \qquad \varphi_{(\mathfrak p\mathfrak q\mathfrak r)(\mathfrak p\mathfrak r)}\circ\varphi_{(\mathfrak p\mathfrak r)\mathfrak p} = \varphi_{(\mathfrak p\mathfrak q\mathfrak r)\mathfrak p}.\end{aligned}$$

Now we calculate:

$$\begin{aligned}\varphi_{\mathfrak p\mathfrak q}\circ\varphi_{\mathfrak q\mathfrak r} &= \varphi^{-1}_{(\mathfrak p\mathfrak q)\mathfrak p}\circ\varphi_{(\mathfrak p\mathfrak q)\mathfrak q}\circ\varphi^{-1}_{(\mathfrak q\mathfrak r)\mathfrak q}\circ\varphi_{(\mathfrak q\mathfrak r)\mathfrak r}\\ &= \varphi^{-1}_{(\mathfrak p\mathfrak q\mathfrak r)\mathfrak p}\circ\varphi_{(\mathfrak p\mathfrak q\mathfrak r)(\mathfrak p\mathfrak q)}\circ\varphi_{(\mathfrak p\mathfrak q)\mathfrak q}\circ\varphi^{-1}_{(\mathfrak q\mathfrak r)\mathfrak q}\circ\varphi^{-1}_{(\mathfrak p\mathfrak q\mathfrak r)(\mathfrak q\mathfrak r)}\circ\varphi_{(\mathfrak p\mathfrak q\mathfrak r)\mathfrak r}\\ &= \varphi^{-1}_{(\mathfrak p\mathfrak q\mathfrak r)\mathfrak p}\circ\varphi_{(\mathfrak p\mathfrak q\mathfrak r)\mathfrak r}\\ &= \varphi^{-1}_{(\mathfrak p\mathfrak r)\mathfrak p}\circ\varphi^{-1}_{(\mathfrak p\mathfrak q\mathfrak r)(\mathfrak p\mathfrak r)}\circ\varphi_{(\mathfrak p\mathfrak q\mathfrak r)(\mathfrak p\mathfrak r)}\circ\varphi_{(\mathfrak p\mathfrak r)\mathfrak r}\\ &= \varphi^{-1}_{(\mathfrak p\mathfrak r)\mathfrak p}\circ\varphi_{(\mathfrak p\mathfrak r)\mathfrak r} = \varphi_{\mathfrak p\mathfrak r}.\end{aligned}$$

Note (2.379) holds everywhere on $V(\mathfrak p(3,i_3);\mathfrak d(3,1,i_3))$. Therefore we can perform the above calculation everywhere on $\varphi_{\mathfrak q\mathfrak r}^{-1}(V_{\mathfrak p\mathfrak q})\cap V_{\mathfrak p\mathfrak r}$. (The maps appearing in the intermidiate stage of the calculation are defined in larger domain.)

The proof of the consistency of the bundle maps $\hat\varphi_{\mathfrak{pq}}$, $\hat\varphi_{\mathfrak{qr}}$, $\hat\varphi_{\mathfrak{pr}}$ is the same by using $\hat\varphi_{(\mathfrak{pqr})\mathfrak{r}}$ etc.

The proof of Theorem 2.3 is now complete. □

3. Appendix

3.1. **Proof of Proposition 2.19.** In this subsection we prove Propositions 2.19, 2.23 and Lemma 2.26. It seems likely that there are several different ways to prove them. We prove the proposition by the alternating method similar to those in the proof of Theorems 1.10, 1.34, 2.70, 2.72.

In view of Lemma 2.151, it suffices to prove them in the case $\mathfrak{x} = \mathfrak{Y}_0$. So we assume it throughout this subsection.

We start with describing the situation. We consider the universal bundle (2.154). The base space $\mathfrak{V}(\mathfrak{x}_{\rm v})$ is a neighborhood of $\mathfrak{x}_{\rm v}$ in the Deligne-Mumford moduli space. Suppose we have two coordinates at infinity, which we write $\mathfrak{w}(j)$, $j = 1, 2$. We denote the universal bundle (2.154) over $\mathfrak{V}(\mathfrak{x}_{\rm v})$ that is a part of $\mathfrak{w}(j)$ by

$$\pi^{(j)} : \mathfrak{M}^{(j)}_{\mathfrak{x}_{\rm v}} \to \mathfrak{V}(\mathfrak{x}_{\rm v}). \tag{3.386}$$

Actually $\pi^{(1)} = \pi^{(2)}$ but we distinguish them.[22] The fiber at the base point $\mathfrak{x}_{\rm v}$ is written as $\Sigma^{(j)}_{\rm v}$ and the fiber at $\rho_{\rm v} \in \mathfrak{V}(\mathfrak{x}_{\rm v})$ is written as $\Sigma^{\rho,(j)}_{\rm v}$.

We have an isomorphism

$$\hat\varphi_{12} : \mathfrak{M}^{(2)}_{\mathfrak{x}_{\rm v}} \to \mathfrak{M}^{(1)}_{\mathfrak{x}_{\rm v}} \tag{3.387}$$

of fiber bundles that preserves fiberwise complex structures and marked points. Such an isomorphism is unique since we assumed $\mathfrak{x}_{\rm v}$ to be stable.

By Definition 2.10 (5) we have a trivialization:

$$\varphi^{(j)}_{\rm v} : \Sigma^{(j)}_{\rm v} \times \mathfrak{V}(\mathfrak{x}_{\rm v}) \to \mathfrak{M}^{(j)}_{\mathfrak{x}_{\rm v}}. \tag{3.388}$$

The map $\varphi^{(j)}_{\rm v}$ is a diffeomorphism of fiber bundles of C^∞-class, and preserves the complex structure on the neck (ends). Moreover it preserves $\Gamma_{\mathfrak{p}}$-action and marked points.

Let $\rho = (\rho_{\rm v})$. The restriction of the composition $(\varphi^{(1)}_{\rm v})^{-1} \circ \hat\varphi_{12} \circ \varphi^{(2)}_{\rm v}$ to the fiber at $\rho_{\rm v} \in \mathfrak{V}(\mathfrak{x}_{\rm v})$ becomes a diffeomorphism

$$u^\rho_{\rm v} : (\Sigma^{(2)}_{\rm v}, j^{(2)}_\rho) \to (\Sigma^{(1)}_{\rm v}, j^{(1)}_\rho). \tag{3.389}$$

We note that $u^\rho_{\rm v}$ is a diffeomorphism and is biholomorphic in the neck region. (Note that the complex structure of the neck region is fixed by the definition of coordinate at infinity.) It is also biholomorphic (everywhere) with respect to the family of complex structures, $j^{(1)}_\rho, j^{(2)}_\rho$ parametrized by ρ.

The map (3.389) preserves the marked points and is $\Gamma_{\mathfrak{x}}$-equivariant. We also assume the image of the neck region by $u^\rho_{\rm v}$ is contained in the neck region. (We can always assume so by extending the neck of the coordinate at infinity $\mathfrak{w}(1)$ of the source.) Hereafter we write $\Sigma^{\rho,(j)}_{\rm v} = (\Sigma^{(j)}_{\rm v}, j^{(j)}_\rho)$ in case we do not need to write $j^{(j)}_\rho$ explicitly.

[22]To prove Lemma 2.26, we need to consider a parametrized family and so the parameter ξ should be added to many of the objects we define. To simplify the notation we omit them.

Remark 3.1. We fix a trivialization as a smooth fiber bundle since it is important to fix a parametrization to study ρ derivative of the ρ-parametrized family of maps from the fibers.

In (2.171) we introduced the map

$$\mathfrak{v}_{(\mathfrak{y}_2,\vec{T}_2,\vec{\theta}_2)} : \Sigma_{(\mathfrak{y}_2,\vec{T}_2,\vec{\theta}_2)} \to \Sigma_{(\mathfrak{y}_1,\vec{T}_1,\vec{\theta}_1)}.$$

Here the marked bordered curves $\Sigma_{(\mathfrak{y}_j,\vec{T}_j,\vec{\theta}_j)}$ $(j = 1, 2)$ are obtained by gluing $\Sigma_{\rm v}^{(j)}$ in a way parametrized by $\mathfrak{y}_j, \vec{T}_j, \vec{\theta}_j$. The idea of the proof is to construct the map $\mathfrak{v}_{(\mathfrak{y}_2,\vec{T}_2,\vec{\theta}_2)}$ by gluing the maps $u_{\rm v}^{\rho}$ using the alternating method. In this subsection we use the notation u, ρ in place of $\mathfrak{v}$, $\mathfrak{y}$.

We introduce several function spaces. Let

$$\mathfrak{y} = \rho = (\rho_{\rm v}) \in \prod_{{\rm v}\in C^0(\mathcal{G}_{\mathfrak{x}})} \mathfrak{V}(\mathfrak{x}_{\rm v}).$$

We write $\Sigma_{\vec{T},\vec{\theta}}^{\rho,(j)}$ using the notation used in the gluing construction in Subsection 2.5.

We use the decomposition (2.217) and (2.220) with coordinate (2.218). The domain (2.222) are also used. We use the bump functions (2.223)-(2.227).

On the function space

$$L^2_{m,\delta}(\Sigma_{\rm v}^{\rho,(2)}; (u_{\rm v}^{\rho})^*T\Sigma_{\rm v}^{\rho',(1)} \otimes \Lambda^{01}) \tag{3.390}$$

we define the norm

$$\|s\|^2_{L^2_{m,\delta}} = \sum_{k=0}^{m} \int_{\Sigma_{\rm v}^{\rho}} e_{{\rm v},\delta}|\nabla^k s|^2 {\rm vol}_{\Sigma_{\rm v}^{\rho}}. \tag{3.391}$$

We modify Definition 2.73 as follows.

Definition 3.2. The Sobolev space

$$L^2_{m+1,\delta}((\Sigma_{\rm v}^{\rho,(2)}, \partial\Sigma_{\rm v}^{\rho,(2)}); (u_{\rm v}^{\rho})^*T\Sigma_{\rm v}^{\rho',(1)}, (u_{\rm v}^{\rho})^*T\partial\Sigma_{\rm v}^{\rho',(1)})$$

consists of elements $(s, \vec{v})$ with the following properties.

(1) $\vec{v} = (v_{\rm e})$ where e runs on the set of edges of v and

$$v_{\rm e} = c_1\frac{\partial}{\partial \tau_{\rm e}} + c_2\frac{\partial}{\partial t_{\rm e}}$$

(in case ${\rm e} \in {\rm C}^1_{\rm c}(\mathcal{G})$) or

$$v_{\rm e} = c\frac{\partial}{\partial \tau_{\rm e}}$$

(in case ${\rm e} \in {\rm C}^1_{\rm o}(\mathcal{G})$). Here $c, c_1, c_2 \in \mathbb{R}$.

(2) The following norm is finite.

$$\begin{aligned} &\|(s,\vec{v})\|^2_{L^2_{m+1,\delta}} \\ &= \sum_{k=0}^{m+1} \int_{K_{\rm v}} |\nabla^k s|^2 {\rm vol}_{\Sigma_i} + \sum_{\text{e: edges of v}} \|v_{\rm e}\|^2 \\ &\quad + \sum_{k=0}^{m+1} \sum_{\text{e: edges of v}} \int_{\text{e-th end}} e_{{\rm v},\delta} |\nabla^k (s - {\rm Pal}(v_{\rm e}))|^2 {\rm vol}_{\Sigma_{\rm v}^{\rho}}. \end{aligned} \tag{3.392}$$

Here Pal is defined by the canonical trivialization of the tangent bundle on the neck region.

In case $\mathrm{v} \in C^0_{\mathrm{s}}(\mathcal{G}_{\mathfrak{r}})$ we use the function space $L^2_{m+1,\delta}(\Sigma^{\rho,(2)}_{\mathrm{v}}; (u^\rho_{\mathrm{v}})^*T\Sigma^{\rho',(1)}_{\mathrm{v}})$ in place of $L^2_{m+1,\delta}((\Sigma^{\rho,(2)}_{\mathrm{v}}, \partial\Sigma^{\rho,(2)}_{\mathrm{v}}); (u^\rho_{\mathrm{v}})^*T\Sigma^{\rho',(1)}_{\mathrm{v}}, (u^\rho_{\mathrm{v}})^*T\partial\Sigma^{\rho',(1)}_{\mathrm{v}})$.

We do not assume any condition similar to Definition 2.75 and put

$$\begin{aligned} &L^2_{m+1,\delta}((\Sigma^{\rho,(2)}, \partial\Sigma^{\rho,(2)}); (u^\rho_{\mathrm{v}})^*T\Sigma^{\rho',(1)}, (u^\rho)^*T\partial\Sigma^{\rho',(1)}) \\ =& \bigoplus_{\mathrm{v}\in C^0_{\mathrm{d}}(\mathcal{G}_{\mathfrak{r}})} L^2_{m+1,\delta}((\Sigma^{\rho,(2)}_{\mathrm{v}}, \partial\Sigma^{\rho,(2)}_{\mathrm{v}}); (u^\rho_{\mathrm{v}})^*T\Sigma^{\rho',(1)}_{\mathrm{v}}, (u^\rho_{\mathrm{v}})^*T\partial\Sigma^{\rho',(1)}_{\mathrm{v}}) \\ &\oplus \bigoplus_{\mathrm{v}\in C^0_{\mathrm{s}}(\mathcal{G}_{\mathfrak{r}})} L^2_{m+1,\delta}(\Sigma^{\rho,(2)}_{\mathrm{v}}; (u^\rho_{\mathrm{v}})^*T\Sigma^{\rho',(1)}_{\mathrm{v}}). \end{aligned} \tag{3.393}$$

The sum of (3.390) over v is denoted by

$$L^2_{m,\delta}(\Sigma^{\rho,(2)}; (u^\rho)^*T\Sigma^{\rho',(1)} \otimes \Lambda^{01}).$$

We next define weighted Sobolev norms for the sections of various bundles on $\Sigma^{\rho,(2)}_{\vec T,\vec\theta}$. Here $\Sigma^{\rho,(2)}_{\vec T,\vec\theta}$ was denoted by $\Sigma^{\rho}_{\vec T,\vec\theta}$ in Subsection 2.5. Let

$$u' : (\Sigma^{\rho,(2)}_{\vec T,\vec\theta}, \partial\Sigma^{\rho,(2)}_{\vec T,\vec\theta}) \to (\Sigma^{\rho',(1)}_{\vec T',\vec\theta'}, \partial\Sigma^{\rho',(1)}_{\vec T',\vec\theta'})$$

be a diffeomorphism that sends each neck region of the source to the corresponding neck region of the target. We first consider the case when all $T_{\mathrm{e}} \ne \infty$. In this case $\Sigma^{\rho,(j)}_{\vec T,\vec\theta}$ is compact. We consider an element

$$s \in L^2_{m+1}((\Sigma^{\rho,(2)}_{\vec T,\vec\theta}, \partial\Sigma^{\rho,(2)}_{\vec T,\vec\theta}); (u')^*T\Sigma^{\rho',(1)}_{\vec T',\vec\theta'}, (u')^*T\partial\Sigma^{\rho',(1)}_{\vec T',\vec\theta'}).$$

Since we take m large the section s is continuous. We take a point $(0,1/2)_{\mathrm{e}}$ in the e-th neck. So $s((0,1/2)_{\mathrm{e}}) \in T_{u'((0,1/2)_{\mathrm{e}})}\Sigma^{\rho',(1)}_{\vec T',\vec\theta'}$ is well-defined.

We use a canonical trivialization of the tangent bundle in the neck regions to define Pal below. We put

$$\begin{aligned} \|s\|^2_{L^2_{m+1,\delta}} =& \sum_{k=0}^{m+1}\sum_{\mathrm{v}} \int_{K_{\mathrm{v}}} |\nabla^k s|^2 \mathrm{vol}_{\Sigma^\rho_{\mathrm{v}}} \\ &+ \sum_{k=0}^{m+1}\sum_{\mathrm{e}} \int_{\text{e-th neck}} e_{\vec T,\delta} |\nabla^k (s - \mathrm{Pal}(s(0,1/2)_{\mathrm{e}}))|^2 dt_{\mathrm{e}} d\tau_{\mathrm{e}} \\ &+ \sum_{\mathrm{e}} \|s((0,1/2)_{\mathrm{e}}))\|^2. \end{aligned} \tag{3.394}$$

For a section $s \in L^2_m(\Sigma^{\rho,(2)}_{\vec T,\vec\theta}; (u')^*T\Sigma^{\rho',(1)}_{\vec T',\vec\theta'} \otimes \Lambda^{01})$ we define

$$\|s\|^2_{L^2_{m,\delta}} = \sum_{k=0}^{m} \int_{\Sigma_T} e_{T,\delta} |\nabla^k s|^2 \mathrm{vol}_{\Sigma_T}. \tag{3.395}$$

We next consider the case when some of the edges e have infinite length, namely $T_{\mathrm{e}} = \infty$. Let $C^{1,\mathrm{inf}}_{\mathrm{o}}(\mathcal{G}_{\mathfrak{r}}, \vec T)$ (resp. $C^{1,\mathrm{inf}}_{\mathrm{c}}(\mathcal{G}_{\mathfrak{r}}, \vec T)$) be the set of elements e in $C^1_{\mathrm{o}}(\mathcal{G}_{\mathfrak{r}})$ (resp. $C^1_{\mathrm{c}}(\mathcal{G}_{\mathfrak{r}})$) with $T_{\mathrm{e}} = \infty$ and $C^{1,\mathrm{fin}}_{\mathrm{o}}(\mathcal{G}_{\mathfrak{r}}, \vec T)$ (resp. $C^{1,\mathrm{fin}}_{\mathrm{c}}(\mathcal{G}_{\mathfrak{r}}, \vec T)$) be the set of elements $C^1_{\mathrm{o}}(\mathcal{G}_{\mathfrak{r}})$ (resp. $C^1_{\mathrm{c}}(\mathcal{G}_{\mathfrak{r}})$) with $T_{\mathrm{e}} \ne \infty$. Note the ends of $\Sigma^{\rho}_{\vec T,\vec\theta}$ correspond two to one to $C_{\rm o}^{1,\rm inf}(\mathcal G_{\mathfrak r},\vec T)\cup C_{\rm c}^{1,\rm inf}(\mathcal G_{\mathfrak r},\vec T)$. The ends that correspond to an element of $C_{\rm o}^{1,\rm inf}(\mathcal G_{\mathfrak r},\vec T)$ is $([-5T_{\rm e},\infty)\times[0,1])\cup(-\infty,5T_{\rm e}]\times[0,1])$ and the ends that correspond to $C_{\rm c}^{1,\rm inf}(\mathcal G_{\mathfrak p},\vec T)$ is $([-5T_{\rm e},\infty)\times S^1)\cup(-\infty,5T_{\rm e}]\times S^1)$. We have a weight function $e_{{\rm v},\delta}(\tau_{\rm e},t_{\rm e})$ on it.

Definition 3.3. An element of

$$L^2_{m+1,\delta}((\Sigma^{\rho,(2)}_{\vec T,\vec\theta},\partial\Sigma^{\rho,(2)}_{\vec T,\vec\theta});(u')^*T\Sigma^{\rho',(1)}_{\vec T',\vec\theta'},(u')^*T\partial\Sigma^{\rho',(1)}_{\vec T',\vec\theta'})$$

is a pair $(s,\vec v)$ such that:

(1) s is a section of $(u')^*T\Sigma^{\rho',(1)}_{\vec T',\vec\theta'}$ on $\Sigma^{\rho,(2)}_{\vec T,\vec\theta}$ minus singular points $z_{\rm e}$ corresponding to the edges e with $T_{\rm e}=\infty$.
(2) s is locally of L^2_{m+1} class.
(3) On $\partial\Sigma^{\rho,(2)}_{\vec T,\vec\theta}$ the restriction of s is in $(u')^*T\partial\Sigma^{\rho',(1)}_{\vec T',\vec\theta'}$.
(4) $\vec v=(v_{\rm e})$ where e runs in $C^{1,\rm inf}(\mathcal G_{\mathfrak p},\vec T)$ and $v_{\rm e}$ is as in Definition 3.2 (1).
(5) For each e with $T_{\rm e}=\infty$ the integral

$$\begin{aligned}&\sum_{k=0}^{m+1}\int_0^\infty\int_{t_{\rm e}} e_{{\rm v},\delta}(\tau_{\rm e},t_{\rm e})|\nabla^k(s(\tau_{\rm e},t_{\rm e})-{\rm Pal}(v_{\rm e}))|^2d\tau_{\rm e}dt_{\rm e}\\&+\sum_{k=0}^{m+1}\int_{-\infty}^0\int_{t_{\rm e}} e_{{\rm v},\delta}(\tau_{\rm e},t_{\rm e})|\nabla^k(s(\tau_{\rm e},t_{\rm e})-{\rm Pal}(v_{\rm e}))|^2d\tau_{\rm e}dt_{\rm e}\end{aligned}\tag{3.396}$$

is finite. (Here we integrate over $t_{\rm e}\in[0,1]$ (resp. $t_{\rm e}\in S^1$) if $\rm e\in C_{\rm o}^{1,\rm inf}(\mathcal G_{\mathfrak p},\vec T)$ (resp. $\rm e\in C_{\rm c}^{1,\rm inf}(\mathcal G_{\mathfrak p},\vec T)$).
(6) The section s vanishes at each marked points.

We define

$$\|(s,\vec v)\|^2_{L^2_{m+1,\delta}}=(3.394)+\sum_{{\rm e}\in C^{1,\rm inf}(\mathcal G_{\mathfrak p},\vec T)}(3.396)+\sum_{{\rm e}\in C^{1,\rm inf}(\mathcal G_{\mathfrak p},\vec T)}\|v_{\rm e}\|^2.\tag{3.397}$$

An element of

$$L^2_{m,\delta}(\Sigma^{\rho,(2)}_{\vec T,\vec\theta};(u')^*T\Sigma^{\rho',(1)}_{\vec T',\vec\theta'}\otimes\Lambda^{01})$$

is a section s of the bundle $(u')^*T\Sigma^{\rho',(1)}_{\vec T',\vec\theta'}\otimes\Lambda^{01}$ such that it is locally of L^2_m-class and

$$\begin{aligned}&\sum_{k=0}^{m}\int_0^\infty\int_{t_{\rm e}} e_{{\rm v},\delta}|\nabla^k s(\tau_{\rm e},t_{\rm e})|^2d\tau_{\rm e}dt_{\rm e}\\&+\sum_{k=0}^{m}\int_{-\infty}^0\int_{t_{\rm e}} e_{{\rm v},\delta}|\nabla^k(s(\tau_{\rm e},t_{\rm e})|^2d\tau_{\rm e}dt_{\rm e}\end{aligned}\tag{3.398}$$

is finite. We define

$$\|s\|^2_{L^2_{m,\delta}}=(3.395)+\sum_{{\rm e}\in C^{1,\rm inf}(\mathcal G_{\mathfrak p},\vec T)}(3.398).\tag{3.399}$$

For a subset W of $\Sigma^{\rho,(2)}_{\rm v}$ or $\Sigma^{\rho,(2)}_{\vec T,\vec\theta}$ we define $\|s\|_{L^2_{m,\delta}(W\subset\Sigma^{\rho,(2)}_{\rm v})}$, $\|s\|_{L^2_{m,\delta}(W\subset\Sigma^{\rho,(2)}_{\vec T,\vec\theta})}$ by restricting the domain of the integration (3.394), (3.395), (3.397) or (3.399) to W.

We consider maps $u^\rho_{\rm v} : (\Sigma^{\rho,(2)}_{\rm v}, \partial\Sigma^{\rho,(2)}_{\rm v}) \to (\Sigma^{\rho,(1)}_{\rm v}, \partial\Sigma^{\rho,(1)}_{\rm v})$ in (3.389), for all v. We write $u^\rho = (u^\rho_{\rm v})$.

We next define a vector space that corresponds to a fiber of the 'obstruction bundle' in our situation. Let $u' : (\Sigma^{\rho,(2)}_{\rm v}, \partial\Sigma^{\rho,(2)}_{\rm v}) \to (\Sigma^{\rho',(1)}_{\rm v}, \partial\Sigma^{\rho',(1)}_{\rm v})$ be a diffeomorphism that sends each of the neck region of the source to the corresponding neck region of the target. We define

$$E^\rho_{\rm v}(u') \subset \Gamma_0(K^{\rho,(2)}_{\rm v}, (u')^*T\Sigma^{\rho',(1)}_{\rm v} \otimes \Lambda^{01})$$

as follows.

We may identify $\mathfrak{V}(\mathfrak{x}_{\rm v})$ as an open subset of certain Euclidean space. Let $\mathfrak{e}_{\rm v} \in T_{\rho'_{\rm v}}\mathfrak{V}(\mathfrak{x}_{\rm v})$. We define

$$\mathfrak{I}^{\rho'}_{\rm v}(u', \mathfrak{e}_{\rm v}) = \frac{d}{dt}(\overline{\partial}^{\rho,\rho'+t\mathfrak{e}_{\rm v}} u^\rho_{\rm v})\Big|_{t=0}. \tag{3.400}$$

Here $\overline{\partial}^{\rho,\rho+t\mathfrak{e}_{\rm v}}$ is the $\overline{\partial}$ operator with respect to the complex structure $j^{(1)}_{\rho'+t\mathfrak{e}_{\rm v}}$ (on the target) and $j^{(2)}_\rho$ (on the source). We thus obtain a map:

$$\mathfrak{I}^{\rho'}_{\rm v}(u', \cdot) : T_{\rho'_{\rm v}}\mathfrak{V}(\mathfrak{x}_{\rm v}) \to L^2_{m,\delta}(\Sigma^{\rho,(2)}_{\rm v}; (u')^*T\Sigma^{\rho',(1)}_{\rm v} \otimes \Lambda^{01}). \tag{3.401}$$

Since the complex structure is independent of ρ on the neck region, the image of (3.401) is contained in $\Gamma_0(K^{\rho,(2)}_{\rm v}, (u')^*T\Sigma^{\rho',(1)}_{\rm v} \otimes \Lambda^{01})$, that is, the set of smooth sections supported on the interior of the core.

Definition 3.4. We denote by $E^\rho_{\rm v}(u')$ the image of (3.401).

We consider the linearization of the Cauchy-Riemann equation associated to the biholomorphic map u' that is

$$\begin{aligned} D_{u'}\overline{\partial} :&L^2_{m+1,\delta}((\Sigma^{\rho,(2)}_{\rm v}, \partial\Sigma^{\rho,(2)}_{\rm v}); (u')^*T\Sigma^{\rho',(1)}_{\rm v}, (u')^*T\partial\Sigma^{\rho',(1)}_{\rm v}) \\ &\to L^2_{m,\delta}(\Sigma^{\rho,(2)}_{\rm v}; (u')^*T\Sigma^{\rho',(1)}_{\rm v} \otimes \Lambda^{01}). \end{aligned} \tag{3.402}$$

Lemma 3.5. *If u' is sufficiently close to $u^\rho_{\rm v}$ then the kernel of (3.402) is zero and we have*

$$\operatorname{Im}(D_{u'}\overline{\partial}) \oplus E^\rho_{\rm v}(u') = L^2_{m,\delta}(\Sigma^{\rho,(2)}_{\rm v}; (u')^*T\Sigma^{\rho',(1)}_{\rm v} \otimes \Lambda^{01}). \tag{3.403}$$

Proof. We first consider the case $u' = u^\rho_{\rm v}$, that is a biholomorphic map. Then the kernel is identified with the set of holomorphic vector fields on $\Sigma^{\rho,(2)}$ that vanish on the singular points and marked points. Such a vector field is necessary zero by stability.

By the standard result of deformation theory, the cokernel is identified with the deformation space of the complex structures, since $u^\rho_{\rm v}$ is biholomorphic. Therefore (3.403) holds.

We then find that the conclusion holds if u' is sufficiently close to $u^\rho_{\rm v}$ so that $D_{u'}\overline{\partial}$ is close to $D_{u^\rho_{\rm v}}\overline{\partial}$ in operator norm and $E^\rho_{\rm v}(u')$ is close to $E^\rho_{\rm v}(u^\rho_{\rm v})$, in the sense that we can choose their orthonormal basis that are close to each other. □

Remark 3.6. 'Sufficently close' is a bit imprecise way to state the lemma. In the case we apply the lemma, we can easily check that the last part of the proof works.

We next take a map

$$\mathrm{E} : \{(z,v) \in T\Sigma^{(1)} \mid |v| \le \epsilon\} \to \Sigma^{(1)} \tag{3.404}$$

such that

(1) $\mathrm{E}(z,0) = z$ and
$$\frac{d}{dt}\mathrm{E}(z,tv)\bigg|_{t=0} = v.$$
(2) If $(z,v) \in T\partial\Sigma^{(1)}$ then $\mathrm{E}(z,v) \in \partial\Sigma^{(1)}$.
(3) $\mathrm{E}(z,v) = z + v$ on the neck region.

Now we start the gluing construction. Let $(\vec T,\vec\theta) \in (\vec T^{\mathrm{o}}_0,\infty] \times ((\vec T^{\mathrm{c}}_0,\infty] \times \vec S^1)$. For $\kappa = 0,1,2,\dots$, we will define a series of maps

$$u^{\rho}_{\vec T,\vec\theta,(\kappa)} \;:\; (\Sigma^{\rho,(2)}_{\vec T,\vec\theta}, \partial\Sigma^{\rho,(2)}_{\vec T,\vec\theta}) \to (\Sigma^{\rho_{(\kappa)},(1)}_{\vec T^{(\kappa)},\vec\theta^{(\kappa)}}, \partial\Sigma^{\rho_{(\kappa)},(1)}_{\vec T^{(\kappa)},\vec\theta^{(\kappa)}}) \tag{3.405}$$
$$\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(\kappa)} \;:\; (\Sigma^{\rho,(2)}_{\mathrm{v}}, \partial\Sigma^{\rho,(2)}_{\mathrm{v}}) \to (\Sigma^{\rho_{(\kappa)},(1)}_{\mathrm{v}}, \partial\Sigma^{\rho_{(\kappa)},(1)}_{\mathrm{v}}), \tag{3.406}$$

(we will explain $\rho_{(\kappa)}$, $\vec T^{(\kappa)}$ and $\vec\theta^{(\kappa)}$ below) and elements

$$\mathfrak{e}^{\rho}_{\mathrm{v},\vec T,\vec\theta,(\kappa)} \in E_{\mathrm{v}}(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(\kappa)}) \tag{3.407}$$
$$\mathrm{Err}^{\rho}_{\mathrm{v},\vec T,\vec\theta,(\kappa)} \in L^2_{m,\delta}(\Sigma^{\rho,(2)}_{\mathrm{v}}; (\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(\kappa)})^* T\Sigma^{\rho_{(\kappa)},(1)}_{\mathrm{v}} \otimes \Lambda^{01}). \tag{3.408}$$

Moreover we will define $V^{\rho}_{\vec T,\vec\theta,\mathrm{v},(\kappa)}$ for $\mathrm{v} \in C^0(\mathcal G)$, $\Delta T^{\rho}_{\vec T,\vec\theta,(\kappa),\mathrm{v},\mathrm{e}} \in \mathbb R$ for $\mathrm{e} \in C^1(\mathcal G)$ and $\Delta\theta^{\rho}_{\vec T,\vec\theta,(\kappa),\mathrm{v},\mathrm{e}} \in \mathbb R$ for $\mathrm{e} \in C^1_{\mathrm{c}}(\mathcal G)$. We put

$$v^{\rho}_{\vec T,\vec\theta,(\kappa),\mathrm{v},\mathrm{e}} = \Delta T^{\rho}_{\vec T,\vec\theta,(\kappa),\mathrm{v},\mathrm{e}} \frac{\partial}{\partial\tau_{\mathrm{e}}}, \qquad \text{for } \mathrm{e} \in C^1_{\mathrm{o}}(\mathcal G),$$
$$v^{\rho}_{\vec T,\vec\theta,(\kappa),\mathrm{v},\mathrm{e}} = \Delta T^{\rho}_{\vec T,\vec\theta,(\kappa),\mathrm{v},\mathrm{e}} \frac{\partial}{\partial\tau_{\mathrm{e}}} + \Delta\theta^{\rho}_{\vec T,\vec\theta,(\kappa),\mathrm{v},\mathrm{e}} \frac{\partial}{\partial t_{\mathrm{e}}} \qquad \text{for } \mathrm{e} \in C^1_{\mathrm{c}}(\mathcal G).$$

The pair $((V^{\rho}_{\vec T,\vec\theta,\mathrm{v},(\kappa)}), (v^{\rho}_{\vec T,\vec\theta,(\kappa),\mathrm{v},\mathrm{e}}))$ becomes an element of

$$L^2_{m+1,\delta}((\Sigma^{\rho,(2)}_{\mathrm{v}}, \partial\Sigma^{\rho,(2)}_{\mathrm{v}}); (\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(\kappa-1)})^* T\Sigma^{\rho_{(\kappa)},(1)}_{\vec T^{(\kappa)},\vec\theta^{(\kappa)}}, (\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(\kappa-1)})^* T\partial\Sigma^{\rho_{(\kappa)},(1)}_{\vec T^{(\kappa)},\vec\theta^{(\kappa)}}).$$

The vectors $\vec T^{(\kappa)}$ and $\vec\theta^{(\kappa)}$ are determined by $\Delta T^{\rho}_{\vec T,\vec\theta,(1),\mathrm{v},\mathrm{e}}, \dots, \Delta T^{\rho}_{\vec T,\vec\theta,(\kappa-1),\mathrm{v},\mathrm{e}}$ and $\Delta\theta^{\rho}_{\vec T,\vec\theta,(1),\mathrm{v},\mathrm{e}}, \dots, \Delta\theta^{\rho}_{\vec T,\vec\theta,(\kappa-1),\mathrm{v},\mathrm{e}}$ as follows. For each e let $\mathrm{v}_{\leftarrow}(\mathrm{e})$ and $\mathrm{v}_{\rightarrow}(\mathrm{e})$ be the verticies for which e is outgoing (resp. incoming) edge. We put:

$$10T^{(\kappa)}_{\mathrm{e}} = 10T_{\mathrm{e}} - \sum_{a=0}^{\kappa} \Delta T^{\rho}_{\vec T,\vec\theta,(a),\mathrm{v}_{\leftarrow}(\mathrm{e}),\mathrm{e}} + \sum_{a=0}^{\kappa} \Delta T^{\rho}_{\vec T,\vec\theta,(a),\mathrm{v}_{\rightarrow}(\mathrm{e}),\mathrm{e}} \tag{3.409}$$

$$\theta^{(\kappa)}_{\mathrm{e}} = \theta_{\mathrm{e}} + \sum_{a=0}^{\kappa} \Delta\theta^{\rho}_{\vec T,\vec\theta,(a),\mathrm{v}_{\leftarrow}(\mathrm{e}),\mathrm{e}} - \sum_{a=0}^{\kappa} \Delta\theta^{\rho}_{\vec T,\vec\theta,(a),\mathrm{v}_{\rightarrow}(\mathrm{e}),\mathrm{e}}. \tag{3.410}$$

Remark 3.7. As induction proceeds, we will modify the length of the neck region a bit from T_{e} to $T^{(\kappa)}_{\mathrm{e}}$. We also modify θ_{e} (that is the parameter to tell how much we twist the S^1 direction when we glue the piecies to obtain our curve) to $\theta^{(\kappa)}_{\mathrm{e}}$.

The elements $\rho_{(\kappa)} = (\rho_{\mathrm{v},(\kappa)})$ is defined from $\mathfrak{e}^{\rho}_{\mathrm{v},\vec T,\vec\theta,(\kappa)}$ inductively as follows.

$$\mathfrak{I}^{\rho_{(\kappa-1)}}_{\mathrm{v}}(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(\kappa-1)}, \rho_{\mathrm{v},(\kappa)} - \rho_{\mathrm{v},(\kappa-1)}) = \mathfrak{e}^{\rho}_{\mathrm{v},\vec T,\vec\theta,(\kappa)}. \tag{3.411}$$

So $T^{(\kappa)}_{\mathrm{e}}, \theta^{(\kappa)}_{\mathrm{e}}$ and $\rho_{\mathrm{v},(\kappa)}$ *depend* on $\rho, \vec T, \vec\theta$.

Remark 3.8. The construction of these objects are very much similar to that of Subsection 2.5. Note that $(\Sigma^{(1)}, \partial\Sigma^{(1)})$ plays the role of (X, L) here. (In fact $\partial\Sigma^{(1)}$ is a Lagrangian submanifold of $\Sigma^{(1)}$.) However the construction here is different from one in Subsection 2.5 in the following two points.

(1) We will construct a map u that not only satifies $\overline{\partial} u \equiv 0 \mod E^{\rho}_{\rm v}$ but is also a genuin holomorphic map. The linearized equation (3.402) is *not* surjective. We will kill the cokernel by deforming the complex structure of the target. Namely $\rho \ne \rho_{(\kappa)}$ in general.
(2) We do *not* require $\Delta T^{\rho}_{\vec T,\vec\theta,(\kappa),{\rm v}_{\leftarrow}({\rm e}),{\rm e}} = \Delta T^{\rho}_{\vec T,\vec\theta,(\kappa),{\rm v}_{\rightarrow}({\rm e}),{\rm e}}$ or $\Delta \theta^{\rho}_{\vec \theta,\vec\theta,(\kappa),{\rm v}_{\leftarrow}({\rm e}),{\rm e}} = \Delta \theta^{\rho}_{\vec \theta,\vec\theta,(\kappa),{\rm v}_{\rightarrow}({\rm e}),{\rm e}}$. This condition corresponds to $Dev_{\mathcal G_{\mathfrak p}}(V, \Delta p) = 0$ that we put in Definition 2.75. Here we did not put a similar condition in (3.393). Instead we deform the complex structure of the target again. Namely $T^{(\kappa)}_{\rm e} \ne T_{\rm e}$, $\theta^{(\kappa)}_{\rm e} \ne \theta_{\rm e}$ in general.

Now we start the construction of the above objects by induction on κ.

Pregluing: Since $u^{\rho}_{\rm v} : \Sigma^{\rho,(2)}_{\rm v} \to \Sigma^{\rho,(1)}_{\rm v}$ is biholomorphic and sends the neck region to the corresponding neck region, there exists $\Delta T^{\rho}_{\vec T,\vec\theta,(0),{\rm v},{\rm e}} \in \mathbb R$ for ${\rm e} \in C^1(\mathcal G)$ and $\Delta \theta^{\rho}_{\vec T,\vec\theta,(0),{\rm v},{\rm e}} \in \mathbb R$ for ${\rm e} \in C^1_{\rm c}(\mathcal G)$ such that

$$|u^{\rho}_{\rm v}(\tau_{\rm e}, t_{\rm e}) - (\tau_{\rm e} + \Delta T^{\rho}_{\vec T,\vec\theta,(0),{\rm v},{\rm e}}, t_{\rm e} + \Delta \theta^{\rho}_{\vec T,\vec\theta,(0),{\rm v},{\rm e}})| \le C_1 e^{-\delta_1|\tau_{\rm e}|}. \tag{3.412}$$

Note that in case ${\rm e} \in C^1_{\rm o}(\mathcal G)$ we put $\Delta \theta^{\rho}_{\vec T,\vec\theta,(0),{\rm v},{\rm e}} = 0$.

We identify the e-th neck region of $\Sigma^{\rho_{(\kappa)},(2)}_{\vec T^{(\kappa)},\vec\theta^{(\kappa)}}$ with

$$[-5T_{\rm e} + \mathfrak s\Delta T^{\leftarrow}_{{\rm e},(\kappa)}, 5T_{\rm e} + \mathfrak s\Delta T^{\rightarrow}_{{\rm e},(\kappa)}] \times [0,1] \text{ or } S^1,$$

where

$$\mathfrak s\Delta T^{\leftarrow}_{{\rm e},(\kappa)} = \sum_{a=0}^{\kappa} \Delta T^{\rho}_{\vec T,\vec\theta,(a),{\rm v}_{\leftarrow}({\rm e}),{\rm e}},$$

$$\mathfrak s\Delta T^{\rightarrow}_{{\rm e},(\kappa)} = \sum_{a=0}^{\kappa} \Delta T^{\rho}_{\vec T,\vec\theta,(a),{\rm v}_{\rightarrow}({\rm e}),{\rm e}}.$$

We also denote

$$\mathfrak s\Delta \theta^{\leftarrow}_{{\rm e},(\kappa)} = \sum_{a=1}^{\kappa} \Delta \theta^{\rho}_{\vec T,\vec\theta,(a),{\rm v}_{\leftarrow}({\rm e}),{\rm e}},$$

$$\mathfrak s\Delta \theta^{\rightarrow}_{{\rm e},(\kappa)} = \sum_{a=1}^{\kappa} \Delta \theta^{\rho}_{\vec T,\vec\theta,(a),{\rm v}_{\rightarrow}({\rm e}),{\rm e}}.$$

We use the symbol $\tau^{(\kappa)}_{\rm e}$ as the coordinate of the first factor. The symbol $t^{(\kappa)}_{\rm e}$ denotes the coordinate of the second factor that is given by

$$t^{(\kappa)}_{\rm e} = t_{\rm e} + \mathfrak s\Delta \theta^{\leftarrow}_{{\rm e},(\kappa)}$$

in case ${\rm e} \in C^1_{\rm c}(\mathcal G_{\mathfrak r})$. Here $t_{\rm e}$ is the canonical coordinate of S^1. In case ${\rm e} \in C^1_{\rm o}(\mathcal G_{\mathfrak r})$, $t^{(\kappa)}_{\rm e} = t_{\rm e}$.

We have

$$\tau^{(\kappa)}_{\rm e} = \tau'_{\rm e} - 5T_{\rm e} + \mathfrak s\Delta T^{\leftarrow}_{{\rm e},(\kappa)} = \tau''_{\rm e} + 5T_{\rm e} + \mathfrak s\Delta T^{\rightarrow}_{{\rm e},(\kappa)}. \tag{3.413}$$

(Hence $\tau'_{\rm e} = \tau''_{\rm e} + 10T_{\rm e} - \mathfrak s\Delta T^{\leftarrow}_{{\rm e},(\kappa)} + \mathfrak s\Delta T^{\rightarrow}_{{\rm e},(\kappa)} = \tau''_{{\rm e},(\kappa)} + 10T^{(\kappa)}_{\rm e}$. See (3.409).)

In case $\mathrm{e} \in C^1_{\mathrm{c}}(\mathcal{G}_{\mathfrak{r}})$ we also have

$$t_{\mathrm{e}}^{(\kappa)} = t'_{\mathrm{e}} + \mathfrak{s}\Delta\theta^{\leftarrow}_{\mathrm{e},(\kappa)} = t''_{\mathrm{e}} - \theta_{\mathrm{e}} + \mathfrak{s}\Delta\theta^{\rightarrow}_{\mathrm{e},(\kappa)}. \tag{3.414}$$

(Hence $t'_{\mathrm{e}} = t''_{\mathrm{e}} - \theta_{\mathrm{e}}^{(\kappa)}$. See (3.410).)

We define the map $\mathrm{id}^{\rho,\vec T,\vec\theta}_{\mathrm{e},(\kappa)}$ from the e-th neck of $\Sigma^{\rho,(2)}_{\vec T,\vec\theta}$ to the e-th neck of $\Sigma^{\rho_{(\kappa)},(1)}_{\vec T^{(\kappa)},\vec\theta^{(\kappa)}}$ by

$$\mathrm{id}^{\rho,\vec T,\vec\theta}_{\mathrm{e},(\kappa)} : (\tau_{\mathrm{e}}, t_{\mathrm{e}}) \mapsto (\tau_{\mathrm{e}}^{(\kappa)}, t_{\mathrm{e}}^{(\kappa)}) = (\tau_{\mathrm{e}}, t_{\mathrm{e}}). \tag{3.415}$$

We now put:

$$u^{\rho}_{\vec T,\vec\theta,(0)} = \begin{cases} \chi^{\leftarrow}_{\mathrm{e},\mathcal{B}}(u^{\rho}_{\mathrm{v}_{\leftarrow}(\mathrm{e})} - \mathrm{id}^{\rho,\vec T,\vec\theta}_{\mathrm{e},(0)}) + \chi^{\rightarrow}_{\mathrm{e},\mathcal{A}}(u^{\rho}_{\mathrm{v}_{\rightarrow}(\mathrm{e})} - \mathrm{id}^{\rho,\vec T,\vec\theta}_{\mathrm{e},(0)}) + \mathrm{id}^{\rho,\vec T,\vec\theta}_{\mathrm{e},(0)} & \text{on the e-th neck} \\ u^{\rho}_{\mathrm{v}} & \text{on } K_{\mathrm{v}}. \end{cases} \tag{3.416}$$

Step 0-4: We next define

$$\mathrm{Err}^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)} = \begin{cases} \chi^{\leftarrow}_{\mathrm{e},\mathcal{X}}\overline{\partial}u^{\rho}_{\vec T,\vec\theta,(0)} & \text{on the e-th neck if e is outgoing} \\ \chi^{\rightarrow}_{\mathrm{e},\mathcal{X}}\overline{\partial}u^{\rho}_{\vec T,\vec\theta,(0)} & \text{on the e-th neck if e is incoming} \\ 0 & \text{on } K_{\mathrm{v}}. \end{cases} \tag{3.417}$$

Step 1-1: Let $\mathrm{id}_{\mathrm{v},\mathrm{e}}$ be the identity map from the neck region of $\Sigma^{(2)}_{\mathrm{v}}$ to the neck region of $\Sigma^{(1)}_{\mathrm{v}}$. (It does not coicide with u^{ρ}_{v} there.) We set:

$$\Delta^{\mathrm{v}_{\leftarrow}(\mathrm{e}),\mathrm{e}}_{\vec T,\vec\theta,(0)} = (\mathfrak{s}\Delta T^{\leftarrow}_{\mathrm{e},(0)}, \mathfrak{s}\Delta\theta^{\leftarrow}_{\mathrm{e},(0)}), \quad \Delta^{\mathrm{v}_{\rightarrow}(\mathrm{e}),\mathrm{e}}_{\vec T,\vec\theta,(0)} = (\mathfrak{s}\Delta T^{\rightarrow}_{\mathrm{e},(0)}, \mathfrak{s}\Delta\theta^{\rightarrow}_{\mathrm{e},(0)}). \tag{3.418}$$

(In case $\mathrm{e} \in C^1_{\mathrm{o}}(\mathcal{G}_{\mathfrak{r}})$ we set $\mathfrak{s}\Delta\theta^{\leftarrow}_{\mathrm{e},(0)} = \mathfrak{s}\Delta\theta^{\rightarrow}_{\mathrm{e},(0)} = 0$.) We then define

$$\mathrm{id}^{\vec T,\vec\theta,(0)}_{\mathrm{v},\mathrm{e}} = \mathrm{id}_{\mathrm{v},\mathrm{e}} + \Delta^{\mathrm{v},\mathrm{e}}_{\vec T,\vec\theta,(0)}. \tag{3.419}$$

Now, we put

$$\begin{aligned} &\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}(z) \\ &= \begin{cases} \chi^{\leftarrow}_{\mathrm{e},\mathcal{B}}(\tau_{\mathrm{e}} - T_{\mathrm{e}}, t_{\mathrm{e}})\, u^{\rho}_{\vec T,\vec\theta,(0)}(\tau_{\mathrm{e}}, t_{\mathrm{e}}) + \chi^{\rightarrow}_{\mathrm{e},\mathcal{B}}(\tau_{\mathrm{e}} - T_{\mathrm{e}}, t_{\mathrm{e}})\mathrm{id}^{\vec T,\vec\theta,(0)}_{\mathrm{v},\mathrm{e}} \\ \qquad\qquad \text{if } z = (\tau_{\mathrm{e}}, t_{\mathrm{e}}) \text{ is on the e-th neck that is outgoing} \\ \chi^{\rightarrow}_{\mathrm{e},\mathcal{A}}(\tau_{\mathrm{e}} - T_{\mathrm{e}}, t_{\mathrm{e}}) u^{\rho}_{\vec T,\vec\theta,(0)}(\tau, t) + \chi^{\leftarrow}_{\mathrm{e},\mathcal{A}}(\tau_{\mathrm{e}} - T_{\mathrm{e}}, t_{\mathrm{e}})\mathrm{id}^{\vec T,\vec\theta,(0)}_{\mathrm{v},\mathrm{e}} \\ \qquad\qquad \text{if } z = (\tau_{\mathrm{e}}, t_{\mathrm{e}}) \text{ is on the e-th neck that is incoming} \\ u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}(z) \qquad \text{if } z \in K_{\mathrm{v}}. \end{cases} \end{aligned} \tag{3.420}$$

Definition 3.9. We define $V^{\rho}_{\vec T,\vec\theta,\mathrm{v},(1)}$ for $\mathrm{v} \in C^0(\mathcal{G}_{\mathfrak{p}})$ and real numbers $\Delta T^{\rho}_{\vec T,\vec\theta,(1),\mathrm{v}_{\leftarrow}(\mathrm{e}),\mathrm{e}}$, $\Delta T^{\rho}_{\vec T,\vec\theta,(1),\mathrm{v}_{\rightarrow}(\mathrm{e}),\mathrm{e}}$ for $\mathrm{e} \in C^1(\mathcal{G}_{\mathfrak{p}})$ and $\Delta\theta^{\rho}_{\vec T,\vec\theta,(1),\mathrm{v}_{\leftarrow}(\mathrm{e}),\mathrm{e}}$, $\Delta\theta^{\rho}_{\vec T,\vec\theta,(1),\mathrm{v}_{\rightarrow}(\mathrm{e}),\mathrm{e}}$ for $\mathrm{e} \in C^1_{\mathrm{c}}(\mathcal{G}_{\mathfrak{p}})$ so that the following conditions are satisfied.

$$D_{\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}}\overline{\partial}(V^{\rho}_{\vec T,\vec\theta,\mathrm{v},(1)}) - \mathrm{Err}^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)} = \mathfrak{e}^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)} \in E_{\mathrm{v}}(\hat u^{\rho}_{\mathrm{v},\vec T,\vec\theta,(0)}) \tag{3.421}$$

and

$$\begin{aligned}
&\lim_{\tau_{\rm e}\to\infty}\left(V^{\rho}_{\vec T,\vec\theta,{\rm v}_{\leftarrow}({\rm e}),(1)}(\tau_{\rm e},t_{\rm e}) - \Delta T^{\rho}_{\vec T,\vec\theta,(1),{\rm v}_{\leftarrow}({\rm e}),{\rm e}}\frac{\partial}{\partial\tau_{\rm e}}\right) = 0,\\
&\lim_{\tau_{\rm e}\to-\infty}\left(V^{\rho}_{\vec T,\vec\theta,{\rm v}_{\rightarrow}({\rm e}),(1)}(\tau_{\rm e},t_{\rm e}) - \Delta T^{\rho}_{\vec T,\vec\theta,(1),{\rm v}_{\rightarrow}({\rm e}),{\rm e}}\frac{\partial}{\partial\tau_{\rm e}}\right) = 0,
\end{aligned} \tag{3.422}$$

if ${\rm e}\in C^1_{\rm o}(\mathcal G_{\mathfrak p})$,

$$\begin{aligned}
&\lim_{\tau_{\rm e}\to\infty}\left(V^{\rho}_{\vec T,\vec\theta,{\rm v}_{\leftarrow}({\rm e}),(1)}(\tau_{\rm e},t_{\rm e}) - \Delta T^{\rho}_{\vec T,\vec\theta,(1),{\rm v}_{\leftarrow}({\rm e}),{\rm e}}\frac{\partial}{\partial\tau_{\rm e}} - \Delta\theta^{\rho}_{\vec T,\vec\theta,(1),{\rm v}_{\leftarrow}({\rm e}),{\rm e}}\frac{\partial}{\partial t_{\rm e}}\right) = 0,\\
&\lim_{\tau_{\rm e}\to-\infty}\left(V^{\rho}_{\vec T,\vec\theta,{\rm v}_{\rightarrow}({\rm e}),(1)}(\tau_{\rm e},t_{\rm e}) - \Delta T^{\rho}_{\vec T,\vec\theta,(1),{\rm v}_{\rightarrow}({\rm e}),{\rm e}}\frac{\partial}{\partial\tau_{\rm e}} - \Delta\theta^{\rho}_{\vec T,\vec\theta,(1),{\rm v}_{\rightarrow}({\rm e}),{\rm e}}\frac{\partial}{\partial t_{\rm e}}\right) = 0,
\end{aligned} \tag{3.423}$$

if ${\rm e}\in C^1_{\rm c}(\mathcal G_{\mathfrak p})$.

The unique existence of such objects is a consequence of Lemma 3.5.
We define $\rho_{(1)}$ by (3.411).

Step 1-2:

Definition 3.10. We define $u^{\rho}_{\vec T,\vec\theta,(1)}(z)$ as follows. (Here E is as in (3.404).)

(1) If $z\in K_{\rm v}$ we put

$$u^{\rho}_{\vec T,\vec\theta,(1)}(z) = {\rm E}(u^{\rho}_{\vec T,\vec\theta,(0)}, V^{\rho}_{\vec T,\vec\theta,{\rm v},(1)}(z)). \tag{3.424}$$

(2) If $z=(\tau_{\rm e},t_{\rm e})\in[-5T_{\rm e},5T_{\rm e}]\times[0,1]$ or S^1, we put

$$\begin{aligned}
&u^{\rho}_{\vec T,\vec\theta,(1)}(\tau_{\rm e},t_{\rm e})\\
&= \chi^{\leftarrow}_{{\rm v}_{\leftarrow}({\rm e}),\mathcal B}(\tau_{\rm e},t_{\rm e})(V^{\rho}_{\vec T,\vec\theta,{\rm v}_{\leftarrow}({\rm e}),(1)}(\tau_{\rm e},t_{\rm e}) - (\Delta T^{\leftarrow}_{{\rm e},(1)},\Delta\theta^{\leftarrow}_{{\rm e},(1)}))\\
&+ \chi^{\rightarrow}_{{\rm v}_{\rightarrow}({\rm e}),\mathcal A}(\tau_{\rm e},t_{\rm e})(V^{\rho}_{\vec T,\vec\theta,{\rm v}_{\rightarrow}({\rm e}),(1)}(\tau_{\rm e},t_{\rm e}) - (\Delta T^{\rightarrow}_{{\rm e},(1)},\Delta\theta^{\rightarrow}_{{\rm e},(1)}))\\
&+ u^{\rho}_{\vec T,\vec\theta,(0)}(\tau_{\rm e},t_{\rm e}).
\end{aligned} \tag{3.425}$$

Here we use the coordinate $(\tau_{\rm e}^{(1)},t_{\rm e}^{(1)})$ given in (3.413) and (3.414) for the *target*.

We remark that $\tau_{\rm e}^{(0)}=\tau_{\rm e}^{(1)}-\Delta T^{\leftarrow}_{{\rm e},(1)}$. Therefore, in a neighborhood of $\{-5T_{\rm e}\}\times[0,1]\times S^1$, (3.424) and (3.425) are consistent.

Step 1-3: We recall that $\rho_{{\rm v},(1)}$ is defined by

$$\mathfrak I^{\rho(0)}_{\rm v}(\hat u^{\rho}_{{\rm v},\vec T,\vec\theta,(0)},\rho_{{\rm v},(1)}-\rho_{{\rm v},(0)}) = \mathfrak e^{\rho}_{{\rm v},\vec T,\vec\theta,(1)}. \tag{3.426}$$

(Note $\rho_{{\rm v},(0)}=0$.)

Step 1-4:

Definition 3.11. We put

$$\mathrm{Err}^{\rho}_{{\rm v},\vec T,\vec\theta,(1)} = \begin{cases}
\chi^{\leftarrow}_{{\rm e},\mathcal X}\overline\partial u^{\rho}_{\vec T,\vec\theta,(1)} & \text{on e-th neck if e is outgoing}\\
\chi^{\rightarrow}_{{\rm e},\mathcal X}\overline\partial u^{\rho}_{\vec T,\vec\theta,(1)} & \text{on e-th neck if e is incoming}\\
\overline\partial u^{\rho}_{\vec T,\vec\theta,(1)} & \text{on } K_{\rm v}.
\end{cases} \tag{3.427}$$

We extend them by 0 outside a compact set and will regard them as elements of the function space $L^2_{m,\delta}(\Sigma^{\rho,(2)}_{\rm v};(\hat u^{\rho}_{{\rm v},\vec T,\vec\theta,(1)})^*T\Sigma^{\rho_{(1)},(1)}_{\vec T^{(1)},\vec\theta^{(1)}}\otimes\Lambda^{01})$, where $\hat u^{\rho}_{{\rm v},\vec T,\vec\theta,(1)}$ will be defined in the next step.

We thus come back to Step 2-1 and continue. We obtain the following estimate by induction on κ. We put $R_{\rm e} = 5T_{\rm e}+1$.

Lemma 3.12. *There exist* $T_m, C_{2,m},\dots,C_{8,m},\epsilon_{1,m} > 0$ *and* $0<\mu<1$ *such that the following inequalities hold if* $T_{\rm e} > T_m$ *for all* e. *We put* $T_{\min} = \min\{T_{\rm e} \mid {\rm e} \in C^1(\mathcal G_{\mathfrak p})\}$.

$$\begin{aligned}
\left\|\left((V^{\rho}_{\vec T,\vec\theta,{\rm v},(\kappa)}),(v^{\rho}_{\vec T,\vec\theta,{\rm v},{\rm e},(\kappa)})\right)\right\|_{L^2_{m+1,\delta}(\Sigma^{\rho,(2)}_{\rm v})} &< C_{2,m}\mu^{\kappa-1}e^{-\delta T_{\min}}, &(3.428)\\
\left\|(v^{\rho}_{\vec T,\vec\theta,{\rm v},{\rm e},(\kappa)})\right\| &< C_{3,m}\mu^{\kappa-1}e^{-\delta T_{\min}}, &(3.429)\\
\left\|u^{\rho}_{\vec T,\vec\theta,(\kappa)} - u^{\rho}_{\vec T,\vec\theta,(0)}\right\|_{L^2_{m+1,\delta}((K^{(2)}_{\rm v})^{+\vec R})} &< C_{4,m}e^{-\delta T_{\min}}, &(3.430)\\
\left\|{\rm Err}^{\rho}_{{\rm v},\vec T,\vec\theta,(\kappa)}\right\|_{L^2_{m,\delta}(\Sigma^{\rho,(2)}_{\rm v})} &< C_{5,m}\epsilon_{1,m}\mu^{\kappa}e^{-\delta T_{\min}}, &(3.431)\\
\left\|\mathfrak e^{\rho}_{\vec T,\vec\theta,(\kappa)}\right\|_{L^2_m((K^{(2)}_{\rm v})^{+\vec R})} &< C_{6,m}\mu^{\kappa-1}e^{-\delta T_{\min}}, &(3.432)\\
\left\|\Delta T^{\rho}_{\vec T,\vec\theta,(\kappa),{\rm v},{\rm e}}\right\| &< C_{7,m}\mu^{\kappa-1}e^{-\delta T_{\min}}, &(3.433)\\
\left\|\Delta \theta^{\rho}_{\vec T,\vec\theta,(\kappa),{\rm v},{\rm e}}\right\| &< C_{8,m}\mu^{\kappa-1}e^{-\delta T_{\min}}. &(3.434)
\end{aligned}$$

The proof is the same as the proof of Proposition 2.87 and so is omitted. We note that (3.432) and (3.411) imply

$$\left\|\rho_{(\kappa)}-\rho\right\| < C_{9,m}\mu^{\kappa-1}e^{-\delta T_{\min}}. \tag{3.435}$$

Therefore the limit

$$\lim_{\kappa\to\infty}\rho_{(\kappa)} = \rho'(\rho,\vec T,\vec\theta)$$

exists. (3.433) and (3.434) imply that

$$\lim_{\kappa\to\infty}\mathfrak s\Delta T^{\rho}_{\vec T,\vec\theta,(\kappa),{\rm v},{\rm e}} = \mathfrak s\Delta T^{\rho}_{\vec T,\vec\theta,(\infty),{\rm v},{\rm e}}$$

and

$$\lim_{\kappa\to\infty}\mathfrak s\Delta \theta^{\rho}_{\vec T,\vec\theta,(\kappa),{\rm v},{\rm e}} = \mathfrak s\Delta \theta^{\rho}_{\vec T,\vec\theta,(\infty),{\rm v},{\rm e}}$$

converge. We put

$$\vec T'(\rho,\vec T,\vec\theta) = \vec T + \mathfrak s\Delta\vec T^{\rho}_{\vec T,\vec\theta,(\infty)}, \quad \vec\theta'(\rho,\vec T,\vec\theta) = \vec\theta + \mathfrak s\Delta\vec\theta^{\rho}_{\vec T,\vec\theta,(\infty)}.$$

Then (3.430) imlies that

$$\lim_{\kappa\to\infty} u^{\rho}_{\vec T,\vec\theta,(\kappa)}$$

converges to a map

$$u^{\rho}_{\vec T,\vec\theta,(\infty)} : (\Sigma^{\rho,(2)}_{\vec T,\vec\theta},\partial\Sigma^{\rho,(2)}_{\vec T,\vec\theta}) \to (\Sigma^{\rho'(\rho,\vec T,\vec\theta),(1)}_{\vec T'(\rho,\vec T,\vec\theta),\vec\theta'(\rho,\vec T,\vec\theta)},\partial\Sigma^{\rho'(\rho,\vec T,\vec\theta),(1)}_{\vec T'(\rho,\vec T,\vec\theta),\vec\theta'(\rho,\vec T,\vec\theta)})$$

in L^2_{m+1} topology. (Note the union of $(K^{(2)}_{\rm v})^{+\vec R}$ for various v covers $\Sigma^{\rho_{(\kappa)},(2)}_{\vec T,\vec\theta}$.) The formula (3.431) then implies that $u^{\rho}_{\vec T,\vec\theta,(\infty)}$ is a biholomorphic map.

Therefore, using the notation in Proposition 2.19 we have

$$\overline\Phi_{12}(\rho,\vec T,\vec\theta) = (\rho'(\rho,\vec T,\vec\theta),\vec T'(\rho,\vec T,\vec\theta'),\vec\theta'(\rho,\vec T,\vec\theta)). \tag{3.436}$$

Using the notation in Proposition 2.23 we have

$$\mathfrak v_{(\rho,\vec T,\vec\theta)} = u^{\rho}_{\vec T,\vec\theta,(\infty)}. \tag{3.437}$$

The $T_{\rm e}$ etc. derivative of the objects we costructed enjoy the following estimate.

Lemma 3.13. *There exist $T_m, C_{10,m}, \dots, C_{16,m}, \epsilon_{2,m} > 0$ and $0 < \mu < 1$ such that the following inequalities hold if $T_{\rm e} > T_m$ for all* e.

Let $\mathrm{e}_0 \in C^1(\mathcal{G}_{\mathfrak p})$. Then for each $\vec k_T$, $\vec k_\theta$ we have

$$\left\Vert \nabla_\rho^n \frac{\partial^{\vert\vec k_T\vert}}{\partial T^{\vec k_T}} \frac{\partial^{\vert\vec k_\theta\vert}}{\partial \theta^{\vec k_\theta}} \frac{\partial}{\partial T_{\mathrm{e}_0}} \Big((V^\rho_{\vec T,\vec\theta,\mathrm{v},(\kappa)}), (v^\rho_{\vec T,\vec\theta,\mathrm{v},\mathrm{e},(\kappa)}) \Big) \right\Vert_{L^2_{m+1-\vert\vec k_T\vert-\vert\vec k_\theta\vert-n-1,\delta}(\Sigma_{\rm v}^{\rho,(2)})} < C_{10,m}\mu^{\kappa-1} e^{-\delta T_{\mathrm{e}_0}}, \tag{3.438}$$

$$\left\Vert \nabla_\rho^n \frac{\partial^{\vert\vec k_T\vert}}{\partial T^{\vec k_T}} \frac{\partial^{\vert\vec k_\theta\vert}}{\partial \theta^{\vec k_\theta}} \frac{\partial}{\partial T_{\mathrm{e}_0}} (v^\rho_{\vec T,\vec\theta,\mathrm{v},\mathrm{e},(\kappa)}) \right\Vert < C_{11,m}\mu^{\kappa-1} e^{-\delta T_{\mathrm{e}_0}}, \tag{3.439}$$

$$\left\Vert \nabla_\rho^n \frac{\partial^{\vert\vec k_T\vert}}{\partial T^{\vec k_T}} \frac{\partial^{\vert\vec k_\theta\vert}}{\partial \theta^{\vec k_\theta}} \frac{\partial}{\partial T_{\mathrm{e}_0}} u^\rho_{\vec T,\vec\theta,(\kappa)} \right\Vert_{L^2_{m+1-\vert\vec k_T\vert-\vert\vec k_\theta\vert-n-1,\delta}((K_{\rm v}^{(2)}))+\vec R)} < C_{12,m} e^{-\delta T_{\mathrm{e}_0}}, \tag{3.440}$$

$$\left\Vert \nabla_\rho^n \frac{\partial^{\vert\vec k_T\vert}}{\partial T^{\vec k_T}} \frac{\partial^{\vert\vec k_\theta\vert}}{\partial \theta^{\vec k_\theta}} \frac{\partial}{\partial T_{\mathrm{e}_0}} \mathrm{Err}^\rho_{\mathrm{v},\vec T,\vec\theta,(\kappa)} \right\Vert_{L^2_{m-\vert\vec k_T\vert-\vert\vec k_\theta\vert-n-1,\delta}(\Sigma_{\rm v}^{\rho,(2)})} < C_{13,m}\epsilon_{2,m}\mu^{\kappa} e^{-\delta T_{\mathrm{e}_0}}, \tag{3.441}$$

$$\left\Vert \nabla_\rho^n \frac{\partial^{\vert\vec k_T\vert}}{\partial T^{\vec k_T}} \frac{\partial^{\vert\vec k_\theta\vert}}{\partial \theta^{\vec k_\theta}} \frac{\partial}{\partial T_{\mathrm{e}_0}} \mathfrak{e}^\rho_{\vec T,\vec\theta,(\kappa)} \right\Vert_{L^2_{m-\vert\vec k_T\vert-\vert\vec k_\theta\vert-n-1}(K_{\rm v}^{(2)})} < C_{14,m}\mu^{\kappa-1} e^{-\delta T_{\mathrm{e}_0}}, \tag{3.442}$$

$$\left\Vert \nabla_\rho^n \frac{\partial^{\vert\vec k_T\vert}}{\partial T^{\vec k_T}} \frac{\partial^{\vert\vec k_\theta\vert}}{\partial \theta^{\vec k_\theta}} \frac{\partial}{\partial T_{\mathrm{e}_0}} \Delta T^\rho_{\vec T,\vec\theta,(\kappa),\mathrm{v},\mathrm{e}} \right\Vert < C_{15,m}\mu^{\kappa-1} e^{-\delta T_{\mathrm{e}_0}}, \tag{3.443}$$

$$\left\Vert \nabla_\rho^n \frac{\partial^{\vert\vec k_T\vert}}{\partial T^{\vec k_T}} \frac{\partial^{\vert\vec k_\theta\vert}}{\partial \theta^{\vec k_\theta}} \frac{\partial}{\partial T_{\mathrm{e}_0}} \Delta \theta^\rho_{\vec T,\vec\theta,(\kappa),\mathrm{v},\mathrm{e}} \right\Vert < C_{16,m}\mu^{\kappa-1} e^{-\delta T_{\mathrm{e}_0}} \tag{3.444}$$

for $\vert\vec k_T\vert + \vert\vec k_\theta\vert + n < m - 11$.

Let $\mathrm{e}_0 \in C^1_{\rm c}(\mathcal{G}_{\mathfrak p})$. Then the same inequalities as above hold if we replace $\frac{\partial}{\partial T_{\mathrm{e}_0}}$ by $\frac{\partial}{\partial \theta_{\mathrm{e}_0}}$.

The proof is mostly the same as that of Proposition 2.88. The difference is the following point only. We remark that in (3.438), (3.439), (3.440), (3.441) the norm is $L^2_{m+1-\vert\vec k_T\vert-\vert\vec k_\theta\vert-n-1,\delta}$ norm. On the other hand, in (2.271), (2.272), (2.273), (2.274), the norm was $L^2_{m+1-\vert\vec k_T\vert-\vert\vec k_\theta\vert-1,\delta}$ norm. The reason is as follows. We remark that in our case

$$T_{\rm e}^{(\kappa)} = T_{\rm e} - \frac{1}{10}\sum_{a=0}^{\kappa} \Delta T^\rho_{\vec T,\vec\theta,(a),\mathrm{v}_\leftarrow(\mathrm{e}),\mathrm{e}} + \frac{1}{10}\sum_{a=0}^{\kappa} \Delta T^\rho_{\vec T,\vec\theta,(a),\mathrm{v}_\rightarrow(\mathrm{e}),\mathrm{e}}$$

is ρ dependent. When we study ρ derivative in the inductive steps, we need to take ρ derivative of

$$\hat u^\rho_{\mathrm{v}',\vec T,\vec\theta,(\kappa)}(\tau'_{\rm e} - 10T_{\rm e}^{(\kappa)}, t'_{\rm e} + \theta_{\rm e}^{(\kappa)})$$

etc.. Then there will be a term including $\tau''_{\rm e}$ or $t''_{\rm e}$ derivative of $\hat u^\rho_{\mathrm{v}',\vec T,\vec\theta,(\kappa)}$.

Except this point the proof of Lemma 3.13 is the same as the proof of Proposition 2.88 and so is omitted.

Proof of Proposition 2.19. We note that (2.275) and (3.411) imply

$$\left\|\nabla_\rho^n \frac{\partial^{|\vec k_T|}}{\partial T^{\vec k_T}} \frac{\partial^{|\vec k_\theta|}}{\partial \theta^{\vec k_\theta}} \frac{\partial}{\partial T_{\mathrm{e}_0}}(\rho_{(\kappa)} - \rho)\right\| < C_{17,m}\mu^{\kappa-1}e^{-\delta T_{\mathrm{e}}} \tag{3.445}$$

and the same formula wtih $\frac{\partial}{\partial T_{\mathrm{e}_0}}$ replaced by $\frac{\partial}{\partial \theta_{\mathrm{e}_0}}$ if $\mathrm{e}_0 \in C^1_{\mathrm{c}}(\mathcal{G}_{\mathfrak p})$. (3.436), (3.445), (3.443) and (3.444) imply (2.169). ☐

Proof of Proposition 2.23. This is an immediate consequence of (3.437) and (3.440). ☐

Proof of Lemma 2.26. This is a parametrized version and the proof is the same as above. ☐

3.2. **From C^m structure to C^∞ structure.** In this subsection we will prove that the Kuranishi structure of C^m-class, which we obtained in Section 2, is actually of C^∞-class.

We consider the embedding $\mathfrak F^{(1)}$ (see the formula (2.336)) which we constructed in the proof of Lemma 2.133. Here we fix m.

Lemma 3.14. *The image of $\mathfrak F^{(1)}$ is a C^∞ submanifold.*

Proof. We first note several obvious facts. Let $\mathfrak M$ be a Banach manifold and $X \subset \mathfrak M$ be a subset. Then the statement that X is a $C^{m'}$-submanifold of finite dimension is well-defined. And the $C^{m'}$-structure of X as a submanifold is unique if exists. Here m' is one of $0, 1, \dots, \infty$. Moreover X is a C^∞-submanifold if and only if for each $p \in X$ and m' there exists a neighborhood U of p such that $U \cap X$ is a submanifold of $C^{m'}$-class.

Now we prove the lemma. Let $\mathfrak q$ be in the image of $\mathfrak F^{(1)}$ and take any m'. Let $\mathfrak w_{\mathfrak p}$ be the stabilization data at $\mathfrak p$ that we used to define $\mathfrak F^{(1)}$. We take the stabilization data $\mathfrak w_{\mathfrak q}$ on $\mathfrak q$ that is induced by $\mathfrak w_{\mathfrak p}$. We define Glue at $\mathfrak q$ using the stabilization data $\mathfrak w_{\mathfrak q}$. Then, as in the proof of Lemma 2.153, we obtain

$$\begin{aligned}\mathfrak F^{(2)} &: \hat V(\mathfrak q, \mathfrak w_{\mathfrak q}; (\mathfrak o', \mathcal T'; \mathfrak A)) \\ &\to \prod_{\mathrm{v}\in C^0(\mathcal G_{\mathfrak q})} C^{m'}((K_{\mathrm v}^{+\vec R}, K_{\mathrm v}^{+\vec R} \cap \partial\Sigma_{\mathfrak q,\mathrm v}), (X, L)) \\ &\times \prod_{\mathrm{v}\in C^0(\mathcal G_{\mathfrak p})} \mathfrak V((\mathfrak x_{\mathfrak p} \cup \vec w_{\mathfrak q})_{\mathrm v}) \times ((\vec{\mathcal T}', \infty] \times (\vec{\mathcal T}', \infty] \times \vec S^1).\end{aligned} \tag{3.446}$$

Let us denote the target of $\mathfrak F^{(j)}$ by $\mathfrak X(j)$. The map $\mathfrak F^{(2)}$ is a $C^{m'}$-embedding. We define $\pi_{m,m'} : \mathfrak X(2) \to \mathfrak X(1)$ so that it is the identity map for the second factor and the inclusion map

$$C^{m'}((K_{\mathrm v}^{+\vec R}, K_{\mathrm v}^{+\vec R} \cap \partial\Sigma_{\mathfrak q,\mathrm v}), (X, L)) \to C^{m}((K_{\mathrm v}^{+\vec R}, K_{\mathrm v}^{+\vec R} \cap \partial\Sigma_{\mathfrak q,\mathrm v}), (X, L))$$

for the first factor. This map is of C^∞ class. We note that

$$\pi_{m,m'} \circ \mathfrak F^{(2)} = \mathfrak F^{(1)} \circ \varphi_{12},$$

since we use the induced stabilization data for $\mathfrak q$. We already proved that φ_{12} is a diffeomorphism of C^m-class to an open subset. Moreover $\mathfrak F^{(2)}$ is an embedding of

$C^{m'}$-class. Therefore a neighborhood of $\mathfrak{q}$ of the image of $\mathfrak{F}^{(1)}$ is a submanifold of $C^{m'}$-class. The proof of Lemma 3.14 is complete. □

We define a C^∞ structure of the Kuranishi neighborhood so that $\mathfrak{F}^{(1)}$ is a diffeomorphism to its image.

Lemma 3.15. *The coordinate change φ_{12} we defined is a diffeomorphism of C^∞-class.*

Proof. We prove the case of φ_{12} in Lemma 2.133. We consider the following commutative diagram.

$$\begin{array}{ccccc}
\hat V(\mathfrak{p},\mathfrak{w}^{(1)}_{\mathfrak{p}};(\mathfrak{o}^{(1)},\mathcal{T}^{(1)});\mathfrak{A})_{\epsilon_0,\vec{\mathcal{T}}^{(1)}} & \xrightarrow{\mathfrak{F}^{(1)}_{m'}} & \mathfrak{X}^{(1)}_{m'} & \xrightarrow[\pi_{m,m'}]{} & \mathfrak{X}^{(1)}_{m} \\
\uparrow \subset & & \uparrow \mathfrak{H}_{12} & & \uparrow \mathfrak{H}_{12} \\
\hat V(\mathfrak{p},\mathfrak{w}^{(2)}_{\mathfrak{p}};(\mathfrak{o}^{(2)},\mathcal{T}^{(2)});\mathfrak{A})_{\epsilon_0,\vec{\mathcal{T}}^{(2)}} & \xrightarrow{\mathfrak{F}^{(2)}_{2m'}} & \mathfrak{X}^{(2)}_{2m'} & \xrightarrow[\pi_{2m,2m'}]{} & \mathfrak{X}^{(2)}_{2m}
\end{array} \tag{3.447}$$

Here

$$\begin{aligned}
\mathfrak{X}^{(2)}_{2m'} := & \prod_{\mathrm{v}\in C^0(\mathcal{G}_{\mathfrak{p}})} C^{2m'}((K^{+\vec R}_{\mathrm{v}}, K^{+\vec R}_{\mathrm{v}} \cap \partial\Sigma_{\mathfrak{p},\mathrm{v}}),(X,L)) \\
& \times \prod_{\mathrm{v}\in C^0(\mathcal{G}_{\mathfrak{p}})} \mathfrak{V}((\mathfrak{x}_{\mathfrak{p}}\cup \vec w_{\mathfrak{p}})_{\mathrm{v}}) \times ((\vec{\mathcal{T}}^{(2)},\infty] \times (\vec{\mathcal{T}}^{(2)},\infty] \times \vec S^1)
\end{aligned} \tag{3.448}$$

is the space appearing in (2.336), (2.337) and the map $\mathfrak{F}^{(2)}_{2m'}$ is defined as in (2.363). (We include $2m'$ in the notation to specify the function space we use.) The space $\mathfrak{X}^{(1)}_{m'}$ and the map $\mathfrak{F}^{(1)}_{m'}$ are similarly defined. The two maps $\mathfrak{H}_{12}$ in the vertical arrow are given by

$$\mathfrak{H}_{12}(u,(\rho,\vec T,\vec\theta))) = (u\circ \mathfrak{v}_{(\rho,\vec T,\vec\theta)}, \overline{\Phi}_{12}(\rho,\vec T,\vec\theta)).$$

The maps in the horizontal lines are of C^∞ class by definition. The map $\mathfrak{H}_{12}$ in the second vertical line is of $C^{m'}$ class by Sublemma 2.134. The map $\mathfrak{H}_{12}$ in the third vertical line is one used in the proof of Lemma 2.133. Therefore φ_{12} is of $C^{m'}$-class at $\mathfrak{p}$. Note we can start at arbitrary point $\mathfrak{q}$ in the image of $\mathfrak{F}^{(2)}$ and prove that φ_{12} is of $C^{m'}$-class for any m' at any point $\mathfrak{q}$, by using the proof of Lemma 3.14. This implies the lemma in the case of φ_{12} in Lemma 2.133.

In the other cases, the proof of the smoothness of the coordinate change is similar. □

We have thus proved that the Kuranishi structure we obtained is of C^∞-class.

3.3. **Proof of Lemma 2.56.**

Proof of Lemma 2.56.

Sublemma 3.16. *There exists a finite dimensional smooth and comapct family $\mathfrak{M}$ of pairs (Σ,u') such that each element of $\mathcal{M}_{k+1,\ell}(\beta)$ appears as its member.*

Proof. Run the gluing argument of Section 2.5 at each point $\mathfrak{p}\in\mathcal{M}_{k+1,\ell}(\beta)$ using an obstruction bundle data given at that point. We then obtain a neighborhood of each $\mathfrak{p}$ in a finite dimensional manifold. We can take finitely many of them to cover $\mathcal{M}_{k+1,\ell}(\beta)$ by compactness. □

We take a finite number of $\mathfrak{p}_c$ so that (2.203) is satisfied. For each c and $N \in \mathbb{Z}_+$ we take $E_{c,N} \subset \bigoplus_{\rm v} \Gamma_0(\operatorname{Int} K_{\rm v}^{\rm obst}; u_{\mathfrak{p}_c}^* TX \otimes \Lambda^{01})$ that is isomorphic to the N copies of E_c as the $\Gamma_{\mathfrak{p}_c}$ vector space and $E_c \subset E_{c,N}$.

We consider the space of $\Gamma_{\mathfrak{p}_c}$-equivariant embeddings $\sigma_c : E_c \to E_{c,N}$ in the neighborhood of the original embedding. Each σ_c determines a perturbed E_c which we write $E_c^{\sigma_c}$.

The condition that $E_c^{\sigma_c}(\mathfrak{q}) \cap E_{c'}^{\sigma_{c'}}(\mathfrak{q}) \ne \{0\}$ for some $\mathfrak{q} \in \mathfrak{M}$ such that $\mathfrak{q} \cup \vec{w}_c'$ is $\epsilon_{\mathfrak{p}_c}$ close to $\mathfrak{p}_c$ defines a subspace of the set of $(\sigma_c)_{c\in\mathfrak{C}}$'s whose codimension depends on the number of c's, the dimension of E_c and the dimension of $\mathfrak{M}$ and N. By taking N huge, we may assume that such $(\sigma_c)_{c\in\mathfrak{C}}$ is nowhere dense. Namely the conclusion holds after perturbing E_c by arbitrary small amount in $E_{c,N}$. □

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