LECTURES ON FLOER THEORY AND SPECTRAL INVARIANTS OF HAMILTONIAN FLOWS

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Abstract. The main purpose of this lecture is to provide a coherent explanation of the chain level Floer theory and its applications to the study of geometry of the Hamiltonian diffeomorphism group of closed symplectic manifolds. In particular, we explain the author's recent construction of spectral invariants of Hamiltonian paths and an invariant norm of the Hamiltonian diffeomorphism group on non-exact symplectic manifolds.

1. Introduction

The main purpose of this lecture note is to provide a coherent explanation on the chain level Floer theory and its applications to the study of geometry of the Hamiltonian diffeomorphism group of closed symplectic manifolds (M, ω), which has been systematically developed in a series of papers [Oh5]-[Oh11]. This study is based on a construction of certain invariants, which we call *spectral invariants*, of one-periodic Hamiltonian functions $H : S^1 \times M \to \mathbb{R}$ satisfying the normalization condition

$$\int_M H_t \, d\mu = 0$$

where $d\mu$ is the Liouville measure of (M, ω). We denote the set of such functions by

$$\mathcal{H}_m := C_m^\infty(S^1 \times M, \mathbb{R})$$

where "m" stands for "mean zero". The construction of these invariants is through a Floer theoretic version of the mini-max theory of the associated perturbed action functional $\mathcal{A}_H$

$$\mathcal{A}_H(\gamma, w) = -\int w^*\omega - \int_0^1 H(t, \gamma(t)) \, dt$$

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defined for the pairs (γ, w) of smooth maps $\gamma : S^1 = \mathbb{R}/\mathbb{Z} \times M$ and $w : D^2 \to M$ satisfying

$$w|_{\partial D^2} = \gamma.$$

These invariants form a function

$$\rho : \mathcal{H}_m \times QH^*(M) \to \mathbb{R}$$

whose value $\rho(H; a)$ is the mini-max value of the action functional over the *Novikov Floer cycles* representing the Floer homology class $a^\flat$ which is 'dual' to the quantum cohomology class a.

In the classical mini-max theory for the *indefinite* functionals as in [Ra], [BnR], there was implicitly used the notion of 'semi-infinite cycles' to carry out the mini-max procedure. There are two essential ingredients needed to prove existence of actual critical values out of the mini-max values: one is the finiteness of the mini-max value, or the *linking property* of the (semi-infinite) cycles associated to the class a and the other is to prove that the corresponding mini-max value is indeed a critical value of the action functional. *When the global gradient flow of the action functional exists* as in the classical critical point theory [BnR] this point is closely related to the well-known Palais-Smale condition and the deformation lemma which are essential ingredients needed to prove the criticality of the mini-max value. Partly because we do not have the global flow, we need to geometrize all these classical mini-max procedures. It turns out that the Floer homology theory in the chain level is the right framework for this purpose.

The idea of construction of spectral invariants is originated from the author's Floer theoretic construction [Oh3] of Viterbo's invariants [V] of Lagrangian submanifolds in the cotangent bundle, and is also based on the framework of the mini-max theory over natural semi-infinite cycles on the covering space $\widetilde{\mathcal{L}}_0(M)$. We call the corresponding semi-infinite cycles the *Novikov Floer cycles.* In this construction, the 'finiteness' condition in the definitions of the Novikov ring and the Novikov Floer cycles is fully exploited in the proofs of various existence results of pseudo-holomorphic curves.

Now the organization of the content of the paper is in order. In section 2, we view the free loop space of the symplectic manifold as an infinite dimensional (weakly) symplectic manifold with the natural symplectic action of S^1 induced by the domain rotation. Its lifted action to the universal covering space then has the associated moment map which is nothing but the *unperturbed* action functional. After then, we compute the first variation and the gradient equation of the action functional with respect the L^2-metric induced by a one-periodic family $J = \{J_t\}_{0 \leq t \leq 1}$ of compatible almost complex structures. We also look at the *non-autonomous version* of the gradient equation associated to each two parameter family

$$j : [0,1] \times S^1 \to \mathcal{J}_\omega$$

and a cut-off function $\rho : \mathbb{R} \to [0,1]$.

In section 3, we review construction of the Floer complex and of the various basic operators in the chain level Floer theory. While these constructions are standard by now (see [Fl2], [SZ]), we add some novelty in our exposition which is needed in our construction of the spectral invariants and their applications.

In section 4, we carefully study the energy estimates and the change of action levels under the Floer trajectories, and explain its relation to the $L^{(1,\infty)}$ norm of Hamiltonian functions which arise naturally in this study of energy estimates.

In section 5, we give the definition of $\rho(H;a)$ and prove their basic properties, especially the well-definedness and the finiteness of its value.

In section 6, we discuss the so called, *spectrality*, i.e., whether the mini-max value $\rho(H;a)$ is indeed a critical value of $\mathcal{A}_H$. We give the proof, coming from [Oh8], of the spectrality for an arbitrary smooth H *on rational symplectic manifolds.* For the non-rational (M,ω), we just state the theorem from [Oh11] that the spectrality holds for *nondegenerate* Hamiltonian function H, whose proof is referred to [Oh11].

In section 7, we follow [Oh4], [Sc] and [En1] and explain the pants product in Floer homology and prove the triangle inequality

$$\rho(H\#K;a\cdot b)\leq \rho(H;a)+\rho(K;b).$$

In section 8, we explain our construction of the *spectral norm*, denoted by $\gamma : Ham(M,\omega)\to \mathbb{R}_+$, which was carried out in [Oh9]. As illustrated by Ostrover [Os], this norm is not the same as but smaller than the Hofer norm. Along the way, we also introduce certain geometric invariants of the pair (H,J) and also their family versions. These geometric invariants play crucial roles in our proof of nondegeneracy of the spectral norm γ. We call these invariants the *ϵ-regularity type invariants* in general because their non-triviality strongly relies on the so called the *ϵ-regularity theorem*, which was first introduced by Sacks and Uhlenbeck [SU] in the context of harmonic maps.

In section 9, we explain a simple criterion for the length minimizing property of the Hamiltonian paths in terms of the spectral invariant $\rho(H;1)$ stated in [Oh7]. An analogous criterion had been used by Hofer [Ho2] and by Bialy-Polterovich [BP] in $\mathbb{C}^n$. We illustrate its application to the study of length minimizing property of some autonomous Hamiltonians. Besides this criterion, this application is based on a construction of an optimal Floer cycle as done in [Po3], [Oh5] and [KL], especially the one used by Kerman and Lalonde in [KL]. We refer readers to section 9 for more detailed explanations.

In Appendix, starting from the definition of the Conley-Zehnder index $\mu_H([z,w])$ given in [SZ], [HS], *but using a different convention of the canonical symplectic form on $\mathbb{C}^n\cong\mathbb{R}^{2n}$ from* [SZ], [HS], we provide complete details of the proof of the following index formula in our convention :

$$\mu_H([z,w'])=\mu_H([z,w])-2c_1([w'\#\overline{w}]) \tag{1.1}$$

or

$$\mu_H([z,A\#w])=\mu_H([z,w])-2c_1(A). \tag{1.2}$$

There are many different conventions used in the literature of symplectic geometry concerning the definitions of Hamiltonian vector fields, the canonical symplectic form on the cotangent bundle, the action functional and others. And partly because there is no literature which provides detailed explanations of the index formula in any fixed convention, this formula has been a source of confusion at least for the present author, especially concerning the sign in front of the first Chern number term in the formula. We set the record straight here once and for all by announcing that *the sign is '-' in our convention* which has been used by the author here and [Oh5]-[Oh11]. And we also emphasize that the form of this index formula has nothing to do with whether one use the homological or the cohomological version of the Floer homology as long as they fix the definition of the Conley-Zehnder index of the symplectic path in $Sp(n)$ as in [SZ].

We like to refer readers to [En2], [EnP] for other interesting applications of the spectral invariants in which a construction of *quasi-morphisms* on $Ham(M,\omega)$ is given for some class of symplectic manifolds. We refer to Polterovich's lecture [Po4] in this volume for a survey of these works.

To get the main stream of ideas transparent without getting bogged down with technicalities related with transversality question of various moduli spaces, we assume that (M,ω) is strongly semi-positive in the sense of [Se], [En1]: A closed symplectic manifold is called *strongly semi-positive* if there is no spherical homology class $A \in \pi_2(M)$ such that

$$\omega(A) > 0, \quad 2 - n \leq c_1(A) < 0.$$

Under this condition, the transversality problem concerning various moduli spaces of pseudo-holomorphic curves is standard. We will not mention this generic transversality question at all in the main body of the paper unless it is absolutely necessary. In section 10, we will briefly explain how this general framework could be incorporated in our proofs in the context of Kuranishi structure [FOn].

In this lecture, we will be very brief in explaining the Fredholm theory and compactness properties of the Floer moduli space the details of which are by now well-known and standard in the literature, at least for the semi-positive cases. We refer readers to the articles [HS], [SZ] for such details in the semi-positive case. Instead, we will put more emphasis on the calculations involved in the analysis of the filtration changes under the chain map, and on explaining the chain level arguments used in our Floer mini-max theory to overcome the difficulties arising from the non-exactness and the non-rationality of general symplectic manifolds. These chain level arguments also require one to closely examine all the basic constructions in Floer theory, especially in the choice of compatible almost complex structures and its relation to the given Hamiltonian functions. These materials have recently appeared in the series of our papers [Oh5]-[Oh11] and are less known in the standard Floer theory. We believe that these details deserve more attention and scrutiny in the future.

Another exposition of spectral invariants, based on the approach using the so called "PSS-isomorphism", has been given by McDuff and Salamon [MSa] for the rational case. However, to make this approach well-founded, it remains to fill some nonstandard analytic details in the proof of isomorphism property of the PSS-map which is used in the various construction carried out in [PSS], [MSa].

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Convention and Notations.

- The Hamiltonian vector field X_f associated to a function f on (M,ω) is defined by $df = \omega(X_f, \cdot)$.
- The multiplication $F\#G$ and the inverse $\overline{G}$ on the set of time periodic Hamiltonians $C^\infty(M \times S^1)$ are defined by

$$\begin{aligned} F\#G(t,x) &= F(t,x) + G(t,(\phi_F^t)^{-1}(x)) \\ \overline{G}(t,x) &= -G(t,\phi_G^t(x)). \end{aligned}$$

- $\mathcal{L}(M) = \mathrm{Map}(S^1, M)$

- $\mathcal{L}_0(M)$= the connected component of $\mathcal{L}_0(M)$ consisting of contractible loops.
- $\widetilde{\mathcal{L}}_0(M)$ = the universal covering space or the Γ-covering space of $\mathcal{L}_0(M)$ depending on the circumstances.
- $\mathcal{H}_m = \mathcal{H}_m(M) = C^\infty(S^1 \times M, \mathbb{R})$
- $\mathcal{J}_\omega = \mathcal{J}_\omega(M)$ = the set of compatible almost complex structures
- $j_\omega = C^\infty(S^1, \mathcal{J}_\omega)$
- $\mathcal{P}(\mathcal{H}_m) = C^\infty([0,1], \mathcal{H}_m)$
- $\mathcal{P}(j_\omega) = C^\infty([0,1], j_\omega)$

2. The free loop space and the action functional

2.1. The free loop space and the S^1-action in general. Let M be a general smooth manifold, not necessarily symplectic. We denote by $\mathcal{L}(M) := Map(S^1, M)$ be the free loop space, i.e., the set of smooth maps

$$\gamma : S^1 = \mathbb{R}/\mathbb{Z} \to M.$$

We emphasize the loops have a marked point $0 \in \mathbb{R}/\mathbb{Z}$ and often parameterize them by the unit interval $[0,1]$ with the periodic boundary condition $\gamma(0) = \gamma(1)$. $\mathcal{L}(M)$ has the distinguished connected component of contractible loops, which we denote by $\mathcal{L}_0(M)$. The universal covering space of $\mathcal{L}_0(M)$, denoted by $\widetilde{\mathcal{L}}_0(M)$, can be expressed by

$$\{[\gamma, w] \mid \gamma \in \mathcal{L}_0(M) \text{ and } w : D^2 \to M \text{ satisfying } \partial w =: w|_{\partial D^2} = \gamma\}$$

where $[\gamma, w]$ is the set of homotopy classes of w relative to $\partial w = \gamma$. Here we identify ∂D^2 with S^1. We call such w a bounding disc of γ. The deck transformation of the universal covering space $\widetilde{\mathcal{L}}_0(M) \to \mathcal{L}_0(M)$ is realized by the operation of "gluing a sphere"

$$(\gamma, w) \mapsto (\gamma, w \# u) \tag{2.1}$$

for a (and so any) sphere $u : S^2 \to M$ representing the given class $A \in \pi_2(M) \cong \pi_1(\mathcal{L}_0(M))$.

There is a natural circle action on $\mathcal{L}(M)$ induced by the time translation

$$\gamma \mapsto \gamma \circ R_\varphi = \gamma(\cdot + \varphi) \tag{2.2}$$

where $R_\varphi : S^1 \to S^1$ is the map given by

$$R_\varphi(t) = t + \varphi, \quad \varphi \in S^1.$$

The infinitesimal generator of this action is the vector field $\mathbb{X}$ on $\mathcal{L}(M)$ provided by

$$\mathbb{X}(\gamma) = \dot\gamma.$$

The fixed point set of this S^1 action is the set of constant loops

$$M \hookrightarrow \mathcal{L}_0(M).$$

This action lifts to an action on the set of pairs

$$(\gamma, w) \mapsto (\gamma \circ R_\varphi, w \circ R_\varphi) \tag{2.3}$$

induced by the complex multiplication, which we again denote by

$$R_\varphi : z \in D^2 \subset \mathbb{C} \mapsto e^{2\pi i \varphi} z.$$

The fixed point set of the induced S^1 action on $\widetilde{\mathcal{L}}_0(M)$ forms a principal bundle of $\pi_2(M)$ over M with $\pi_2(M)$-action induced by (2.3): Obviously S^1 acts trivially on the constant loops $\gamma \equiv x \in M$. On the other hand, it acts trivially on the homotopy class of the pairs (x, w) because the pair (x, w) and $(x \circ R_\varphi, w \circ R_\varphi)$ are homotopic as R_φ is homotopic to the identity.

2.2. **The free loop space of symplectic manifolds.** Now we specialize our discussion on the loop space to the case of a symplectic manifold (M, ω). In this case, $\mathcal{L}(M)$ carries a canonical (weak) symplectic form defined by

$$\Omega(\xi_1, \xi_2) := \int_0^1 \omega(\xi_1(t), \xi_2(t))\, dt : \tag{2.4}$$

the closedness of Ω is a consequence of the closedness of ω *together with the fact that S^1 has no boundary*, and the (weak) nondegeneracy follows from the nondegeneracy of ω. The S^1 action (2.2) is symplectic, i.e., preserves Ω $L_{\mathbb{X}}\Omega = 0$. Obviously Ω induces a symplectic form on the covering space $\widetilde{\mathcal{L}}_0(M)$ by the pull-back under the projection $\widetilde{\mathcal{L}}_0(M) \to \mathcal{L}_0(M)$, which we denote by $\widetilde{\Omega}$.

Lemma 2.1. *The form $\mathbb{X} \rfloor \Omega$ is a closed one form on $\mathcal{L}(M)$.*

Proof. Since the closedness is local, it is enough to construct a function $\mathcal{A} = \mathcal{A}_0$ defined in a neighborhood of any given loop γ_0 that satisfies

$$d\mathcal{A} = \mathbb{X} \rfloor \Omega. \tag{2.5}$$

Note that for any path γ sufficiently C^∞ close to a given γ_0, we have a distinguished path to γ defined by

$$u_{\gamma_0\gamma} : s \in [0,1] \mapsto \exp_{\gamma_0}(sE(\gamma_0, \gamma)), \quad E(\gamma_0, \gamma) := (\exp_{\gamma_0})^{-1}(\gamma)$$

which defines a distinguished homotopy class of paths $[u_{\gamma_0\gamma}]$ with fixed ends,

$$u(0) = \gamma_0,\ u(1) = \gamma.$$

We ambiguously denote the associated map

$$u_{\gamma_0\gamma} : [0,1] \times S^1 \to M$$

also by $u_{\gamma_0\gamma}$. We then define the locally defined function $\mathcal{A}$ by the formula

$$\mathcal{A}(\gamma; \gamma_0) = 0 - \int u_{\gamma_0\gamma}^* \omega \tag{2.6}$$

where '0' should be regarded as the value $\mathcal{A}(\gamma_0; \gamma_0)$, which can be chosen arbitrarily. Now we verify (2.5). We first note that for any loop γ nearby γ_0 and for a tangent vector $\xi = \frac{d}{ds}\Big|_{s=0} u \in T_\gamma \mathcal{L}(M)$, we have

$$d\mathcal{A}(\gamma)(\xi) = \frac{d}{ds}\Big|_{s=0} \Big(- \int u_{\gamma\gamma_s}^* \omega \Big) \tag{2.7}$$

But we derive, after a change of variables,

$$\begin{aligned} -\frac{d}{ds}\Big|_{s=0} \int u_{\gamma\gamma_s}^* \omega &= -\frac{d}{du}\Big|_{u=0} \int_0^s \int_0^1 \omega\Big(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}\Big)\, dt\, ds \\ &= \int_0^1 -\omega(\xi(t), \dot\gamma(t))\, dt = \mathbb{X}(\gamma) \rfloor \Omega. \end{aligned}$$

This combined with (2.7) finishes the proof of (2.5) and hence the lemma. $\square$

Remark 2.1. In the point of view of de Rham theory of the loop space [Ch], [GJP], a symplectic form ω on M induces a canonical cohomology class of degree one induced by the closed one form $\mathbb{X}\rfloor\Omega$, which is obtained by *the iterated integrals*. This one form is *not exact* in general. Exactness of this one form is precisely the so called the *weakly exactness* of the symplectic form ω. The Floer homology can be considered as a version of the Novikov Morse homology of this closed one form, or the Morse homology of the circle valued functions on $\mathcal{L}(M)$.

If we restrict this closed one-form to $\mathcal{L}_0(M)$ and consider its lifting to the universal covering space $\widetilde{\mathcal{L}}_0(M)$, the formula (2.6) has a *global* lifting induced by the function of the pairs (γ, w), again denoted by $\mathcal{A} = \mathcal{A}_0$

$$\mathcal{A}_0(\gamma, w) = -\int w^*\omega$$

considering w as a path from a constant path $w(0)$ to $\gamma = \partial w$. We call this the *unperturbed action functional.* It satisfies

$$d\mathcal{A}_0 = \mathbb{X}\rfloor\widetilde{\Omega} \tag{2.8}$$

on $\widetilde{\mathcal{L}}_0(M)$. In other words, the S^1-action on $\widetilde{\mathcal{L}}_0(M)$ is Hamiltonian and its associated moment map is nothing but the function $\mathcal{A}_0 : \widetilde{\mathcal{L}}_0(M) \to \mathbb{R}$ (see [W] for more detailed discussions).

2.3. **The Novikov covering.** Following [Fl2], [HS], we now introduce a notion of the *Novikov covering space* of $\mathcal{L}_0(M)$.

Definition 2.2. Let (γ, w) be a pair of $\gamma \in \mathcal{L}_0(M)$ and w be a disc bounding γ. We say that (γ, w) is Γ-*equivalent* to (γ, w') if and only if

$$\omega([w'\#\overline{w}]) = 0 \quad \text{and} \quad c_1([w'\#\overline{w}]) = 0$$

where $\overline{w}$ is the map with opposite orientation on the domain and $w'\#\overline{w}$ is the obvious glued sphere. Here Γ stands for the group

$$\Gamma = \frac{\pi_2(M)}{\ker\,(\omega|_{\pi_2(M)}) \cap \ker\,(c_1|_{\pi_2(M)})}.$$

We denote

$$\Gamma_\omega := \omega(\Gamma) = \omega(\pi_2(M)) \subset \mathbb{R}$$

and call it the (spherical) *period group* of (M, ω).

Definition 2.3. We call (M, ω) *rational* if $\Gamma_\omega \subset \mathbb{R}$ is a discrete subgroup, and *irrational* otherwise.

Example 2.4. The product $S^2(r_1) \times S^2(r_2)$ with the product symplectic form $\omega_1 \oplus \omega_2$ is rational if and only if the ratio r_2^2/r_1^2 is rational.

Remark 2.5. We note that for an irrational (M, ω), the period group is a countable dense subset of $\mathbb{R}$. In general, the dynamical behavior of the Hamiltonian flow on an irrational symplectic manifold is expected to become much more complex than on a rational symplectic manifold. The period group Γ_ω is the simplest indicator of this distinct dynamical behavior.

From now on, we exclusively denote by $[\gamma, w]$ the Γ-equivalence class of (γ, w) and by $\widetilde{\mathcal{L}}_0(M)$ the set of Γ-equivalence classes. We denote by $\pi : \widetilde{\mathcal{L}}_0(M) \to \mathcal{L}_0(M)$ the canonical projection. We call $\widetilde{\mathcal{L}}_0(M)$ the Γ-*covering space* of $\mathcal{L}_0(M)$. We denote by A or q^A the image of $A \in \pi_2(M)$ under the projection $\pi_2(M) \to \Gamma$. There are two natural invariants associated to A: the *valuation* $v(A)$

$$v : \Gamma \to \mathbb{R}; \quad v(A) = \omega(A) \tag{2.9}$$

and the *degree* $d(A)$

$$d : \Gamma \to \mathbb{Z}; \quad d(A) = c_1(A). \tag{2.10}$$

In general these two invariants are independent and so q^A is a formal parameter depending on two variables. In that sense, we may also denote

$$q^A = T^{\omega(A)} e^{c_1(A)}$$

with two different formal parameters T and e.

The (unperturbed) action functional $\mathcal{A}_0$ defined above obviously projects down to the Γ-covering space by the same formula

$$\mathcal{A}_0([\gamma, w]) = -\int w^*\omega$$

as in subsection 2.2. This functional provides a natural increasing filtration on the space $\widetilde{\mathcal{L}}_0(M)$: for each $\lambda \in \mathbb{R}$, we define

$$\widetilde{\mathcal{L}}_0^\lambda(M) := \{[z, w] \in \widetilde{\mathcal{L}}_0(M) \mid \mathcal{A}_0([z, w]) \leq \lambda\}.$$

We note that

$$\widetilde{\mathcal{L}}_0^\lambda(M) \subset \widetilde{\mathcal{L}}_0^{\lambda'}(M) \quad \text{if } \lambda \leq \lambda'.$$

It follows from (2.8) that the critical set, denoted by $\mathrm{Crit}\mathcal{A}_0$, of $\mathcal{A}_0 : \widetilde{\mathcal{L}}_0(M) \to \mathbb{R}$ is the disjoint union of copies of M

$$\mathrm{Crit}\mathcal{A}_0(M) = \bigcup_{g \in \Gamma} g \cdot M$$

where $M \hookrightarrow \mathcal{L}_0(M); x \mapsto [x, \widehat{x}]$ is the canonical inclusion, where $\widehat{x}$ is the constant disc $\widehat{x} \equiv x$. The following is well-known and straightforward to check.

Lemma 2.2. *At each* $[x, \widehat{x}\#A] \in \mathrm{Crit}\mathcal{A}_0$*, the Hessian* $d^2\mathcal{A}_0$ *defines a bilinear form on*

$$T_{[x,\widehat{x}\#A]}\widetilde{\mathcal{L}}_0(M) \cong T_x\mathcal{L}_0(M)$$

which is (weakly) nondegenerate in the normal direction to $\mathrm{Crit}\mathcal{A}_0$*. In particular,* $\mathcal{A}_0$ *is a Bott-Morse function.*

For the convenience of notations, we also denote

$$[x, \widehat{x}] = \widehat{x}, \quad [x, \widehat{x}\#A] = \widehat{x} \otimes q^A.$$

2.4. **Perturbed action functionals and their action spectra.** When a one-periodic Hamiltonian $H : (\mathbb{R}/\mathbb{Z}) \times M \to \mathbb{R}$ is given, we consider the perturbed functional $\mathcal{A}_H : \widetilde{\mathcal{L}}_0(M) \to \mathbb{R}$ defined by

$$\mathcal{A}_H([\gamma, w] = \mathcal{A}_0 - \int H(t, \gamma(t))dt = -\int w^*\omega - \int H(t, \gamma(t))dt. \tag{2.11}$$

Unless otherwise stated, we will always consider one-periodic normalized Hamiltonian functions $H : [0, 1] \times M \to \mathbb{R}$.

Lemma 2.3. *The set of critical points of $\mathcal{A}_H$ is given by*

$$\mathrm{Crit}(\mathcal{A}_H) = \{[z,w] \mid z \in Per(H),\, \partial w = z\}$$

to which the Γ *action on* $\widetilde{\mathcal{L}}_0(M)$ *canonically restricts.*

Definition 2.6. We define the *action spectrum* of H by

$$\mathrm{Spec}(H) := \{\mathcal{A}_H(z,w) \in \mathbb{R} \mid [z,w] \in \widetilde{\Omega}_0(M),\, z \in \mathrm{Per}(H)\},$$

i.e., the set of critical values of $\mathcal{A}_H : \widetilde{\mathcal{L}}(M) \to \mathbb{R}$. For each given $z \in \mathrm{Per}(H)$, we denote

$$\mathrm{Spec}(H;z) = \{\mathcal{A}_H(z,w) \in \mathbb{R} \mid (z,w) \in \pi^{-1}(z)\}.$$

Note that $\mathrm{Spec}(H;z)$ is a principal homogeneous space modelled by the period group Γ_ω. We then have

$$\mathrm{Spec}(H) = \cup_{z\in \mathrm{Per}(H)} \mathrm{Spec}(H;z).$$

Recall that Γ_ω is either a discrete or a countable dense subgroup of $\mathbb{R}$. The following was proven in [Oh5].

Proposition 2.4. *Let H be any periodic Hamiltonian.* $\mathrm{Spec}(H)$ *is a measure zero subset of* $\mathbb{R}$ *for any H.*

We note that when $H = 0$, we have

$$\mathrm{Spec}(H) = \Gamma_\omega.$$

The following definition is standard.

Definition 2.7. We say that two Hamiltonians H and F are *homotopic* if $\phi_H^1 = \phi_F^1$ and their associated Hamiltonian paths $\phi_H,\, \phi_K \in \mathcal{P}(Ham(M,\omega), id)$ are path-homotopic relative to the boundary. In this case we denote $H \sim F$ and denote the set of equivalence classes by $\widetilde{Ham}(M,\omega)$.

The following lemma was proven in the aspherical case in [Sc], [Po3]. We refer the reader to [Oh6] for complete details of its proof in the general case.

Proposition 2.5. *Suppose that F, G are normalized. If $F \sim G$, we have*

$$\mathrm{Spec}\ (F) = \mathrm{Spec}\ (G)$$

as a subset of $\mathbb{R}$.

This enables us to make this definition

Definition 2.8. For any $H \in C_m^\infty(S^1 \times M)$, we define the spectrum of $h \in \widetilde{Ham}(M,\omega)$

$$\mathrm{Spec}(h) := \mathrm{Spec}(F)$$

for a (and so any) normalized Hamiltonian F with $[\phi, F] = h$.

2.5. **The L^2-gradient flow and perturbed Cauchy-Riemann equations.** The Floer homology theory [Fl2], [HS] is the semi-infinite version of the Novikov's circle valued Morse theory [N1], [N2] of $\mathcal{A}_H$ on the space $\mathcal{L}_0(M)$ of contractible free loops. To do the Morse theory of $\mathcal{A}_H$, we need to provide a metric on $\widetilde{\mathcal{L}}_0(M)$. We do this by first defining a metric on $\mathcal{L}_0(M)$ and then pulling it back to $\widetilde{\mathcal{L}}_0(M)$. Note that any S^1-family $\{g_t\}_{t\in S^1}$ of Riemannian metrics on M induces an L^2-type metric on $\mathcal{L}(M)$ by the formula

$$\ll \xi_1, \xi_2 \gg = \int_0^1 g_t(\xi_1(t), \xi_2(t))\, dt \tag{2.12}$$

for $\xi_1,\, \xi_2 \in T_\gamma \mathcal{L}(M)$. On the symplectic manifold (M, ω), we will particularly consider the family of *almost Kähler metrics* induced by the almost complex structures compatible to the symplectic form ω. Following [Gr], we give the following definition.

Definition 2.9. An almost complex structure J on M is called *compatible* to ω, if J satisfies

(1) (Tameness) $\omega(v, Jv) \geq 0$ and equality holds only when $v = 0$
(2) (Symmetry) $\omega(v_1, Jv_2) = \omega(v_2, Jv_1)$.

We denote by $\mathcal{J}_\omega = \mathcal{J}_\omega(M)$ the set of compatible almost complex structures.

Gromov's lemma [Gr] says that $\mathcal{J}_\omega$ is a contractible infinite dimensional (Frechêt) manifold.

We denote the associated family of metrics on M by

$$g_J = \omega(\cdot, J\cdot)$$

and its associated norm by $|\cdot|_J$. When we are given a one-periodic family $J = \{J_t\}_{t\in S^1}$, it induces the associated L^2-metric on $\mathcal{L}(M)$ by $\ll \cdot, \cdot \gg$ which can be written as

$$\ll \xi_1, \xi_2 \gg_J = \int_0^1 \omega(\xi_1(t), J_t\, \xi_2(t))\, dt. \tag{2.13}$$

From now on, we will always denote by J an S^1-*family* of compatible almost complex structures unless otherwise stated, and denote

$$j_\omega := C^\infty(S^1, \mathcal{J}_\omega).$$

If we denote by $\mathrm{grad}_J \mathcal{A}_H$ the associated L^2-gradient vector field, (2.8) and (2.13) imply that $\mathrm{grad}_J \mathcal{A}_H$ has the form

$$\mathrm{grad}_J \mathcal{A}_H([\gamma, w])(t) = J_t\Big(\dot\gamma(t) - X_H(t, \gamma(t))\Big) \tag{2.14}$$

which we will simply write $J(\dot\gamma - X_H(\gamma))$. It follows from this formula that the gradient is projectable to $\mathcal{L}_0(M)$. Therefore when we project the *negative* gradient flow equation of a path $u : \mathbb{R} \to \widetilde{\mathcal{L}}_0(M)$ to $\mathcal{L}_0(M)$, it has the form

$$\frac{\partial u}{\partial \tau} + J\Big(\frac{\partial u}{\partial t} - X_H(u)\Big) = 0 \tag{2.15}$$

if we regard u as a map $u : \mathbb{R} \times S^1 \to M$. We call this equation *Floer's perturbed Cauchy-Riemann equation* or simply as the perturbed Cauchy-Riemann equation associated to the pair (H, J).

The Floer theory largely relies on the study of the moduli spaces of *finite energy* solutions $u : \mathbb{R} \times S^1 \to M$ of the kind (2.15) of perturbed Cauchy-Riemann equations. The relevant energy function is given by

Definition 2.10. **[Energy]** For a given smooth map $u : \mathbb{R} \times S^1 \to M$, we define the energy, denoted by $E_{(H,J)}(u)$, of u by

$$E_{(H,J)}(u) = \frac{1}{2} \int \Big(\Big|\frac{\partial u}{\partial \tau}\Big|^2_{J_t} + \Big|\frac{\partial u}{\partial t} - X_H(u)\Big|^2_{J_t} \Big)\, dt\, d\tau.$$

The following lemma exemplifies significance of the finite energy condition. Although the proof is standard, we provide details of the proof for the reader's convenience to illustrate the kind of analytic arguments used in the study of perturbed Cauchy-Riemann equations.

Proposition 2.6. *Let $H : S^1 \times M \to \mathbb{R}$ be any Hamiltonian. Suppose that $u : \mathbb{R} \times S^1 \to M$ is a finite energy solution of (2.15). Then there exists a sequence $\tau_k \to \infty$ (respectively $\tau_k \to -\infty$) such that the loop $z_k := u(\tau_k) = u(\tau_k, \cdot)$ C^∞ converges to a one-periodic solution $z : S^1 \to M$ of the Hamilton equation $\dot{x} = X_H(x)$.*

Proof. Since u satisfies (2.15), the energy of u can be re-written as

$$E_{(H,J)}(u) = \int \int \Big|\frac{\partial u}{\partial t} - X_H(u)\Big|^2_{J_t}\, dt\, d\tau.$$

Therefore the finite energy condition, in particular, implies existence of $\tau_k \nearrow \infty$ such that

$$\int_0^1 \Big|\frac{\partial u}{\partial t}(\tau_k, \cdot) - X_H(u(\tau_k, \cdot))\Big|^2_{J_t}\, dt \to 0 \tag{2.16}$$

as $k \to \infty$. Since M is compact, X_H is bounded and so (2.16) implies

$$\int_0^1 |\dot{z}_k|^2_{J_t}\, dt \to 0 \tag{2.17}$$

for some $C > 0$ independent of k. (2.17) implies the equicontinuity of z_k and so there exists a subsequence, which we still denote by τ_k, such that $z_k \to z_\infty$ in C^0-topology. Furthermore Fatou's lemma implies

$$\int_0^1 |\dot{z} - X_H(z)|^2_{J_t}\, dt \leq \liminf_k \int_0^1 |\dot{z}_k - X_H(z_k)|^2_{J_t}\, dt \to 0.$$

Therefore z is a weak solution of $\dot{x} = X_H(x)$, which lies in $W^{1,2}$. In particular, $\dot{z}$ lies in $W^{2,2}(S^1)$, which follows from differentiating $\dot{z} = X_H(z)$. Then the Sobolev embedding $W^{2,2}(S^1) \hookrightarrow C^1(S^1)$ implies that z is C^1 and satisfies $\dot{x} = X_H(x)$. Once we know z is C^1, the boot-strap argument by differentiating $\dot{z} = X_H(z)$ implies z is smooth.

Finally since $z_k \to z$ in C^0, so does $X_H(z_k) \to X_H(z)$, which in turn implies $z_k \to z$ in C^1. Similar boot-strap argument then implies the C^∞-convergence of $z_k \to z$. This finishes the proof. $\square$

We denote by

$$\mathcal{M}(H, J) = \mathcal{M}(H, J; \omega)$$

the set of finite energy solutions of (2.15) for general H not necessarily nondegenerate.

Similar discussion can be carried out for the *non-autonomous* version of (2.15), which we now describe. We first denote

$$\mathcal{H}_m = \mathcal{H}_m(M) := \{H : S^1 \times M \to \mathbb{R} \mid H \text{ is normalized}\}.$$

Consider the $\mathbb{R}$-family

$$\begin{aligned} \mathcal{H}_{\mathbb{R}} &: \mathbb{R} \to \mathcal{H}_m; \quad \tau \mapsto H(\tau) \\ j_{\mathbb{R}} &: \mathbb{R} \to \mathcal{J}_\omega; \quad \tau \mapsto J(\tau) \end{aligned}$$

that are asymptotically constant, i.e.,

$$H(\tau) = H^{\pm\infty}, J(\tau) = J^{\pm\infty}$$

for some $H^{\pm\infty} \in \mathcal{H}_m$ and $J^{\pm\infty} \in j_\omega$ if $|\tau| > R$ for a sufficiently large constant R. To any such pair is associated the following *non-autonomous* version of (2.15)

$$\frac{\partial u}{\partial \tau} + J(\tau)\Big(\frac{\partial u}{\partial t} - X_{H(\tau)}(u)\Big) = 0. \tag{2.18}$$

The associated energy function is given by

$$E_{(\mathcal{H}_{\mathbb{R}}, j_{\mathbb{R}})}(u) = \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 \Big(\Big|\frac{\partial u}{\partial \tau}\Big|^2_{J(\tau)} + \Big|\frac{\partial u}{\partial t} - X_{H(\tau)}(u)\Big|^2_{J(\tau)}\Big)\, dt\, d\tau.$$

We denote by

$$\mathcal{M}(\mathcal{H}_{\mathbb{R}}, j_{\mathbb{R}}) = \mathcal{M}(\mathcal{H}_{\mathbb{R}}, j_{\mathbb{R}}; \omega)$$

the set of finite energy solutions of (2.18).

Here is the analog to Proposition 2.6, whose proof is essentially the same as Proposition 2.6 due to the asymptotically constant condition on $(\mathcal{H}, j)$.

Proposition 2.7. *Let $\mathcal{H}_{\mathbb{R}}$ and $j_{\mathbb{R}}$ be as above. Suppose that $u : \mathbb{R} \times S^1 \to M$ is a finite energy solution of (2.18). Then there exists a sequence $\tau_k \to \infty$ (respectively $\tau_k \to -\infty$) such that the loop $z_k := u(\tau_k) = u(\tau_k, \cdot)$ C^∞ converges to a one-periodic solution $z : S^1 \to M$ of the Hamilton equation $\dot{x} = X_{H^{\pm\infty}}(x)$ respectively.*

A typical way how such an asymptotically constant family appears is through an *elongation* of a given smooth one-parameter family over $[0, 1]$.

Definition 2.11. A continuous map $f : [0, 1] \to T$ for any topological space T is said to be *boundary flat* if the map is constant near the boundary $\partial[0, 1] = \{0, 1\}$.

Let $\mathcal{H} : [0, 1] \to \mathcal{H}_m$ be a homotopy connecting two Hamiltonians H_α, $H_\beta \in \mathcal{H}_m$, and $j : [0, 1] \to \mathcal{J}_\omega$ connecting J_α, $J_\beta \in \mathcal{J}_\omega$. We denote

$$\begin{aligned} \mathcal{P}(j_\omega) &:= C^\infty([0, 1], j_\omega) \\ \mathcal{P}(\mathcal{H}_m) &:= C^\infty([0, 1], \mathcal{H}_m). \end{aligned}$$

We define a function $\rho : \mathbb{R} \to [0, 1]$ of the type

$$\rho(\tau) = \begin{cases} 0 & \text{for } \tau \leq -R \\ 1 & \text{for } \tau \geq R \end{cases} \tag{2.19}$$

for some $R > 0$. We call ρ a (positively) *monotone* cut-off function if it satisfies $\rho'(\tau) \geq 0$ for all τ's in addition.

Each such pair $(\mathcal{H}, j)$, combined with a cut-off function ρ, defines a pair $(\mathcal{H}^\rho, j^\rho)$ of asymptotically constant $\mathbb{R}$-families

$$\mathcal{H}_{\mathbb{R}} = \mathcal{H}^\rho, \quad j_{\mathbb{R}} = j^\rho$$

where $\mathcal{H}^\rho$ is the reparameterized homotopy $\mathcal{H}^\rho = \{H^\rho\}_{\tau\in\mathbb{R}}$ defined by

$$\tau \mapsto H^\rho(\tau,t,x) = H(\rho(\tau),t,x).$$

We call $\mathcal{H}^\rho$ the *ρ-elongation* of $\mathcal{H}$ or the *ρ-elongated homotopy* of $\mathcal{H}$. The same definition applies to j. Therefore such a triple $(\mathcal{H}, j; \rho)$ gives rise to the *non-autonomous* equation

$$\frac{\partial u}{\partial \tau} + J^{\rho(\tau)}\Big(\frac{\partial u}{\partial t} - X_{H^{\rho(\tau)}}(u)\Big) = 0. \tag{2.20}$$

We denote by

$$\mathcal{M}(\mathcal{H}, j; \rho)$$

the set of finite energy solutions of (2.20).

2.6. **Comparison of two Cauchy-Riemann equations.** In this subsection, we explain the relation between Floer's standard perturbed Cauchy-Riemann equation (2.15) for $u : \mathbb{R}\times S^1 \to M$ and its mapping cylinder version $v : \mathbb{R}\times\mathbb{R} \to M$

$$\begin{cases} \frac{\partial v}{\partial \tau} + J'_t \frac{\partial v}{\partial t} = 0 \\ \phi(v(\tau, t+1)) = v(\tau,t), \quad \int |\frac{\partial v}{\partial \tau}|^2_{J'_t} < \infty \end{cases} \tag{2.21}$$

where $\phi = \phi_1^H$. We often restrict v to $\mathbb{R}\times[0,1]$ and consider it as a map from $\mathbb{R}\times[0,1]$ that satisfies $\phi(v(\tau,1)) = v(\tau,0)$. A similar correspondence had been exploited in [Oh2], [Oh3] in the 'open string' context of Lagrangian submanifolds for the same purpose, and call the former version of Floer homology the *dynamical* and the latter *geometric*. We do the same here.

For any given solution $u = u(\tau,t) : \mathbb{R}\times S^1 \to M$, we 'open up' u along $t = 0 \equiv 1$ and define the map

$$v : \mathbb{R}\times[0,1] \to M$$

by

$$v(\tau,t) = (\phi_H^t)^{-1}(u(\tau,t)) \tag{2.22}$$

and then extend to $\mathbb{R}$ so that $\phi(v(\tau,t+1)) = v(\tau,t)$. A simple computation shows that when u satisfies (2.15) the map v satisfies (2.21), *provided* the family $J' = \{J'_t\}_{0\le t\le 1}$ is defined by

$$J'_t = (\phi_H^t)^* J_t$$

for the given periodic family J used for the equation (2.15), and vice versa. By definition, this family J' of almost complex structure satisfies

$$J'(t+1) = \phi^* J'(t). \tag{2.23}$$

One can even fix $J(0) = J_0$ for any given almost complex structure J_0 which leads to the following definition [Oh8]

Definition 2.12. Let $J_0 \in \mathcal{J}_\omega$ and $\phi \in Ham(M,\omega)$. We define $j_{(\phi,J_0)}$ by

$$j_{(\phi,J_0)} := \{J' : [0,1] \to \mathcal{J}_\omega \mid J'(t+1) = \phi^* J'(t), \quad J'(0) = J_0\}. \tag{2.24}$$

The condition

$$\phi(v(\tau,t+1)) = v(\tau,t) \tag{2.25}$$

enables us to consider the map

$$v : \mathbb{R}\times\mathbb{R} \to M$$

as a pseudo-holomorphic section of the 'mapping cylinder'

$$E_\phi := \mathbb{R}\times M_\phi = \mathbb{R}\times\mathbb{R}\times M/(\tau, t+1, \phi(x)) \sim (\tau,t,x)$$

where M_ϕ is the mapping circle defined by

$$M_\phi := \mathbb{R} \times M/(t+1, \phi(x)) \sim (t, x).$$

Note that the product symplectic form $d\tau \wedge dt + \omega$ on $\mathbb{R} \times \mathbb{R} \times M$ naturally projects to E_ϕ since ϕ is symplectic, and so E_ϕ has the structure of a Hamiltonian fibration. In this setting, $v : \mathbb{R} \times \mathbb{R} \to M$ can be regarded as the section $s : \mathbb{R} \times S^1 \to E_\phi$ defined by

$$s(\tau, t) = [\tau, t, v(\tau, t)]$$

which becomes a pseudo-holomorphic section of E_ϕ for a suitably defined almost complex structure.

One advantage of the mapping cylinder version over the more standard dynamical version (2.15) is that its dependence on the Hamiltonian H is much weaker than in the latter. Indeed, this mapping cylinder version can be put into the general framework of Hamiltonian fibrations with given fixed monodromy of the fibration at infinity as in [En1]. This framework turns out to be essential to prove the triangle inequality of the spectral invariants. (See [Sc], [Oh8] or section 7 later in this paper).

Another important ingredient is the comparison of two different energies $E_{(H,J)}(u)$ and $E_{J'}(v)$: for the given $J' = \{J'_t\}_{0\leq t\leq 1} \in j_{(J_0,\phi)}$, we define the energy of the map $v : \mathbb{R} \times [0,1] \to M$ by

$$E_{J'}(v) = \frac{1}{2}\int_{\mathbb{R}\times[0,1]} \Big(\Big|\frac{\partial v}{\partial \tau}\Big|^2_{J'_t} + \Big|\frac{\partial v}{\partial t}\Big|^2_{J'_t}\Big) dt\, d\tau.$$

This energy is the *vertical* part of the energy of the section $s : \mathbb{R} \times S^1 \to E_\phi$ defined above with respect to a suitably chosen almost complex structure $\widetilde{J}$ on E_ϕ. (See section 3 [Oh9] for more explanation.) Note that because of (2.23)-(2.25), one can replace the domain of integration $\mathbb{R} \times [0,1]$ by any *fundamental domain* of the covering projection

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R} \times (\mathbb{R}/\mathbb{Z})$$

without changing the integral. The choice of $\mathbb{R} \times [0,1]$ is one such choice.

The following identity plays an important role in the proof of the nondegeneracy of the invariant norm we construct later. The proof is a straightforward computation left to the readers.

Lemma 2.8. *Let $J = \{J_t\}_{0\leq t\leq 1}$ be a periodic family and define $J' = \{J'_t\}_{0\leq t\leq 1}$ by*

$$J'_t = (\phi_H^t)^* J_t.$$

Let $u : \mathbb{R} \times S^1 \to M$ be any smooth map and $v : \mathbb{R} \times [0,1] \to M$ be the map defined by

$$v(\tau, t) = (\phi_H^t)^{-1}(u(\tau, t)).$$

Then we have

$$E_{(H,J)}(u) = E_{J'}(v).$$

3. Floer complex and the Novikov ring

In this section we provide the details of construction of the Floer complex and its basic operators. The details of construction are given in [Fl2] and [SZ], for example. But we closely follow the exposition given in [Oh8].

3.1. **Novikov Floer chains and the Novikov ring.** Suppose that $\phi \in \mathcal{H}am(M,\omega)$ is nondegenerate. For each nondegenerate $H : S^1 \times M \to \mathbb{R}$, we know that the cardinality of $\mathrm{Per}(H)$ is finite. We consider the free $\mathbb{Q}$ vector space generated by the critical set of $\mathcal{A}_H$

$$\mathrm{Crit}\mathcal{A}_H = \{[z,w] \in \widetilde{\Omega}_0(M) \mid z \in \mathrm{Per}(H)\}.$$

To be able to define the Floer boundary operator correctly, we need to complete this vector space downward with respect to the real filtration provided by the action $\mathcal{A}_H([z,w])$ of the element $[z,w]$ as in [Fl2], [HS]. More precisely, we give the following definitions slightly streamlining those from [Oh5].

Definition 3.1. Consider the formal sum

$$\beta = \sum_{[z,w]\in \mathrm{Crit}\mathcal{A}_H} a_{[z,w]}[z,w],\ a_{[z,w]} \in \mathbb{Q} \tag{3.1}$$

(1) We call those $[z,w]$ with $a_{[z,w]} \neq 0$ *generators* of the sum β and write
$$[z,w] \in \beta.$$
We also say that $[z,w]$ *contributes* to β in that case.

(2) We define the *support* of β by
$$\mathrm{supp}(\beta) := \{[z,w] \in \mathrm{Crit}\mathcal{A}_H \mid a_{[z,w]} \neq 0 \ \text{ in the sum (3.1)}\}.$$

(3) We call the formal sum β a *Novikov Floer chain* (or simply a *Floer chain*) if
$$\#\Big(\mathrm{supp}(\beta) \cap \{[z,w] \mid \mathcal{A}_H([z,w]) \geq \lambda\}\Big) < \infty \tag{3.2}$$
for any $\lambda \in \mathbb{R}$. We denote by $CF_*(H)$ the set of Floer chains.

Note that $CF_*(H)$ is a $\mathbb{Q}$-vector space which is always infinite dimensional in general, unless (M,ω) is symplectically aspherical. Since the aspherical case was studied in [Oh4], [Sc] before, we will focus on the general case where the quantum contributions could be present. There is a natural grading on $CF_*(H):$ we associate the *Conley-Zehnder index*, denote by $\mu_H([z,w])$ to each generator $[z,w] \in \mathrm{Crit}\mathcal{A}_H$. We refer to [CZ], [SZ], [HS] for the definition of $\mu_H([z,w])$. For readers' convenience, we recall the definition in Appendix in the course of proving the index formula in our convention.

Now consider a Floer chain

$$\beta = \sum a_{[z,w]}[z,w], \quad a_{[z,w]} \in \mathbb{Q}.$$

Following [Oh5], we introduce the following notion which is a crucial concept for the mini-max argument that we carry out in this paper.

Definition 3.2. Let $\beta \neq 0$ be a Floer chain in $CF_*(H)$. We define the *level* of the chain β and denote it by

$$\lambda_H(\beta) = \max_{[z,w]}\{\mathcal{A}_H([z,w]) \mid [z,w] \in \mathrm{supp}(\beta)\},$$

and set $\lambda_H(0) = -\infty$. We call a generator $[z,w] \in \beta$ satisfying $\mathcal{A}_H([z,w]) = \lambda_H(\beta)$ a *peak* of β, and denote it by $\mathrm{peak}(\beta)$.

We emphasize that it is the Novikov condition (3) of Definition 3.1 that guarantees that $\lambda_H(\beta)$ is well-defined. The following lemma illustrates optimality of the definition of the Novikov covering space.

Lemma 3.1. *Let $\beta \neq 0$ be a homogeneous Floer chain. Then the peak of β over a fixed periodic orbit is unique.*

Proof. By the assumption of homogeneity, the generators of β have the same Conley-Zehnder indices. Let $[z, w]$ and $[z, w']$ be two such peaks of β. Then we have

$$\mathcal{A}_H([z, w]) = \lambda_H(\beta) = \mathcal{A}_H([z, w'])$$

which in turn implies $\omega([w]) = \omega([w'])$. By the homogeneity assumption, we also have

$$\mu_H([z, w]) = \mu_H([z, w']).$$

It follows from the definition of Γ-equivalence classes that $[z, w] = [z, w']$, which finishes the proof. □

So far we have defined $CF_*(H)$ as a $\mathbb{Z}$-graded $\mathbb{Q}$-vector space with $\mathrm{Crit}\mathcal{A}_H$ as its generating set which has infinitely many elements, unless (M, ω) is symplectically aspherical. We now explain the description of $CF(H)$ as a module over the *Novikov ring* as in [Fl2], [HS].

We consider the group ring $\mathbb{Q}[\Gamma]$ consisting of the finite sum

$$R = \sum_{i=1}^{k} r_i q^{A_i} \in \mathbb{Q}([\Gamma])$$

and define its support by

$$\mathrm{supp}\ R = \{A \in \Gamma \mid A = A_i \text{ in this sum }\}.$$

We recall the valuation $v : \Gamma \to \mathbb{R}$ and the degree map $d : \Gamma \to \mathbb{R}$. We now define a valuation $v : \mathbb{Q}[\Gamma] \to \mathbb{R}$ by

$$v(R) = v^{\downarrow}(R) = v^{\downarrow}(\sum_{i=1}^{k} r_i q^{A_i}) := \max\{\omega(A_i) \mid A_i \in \mathrm{supp}\ R\}.$$

This satisfies the following Non-Archimedean triangle inequality

$$v(R_1 + R_2) \leq \max\{v(R_1), v(R_2)\} \tag{3.3}$$

and so induces a natural metric topology on $\mathbb{Q}[\Gamma]$.

Definition 3.3. The (downward) Novikov ring is the *downward completion* $\mathbb{Q}[[\Gamma]$ of $\mathbb{Q}[\Gamma]$ with respect to the valuation $v : \mathbb{Q}[\Gamma] \to \mathbb{R}$. We denote it by $\Lambda_\omega^{\downarrow}$.

More concretely we have the description

$$\Lambda_\omega^{\downarrow} = \{\sum_{A \in \Gamma} r_A q^A \mid \forall \lambda \in \mathbb{R}, \#\{A \in \Gamma \mid r_A \neq 0, \omega(A) > \lambda\} < \infty\}.$$

Similarly we define the *upward Novikov ring*, denoted by $\Lambda_\omega^{\uparrow}$, by

$$\Lambda_\omega^{\uparrow} = \{\sum_{A \in \Gamma} r_A q^{-A} \mid \forall \lambda \in \mathbb{R}, \#\{A \in \Gamma \mid r_A \neq 0, \omega(-A) < \lambda\} < \infty\}.$$

Since we will mostly use the downward Novikov ring in this lecture, we will just denote $\Lambda_\omega = \Lambda_\omega^{\downarrow}$ dropping the arrow. Then we have the valuation on Λ_ω given by

$$v(R) = \max\{\omega(A) \mid A \in \mathrm{supp}\ R\}. \tag{3.4}$$

We recall that Γ induces a natural action on $\mathrm{Crit}\mathcal{A}_H$ by 'gluing a sphere'

$$[z, w] \mapsto [z, w\#A]$$

which in turn induces the multiplication of Λ_ω on $CF(H)$ by the convolution product. This enables one to regard $CF(H)$ as a Λ_ω-module. We will try to consistently denote by $CF(H)$ as a Λ_ω-module, and by $CF_*(H)$ as a graded $\mathbb{Q}$ vector space.

The action functional provides a natural filtration on $CF_*(H)$: for any given $\lambda \in \mathbb{R} \setminus \mathrm{Spec}(H)$, we define

$$CF_*^\lambda(H) = \{\alpha \in CF_*(H) \mid \mathcal{A}_H(\mathrm{peak}(\alpha)) \leq \lambda\}$$

and denote the natural inclusion homomorphism by

$$i_\lambda : CF_*^\lambda(H) \to CF_*(H).$$

3.2. **Definition of the Floer boundary map.** Suppose H is a nondegenerate one-periodic Hamiltonian function and J a one-periodic family of compatible almost complex structures. We first recall Floer's construction of the Floer boundary map, and the transversality conditions needed to define the Floer homology $HF_*(H, J)$ of the pair.

The following definition is useful for the later discussion.

Definition 3.4. Let z, $z' \in \mathrm{Per}(H)$. We denote by $\pi_2(z, z')$ the set of homotopy classes of smooth maps

$$u : [0,1] \times S^1 := T \to M$$

relative to the boundary

$$u(0,t) = z(t), \quad u(1,t) = z'(t).$$

We denote by $[u] \in \pi_2(z, z')$ its homotopy class and by C a general element in $\pi_2(z, z')$.

We define by $\pi_2(z)$ the set of relative homotopy classes of the maps

$$w : D^2 \to M; \quad w|_{\partial D^2} = z.$$

We note that there is a natural action of $\pi_2(M)$ on $\pi_2(z)$ and $\pi_2(z, z')$ by the obvious operation of a 'gluing a sphere'. Furthermore there is a natural map of $C \in \pi_2(z, z')$

$$(\cdot)\#C : \pi_2(z) \to \pi_2(z')$$

induced by the gluing map

$$w \mapsto w\#u.$$

More specifically we will define the map $w\#u : D^2 \to M$ in the polar coordinates (r, θ) of D^2 by the formula

$$w\#u : (r,\theta) = \begin{cases} w(2r, \theta) & \text{for } 0 \leq r \leq \frac{1}{2} \\ w(2r-1, \theta) & \text{for } \frac{1}{2} \leq r \leq 1 \end{cases} \tag{3.5}$$

once and for all. There is also the natural gluing map

$$\pi_2(z_0, z_1) \times \pi_2(z_1, z_2) \to \pi_2(z_0, z_2)$$

$$(u_1, u_2) \mapsto u_1\#u_2.$$

We also explicitly represent the map $u_1\#u_2 : T \to M$ in the standard way once and for all similarly to (3.5).

Definition 3.5. We define the *relative Conley-Zehnder index* of $C \in \pi_2(z, z')$ by

$$\mu_H(z, z'; C) = \mu_H([z, w]) - \mu_H([z', w\#C])$$

for a (and so any) representative $u : [0,1] \times S^1 \times M$ of the class C. We will also write $\mu_H(C)$, when there is no danger of confusion on the boundary condition.

It is easy to see that this definition does not depend on the choice of bounding disc w of z, and so the function

$$\mu_H : \pi_2(z, z') \to \mathbb{Z}$$

is well-defined.

Remark 3.6. In fact, the function $\mu_H : \pi_2(z, z') \to \mathbb{Z}$ can be defined without assuming z_0, z_1 being contractible, as long as z_0 and z_1 lie in the same component of $\Omega(M)$: For any given map $u : T \to M$, choose a *marked* symplectic trivialization

$$\Phi : u^*TM \to T \times \mathbb{R}^{2n}$$

that satisfies

$$\Phi \circ \Phi^{-1}|_{[0,1]\times\{1\}} = id.$$

We know that $z_0(t) = \phi_H^t(p_0)$ and $z_1(t) = \phi_H^t(p_1)$ for p_0, $p_1 \in \mathrm{Fix}(\phi_H^1)$. Then we have two maps

$$\alpha_{\Phi,i} : [0,1] \to Sp(2n), \quad i = 0,\, 1$$

such that

$$\Phi \circ d\phi_H^t(p_i) \circ \Phi^{-1}(i, t, v) = (i, t, \alpha_{\Phi,i}(t)v)$$

for $v \in \mathbb{R}^{2n}$ and $t \in [0,1]$. By the nondegeneracy of H, the maps $\alpha_{\Phi,i}$ define elements in $\mathbb{S}P^*(1)$. Then we define

$$\mu_H(z, z'; C) := \mu_{CZ}(\alpha_{\Phi,0}) - \mu_{CZ}(\alpha_{\Phi,1}).$$

It is easy to check that this definition does not depend on the choice of marked symplectic trivializations.

We now denote by

$$\mathcal{M}(H, J; z, z'; C)$$

the set of finite energy solutions of (2.15) with the asymptotic condition and the homotopy condition

$$u(-\infty) = z, \quad u(\infty) = z'; \quad [u] = C. \tag{3.6}$$

Here we remark that although u is a priori defined on $\mathbb{R}\times S^1$, it can be compactified into a continuous map $\overline{u} : [0,1] \times S^1 \to M$ with the corresponding boundary condition

$$\overline{u}(0) = z, \quad \overline{u}(1) = z'$$

due to the exponential decay property of solutions u of (4.2), recalling we assume H is nondegenerate. We will call $\overline{u}$ the *compactified map* of u. By some abuse of notation, we will also denote by $[u]$ the class $[\overline{u}] \in \pi_2(z, z')$ of the compactified map $\overline{u}$.

The Floer boundary map

$$\partial_{(H,J)}; CF_{k+1}(H) \to CF_k(H)$$

is defined under the following conditions. (See [Fl2], [HS].)

Definition 3.7. [The boundary map] Let H be nondegenerate. Suppose that J satisfies the following conditions:

(1) For any pair $(z_0, z_1) \subset \mathrm{Per}(H)$ satisfying
$$\mu_H(z_0, z_1; C) = \mu_H([z_0, w_0]) - \mu_H([z_1, w_0 \# C]) = 0,$$
$\mathcal{M}(H, J; z_0, z_1; C) = \emptyset$ unless $z_0 = z_1$ and $C = 0$. When $z_0 = z_1$ and $C = 0$, the only solutions are the stationary solution, i.e., $u(\tau) \equiv z_0 = z_1$ for all $\tau \in \mathbb{R}$.

(2) For any pair $(z_0, z_1) \subset \mathrm{Per}(H)$ and a homotopy class $C \in \pi_2(z_0, z_1)$ satisfying
$$\mu_H(z_0, z_1; C) = 1,$$
$\mathcal{M}(H, J; z_0, z_1; C)/\mathbb{R}$ is transverse and compact and so a finite set. We denote
$$n(H, J; z_0, z_1; C) = \#(\mathcal{M}(H, J; z_0, z_1; C)/\mathbb{R})$$
the algebraic count of the elements of the space $\mathcal{M}(H, J; z_0, z_1; C)/\mathbb{R}$. We set $n(H, J; z_0, z_1 : C) = 0$ otherwise.

(3) For any pair $(z_0, z_2) \subset \mathrm{Per}(H)$ and $C \in \pi_2(z_0, z_2)$ satisfying
$$\mu_H(z_0, z_2; C) = 2,$$
$\mathcal{M}(H, J; z_0, z_2; C)/\mathbb{R}$ can be compactified into a smooth one-manifold with boundary comprising the collection of the broken trajectories
$$[u_1] \#_\infty [u_2]$$
where $u_1 \in \mathcal{M}(H, J; z_0, y : C_1)$ and $u_2 \in \mathcal{M}(H, J; y, z_2 : C^2)$ for all possible $y \in \mathrm{Per}(H)$ and $C_1 \in \pi_2(z_0, y)$, $C_2 \in \pi_2(y, z_2)$ satisfying
$$C_1 \# C_2 = C \quad ; \quad \begin{array}{l} [u_1] \in \mathcal{M}(H, J; z_0, y; C_1)/\mathbb{R}, \\ [u_2] \in \mathcal{M}(H, J; y, z_2; C_2)/\mathbb{R} \end{array}$$
and
$$\mu_H(z_0, y; C_1) = \mu_H(y, z_2; C_2) = 1.$$
Here we denote by $[u]$ the equivalence class represented by u.

We call any such J *H-regular* and call such a pair (H, J) *Floer regular*.

The upshot is that for a Floer regular pair (H, J) the Floer boundary map
$$\partial = \partial_{(H,J)} : CF_*(H) \to CF_*(H)$$
is defined and satisfies $\partial\partial = 0$, which enables us to take its homology.

We now explain this construction in detail. For each given $[z^-, w^-]$ and $[z^+, w^+]$, we collect the elements $C \in \pi_2(z^-, z^+)$ satisfying
$$[z^+, w^+] = [z^+, w^- \# C] \quad \text{in } \widetilde{\mathcal{L}}_0(M) \tag{3.7}$$
and define the moduli space
$$\mathcal{M}(H, J; [z^-, w^-], [z^+, w^+]) := \bigcup_C \{\mathcal{M}(H, J; z^-, z^+; C) \mid C \text{ satisfies (3.7)}\}.$$
We like to note that there could be more than one $C \in \pi_2(z^-, z^+)$ that satisfies (3.7) according to the definition of the Γ-covering space $\widetilde{\mathcal{L}}_0(M)$. The following lemma is an easy consequence of a standard compactness argument.

Lemma 3.2. *This union is a finite union. In other words, for any given pair* $([z^-, w^-], [z^+, w^+])$*, there are only a finite number of* $C \in \pi_2(z^-, z^+)$ *that satisfies (3.7) and* $\mathcal{M}(H, J; z^-, z^+; C) \neq \emptyset$.

Now considering u as a path in the covering space $\widetilde{\mathcal{L}}_0(M)$, we write the asymptotic condition of $u \in \mathcal{M}(H, J; [z^-, w^-], [z^+, w^+])$ as

$$u(-\infty) = [z^-, w^-], \quad u(\infty) = [z^+, w^+]. \tag{3.8}$$

The Floer boundary map $\partial = \partial_{(H,J)} : CF_*(H) \to CF_*(H)$ is defined by its matrix coefficient

$$\langle \partial([z^-, w^-]), [z^+, w^+] \rangle := \sum_C n_{(H,J)}(z^-, z^+; C),$$

where C is as in (3.7) and the Conley-Zehnder indices of $[z^-, w^-]$ and $[z^+, w^+]$ satisfy

$$\mu_H([z^-, w^-]) - \mu_H([z^+, w^+]) = \mu(z^-, z^+; C) = 1,$$

We set the matrix coefficient to be zero otherwise. $\partial = \partial_{(H,J)}$ has degree -1 and satisfies $\partial \circ \partial = 0$.

Definition 3.8. We say that a Floer chain $\beta \in CF(H)$ is *Floer cycle* of (H, J) if $\partial \beta = 0$, i.e., if $\beta \in \ker \partial_{(H,J)}$, and a *Floer boundary* if $\beta \in \operatorname{Im} \partial_{(H,J)}$. Two Floer chains β, β' are said to be *homologous* if $\beta' - \beta$ is a boundary.

We denote

$$ZF_*(H, J) = \ker \partial, \quad BF_*(H, J) = \operatorname{im} \partial$$

and then the Floer homology is defined by

$$HF_*(H, J) := ZF_*(H, J)/BF_*(H, J).$$

One may regard this either as a graded $\mathbb{Q}$-vector space or as a Λ_ω-module. We will mostly consider it as a graded $\mathbb{Q}$-vector space in this lecture, because it well suits the mini-max theory of the action functional on $\widetilde{\mathcal{L}}_0(M)$.

3.3. **Definition of the Floer chain map.** When we are given a family $(\mathcal{H}, j)$ with $\mathcal{H} = \{H^s\}_{0 \leq s \leq 1}$ and $j = \{J^s\}_{0 \leq s \leq 1}$ and a cut-off function $\rho : \mathbb{R} \to [0, 1]$, the chain homomorphism

$$h_{\mathcal{H}} = h_{(\mathcal{H},j)} : CF_*(H_\alpha) \to CF_*(H_\beta)$$

is defined by considering the non-autonomous equation (2.18). It may be instructive to mention that (2.12) is *not* a gradient-like flow unlike (2.15). We now provide the details.

Consider the pair $(\mathcal{H}_\mathbb{R}, j_\mathbb{R})$ that are *asymptotically constant*, i.e., there exists $R > 0$ such that

$$J(\tau) \equiv J(\infty), \quad H(\tau) \equiv H(\infty)$$

for all τ with $|\tau| \geq R$.

Definition 3.9. [The chain map] We say that $(\mathcal{H}_\mathbb{R}, j_\mathbb{R})$ *is Floer regular* if the following holds:

(1) For any pair $z_0 \in \operatorname{Per}(H_0)$ and $z_1 \subset \operatorname{Per}(H_1)$ satisfying

$$\mu_{\mathcal{H}_\mathbb{R}}(z_0, z_1; C) = 0,$$

$\mathcal{M}(\mathcal{H}_{\mathbb{R}}, j_{\mathbb{R}}; z_0, z_1; C)$ is transverse and compact, and so a finite set. We denote

$$n(\mathcal{H}_{\mathbb{R}}, j_{\mathbb{R}}; z_0, z_1; C) := \#(\mathcal{M}(\mathcal{H}_{\mathbb{R}}, j_{\mathbb{R}}; z_0, z_1; C))$$

the algebraic count of the elements in $\mathcal{M}(\mathcal{H}_{\mathbb{R}}, j_{\mathbb{R}}; z_0, z_1; C)$. Otherwise, we set

$$n(\mathcal{H}_{\mathbb{R}}, j_{\mathbb{R}}; z_0, z_1 : C) = 0.$$

(2) For any pair $z_0 \in \mathrm{Per}(H_0)$ and $z_1 \in \mathrm{Per}(H_1)$ satisfying

$$\mu_{\mathcal{H}_{\mathbb{R}}}(z_0, z_2; C) = 1,$$

$\mathcal{M}(H, J; z_0, z_2; C)$ is transverse and can be compactified into a smooth one-manifold with boundary comprising the collection of the broken trajectories

$$u_1 \#_\infty u_2$$

where

$$\begin{aligned}(u_1, u_2) \in \quad & \mathcal{M}(\mathcal{H}_{\mathbb{R}}, j_{\mathbb{R}}; z_0, y : C_1) \times \mathcal{M}(H(\infty), J(\infty); y, z_2 : C^2);\\ & \mu_{\mathcal{H}_{\mathbb{R}}}(z_0, y; C_1) = 0,\ \mu_H(y, z_2; C_2) = 1\end{aligned}$$

or

$$\begin{aligned}(u_1, u_2) \in \quad & \mathcal{M}(H(-\infty), J(-\infty); z_0, y : C^1) \times \mathcal{M}(\mathcal{H}_{\mathbb{R}}, j_{\mathbb{R}}; y, z_2 : C_1);\\ & \mu_{\mathcal{H}_{\mathbb{R}}}(z_0, y; C_1) = 1,\ \mu_H(y, z_2; C_2) = 0\end{aligned}$$

and $C_1 \# C_2 = C$ for all possible such $y \in \mathrm{Per}(H)$ and $C_1 \in \pi_2(z_0, y)$, $C_2 \in \pi_2(y, z_2)$.

We say that $(\mathcal{H}_{\mathbb{R}}, j_{\mathbb{R}})$ are *Floer regular* if it satisfies these conditions.

Now suppose that $(\mathcal{H}, j)$ is a homotopy connecting two Floer regular pairs (H_α, J_α) and (H_β, J_β). Choose a cut-off function $\rho : \mathbb{R} \to [0, 1]$.

For each such pair $(\mathcal{H}, j)$ and a cut-off function ρ, we consider the ρ-elongations $\mathcal{H}^\rho$ and j^ρ respectively. Therefore to such a triple $(\mathcal{H}, j; \rho)$ is associated the *non-autonomous* equation (2.18) with the boundary condition

$$u(-\infty) = z_0, \quad u(\infty) = z_1 \tag{3.9}$$

and the homotopy condition $[u] = C \in \pi_2(z_0, z_1)$ for a fixed C. Now for each given pair of $[z_\alpha, w_\alpha] \in \mathrm{Crit}\mathcal{A}_{H_\alpha}$ and $[z_\beta, w_\beta] \in \mathrm{Crit}\mathcal{A}_{H_\beta}$, we define

$$\mathcal{M}((\mathcal{H}, j; \rho); [z_\alpha, w_\alpha], [z_\beta, w_\beta]) := \bigcup_C \mathcal{M}((\mathcal{H}, j; \rho); z_\alpha, z_\beta; C)$$

where $C \in \pi_2(z_\alpha, z_\beta)$ are the elements satisfying

$$[z_\beta, w_\beta] = [z_\beta, w_\alpha \# C] \tag{3.10}$$

similarly as in (3.7). We say that $(\mathcal{H}, j; \rho)$ is *Floer regular* if the ρ-elongation $(\mathcal{H}^\rho, j^\rho)$ is Floer regular in the sense of Definition 3.9.

Under the condition in Definition 3.9, we can define a map of degree zero

$$h_{(\mathcal{H}, j; \rho)} : CF(H_\alpha) \to CF(H_\beta)$$

by the matrix element $n_{(\mathcal{H}, j; \rho)}([z_\alpha, w_\alpha], [z_\beta, w_\beta])$ similarly as for the boundary map. The conditions in Definition 3.9 then also imply that $h_{(\mathcal{H}, j)}$ has degree 0 and satisfies the identity

$$h_{(\mathcal{H}, j; \rho)} \circ \partial_{(H_\alpha, J_\alpha)} = \partial_{(H_\beta, J_\beta)} \circ h_{(\mathcal{H}, j; \rho)}.$$

Two such chain maps $h_{(j^1, \mathcal{H}^1)}$, $h_{(j^2, \mathcal{H}^2)}$ are also chain homotopic [Fl2].

3.4. **Semi-positivity and transversality.** In this subsection, we briefly discuss the hypotheses imposed in Definition 3.7 and 3.9.

For the case of the boundary map $\partial_{(H,J)}$, Hofer and Salamon [HS] prove the following

Theorem 3.3. *Suppose (M, ω) satisfies that there is no $A \in \pi_2(M)$ such that*

$$\omega(A) > 0 \quad \textit{and} \quad 4 - n \leq c_1(A) < 0.$$

Then the hypotheses stated in Definition 3.7 hold.

For the case of the chain map $h_{(\mathcal{H},j;\rho)}$, they prove

Theorem 3.4. *Suppose (M, ω) satisfies that there is no $A \in \pi_2(M)$ such that*

$$\omega(A) > 0 \quad \textit{and} \quad 3 - n \leq c_1(A) < 0.$$

Then the hypotheses stated in Definition 3.9 hold.

This leads one to introduce the following definition.

Definition 3.10. A symplectic manifold (M, ω) is called *semi-positive* if it satisfies that there is no $A \in \pi_2(M)$ such that

$$\omega(A) > 0 \quad \text{and} \quad 3 - n \leq c_1(A) < 0.$$

For the later purpose of studying the pants product on the Floer complex, following Seidel [Se] and Entov [En1], we introduce

Definition 3.11. A symplectic manifold (M, ω) is called *strongly semi-positive* if it satisfies that there is no $A \in \pi_2(M)$ such that

$$\omega(A) > 0 \quad \text{and} \quad 2 - n \leq c_1(A) < 0.$$

For the general symplectic manifolds, one needs to use the concept of virtual moduli cycle and abstract multivalued perturbations in the context of the Kuranishi structure [FOn]. We will make further remarks in section 10 in relation to the numerical estimates concerning the energy of solutions and the levels of the Novikov Floer cycles.

3.5. **Composition law of Floer's chain maps.** In this section, we examine the composition raw

$$h_{\alpha\gamma} = h_{\beta\gamma} \circ h_{\alpha\beta}$$

of the Floer's canonical isomorphism

$$h_{\alpha\beta} : HF_*(H_\alpha) \to HF_*(H_\beta). \tag{3.11}$$

Although the above isomorphism *in homology* depends only on the end Hamiltonians H_α and H_β, the corresponding chain map depends on the homotopy $\mathcal{H} = \{H(\eta)\}_{0 \leq \eta \leq 1}$ between H_α and H_β, and also on the homotopy $j = \{J(\eta)\}_{0 \leq \eta \leq 1}$. Let us fix nondegenerate Hamiltonians H_α, H_β and a homotopy $\mathcal{H}$ between them. We then fix a homotopy $j = \{J(\eta)\}_{0 \leq \eta \leq 1}$ of compatible almost complex structures and a cut-off function $\rho : \mathbb{R} \to [0, 1]$.

We recall that we have imposed the homotopy condition

$$[w^+] = [w^- \# u]; \quad [u] = C \quad \text{in} \quad \pi_2(z^-, z^+) \tag{3.12}$$

in the definition of $\mathcal{M}(H, J; [z^-, w^-], [z^+, w^+])$ and of $\mathcal{M}((\mathcal{H}, j; \rho); [z_\alpha, w_\alpha], [z_\beta, w_\beta])$. One consequence of (3.12) is

$$[z^+, w^+] = [z^+, w^- \# u] \quad \text{in} \quad \Gamma$$

but the latter is a weaker condition than the former. In other words, there could be more than one distinct elements C_1, $C_2 \in \pi_2(z^-, z^+)$ such that

$$\mu(z^-, z^+; C_1) = \mu(z^-, z^+; C_2), \quad \omega(C_1) = \omega(C_2).$$

When we are given a homotopy $(\overline{j}, \overline{\mathcal{H}})$ of homotopies with $\overline{j} = \{j_\kappa\}$, $\overline{\mathcal{H}} = \{\mathcal{H}_\kappa\}$, we also define the elongations $\mathcal{H}^{\overline{\rho}}$ of $\mathcal{H}_\kappa$ by a homotopy of cut-off functions $\overline{\rho} = \{\rho_\kappa\}$: we have

$$\mathcal{H}^{\overline{\rho}} = \{\mathcal{H}_\kappa^{\rho_\kappa}\}_{0 \le \kappa \le 1}.$$

Consideration of the parameterized version of (2.20) for $0 \le \kappa \le 1$ defines the chain homotopy map

$$H_{\overline{\mathcal{H}}} : CF_*(H_\alpha) \to CF_*(H_\beta)$$

which has degree $+1$ and satisfies

$$h_{(j_1, \mathcal{H}_1; \rho_1)} - h_{(j_0, \mathcal{H}_0 : \rho_0)} = \partial_{(J^1, H^1)} \circ H_{\overline{\mathcal{H}}} + H_{\overline{\mathcal{H}}} \circ \partial_{(J^0, H^0)}. \tag{3.13}$$

Again the map $H_{\overline{\mathcal{H}}}$ depends on the choice of a homotopy $\overline{j}$ and $\overline{\rho} = \{\rho_\kappa\}_{0 \le \kappa \le 1}$ connecting the two functions ρ_0, ρ_1. Therefore we will denote

$$H_{\overline{\mathcal{H}}} = H_{(\overline{\mathcal{H}}, \overline{j}; \overline{\rho})}$$

as well. Equation (3.13) in particular proves that two chain maps for different homotopies $(j_0, \mathcal{H}_0; \rho_0)$ and $(j_1, \mathcal{H}_1; \rho_1)$ connecting the same end points are chain homotopic [Fl2] and so proves that the isomorphism (3.11) in homology is independent of the homotopies $(\overline{\mathcal{H}}, \overline{j})$ or of $\overline{\rho}$. Now we re-examine the equation (2.18). One key analytic fact in the study of the Floer moduli spaces is an a priori upper bound of the energy, which we will explain in the next section.

Next, we consider the triple

$$(H_\alpha,\, H_\beta,\, H_\gamma)$$

of Hamiltonians and homotopies $\mathcal{H}_1$, $\mathcal{H}_2$ connecting from H_α to H_β and H_β to H_γ respectively. We define their concatenation $\mathcal{H}_1 \# \mathcal{H}_2 = \{H_3(s)\}_{1 \le s \le 1}$ by

$$H_3(s) = \begin{cases} H_1(2s) & 0 \le s \le \frac{1}{2} \\ H_2(2s-1) & \frac{1}{2} \le s \le 1. \end{cases}$$

We note that due to the choice of the cut-off function ρ, the continuity equation (2.18) is *autonomous* for the region $|\tau| > R$ i.e., is invariant under the translation by τ. When we are given a triple $(H_\alpha,\, H_\beta,\, H_\gamma)$, this fact enables us to glue solutions of two such equations corresponding to the pairs (H_α, H_β) and (H_β, H_γ) respectively.

Now a more precise explanation is in order. For a given pair of cut-off functions

$$\rho = (\rho_1, \rho_2)$$

and a positive number $R > 0$, we define an elongated homotopy of $\mathcal{H}_1 \# \mathcal{H}_2$

$$\mathcal{H}_1 \#_{(\rho;R)} \mathcal{H}_2 = \{H_{(\rho;R)}(\tau)\}_{-\infty < \tau < \infty}$$

by

$$H_{(\rho;R)}(\tau, t, x) = \begin{cases} H_1(\rho_1(\tau + 2R), t, x) & \tau \le 0 \\ H_2(\rho_2(\tau - 2R), t, x) & \tau \ge 0. \end{cases}$$

Note that

$$H_{(\rho;R)} \equiv \begin{cases} H_\alpha & \text{for } \tau \leq -(R_1+2R) \\ H_\beta & \text{for } -R \leq \tau \leq R \\ H_\gamma & \text{for } \tau \geq R_2+2R \end{cases}$$

for some sufficiently large R_1, $R_2 > 0$ depending on the cut-off functions ρ_1, ρ_2 and the homotopies $\mathcal{H}_1$, $\mathcal{H}_2$ respectively. *In particular this elongated homotopy is always smooth, even when the usual glued homotopy $\mathcal{H}_1\#\mathcal{H}_2$ may not be so.* We define the elongated homotopy $j_1\#_{(\rho;R)}j_2$ of $j_1\#j_2$ in a similar way.

For an elongated homotopy $(j_1\#_{(\rho;R)}j_2, \mathcal{H}_1\#_{(\rho,R)}\mathcal{H}_2)$, we consider the associated perturbed Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J_3^{\rho(\tau)}\Big(\frac{\partial u}{\partial t} - X_{H_3^{\rho(\tau)}}(u)\Big) = 0 \\ \lim_{\tau\to-\infty} u(\tau) = z^-, \lim_{\tau\to\infty} u(\tau) = z^+ \end{cases}$$

with the condition (3.12).

Now let u_1 and u_2 be given solutions of (2.20) associated to ρ_1 and ρ_2 respectively. If we define the pre-gluing map $u_1\#_R u_2$ by the formula

$$u_1\#_R u_2(\tau,t) = \begin{cases} u_1(\tau+2R,t) & \text{for } \tau \leq -R \\ u_2(\tau-2R,t) & \text{for } \tau \geq R \end{cases}$$

and a suitable interpolation between them by a partition of unity on the region $-R \leq \tau \leq R$, the assignment defines a diffeomorphism

$$(u_1,u_2,R) \to u_1\#_R u_2$$

from

$$\mathcal{M}\Big(j_1,\mathcal{H}_1;[z_1,w_1],[z_2,w_2]\Big) \times \mathcal{M}\Big(j_2,\mathcal{H}_2;[z_2,w_2],[z_3,w_3]\Big) \times (R_0,\infty)$$

onto its image, provided R_0 is sufficiently large. Denote by $\overline{\partial}_{(\mathcal{H},j;\rho)}$ the corresponding perturbed Cauchy-Riemann operator

$$u \mapsto \frac{\partial u}{\partial \tau} + J_3^{\rho(\tau)}\Big(\frac{\partial u}{\partial t} - X_{H_3^{\rho(\tau)}}(u)\Big)$$

acting on the maps u satisfying the asymptotic condition $u(\pm\infty) = z^\pm$ and fixed homotopy condition $[u] = C \in \pi_2(z^-,z^+)$. By perturbing $u_1\#_R u_2$ by the amount that is smaller than the error for $u_1\#_R u_2$ to be a genuine solution, i.e., less than a weighted L^p-norm, for $p > 2$,

$$\|\overline{\partial}_{(\mathcal{H},j;\rho)}(u_1\#_{(\rho;R)}u_2)\|_p$$

in a suitable $W^{1,p}$ space of u's (see [Fl1], [Fl2]), one can construct a unique genuine solution near $u_1\#_R u_2$. By an abuse of notation, we will denote this genuine solution also by $u_1\#_R u_2$. Then the corresponding map defines an embedding

$$\mathcal{M}\Big(j_1,\mathcal{H}_1;[z_1,w_1],[z_2,w_2]\Big) \times \mathcal{M}\Big(j_2,\mathcal{H}_2;[z_2,w_2],[z_3,w_3]\Big) \times (R_0,\infty) \to$$
$$\to \mathcal{M}\Big(j_1\#_{(\rho;R)}j_2, \mathcal{H}_1\#_{(\rho;R)}\mathcal{H}_2;[z_1,w_1],[z_3,w_3]\Big).$$

Especially when we have

$$\mu_{H_\beta}([z_2,w_2]) - \mu_{H_\alpha}([z_1,w_1]) = \mu_{H_\gamma}([z_3,w_3]) - \mu_{H_\beta}([z_2,w_2]) = 0$$

both $\mathcal{M}(j_1, \mathcal{H}_1; [z_1, w_1], [z_2, w_2])$ and $\mathcal{M}(j_2, \mathcal{H}_2; [z_2, w_2], [z_3, w_3])$ are compact, and so consist of a finite number of points. Furthermore the image of the above mentioned embedding exhausts the 'end' of the moduli space

$$\mathcal{M}\Big(j_1 \#_{(\rho;R)} j_2, \mathcal{H}_1 \#_{(\rho;R)} \mathcal{H}_2; [z_1, w_1], [z_3, w_3]\Big)$$

and the boundary of its compactification consists of the broken trajectories

$$u_1 \#_{(\rho;\infty)} u_2 = u_1 \#_\infty u_2.$$

This then proves the following gluing identity

Proposition 3.5. *There exists $R_0 > 0$ such that for any $R \geq R_0$ we have*

$$h_{(\mathcal{H}_1, j_1)\#_{(\rho;R)}(\mathcal{H}_2, j_2)} = h_{(\mathcal{H}_1, j_1; \rho_1)} \circ h_{(\mathcal{H}_2, j_2; \rho_2)}$$

as a chain map from $CF_*(H_\alpha)$ to $CF_*(H_\gamma)$.

Here we remind the readers that the homotopy $\mathcal{H}_1 \#_{(\rho;R)} \mathcal{H}_2$ itself is an elongated homotopy of the glued homotopy $\mathcal{H}_1 \# \mathcal{H}_2$. This proposition then gives rise to the composition law $h_{\alpha\gamma} = h_{\beta\gamma} \circ h_{\alpha\beta}$ in homology.

4. Energy estimates and Hofer's geometry

4.1. **Energy estimates and the action level changes.** Let us fix the Hamiltonians H_α, H_β and a homotopy $\mathcal{H}$ between them. We emphasize that H_α and H_β are *not* necessarily nondegenerate for the discussion of this section.

We choose a homotopy $j = \{J(\eta)\}_{0 \leq \eta \leq 1}$ of compatible almost complex structures and a cut-off function $\rho : \mathbb{R} \to [0, 1]$. We would like to mention that the homotopies can be constant when $H_\alpha = H_\beta$.

Now we re-examine the equation (2.18) (also (2.15) as a special case where $\mathcal{H} \equiv H$ and $j \equiv J$). One key analytic fact on the study of moduli spaces of the equations is an a priori upper bound of the energy

$$E_{(\mathcal{H},j;\rho)}(u) := \frac{1}{2} \int_{-\infty}^{\infty} \int_0^1 \Big(\Big|\frac{\partial u}{\partial \tau}\Big|^2_{J^{\rho(\tau)}} + \Big|\frac{\partial u}{\partial t} - X_{H^{\rho(\tau)}}(u)\Big|^2_{J^{\rho(\tau)}} \Big)\, dt\, d\tau$$

for the solutions u of (2.18) with (3.12). In this respect, the following identity is crucial.

Lemma 4.1. *Let $(\mathcal{H}, j)$ be any pair and ρ be any cut-off function as above. Suppose that u satisfies (2.18) with (3.12), has finite energy and satisfies*

$$\lim_{j\to\infty} u(\tau_j^-) = z^-, \quad \lim_{j\to\infty} u(\tau_j^+) = z^+$$

for some sequences $\tau_j^\pm$ with $\tau_j^- \to -\infty$ and $\tau_j^+ \to \infty$. Then we have

$$\begin{aligned} \mathcal{A}_F([z^+, w^+]) \quad - \quad & \mathcal{A}_H([z^-, w^-]) = -\int \Big|\frac{\partial u}{\partial \tau}\Big|^2_{J^{\rho(\tau)}} \\ - \quad & \int_{-\infty}^{\infty} \rho'(\tau) \int_0^1 \Big(\frac{\partial H^s}{\partial s}\Big|_{s=\rho(\tau)} (t, u(\tau, t)) \Big)\, dt\, d\tau \end{aligned} \tag{4.1}$$

Corollary 4.2. *Let $(\mathcal{H}, j; \rho)$ and u be as in Lemma 4.1.*

(1) *Suppose that ρ is monotone. Then we have*

$$\begin{aligned} \mathcal{A}_F([z^+,w^+]) - \mathcal{A}_H([z^-,w^-]) &\leq -\int \left|\frac{\partial u}{\partial \tau}\right|^2_{J^{\rho_1(\tau)}} + \int_0^1 -\min_{x,s}\left(\frac{\partial H_t^s}{\partial s}\right) dt \qquad (4.2) \\ &\leq \int_0^1 -\min_{x,s}\left(\frac{\partial H_t^s}{\partial s}\right) dt. \qquad (4.3) \end{aligned}$$

And (4.2) can be rewritten as the upper bound for the energy

$$\begin{aligned} \int \left|\frac{\partial u}{\partial \tau}\right|^2_{J^{\rho_1(\tau)}} &\leq \mathcal{A}_H([z^+,w^+]) - \mathcal{A}_F([z^-,w^-]) \\ &\quad + \int_0^1 -\min_{x,s}\left(\frac{\partial H_t^s}{\partial s}\right) dt. \qquad (4.4) \end{aligned}$$

(2) *For a general ρ, we instead have*

$$\begin{aligned} \mathcal{A}_F([z^+,w^+]) - \mathcal{A}_H([z^-,w^-]) &\leq -\int \left|\frac{\partial u}{\partial \tau}\right|^2_{J^{\rho_1(\tau)}} + \int_0^1 \max_{x,s}\left|\frac{\partial H_t^s}{\partial s}\right| dt \qquad (4.5) \\ &\leq \int_0^1 \max_{x,s}\left|\frac{\partial H_t^s}{\partial s}\right| dt. \qquad (4.6) \end{aligned}$$

And (4.6) can be rewritten as the upper bound for the energy

$$\int \left|\frac{\partial u}{\partial \tau}\right|^2_{J^{\rho_1(\tau)}} \leq \mathcal{A}_H([z^+,w^+]) - \mathcal{A}_F([z^-,w^-]) + \int_0^1 \max_{x,s}\left|\frac{\partial H_t^s}{\partial s}\right| dt. \qquad (4.7)$$

The proof is an immediate consequence of (4.1) and omitted.

Here we would like to emphasize that *the above various energy upper bounds do not depend on u or on the choice of j or ρ, but depend only on the homotopy $\mathcal{H}$ itself* and the asymptotic condition of u.

Motivated by the upper estimate (4.3), we introduce the following definition

Definition 4.1. Let $\mathcal{H} = \{H(s)\}_{0\leq s\leq 1}$ be a homotopy of Hamiltonians. We define the *negative part of the variation* and the *positive part of the variation* of $\mathcal{H}$ by

$$\begin{aligned} E^-(\mathcal{H}) &:= \int_0^1 -\min_{x,s}\left(\frac{\partial H_t^s}{\partial s}\right) dt \\ E^+(\mathcal{H}) &:= \int_0^1 \max_{x,s}\left(\frac{\partial H_t^s}{\partial s}\right) dt. \end{aligned}$$

And we define the *total variation* $E(\mathcal{H})$ of $\mathcal{H}$ by

$$E(\mathcal{H}) = E^-(\mathcal{H}) + E^+(\mathcal{H}).$$

If we denote by $\mathcal{H}^{-1}$ the time reversal of $\mathcal{H}$, i.e., the homotopy given by

$$\mathcal{H}^{-1} : s \in [0,1] \mapsto H^{1-s}$$

then we have the identity

$$E^\pm(\mathcal{H}^{-1}) = E^\pm(\mathcal{H}) \quad \text{and } E(\mathcal{H}^{-1}) = E(\mathcal{H}).$$

With these definitions, applied to a pair $(\mathcal{H}, j)$ such that their ends $H(0)$ and $H(1)$ are nondegenerate, the a priori energy estimate (4.3) can be written as

$$\int \left|\frac{\partial u}{\partial \tau}\right|^2_{J^{\rho(\tau)}} \leq -\mathcal{A}_F(u(\infty)) + \mathcal{A}_H(u(-\infty)) + E^-(\mathcal{H})$$

for a monotone ρ, and (4.6) as

$$\int \Big|\frac{\partial u}{\partial \tau}\Big|^2_{J^{\rho(\tau)}} \leq -\mathcal{A}_F(u(\infty)) + \mathcal{A}_H(u(-\infty)) + E(\mathcal{H})$$

for a general ρ. Here we recall that, when the Hamiltonian is nondegenerate, any finite energy solution has well-defined asymptotic limits as $\tau \to \pm$ [Fl1].

Corollary 4.3. *Let (H, J) and be given. Then for any finite energy solution u of (2.15) with (3.12), we have*

$$\mathcal{A}_H([z^+, w^+]) - \mathcal{A}_H([z^-, w^-]) \leq -\int \Big|\frac{\partial u}{\partial \tau}\Big|^2_J \leq 0. \tag{4.8}$$

In particular, when (H, J) is Floer-regular, then the associated boundary map $\partial_{(H,J)}$ satisfies

$$\partial_{(H,J)}(CF_*^\lambda(H)) \subset CF_*^\lambda(H)$$

and hence canonically restricts to a boundary map

$$\partial_{(H,J)} : (CF_*^\lambda(H), \partial_{(H,J)}) \to (CF_*^\lambda(H), \partial_{(H,J)})$$

for any real number $\lambda \in \mathbb{R}$.

We denote by $HF_*^\lambda(H, J)$ the associated filtered homology and call it the filtered Floer homology group.

Corollary 4.4. *Suppose (H^0, J^0) and (H^1, J^1) are Floer regular, $(\mathcal{H}, j)$ is a Floer-regular path between them, and ρ is as before. Then the chain map $h_{(\mathcal{H},j;\rho)}$ satisfies*

$$h_{(\mathcal{H},j;\rho)}(CF_*^\lambda(H^0)) \subset CF_*^{\lambda+E^-(\mathcal{H})}(H^1)$$

and so canonically restricts to a chain map

$$h_{(\mathcal{H},j;\rho)} : (CF_*^\lambda(H^0), \partial_{(H^0,J^0)}) \to (CF_*^{\lambda+E^-(\mathcal{H})}(H^1), \partial_{(H^1,J^1)}).$$

One particular case of Corollary 4.2 and Corollary 4.4 is worthwhile to mention separately which will be used in the construction of the spectral invariants $\rho(H; a)$ later. The same result was used in [Oh3] for the spectral invariants of Lagrangian submanifolds on the cotangent bundle.

Corollary 4.5. *Let H be given. Consider two J^0 and J^1, a cut-off function ρ and the homotopy $(\mathcal{H}, j)$ between (H, J^0) and (H, J^1) satisfying $\mathcal{H} \equiv H$. Then for any finite energy solution u of (2.18) with (3.12), we have*

$$\mathcal{A}_H([z^+, w^+]) - \mathcal{A}_H([z^-, w^-]) \leq -\int \Big|\frac{\partial u}{\partial \tau}\Big|^2_{J^{\rho(\tau)}} \leq 0. \tag{4.9}$$

In particular, when H is nondegenerate and J^0, J^1 are H-regular and (H, j) is generic, then the associate chain map $h_{(H,j);\rho}$ satisfies

$$h_{(\mathcal{H},j);\rho}(CF_*^\lambda(H)) \subset CF_*^\lambda(H)$$

and hence canonically restricts to a chain map

$$h^\lambda_{(H,j);\rho} : (CF_*^\lambda(H), \partial_{(H,J^0)}) \to (CF_*^\lambda(H), \partial_{(H,J^1)})$$

and induces an isomorphism in homology for any real number $\lambda \in \mathbb{R}$.

Proof. It remains to prove that $h^\lambda_{(H,j);\rho}$ induces an isomorphism in homology. For this, we choose any homotopy j' connecting from J^1 to J^0 such that (H,j) is Floer-regular, and a cut-off function ρ. Then we consider the $j\#j'$ which connects from J^0 to J^0. Now we deform $j\#j'$ to the constant homotopy $j_{const} \equiv J^0$. We denote the homotopy of homotopy by $\overline{j}$ connecting from j_{const} to $j\#j'$. Then by (4.9), $(H,\overline{j})$ provides a chain homotopy from $h_{(H,j_{const};\rho)}$ and $h_{(H,j)\#_{(\rho;R)}(H,j')}$. We note that since $j_{const} \equiv J^0$, the elongated homotopy of (H^ρ, j^ρ_{const}) becomes the constant homotopy (H, J^0). Therefore by the Floer-regularity hypothesis of (H, J^0) *as a family*, we derive $h_{(H,j_{const};\rho)} = id$. On the other hand, by choosing $R > 0$ sufficiently large, we have the gluing identity

$$h_{(H,j)\#_{(\rho;R)}(H,j')} = h_{(H,j;\rho)} \circ h_{(H,j';\rho)}$$

Therefore we have proved $h_{(H,j;\rho)} \circ h_{(H,j';\rho)} = id$ on $HF^\lambda_*(H,J^0)$. By the same argument, we also have $h_{(H,j';\rho)} \circ h_{(H,j;\rho)} = id$ on $HF^\lambda_*(H,J^1)$. □

4.2. **Energy estimates and Hofer's norm.** We first recall some basic definitions and facts used in Hofer's geometry of the Hamiltonian diffeomorphism group. In this section, we consider general time dependent Hamiltonian functions which are *not necessarily* one-periodic.

We call a smooth map $\lambda : [0,1] \to Ham(M,\omega)$ a *Hamiltonian path.* According to Banyaga's theorem [Ba], to any such path issued at the identity of $Ham(M,\omega)$ is associated a *unique* normalized smooth Hamiltonian $H : [0,1] \times M \to \mathbb{R}$ satisfying

$$\lambda(t) = \phi^t_H.$$

We will denote the Hamiltonian path generated by H by ϕ_H. Therefore there is a one-one correspondence

$$C^\infty_m([0,1] \times M, \mathbb{R}) \longleftrightarrow \mathcal{P}(Ham(M,\omega), id). \tag{4.10}$$

For a given Hamiltonian diffeomorphism $\phi \in Ham(M,\omega)$, we denote

$$H \mapsto \phi$$

if $\phi = \phi^1_H$.

Remark 4.2. We remind the readers that the one-one correspondence (4.10) holds only in the *smooth* category. It is a fundamental task to understand what is happening when we go down to the Hamiltonians with low regularity, especially in the continuous category. We refer to [Oh10] for a detailed study of this issue.

We recall the standard definitions

$$\begin{aligned} E^-(H) &= \int_0^1 -\min_x H_t\, dt, \quad E^+(H) = \int_0^1 \max_x H_t\, dt \\ \|H\| \,(= E(H)) &= E^+(H) + E^-(H) = \int_0^1 (\max_x H_t - \min_x H_t)\, dt \end{aligned}$$

used in Hofer's geometry. (See [Po3] for example.)

Note that when $\mathcal{H}$ is the linear homotopy

$$\mathcal{H}^{lin} : s \mapsto (1-s)H_1 + sH_2$$

between H_1 and H_2, $E^\pm(\mathcal{H}^{lin})$ and $E(\mathcal{H}^{lin})$ just become $E^\pm(H_2 - H_1)$, and $\|H_2 - H_1\|$, respectively. In fact, $E^\pm(H)$ or $E(H)$ correspond to the variations of the *linear path*

$$s \in [0,1] \mapsto sH$$

in the sense of Definition 4.1. On the other hand, when H is non-autonomous, this linear path does not seem to have much intrinsic meaning in terms of the geometry of $Ham(M,\omega)$ itself. It would be desirable to discover more intrinsic invariants attached to a Hamiltonian path $\lambda \in \mathcal{P}(Ham(M,\omega), id)$.

We first state the following basic facts in the algebra of Hamiltonian functions. (See [Ho1]).

Proposition 4.6. *Let H and F be arbitrary Hamiltonians, not necessarily one-periodic.*

(1) *If $H \mapsto \phi$, $\overline{H} \mapsto \phi^{-1}$ where $\overline{H}$ is defined by*

$$\overline{H}(t,x) := -H(t, \phi_H^t(x)).$$

(2) *If $H \mapsto \phi$, $F \mapsto \psi$, then we have*

$$H\#F \mapsto \phi \circ \psi$$

where $H\#F$ is the Hamiltonian defined by

$$(H\#F)(t,x) := H(t,x) + F(t, (\phi_H^t)^{-1}(x)).$$

Corollary 4.7. *$Ham(M,\omega) \subset Symp(M,\omega)$ forms a Lie subgroup and its associated Lie algebra is (anti)-isomorphic to $(C_m^\infty(M), \{\cdot,\cdot\})$ where $\{\cdot,\cdot\}$ is the Poisson bracket associated to ω.*

Remark 4.3. We would like to mention that, even when $H \sim F$,

$$\|H\| \neq \|F\|.$$

Therefore the map $H \mapsto \|H\|$ does not push down to the universal (étale) covering space $\pi : \widetilde{Ham}(M,\omega) \to Ham(M,\omega)$. One standard way of defining an invariant for the elements $h \in \widetilde{Ham}(M,\omega)$ is by taking the infimum

$$\|h\| := \inf_{[H]=h} \|H\| = \inf_{[H]=h} \operatorname{leng}(\phi_H). \tag{4.11}$$

This function

$$h \in \widetilde{Ham}(M,\omega) \mapsto \|h\| \in \mathbb{R}_+$$

is *not* a priori continuous with respect to the natural topology on $\widetilde{Ham}(M,\omega)$. However we will see later that our construction in [Oh9], [Oh11] naturally provides a *continuous* invariant of $h \in \widetilde{Ham}(M,\omega)$.

Definition 4.4. [The Hofer norm] For $\phi \in Ham(M,\omega)$, the Hofer norm, denoted by $\|\phi\|$, is defined by

$$\|\phi\| := \inf_{H \mapsto \phi} \|H\| (= \inf_{\pi(h)=\phi} \|h\|).$$

Then except the proof of nondegeneracy, the proof of the following theorem is straightforward. Nondegeneracy was proven by Hofer [Ho1] for $\mathbb{C}^n$, by Polterovich [Po1] for tame rational symplectic manifolds, and by Lalonde-McDuff [LM1] in complete generality.

Theorem 4.8. *Let ϕ, $\psi \in Ham(M,\omega)$. Then we have*

(1) (Symmetry) $\|\phi^{-1}\| = \|\phi\|$
(2) (Triangle inequality) $\|\phi\psi\| \leq \|\phi\| + \|\psi\|$
(3) (Symplectic invariance) $\|\eta\phi\eta^{-1}\| = \|\phi\|$ *for any symplectic diffeomorphism* η.
(4) (Nondegeneracy) $\|\phi\| = 0$ *if and only if* $\phi = id$.

We now note that for the linear path

$$\mathcal{H}^{lin} : s \mapsto (1-s)H_\alpha + sH_\beta$$

we have

$$E^\pm(\mathcal{H}^{lin}) = E^\pm(H_\beta - H_\alpha)$$

and in particular, the pseudo-norms $E^\pm(H)$ and $\|H\|$ correspond to the variation of the linear path

$$s \mapsto sH$$

connecting the zero Hamiltonian to H.

Taking the infimum of $E(\mathcal{H})$ over all $\mathcal{H}$ with fixed end points $H(0) = H^0$ and $H(1) = H^1$, we have the inequality

$$\inf_{\mathcal{H}} \left\{ E(\mathcal{H}) \mid H(0) = H^0,\, H(1) = H^1 \right\} \leq \|H^1 - H^0\|$$

which is a strict inequality in general. It seems to be an interesting problem to investigate the geometric meaning of the quantity in the left hand side.

Next, we consider the triple

$$(H_\alpha,\, H_\beta,\, H_\gamma)$$

of Hamiltonians and homotopies $\mathcal{H}_1$, $\mathcal{H}_2$ connecting from H_α to H_β and H_β to H_γ respectively. We define their concatenation $\mathcal{H}_1 \# \mathcal{H}_2$ as defined in subsection 3.5. From the definitions of $E^\pm$ and E for the homotopy $\mathcal{H}$ above, we immediately have the following lemma

Lemma 4.9. *All $E^\pm$ and E are additive under the concatenation of homotopies. In other words, for any triple $(H_\alpha, H_\beta, H_\gamma)$ and homotopies $\mathcal{H}_1$, $\mathcal{H}_2$ as above, we have*

$$E^\pm(\mathcal{H}_1 \# \mathcal{H}_2) = E^\pm(\mathcal{H}_1) + E^\pm(\mathcal{H}_2).$$

The same additivity holds for E.

4.3. **Level changes of Floer chains under the homotopy.** In this subsection, we consider nondegenerate Hamiltonians H and the Floer regular pairs (H, J). Similarly we will only consider the Floer regular homotopy $(\mathcal{H}, j)$ connecting those Floer regular pairs. We also consider homotopy of homotopies, $(\overline{\mathcal{H}}, \overline{j})$ with $\overline{\mathcal{H}} = \{\mathcal{H}_\kappa\}_{0 \leq \kappa \leq 1}$ a nd $\overline{j} = \{j_\kappa\}_{0 \leq \kappa \leq 1}$ and the induced chain homotopy map $H_{\mathcal{H}} = \mathcal{H}_{(\overline{\mathcal{H}}, \overline{j}; \rho)}$.

The following proposition shows how the level of α changes under the various Floer operators.

Proposition 4.10. *Suppose that ρ is a (positively) monotone cut-off function.*

(1) $\lambda_H(\partial_{(H,J)}(\alpha)) < \lambda_H(\alpha)$ *for an arbitrary Floer chain* α.
(2) $\lambda_{H^1}(h_{(\mathcal{H},j;\rho)}(\alpha)) \leq \lambda_{H^0}(\alpha) + E^-(\mathcal{H})$ *for an arbitrary choice of* ρ
(3) $\lambda_{H^1}(H_{\overline{\mathcal{H}}}(\alpha)) \leq \lambda_{H^0}(\alpha) + \max_{\kappa \in [0,1]} E^-(\mathcal{H}_\kappa)$.

Proof. (1) and (2) are immediate consequences of Corollary 4.2.

For the proof of (3), let $[z', w'] \in \mathrm{Crit}\mathcal{A}_{H^1}$ be the peak of the chain. By the definition of the chain map $H_{\mathcal{H}}(\alpha)$, there exists a generator $[z, w] \in \mathrm{Crit}\mathcal{A}_{H^0}$ and a parameter $\kappa \in (0,1)$ such that the equation

$$\frac{\partial u}{\partial \tau} + J^{\kappa,\rho}\Big(\frac{\partial u}{\partial t} - X_{H^{\kappa,\rho}}(u)\Big) = 0 \tag{4.12}$$

with the asymptotic condition

$$u(-\infty) = [z, w], \quad u(\infty) = [z', w']$$

has a solution for some generator $[z, w]$ of α. Then by (4.5), we derive

$$\mathcal{A}_{H^1}([z', w']) - \mathcal{A}_{H^0}([z, w]) \leq E^-(\mathcal{H}_\kappa)$$

i.e.,

$$\mathcal{A}_{H^1}([z', w']) \leq \mathcal{A}_{H^0}([z, w]) + E^-(\mathcal{H}_\kappa). \tag{4.13}$$

Since we have chosen $[z', w']$ to be the peak of $H_{\mathcal{H}}(\alpha)$, applying (4.5) for the pair $(\mathcal{H}_\kappa, j_\kappa)$ using the arguments similar to the above, we prove

$$\mathcal{A}_{H^1}(H_{\overline{\mathcal{H}}}(\alpha)) \leq \lambda_{H^0}(\alpha) + E^-(\mathcal{H}_\kappa).$$

By taking the supremum of the right hand side of this inequality over $\kappa \in (0,1)$, we have proved (3). ☐

Remark 4.5. We would like to emphasize that the $(\mathcal{H}_\kappa, j_\kappa)$ is not Floer-regular, but a minimally *degenerate* pair, and that (3) is not a special case of (2).

We denote

$$E^-(\overline{\mathcal{H}}) := \max_{\kappa \in [0,1]} E^-(\mathcal{H}_\kappa).$$

Then we have the following corollary of Proposition 4.10 (3).

Corollary 4.11. *Let (H^0, J^0) and (H^1, J^1) be two Floer regular pairs. Consider a generic homotopy of homotopies, $(\overline{\mathcal{H}}, \overline{j})$ with*

$$\overline{\mathcal{H}} = \{\mathcal{H}_\kappa\}_{0 \leq \kappa \leq 1}, \overline{j} = \{j_\kappa\}_{0 \leq \kappa \leq 1}$$

where each $\mathcal{H}_\kappa$ is a homotopy connecting (H^0, J^0) and (H^1, J^1). Then the induced chain homotopy map $H_{\mathcal{H}} = \mathcal{H}_{(\overline{\mathcal{H}}, \overline{j}; \rho)}$ satisfies

$$H_{\mathcal{H}}(CF^\lambda(H^0)) \subset CF^{\lambda + E^-(\overline{\mathcal{H}})}(H^1).$$

4.4. **The ϵ-regularity type invariants.** We recall a well-known invariant of the almost Kähler structure (M, ω, J_0) defined by

$$A(\omega, J_0) := \inf\{\omega(v) \mid v : S^2 \to M \text{ is non-constant and } \overline{\partial}_{J_0} v = 0\}.$$

This is known to be positive. We call $A(\omega, J_0)$ the *ϵ-regularity invariant* because its positivity is a consequence of the so called the ϵ-regularity theorem in the geometric analysis [SU]. We refer to [Oh9] for the details of such a proof.

We now generalize this invariant for any compact family $K \subset \mathcal{J}_\omega$ of compatible almost complex structures. Let

$$K : [0,1]^n \to \mathcal{J}_\omega$$

be a continuous n-parameter family in the C^1-topology, and define $A(\omega; K)$ be the constant

$$A(\omega; K) = \inf_{\kappa \in [0,1]^n} \Big\{ A(\omega, J(\kappa)) \Big\}.$$

This is always positive (see Proposition 8.5), and enjoys the following lower semi-continuity property. We refer to [Oh11] for its proof.

Proposition 4.12. *$A(\omega;K)$ is lower semi-continuous in K. In other words, for any given K and $0<\epsilon<A(\omega;K)$, there exists some $\delta=\delta(K,\epsilon)>0$ such that for any K' with $\|K'-K\|_{C^1}\leq\delta$ we have*

$$A(\omega;K')\geq A(\omega;K)-\epsilon.$$

We now introduce two other invariants of the ϵ-regularity type associated to the perturbed Cauchy-Riemann equations. We first remark that our family $J=\{J_t\}_{0\leq t\leq 1}$ is a special case of the compact family K above with $n=1$.

Let H be a given nondegenerate Hamiltonian function and consider the perturbed Cauchy- Riemann equation

$$\frac{\partial u}{\partial\tau}+J\Big(\frac{\partial u}{\partial t}-X_H(u)\Big)=0$$

for each H-regular J. We call a solution u *stationary* if it is τ-independent. We define

$$A_{(H,J)}:=\inf\Big\{\int\Big|\frac{\partial u}{\partial\tau}\Big|_J^2\,\Big|\,u\text{ satisfies (2.15) and is not stationary }\Big\}$$

and

$$A^\mu_{(H,J)}:=\inf\Big\{\int\Big|\frac{\partial u}{\partial\tau}\Big|_J^2\,\Big|\,u\text{ satisfies (2.15) and }\mu_H(u)=1\Big\}.$$

The positivity of $A_{(H,J)}$ is an easy consequence of the Gromov compactness type theorem (see [Oh9] for details of such a proof). Obviously we have $A^\mu_{(H,J)}\geq A_{(H,J)}$.

Then we can strengthen the statement (1) of Proposition 4.10 to the following inequality

$$\lambda_H(\partial_{(H,J)}(\alpha))\leq\lambda_H(\alpha)-A^\mu_{(H,J)}\tag{4.14}$$

for an arbitrary Floer chain α.

5. Definition of spectral invariants and their axioms

5.1. **Floer complex of a small Morse function.**

We start this section with the study of the Floer complex $(CF(H),\partial_{(H,J)})$, as a complex with the Novikov ring as its coefficients, for the special case

$$H=\epsilon f,\quad J\equiv J_0$$

when $\epsilon>0$ is sufficiently small. Here f is any Morse function. The following theorem was essentially proven by Floer in [Fl2]. Also see [HS], [FOn], [LT1].

Theorem 5.1. *Let f be any small Morse function on M and J_0 be a compatible almost complex structure such that f is Morse-Smale with respect to the metric g_{J_0}. Then there exists $\epsilon_0>0$ such that for all $0<\epsilon\leq\epsilon_0$ we have the chain isomorphism*

$$(CF_*(\epsilon f),\partial_{(\epsilon f,J_0)})\cong(CM_*(-\epsilon f),\partial^{Morse}_{(-\epsilon f,g_{J_0})})\otimes\Lambda^\downarrow_\omega.$$

Once we have this theorem, applying the Poincaré duality

$$(CM^*(-\epsilon f),\delta^{Morse}_{(-\epsilon f,g_{J_0})})\cong(CM_{2n-*}(-\epsilon f),\partial^{Morse}_{(-\epsilon f,g_{J_0})}),$$

we have the natural canonical isomorphism

$$H^*(M)\otimes\Lambda^\uparrow\cong HF_*(\epsilon f,J_0)=:HF_*(\epsilon f)$$

as a $\mathbb{Q}$-vector space. Here the grading $*$ in HF_* stands for the degree of the Floer cycle α which is provided by the Conley-Zehnder index of its generators. We refer to [Oh8] (and also to 7.2) for a detailed discussion on this grading problem.

We also recall that $H^*(M) \otimes \Lambda_\omega^\uparrow$ is isomorphic to the quantum cohomology $QH^*(M)$ as a $\Lambda_\omega^\uparrow$-module, by definition. In this sense, the complex

$$(CM^*(-\epsilon f), \delta^{Morse}_{(-\epsilon f, g_{J_0})}) \otimes \Lambda_\omega^\uparrow$$

provides the chain complex of the quantum cohomology. Composing the isomorphism

$$QH^*(M) \cong H^*(M) \otimes \Lambda^\uparrow \cong HF_*(\epsilon f) \cong HF_*(H)$$

after incorporating a grading consideration, we obtain a natural isomorphism

$$QH^{n-k}(M) \cong HF_k(H)$$

as a graded $\mathbb{Q}$-vector space. We would like to emphasize that *for non-exact* (M, ω), there will be *no* isomorphism between them, *unless we reverse the direction of the Novikov rings*. We recall that $QH^*(M)$ is a module over $\Lambda_\omega^\uparrow$, while $HF_*(H)$ is one over $\Lambda_\omega^\downarrow$.

5.2. **Definition of spectral invariants.** For each given (homogeneous) quantum cohomology class $a \in QH^*(M)$, we denote by $a^\flat = a_H^\flat \in HF_*(H, J)$ the image under the above isomorphism. We denote by

$$i_\lambda : HF_*^\lambda(H, J) \to HF_*(H, J)$$

the canonical inclusion induced homomorphism.

Definition 5.1. Let H be a nondegenerate Hamiltonian and J be H-regular. For any given $0 \neq a \in QH^*(M)$, we consider Floer cycles $\alpha \in ZF_*(H, J) \subset CF_*(H)$ of the pair (H, J) representing $a^\flat$. Then we define

$$\rho((H, J); a) := \inf_{\alpha; [\alpha] = a^\flat} \lambda_H(\alpha),$$

or equivalently

$$\rho((H, J); a) := \inf\{\lambda \in \mathbb{R} \mid a^\flat \in \operatorname{Im} i_\lambda \subset HF_*(H, J)\}.$$

We will mostly use the first definition in our exposition, which is more intuitive and flexible to use in practice. The following theorem was proved in [Oh8]. Because its proof illustrates the typical argument in our chain level mini-max theory, we provide more details of the proof than we did in [Oh8].

Theorem 5.2. *Suppose that H is nondegenerate and let $0 \neq a \in QH^*(M)$.*

(1) *We have $\rho((H, J); a) > -\infty$ for any H-regular J.*
(2) *The definition of $\rho((H, J); a)$ does not depend on the choice of H-regular J's. We denote by $\rho(H; a)$ the common value.*

Proof. We will give the proof in several steps. We write the quantum cohomology class a as

$$a = \sum_{A \in \Gamma} a_A q^{-A}.$$

Let $\Gamma(a) \subset \Gamma$ be the support of a, i.e., the set of $A \in \Gamma$ with $a_A \neq 0$ in this sum. By the definitions of the quantum cohomology and of the Novikov ring, we can enumerate $\Gamma(a) = \{A_j\}_{j \in \mathbb{Z}_+}$ so that

$$\omega(-A_1) < \omega(-A_2) < \cdots < \omega(-A_j) < \cdots \to \infty$$

or equivalently

$$\omega(A_1) > \omega(A_2) > \cdots > \omega(A_j) \to -\infty$$

In that case, we denote $A_1 =: A_a$.

Definition 5.2. For each homogeneous element

$$a = \Sigma_{A\in\Gamma} a_A q^{-A} \in QH^k(M), \quad a_A \in H^*(M, \mathbb{Q}) \tag{5.1}$$

of degree k, we define $v(a)$ by

$$v(a) = \omega(-A_1)$$

and call it the *level* of a. And we define the *leading order term* of a by

$$\sigma(a) := a_{A_1} q^{-A_1}.$$

We also call a_{A_1} the *leading order coefficient* of a.

Note that the leading order term $\sigma(a)$ of a *homogeneous* element a is unique among the summands in the sum by the definition of Γ.

Step 1. We first prove the following proposition.

Proposition 5.3. *Suppose that J is H-regular and $\rho((H,J);a)$ is finite, i.e., $\rho((H,J);a) > -\infty$. Then $\rho((H,J');a)$ is also finite for any other H-regular J' and satisfies*

$$\rho((H,J);a) = \rho((H,J');a).$$

Proof. Let $\alpha' \in CF(H)$ be a Floer cycle of (H,J') with $[\alpha'] = a^\flat$. We choose a generic homotopy $j' = \{J'(s)\}_{0\leq s\leq 1}$ satisfying $J'(0) = J'$ and $J'(1) = J$ the constant homotopy $\mathcal{H} = H$, and pick a cut-off function ρ'. We then consider the corresponding chain map $h_{(H,j');\rho'} : CF(H) \to CF(H)$ and the image cycle $h_{(H,j');\rho}(\alpha')$ of (H,J). Since $[h_{(H,j);\rho}(\alpha')] = a^\flat$ in $HF_*(H,J)$, we have

$$\lambda_H(h_{(H,j');\rho}(\alpha')) \geq \rho((H,J);a). \tag{5.2}$$

On the other hand, Corollary 4.5 implies

$$\lambda_H(h_{(H,j');\rho}(\alpha')) \leq \lambda_H(\alpha') \tag{5.3}$$

Combining (5.2) and (5.3), we have obtained

$$\lambda_H(\alpha') \geq \rho((H,J);a).$$

By taking the infimum over all Floer cycles α' of (H,J'), we obtain

$$\rho((H,J');a) \geq \rho((H,J);a). \tag{5.4}$$

In particular, we have also proven that $\rho((H,J');a)$ is finite. Once we have proved finiteness of $\rho((H,J');a)$, we can change the role of J and J', we have also obtain the opposite inequality

$$\rho((H,J);a) \geq \rho((H,J');a)$$

and hence $\rho((H,J');a) = \rho((H,J);a)$. This finishes the proof. □

Step 2. Let f be any Morse function and J_0 be a compatible almost complex structure such that the pair $(-\epsilon f, g_{J_0})$ is Morse-Smale. We fix a sufficiently small $\epsilon > 0$ so that Theorem 5.1 holds. We will prove the finiteness of $\rho((\epsilon f, J_0);a)$, which corresponds to the *linking property* of the classical critical point theory (see [BnR] for example).

Let $\alpha \in ZF(\epsilon f, J_0) \subset CF(\epsilon f)$ be a Floer cycle representing $a^\flat$. It follows from Theorem 5.1 that α has the form

$$\alpha = \sum_{A \in \Gamma} \alpha_A q^A$$

where $\alpha_A \in CM_*(-\epsilon f)$ is a Morse cycle of $(-\epsilon f, g_{J_0})$, i.e.,

$$\partial^{Morse}_{(-\epsilon f, g_{J_0})} \alpha_A = 0$$

and its corresponding homology class satisfies $[\alpha_A] = PD(a_A)$, the Poincaré dual to a_A. Since $[\alpha] = a^\flat \neq 0$, there is at least one α_A whose Morse homology class of $-\epsilon f$ is not zero. By removing the boundary terms from α, which only possibly *decreases* the level of chains, we obtain the following lemma, whose proof we refer to [Oh8].

Lemma 5.4. *There exists another Floer cycle $\alpha' \in ZF_*(\epsilon f, J_0)$ such that α' and α are homologous, and α' has the form*

$$\alpha' = \sum_{A \in \Gamma(a)} \alpha'_A q^A$$

such that $[\alpha'_A] = PD(a_A)$ and $\lambda_{\epsilon f}(\alpha') \leq \lambda_{\epsilon f}(\alpha)$.

The upshot of this lemma is that, as far as the mini-max process is concerned, we can safely *fix the support of* α to be the set $\Gamma(a) \subset \Gamma$ when we choose the mini-maxing cycle α in the class $a^\flat$, which does not depend on α but depends only on the class a.

The following is a standard fact in the finite dimensional critical point theory.

Lemma 5.5. *For a given singular homology class $B \in H_*(M)$, we have*

$$\lambda^{Morse}_{-\epsilon f}(\gamma) \geq \min(-\epsilon f) \geq -\epsilon \max f$$

for any Morse cycle γ with $[\gamma] = B$.

Lemma 5.5 and 5.4 then imply

$$\lambda_{\epsilon f}(\alpha) \geq \lambda_{\epsilon f}(\alpha') \geq -\epsilon \max f - \omega(A_a) > -\infty.$$

Then by taking the infimum over all α with $[\alpha] = a$, we have obtained

$$\rho((\epsilon f, J_0); a) = \inf_{[\alpha]=a} \lambda_{\epsilon f}(\alpha) \geq -\epsilon \max f - \omega(A_a) > -\infty.$$

Once have proven the finiteness of this infimum for the pair $(\epsilon f, J_0)$, Proposition 5.3 implies that $\rho((\epsilon f, J); a)$ does not depend on the choice ϵf-regular J.

Step 3. Let (H, J) be any Floer-regular pair. We consider any generic path $\mathcal{H}$ satisfying $H(0) = H$ and $H(1) = \epsilon f$, j with $J(0) = J$ and $J(1) = J_0$ and a cut-off function ρ, such that $(\mathcal{H}, j; \rho)$ is Floer-regular. Let $h_{(\mathcal{H},j);\rho)} : CF(H) \to CF(\epsilon f)$ be the associated chain map. By the similar argument used in Step 1 using (4.6) applied to the homotopy $\mathcal{H}$, we have obtain

$$\rho((\epsilon f, J_0); a) \leq \rho((H, J); a) + E^-(\mathcal{H})$$

and so

$$\rho((H, J); a) \geq \rho((\epsilon f, J_0); a) - E^-(\mathcal{H}) > -\infty. \tag{5.5}$$

This finishes the proof of finiteness of $\rho((H, J); a)$. Then Proposition 5.3 proves that $\rho((H, J); a)$ does not depend on the choice of H-regular pair. Hence the proof of the theorem. $\square$

The following proposition can be proven by the similar arguments used in the proof of (5.5) by considering the homotopy connecting H and K that is arbitrarily close to the linear homotopy

$$s \mapsto (1-s)H + sK.$$

We omit its proof referring to [Oh8] for the details of the proof.

Proposition 5.6. *For any nondegenerate H, K, we have*

$$-E^+(H-K) \leq \rho(H;a) - \rho(K;a) \leq E^-(H-K).$$

In particular, $\rho_a : H \mapsto \rho(H;a)$ is continuous in the C^0-topology or (in the $L^{(1,\infty)}$-topology) and hence can be continuously extended to $C^0_m([0,1]\times M;\mathbb{R})$.

5.3. **Axioms of spectral invariants.** In this subsection, we state basic properties of the function ρ in a list of axioms.

Theorem 5.7. *Let (M,ω) be arbitrary closed symplectic manifold. For any given quantum cohomology class $0 \neq a \in QH^*(M)$, we have a continuous function denoted by*

$$\rho : \mathcal{H}_m \times QH^*(M) \to \mathbb{R}$$

such that they satisfy the following axioms: Let H, $F \in \mathcal{H}_m$ be smooth Hamiltonian functions and $a \neq 0 \in QH^*(M)$. Then ρ satisfies the following axioms:

(1) **(Projective invariance)** $\rho(H;\lambda a) = \rho(H;a)$ *for any* $0 \neq \lambda \in \mathbb{Q}$.
(2) **(Normalization)** *For* $a = \sum_{A\in\Gamma} a_A q^{-A}$, *we have* $\rho(\underline{0};a) = v(a)$ *where* $\underline{0}$ *is the zero function and*

$$v(a) := \min\{\omega(-A) \mid a_A \neq 0\} = -\max\{\omega(A) \mid a_A \neq 0\}.$$

is the (upward) valuation of a.
(3) **(Symplectic invariance)** $\rho(\eta^*H;a) = \rho(H;a)$ *for any symplectic diffeomorphism* η
(4) **(Triangle inequality)** $\rho(H\#F;a\cdot b) \leq \rho(H;a) + \rho(F;b)$
(5) **(C^0-continuity)** $|\rho(H;a) - \rho(F;a)| \leq \|H\#\overline{F}\| = \|H-F\|$ *where $\|\cdot\|$ is the Hofer's pseudo-norm on $\mathcal{H}_m$. In particular, the function $\rho_a : H \mapsto \rho(H;a)$ is C^0-continuous.*

Proof. The projective invariance is obvious from the construction. The C^0-continuity is an immediate consequence of Proposition 5.6. We postpone the proof of triangle inequality to section 7. For the proof of symplectic invariance, we consider the symplectic conjugation

$$\phi \mapsto \eta\phi\eta^{-1}; \quad Ham(M,\omega) \to Ham(M,\omega)$$

for any symplectic diffeomorphism $\eta : (M,\omega) \to (M,\omega)$. Recall that the pull-back function η_*H given by

$$\eta_*H(t,x) = H(t,\eta^{-1}(x))$$

generates the conjugation $\eta\phi\eta^{-1}$ when $H \mapsto \phi$. We summarize some basic facts on this conjugation relevant to the filtered Floer homology here:

(1) when $H \mapsto \phi$, $\eta_*H \mapsto \eta\phi\eta^{-1}$,
(2) if H is nondegenerate, η_*H is also nondegenerate,
(3) if (H,J) is Floer-regular, then so is (η_*J,η_*H),

(4) there exists natural bijection $\eta_* : \Omega_0(M) \to \Omega_0(M)$ defined by

$$\eta_*([z,w]) = ([\eta \circ z, \eta \circ w])$$

under which we have the identity

$$\mathcal{A}_H([z,w]) = \mathcal{A}_{\eta_* H}(\eta_*[z,w]).$$

(5) the L^2-gradients of the corresponding action functionals satisfy

$$\eta_*(\mathrm{grad}_J \mathcal{A}_H)([z,w]) = \mathrm{grad}_{\eta_* J}(\mathcal{A}_{\eta_* H})(\eta_*([z,w]))$$

(6) if $u : \mathbb{R} \times S^1 \to M$ is a solution of perturbed Cauchy-Riemann equation for (H,J), then $\eta_* u = \eta \circ u$ is a solution for the pair $(\eta_* J, \eta_* H)$. In addition, all the Fredholm properties of (J,H,u) and $(\eta_* J, \eta_* H, \eta_* u)$ are the same.

These facts imply that the conjugation by η induces a canonical filtration preserving chain isomorphism

$$\eta_* : (CF_*^\lambda(H), \partial_{(H,J)}) \to (CF_*^\lambda(\eta_* H), \partial_{(\eta_* H, \eta_* J)})$$

for any $\lambda \in \mathbb{R} \setminus \mathrm{Spec}(H) = \mathbb{R} \setminus \mathrm{Spec}(\eta_* H)$. In particular it induces a filtration preserving isomorphism

$$\eta_* : HF_*^\lambda(H,J) \to HF_*^\lambda(\eta_* H, \eta_* J).$$

in homology. The symplectic invariance is then an immediate consequence of our mini-max procedure used in the construction of $\rho(H;a)$. $\square$

By the one-one correspondence between (normalized) H and its associated Hamiltonian path $\phi_H : t \mapsto \phi_H^t$, one can regard the spectral function

$$\rho_a : C_m^\infty([0,1] \times M) \to \mathbb{R}$$

as a function defined on $\mathcal{P}(Ham(M,\omega); id)$, i.e.,

$$\rho_a : \mathcal{P}(Ham(M,\omega), id) \to \mathbb{R}$$

as described in [Oh10]. Here we denote by $\mathcal{P}(Ham(M,\omega), id)$ the set of smooth Hamiltonian paths in $Ham(M,\omega)$ and by $\widetilde{Ham}(M,\omega)$ the set of path homotopy classes on $\mathcal{P}(Ham(M,\omega), id)$, i.e., the (étale) universal covering space of $Ham(M,\omega)$ in the sense of Appendix 2 [Oh10]. We equip $\widetilde{Ham}(M,\omega)$ with the quotient topology.

An important question to ask is then whether we have the equality

$$\rho(H;a) = \rho(K;a)$$

for any smooth functions $H \sim K$ satisfying $[H] = [K]$ so that ρ_a pushes down to $\widetilde{Ham}(M,\omega)$. We will discuss this in the next section.

6. The Spectrality Axiom

One of the most nontrivial properties of the spectral invariants $\rho(H;a)$ is the following property.

(Spectrality) *For any H and $a \in QH^*(M)$, we have*

$$\rho(H;a) \in \mathit{Spec}(H). \tag{6.1}$$

We proved this spectrality axiom on any *rational* symplectic manifold in [Oh8]. On the other hand, we have proven only the following weaker version on *irrational* symplectic manifolds [Oh11]. We suspect that the spectrality could fail if the Hamiltonian is highly degenerate on nonrational symplectic manifolds.

(Nondegenerate Spectrality) *For any nondegenerate H and $a \in QH^*(M)$, (6.1) holds.*

Before studying these axioms in general, let us state one important consequence thereof.

6.1. **A consequence of the nondegenerate spectrality axiom.** The following proposition shows that the function ρ_a pushes down to $\widetilde{Ham}(M,\omega)$ as a continuous function.

Theorem 6.1 (**Homotopy invariance**). *Let (M,ω) be an arbitrary closed symplectic manifold. Suppose that Nondegenerate Spectrality Axiom holds for (M,ω). Then we have*

$$\rho(H;a) = \rho(K;a) \tag{6.2}$$

for any smooth functions $H \sim K$ satisfying $[H]=[K]$.

Proof. We first consider nondegenerate Hamiltonians H, K with $H \sim K$. We now recall the following basic facts:

1. (1) Nondegeneracy of a Hamiltonian function H depends only on its time one map $\phi = \phi_H^1$.
2. (2) The set $\mathrm{Spec}(H) \subset \mathbb{R}$, which is the set of critical values of the action functional $\mathcal{A}_H$ is a set of measure zero (see [Lemma 2.2, Oh5]).
3. (3) For any two Hamiltonian functions H, $H' \mapsto \phi$ such that $[\phi, H] = [\phi, H']$, we have
$$\mathrm{Spec}(H) = \mathrm{Spec}(H')$$
as a subset of $\mathbb{R}$ provided H, H' satisfy the normalization condition (1.2) (see [Oh6] for the proof).
4. (4) The function $H \mapsto \rho(H;a)$ is continuous with respect to the smooth topology on $C_m^\infty(S^1 \times M)$ (see [Oh5] for its proof).
5. (5) The only continuous functions on a connected space (e.g., the interval $[0,1]$) to $\mathbb{R}$, whose values lie in a measure zero subset, are constant functions.

Since $H \sim K$, we have a smooth family $\mathcal{H} = \{H(s)\}_{0\le s\le 1}$ with $H(0)=H$ and $H(1)=K$. We define a function $\lambda : [0,1] \to \mathbb{R}$ by

$$\lambda(s) = \rho(H(s);a).$$

Note that $H(s)$ is nondegenerate since their time one map is $\phi^1_{H(s)} = \phi^1_H$ for all $s \in [0,1]$, and that its image is contained in the *fixed* subset

$$\mathrm{Spec}(h) \subset \mathbb{R}$$

independent of s, where h is the path homotopy class $[H]=[K]$. This subset has measure zero by (1) above and so totally disconnected. Therefore since the function λ is continuous by the C^0-continuity axiom, λ must be constant and so $\rho(H;a) = \lambda(0) = \lambda(1) = \rho(K;a)$, which finishes the above proof for the nondegenerate Hamiltonians.

We like to emphasize that at this moment, because we do not know validity of the spectrality axiom for degenerate Hamiltonians, the scheme of the above proof used for the nondegenerate case cannot be applied to degenerate Hamiltonians.

Suppose $H \sim K$ which are not-necessarily nondegenerate. We approximate H and K by sequences of nondegenerate Hamiltonians H_i and K_i in the C^∞ topology respectively. We note that the Hamiltonian

$$K \# H_i \# \overline{K}$$

generates the flow $\phi_K^t \circ \phi_{H_i}^t \circ (\phi_K^t)^{-1}$, which is conjugate to the flow $\phi_{H_i}^t$ and is nondegenerate. Therefore we have

$$\rho(H_i; a) = \rho(K \# H_i \# \overline{K}; a) \tag{6.3}$$

by the symplectic invariance of ρ. On the other hand, since $H \sim K$, we have

$$K \# H_i \# \overline{K} \sim K \# H_i \# \overline{H}.$$

Since both are nondegenerate, the above proof of (6.2) for the nondegenerate Hamiltonians implies

$$\rho(K \# H_i \# \overline{K}; a) = \rho(K \# H_i \# \overline{H}; a). \tag{6.4}$$

By taking the limits of (6.3) and (6.4) and using the continuity of $\rho(\cdot; a)$, we get

$$\rho(H; a) = \rho(K \# H \# \overline{K}; a) = \rho(K \# H \# \overline{H}; a) = \rho(K; a)$$

where the last equality comes since $H \# \overline{H} = 0$. Hence the proof. □

Therefore we can define the function $\rho_a : \widetilde{Ham}(M, \omega) \to \mathbb{R}$ by setting

$$\rho(h; a) := \rho(H; a) \tag{6.5}$$

for a (and so any) H satisfying $[H] = h$, whether h is nondegenerate or not. This defines a well-defined function

$$\rho_a : \widetilde{Ham}(M, \omega) \to \mathbb{R}.$$

Theorem 6.2. *The function ρ_a defined by (6.5) is continuous to $\widetilde{Ham}(M, \omega)$ in the quotient topology of $\widetilde{Ham}(M, \omega)$ induced from $\mathcal{P}(Ham(M, \omega), id)$.*

Proof. Recall the definition of the quotient topology under the projection

$$\pi : \mathcal{P}(Ham(M, \omega), id) \to \widetilde{Ham}(M, \omega).$$

We have proved that the assignment

$$H \mapsto \rho(H; a) \tag{6.6}$$

is continuous on $C^\infty([0, 1] \times M)$. By the definition of the quotient topology, the function

$$\rho_a : \widetilde{Ham}(M, \omega) \to \mathbb{R}$$

is continuous, because the composition

$$\rho_a \circ \pi : \mathcal{P}(Ham(M, \omega), id) \to \mathbb{R},$$

which is nothing but (6.6), is continuous. □

6.2. **Spectrality axiom for the rational case.** In this subsection, we will prove the full Spectrality Axiom for the rational symplectic manifolds [Oh8].

We first recall a useful notion of *canonical thin cylinder* between two nearby loops. For the reader's convenience, we provide its precise description following [Oh8]. We denote by J_{ref} a fixed compatible almost complex structure and by exp the exponential map of the metric

$$g := \omega(\cdot, J_{ref}\cdot).$$

Let $\iota(g)$ be the injectivity radius of the metric g. As long as $d(x,y) < \iota(g)$ for the given two points of M, we can write

$$y = \exp_x(\xi)$$

for a unique vector $\xi \in T_xM$. As usual, we write the unique vector ξ as

$$\xi = (\exp_x)^{-1}(y).$$

Therefore if the C^0 distance $d_{C^0}(z,z')$ between the two loops

$$z,\, z' : S^1 \to M$$

is smaller than $\iota(g)$, we can define the canonical map

$$u^{can}_{zz'} : [0,1] \times S^1 \to M$$

by

$$u^{can}_{zz'}(s,t) = \exp_{z(t)}(\xi_{zz'}(t)), \quad \text{or} \quad \xi_{zz'}(t) = (\exp_{z(t)})^{-1}(z'(t)). \tag{6.7}$$

It is important to note that the image of $u^{can}_{zz'}$ is contained in a small neighborhood of z (or z'), and uniformly converges to z_∞ when z and z' converge to a loop z_∞ in the C^1 topology. Therefore $u^{can}_{zz'}$ also picks out a canonical homotopy class, denoted by $[u^{can}_{zz'}]$, among the set of homotopy classes of the maps $u : [0,1] \times S^1 \to M$ satisfying the given boundary condition

$$u(0,t) = z(t), \quad u(1,t) = z'(t).$$

The following lemma is an important ingredient in our proof.

Lemma 6.3. *Let $z,\, z' : S^1 \to M$ be two smooth loops and u^{can} be the above canonical cylinder. Then as $d_{C^1}(z,z') \to 0$, then the map $u^{can}_{zz'}$ converges in the C^1-topology, and its geometric area $Area(u^{can})$ converges to zero. In particular, we have the followings:*

(1) *For any bounding disc w of z, the bounding disc*

$$w' := w \# u^{can}_{zz'}$$

of w' is pre-compact in the C^1-topology of the maps from the unit disc.

(2)

$$\int_{u^{can}_{zz'}} \omega \to 0 \tag{6.8}$$

as $d_{C^1}(z,z') \to 0$ as $z' \to z$.

Proof. (1) is an immediate consequence of the explicit form of $u^{can}_{zz'}$ above and from the standard property of the exponential map.

On the other hand, from the explicit expression of the canonical thin cylinder and from the property of the exponential map, it follows that the geometric area

$\mathrm{Area}(u_{i\infty}^{can})$ converges to zero as $d_{C^1}(z,z')\to 0$ by an easy area estimate. Since z, z' are assumed to be C^1, it follows $u_{zz'}^{can}$ is C^1 and hence the inequality

$$\mathrm{Area}(u_{i\infty}^{can}) \geq \Big|\int_{u_{i\infty}^{can}} \omega\Big|.$$

This implies

$$\lim_{j\to\infty}\int_{u_{i\infty}^{can}} \omega = 0,$$

which finishes the proof. □

The following theorem was proved by the author in [Oh8] from which we borrow its proof verbatim.

Theorem 6.4. *Suppose that (M,ω) is rational. Then for any smooth one-periodic Hamiltonian function $H : S^1\times M\to\mathbb{R}$, we have*

$$\rho(H;a)\in \mathrm{Spec}(H)$$

for each given quantum cohomology class $0\neq a\in QH^*(M)$.

Proof. We need to show that the mini-max value $\rho(H;a)$ is a critical value, or that there exists $[z,w]\in\widetilde\Omega_0(M)$ such that

$$\begin{aligned} \mathcal{A}_H([z,w]) &= \rho(H;a)\\ d\mathcal{A}_H([z,w]) &= 0, \quad \text{i.e.,} \quad \dot z = X_H(z). \end{aligned}$$

The finiteness of the value $\rho(H;a)$ was already proved in subsection 5.2. If H is nondegenerate, we just use the fixed Hamiltonian H. If H is degenerate, we approximate H by a sequence of nondegenerate Hamiltonians H_i in the C^2 topology. Let $\mathrm{peak}(\alpha_i)=[z_i,w_i]\in \mathrm{Crit}\mathcal{A}_{H_i}$ be the peak of the Floer cycle $\alpha_i\in CF_*(H_i)$, such that

$$\lim_{j\to\infty}\mathcal{A}_{H_i}([z_i,w_i])=\rho(H;a). \tag{6.9}$$

Such a sequence can be chosen by the definition of $\rho(\cdot;a)$ and its finiteness property.

Since M is compact and $H_i\to H$ in the C^2 topology, and $\dot z_i = X_{H_i}(z_i)$ for all i, it follows from the standard boot-strap argument that z_i has a subsequence, which we still denote by z_i, converging to some loop $z_\infty : S^1\to M$ satisfying $\dot z = X_H(z)$. Now we show that the sequence $[z_i,w_i]$ are pre-compact on $\widetilde\Omega_0(M)$. Since we fix the quantum cohomology class $0\neq a\in QH^*(M)$ (or more specifically since we fix its degree) and since the Floer cycle is assumed to satisfy $[\alpha_i]=a^\flat$, we have

$$\mu_{H_i}([z_i,w_i])=\mu_{H_j}([z_j,w_j]).$$

Lemma 6.5. *When (M,ω) is rational,* $\mathrm{Crit}\mathcal{A}_K\subset\widetilde\Omega_0(M)$ *is a closed subset of* $\mathbb{R}$ *for any smooth Hamiltonian K, and is locally compact in the subspace topology of the covering space*

$$\pi:\widetilde\Omega_0(M)\to\Omega_0(M).$$

Proof. First note that when (M,ω) is rational, the covering group Γ of π above is discrete. Together with the fact that the set of solutions of $\dot z = X_K(z)$ is compact (on compact M), it follows that

$$\mathrm{Crit}(\mathcal{A}_K)=\{[z,w]\in\widetilde\Omega_0(M)\mid \dot z = X_K(z)\}$$

is a closed subset which is also locally compact. □

Now consider the bounding discs of z_∞ given by

$$w_i' = w_i \# u_{i\infty}^{can}$$

for all sufficiently large i, where $u_{i\infty}^{can} = u_{z_i z_\infty}^{can}$ is the canonical thin cylinder between z_i and z_∞. We note that as $i \to \infty$ the geometric area of $u_{i\infty}^{can}$ converges to 0.

We compute the action of the critical points $[z_\infty, w_i'] \in \mathrm{Crit}\mathcal{A}_H$,

$$\begin{aligned}
\mathcal{A}_H([z_\infty, w_i']) &= -\int_{w_i'} \omega - \int_0^1 H(t, z_\infty(t))\, dt \qquad (6.10)\\
&= -\int_{w_i} \omega - \int_{u_{i\infty}^{can}} \omega - \int_0^1 H(t, z_\infty(t))\, dt\\
&= \Big(-\int_{w_i} \omega - \int_0^1 H_i(t, z_i(t))\, dt\Big)\\
&\quad - \int_{u_{i\infty}^{can}} \omega - \Big(\int_0^1 H(t, z_\infty(t))\, dt - \int_0^1 H_i(t, z_i(t))\, dt\Big)\\
&= \mathcal{A}_{H_i}([z_i, w_i]) - \int_{u_{i\infty}^{can}} \omega\\
&\quad - \Big(\int_0^1 H(t, z_\infty(t))\, dt - \int_0^1 H_i(t, z_i(t))\, dt\Big). \qquad (6.11)
\end{aligned}$$

Since z_i converges to z_∞ uniformly and $H_i \to H$, we have

$$-\Big(\int_0^1 H(t, z_\infty(t))\, dt - \int_0^1 H(t, z_i(t))\, dt\Big) \to 0. \qquad (6.12)$$

Therefore combining (6.9), (6.10) and (6.12), we derive

$$\lim_{i\to\infty} \mathcal{A}_H([z_\infty, w_i']) = \rho(H; a).$$

In particular $\mathcal{A}_H([z_\infty, w_i'])$ is a Cauchy sequence, which implies

$$\Big|\int_{w_i'} \omega - \int_{w_j'} \omega\Big| = \Big|\mathcal{A}_H([z_\infty, w_i']) - \mathcal{A}_H([z_\infty, w_j'])\Big| \to 0$$

i.e.,

$$\int_{w_i' \# \overline{w}_j'} \omega \to 0.$$

Since Γ is discrete and $\int_{w_i' \# \overline{w}_j'} \omega \in \Gamma$, this indeed implies that

$$\int_{w_i' \# \overline{w}_j'} \omega = 0 \qquad (6.13)$$

for all sufficiently large $i, j \in \mathbb{Z}_+$. Since the set $\Big\{\int_{w_i'} \omega\Big\}_{i \in \mathbb{Z}_+}$ is bounded, we conclude that the sequence $\int_{w_i'} \omega$ eventually stabilize, by choosing a subsequence if necessary. Going back to (6.10), we derive that the actions

$$\mathcal{A}_H([z_\infty, w_i'])$$

themselves stabilize and so we have

$$\mathcal{A}_H([z_\infty, w_N']) = \lim_{i\to\infty} \mathcal{A}_H([z_\infty, w_i']) = \rho(H; a)$$

for a fixed sufficiently large $N \in \mathbb{Z}_+$. This proves that $\rho(H;a)$ is indeed the value of $\mathcal{A}_H$ at the critical point $[z_\infty, w'_N]$. This finishes the proof. $\square$

6.3. **Spectrality for the irrational case.** In fact, an examination of the proof of Theorem 6.4 proves a stronger fact which we now explain. We recall that if H, H' are nondegenerate and sufficiently C^2-close, there exists a canonical one-one correspondence between the sets of associated Hamiltonian periodic orbits. We call an *associated pair* any such pair (z, z') of Hamiltonian periodic orbits of H, H' mapped to each other under this correspondence. The following is proved in the Appendix of [Oh8] whose proof we omit.

Proposition 6.6. *Suppose that H, H' are nondegenerate and sufficiently C^2 close. Let (z, z') be an associated pair of H, H'. Then we have*

$$\mu_H([z,w]) = \mu_{H'}([z', w\# u^{can}_{zz'}]). \tag{6.14}$$

We derive

$$\begin{aligned} 2c_1([w'_i \# \overline{w}'_j]) &= 2c_1([w_i \# u^{can}_{i\infty} \# \overline{w_j \# u^{can}_{j\infty}}]) \\ &= 2c_1([w_i \# u^{can}_{i\infty} \# \overline{u}^{can}_{j\infty} \# \overline{w}_j]) \\ &= \mu_{H_i}([z_i, w_i]) - \mu_{H_i}([z_i, w_j \# u^{can}_{j\infty} \# \overline{u}^{can}_{i\infty}]). \end{aligned} \tag{6.15}$$

The third equality comes from the index formula

$$\mu_H([z, w\# A]) = \mu_H([z,w]) - 2c_1(A).$$

On the other hand, we derive

$$\mu_{H_i}([z_i, w_j \# u^{can}_{j\infty} \# \overline{u}^{can}_{i\infty}]) = \mu_{H_i}([z_i, w_j \# u^{can}_{z_j z'_i}]) = \mu_{H_j}([z_j, w_j])$$

when i, j are sufficiently large. Here the first equality follows since $u^{can}_{i\infty} \# \overline{u}^{can}_{i\infty}$ is homotopic to the canonical thin cylinder $u^{can}_{z_j z'_i}$, and the second comes from (6.14). On the other hand, $[z_i, w_i]$ and $[z_j, w_j]$ satisfy

$$\mu_{H_i}([z_i, w_i]) = \mu_{H_j}([z_j, w_j]) \tag{6.16}$$

because they are generators of Floer cycles α_i and α_j both representing the same Floer homology class $a^\flat$ and so having the same degree. Hence combining (6.14)-(6.16), we obtain

$$c_1([w'_i \# \overline{w}'_j]) = 0 \tag{6.17}$$

for all sufficiently large i, j. Combining (6.13) and (6.17), we have proved

$$[z_\infty, w'_i] = [z_\infty, w'_j] \quad \text{in } \widetilde{\Omega}_0(M).$$

If we denote by $[z_\infty, w_\infty]$ this common element of $\widetilde{\Omega}_0(M)$, we have proven that the sequence $[z_i, w_i]$ converges to a critical point $[z_\infty, w_\infty]$ of $\mathcal{A}_H$ in the topology of the covering space $\pi : \widetilde{\Omega}_0(M) \to \Omega_0(M)$. This finishes the proof.

For the irrational case, the sequence $[z_\infty, w'_i]$ used in the above proof will not stabilize, and more seriously the action values $\mathcal{A}_H([z_\infty, w'_i])$ may accumulate at a value in $\mathbb{R} \setminus \mathrm{Spec}(H)$. Recall that in the irrational case, $\mathrm{Spec}(H)$ is a dense subset of $\mathbb{R}$. Therefore in the irrational case, one needs to directly prove that the sequence has a convergent subsequence in the natural topology of $\widetilde{\Omega}_0(M)$. It turns out that the above limiting arguments used for the rational case cannot be carried out due to the possibility that the discs w_i could behave wildly in the limiting process. We emphasize that in the irrational case, the projection $\pi : \widetilde{\Omega}_0(M) \to \Omega_0(M)$ defines a covering only in the étale sense (see Appendix [Oh10] for the precise meaning of this), but not in the ordinary sense. As a result, proving such a convergence is not possible in general even for the nondegenerate case for a given mini-max sequence of *critical points* $[z_i, w_i]$ satisfying (6.10). One needs to use a mini-max sequence of *cycles* instead. This scheme is exactly what we have carried out in [Oh11], which however turns out to be a highly nontrivial matter to carry out.

Theorem 6.7. *Let (M, ω) be an arbitrary closed symplectic manifold. Then the Nondegenerate Spectrality Axiom holds.*

We refer to [Oh11] for the complete details of the proof and many other basic ingredients in the chain level Floer theory

To go to the case of degenerate Hamiltonians from this theorem, it is unavoidable to use the approximation arguments above as in the rational case. Therefore one has to work with the action functional $\mathcal{A}_H$ in the spirit of the general critical point theory as in [BnR]. One important point of the *chain level* theory we develop in [Oh5]-[Oh11] is that it has certain continuity property when Hamiltonian functions become degenerate, even in the irrational case where $\mathrm{Spec}(H)$ is a dense subset of $\mathbb{R}$. Our chain level Floer theory developed in [Oh5]-[Oh11] should be regarded as the *mini-max theory* of the action functional, while the usual Floer homology theory is the *Morse theory* of the action functional. For this reason, we call our chain level theory the *Floer mini-max theory*. However this mini-max theory still meets the same kind of difficulty mentioned above, and cannot prove the spectrality axiom for general degenerate Hamiltonians on irrational symplectic manifolds. (See Remark 2.5 for some related comments.) It would be very interesting to see if this difficulty is something intrinsic for this case.

We summarize the basic axioms of the invariant $\rho : \widetilde{Ham}(M, \omega) \times QH^*(M) \to \mathbb{R}$ in the following theorem, whose proofs immediately follow from Theorem 5.7 and 6.7

Theorem 6.8. *Let (M, ω) be any closed symplectic manifold. Let h, $k \in \widetilde{Ham}(M, \omega)$ and $0 \neq a \in QH^*(M)$. Then for each $0 \neq a \in QH^*(M)$, the function*

$$\rho_a : \widetilde{Ham}(M, \omega) \to \mathbb{R}$$

is continuous, and the function

$$\rho : \widetilde{Ham}(M, \omega) \times QH^*(M) \to \mathbb{R}$$

satisfies the following axioms:

1. **(Nondegenerate spectrality)** *For any nondegenerate h, $\rho(h; a) \in Spec(h)$ for all $0 \neq a \in QH^*(M)$.*
2. **(Projective invariance)** $\rho(\widetilde{\phi}; \lambda a) = \rho(\widetilde{\phi}; a)$ *for any $0 \neq \lambda \in \mathbb{Q}$.*
3. **(Normalization)** *For $a = \sum_{A \in \Gamma} a_A q^{-A}$, we have $\rho(\underline{0}; a) = v(a)$ where $\underline{0}$ is the identity in $\widetilde{Ham}(M, \omega)$ and*
$$v(a) := \min_A \{\omega(-A) \mid a_A \neq 0\} = -\max\{\omega(A) \mid a_A \neq 0\}.$$
is the (upward) valuation of a.
4. **(Symplectic invariance)** $\rho(\eta h \eta^{-1}; a) = \rho(h; a)$ *for any symplectic diffeomorphism η*
5. **(Triangle inequality)** $\rho(h \cdot k; a \cdot b) \leq \rho(h; a) + \rho(k; b)$

(6) **(C^0-continuity)** $|\rho(h;a)-\rho(k;a)| \leq \|h\circ k^{-1}\|$ *where* $\|\cdot\|$ *is the Hofer's pseudo-norm on* $\widetilde{Ham}(M,\omega)$. *In particular, the function* $\rho_a : h \mapsto \rho(h;a)$ *is* C^0*-continuous.*

7. Pants product and the triangle inequality

7.1. **Quantum cohomology in the chain level.** We first recall the definition of the quantum cohomology ring $QH^*(M)$. As a module, it is defined as

$$QH^*(M) = H^*(M,\mathbb{Q}) \otimes \Lambda^{\uparrow}_{\omega}$$

where $\Lambda^{\uparrow}_{\omega}$ is the (upward) Novikov ring

$$\Lambda^{\uparrow}_{\omega} = \Big\{ \sum_{A\in\Gamma} a_A q^{-A} \mid a_A \in \mathbb{Q},\ \#\{A \mid a_i \neq 0,\ \int_{-A}\omega < \lambda\} < \infty,\ \forall\lambda\in\mathbb{R}\Big\}.$$

Due to the finiteness assumption on the Novikov ring, we have the natural (upward) valuation $v : QH^*(M) \to \mathbb{R}$ defined by

$$v\Big(\sum_{A\in\Gamma_\omega} a_A q^{-A}\Big) = \min\{\omega(-A) : a_A \neq 0\} \tag{7.1}$$

which satisfies that for any $a,\ b \in QH^*(M)$

$$v(a+b) \geq \min\{v(a), v(b)\}.$$

The product on $QH^*(M)$ is defined by the usual quantum cup product, which we denote by "$\cdot$" and which preserves the grading, i.e, satisfies

$$QH^k(M) \times QH^\ell(M) \to QH^{k+\ell}(M).$$

Often the homological version of the quantum cohomology is also useful, sometimes called the quantum homology, which is defined by

$$QH_*(M) = H_*(M) \otimes \Lambda^{\downarrow}_{\omega}.$$

We define the corresponding (downward) valuation by

$$v\Big(\sum_{B\in\Gamma} a_B q^{B}\Big) = \max\{\omega(B) : a_B \neq 0\} \tag{7.2}$$

which satisfies that for $f,\ g \in QH_*(M)$

$$v(f+g) \leq \max\{v(f), v(g)\}.$$

We like to point out that the summand in $\Lambda^{\downarrow}_{\omega}$ is written as $b_B q^B$ while the one in $\Lambda^{\uparrow}_{\omega}$ as $a_A q^{-A}$ with the minus sign. This is because we want to clearly show which one we use. Obviously v satisfies the axiom of non-Archimedean norm which induce a topology on $QH^*(M)$ and $QH_*(M)$ respectively. The finiteness assumption in the definition of the Novikov ring allows us to enumerate supp(a) so that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_j > \cdots \to -\infty$$

with $\lambda_j = \omega(B_j)$ for $B_j \in \text{supp}(a)$ when $a \in QH^*(M)$

We have a canonical isomorphism

$$\flat : QH^*(M) \to QH_*(M); \quad \sum a_i q^{-A_i} \to \sum PD(a_i) q^{A_i}$$

and its inverse

$$\sharp : QH_*(M) \to QH^*(M); \quad \sum b_j q^{B_j} \to \sum PD(b_j) q^{-B_j}.$$

We denote by $a^\flat$ and $b^\#$ the images under these maps.

There exists the canonical nondegenerate pairing

$$\langle \cdot, \cdot \rangle : QH^*(M) \otimes QH_*(M) \to \mathbb{Q}$$

defined by

$$\langle \sum a_i q^{-A_i}, \sum b_j q^{B_j} \rangle = \sum (a_i, b_j) \delta_{A_i B_j} \tag{7.3}$$

where $\delta_{A_i B_j}$ is the delta-function and (a_i, b_j) is the canonical pairing between $H^*(M, \mathbb{Q})$ and $H_*(M, \mathbb{Q})$. Note that this sum is always finite by the finiteness condition in the definitions of $QH^*(M)$ and $QH_*(M)$ and so is well-defined. This is equivalent to the Frobenius pairing in the quantum cohomology ring. However we would like to emphasize that the dual vector space $(QH_*(M))^*$ of $QH_*(M)$ is *not* isomorphic to $QH^*(M)$ even as a $\mathbb{Q}$-vector space. Rather the above pairing induces an injection

$$QH^*(M) \hookrightarrow (QH_*(M))^*$$

whose images lie in the set of *continuous* linear functionals on $QH_*(M)$ with respect to the topology induced by the valuation v (7.2) on $QH_*(M)$. We refer to the Appendix of [Oh8] for further discussions on this matter.

Let (C_*, ∂) be any chain complex on M whose homology is the singular homology $H_*(M)$. One may take for C_* the usual singular chain complex or a Morse chain complex. However since we need to take a nondegenerate pairing in the chain level, we should use a model which is *finitely generated*. We will always prefer to use the Morse homology complex

$$(CM_*(-\epsilon f), \partial^{Morse}_{(-\epsilon f, g_{J_0})})$$

of the pair $(-\epsilon f, g_{J_0})$ for a sufficiently small $\epsilon > 0$, because it is finitely generated and avoids some technical issue related to singular degeneration problem of the type studied in [FOh]. The negative sign in $(CM_*(-\epsilon f), \partial^{Morse}_{(-\epsilon f, g_{J_0})})$ is put to make the correspondence between the Morse homology and the Floer homology consistent with our conventions of the Hamiltonian vector field and the action functional. In our conventions, solutions of negative gradient of $-\epsilon f$ correspond to ones for the negative gradient flow of the action functional $\mathcal{A}_{\epsilon f}$. We denote by

$$(CM^*(-\epsilon f), \delta^{Morse}_{(-\epsilon f, g_{J_0})})$$

the corresponding cochain complex, i.e,

$$CM^k := \mathrm{Hom}(CM_k, \mathbb{Q}), \quad \delta_{-\epsilon f} = \partial^*_{(-\epsilon f, g_{J_0})}.$$

Now we extend the complex $(CM_*(-\epsilon f), \partial^{Morse}_{(-\epsilon f, g_{J_0})})$ to the *quantum chain complex*, denoted by

$$(CQ_*(-\epsilon f), \partial_Q)$$

$$CQ_*(-\epsilon f) := CM_*(-\epsilon f) \otimes \Lambda_\omega, \quad \partial_Q := \partial^{Morse}_{(-\epsilon f, g_{J_0})} \otimes \Lambda_\omega.$$

This coincides with the Floer complex $(CF_*(\epsilon f), \partial)$ as a chain complex if ϵ is sufficiently small (Theorem 5.1). Similarly we define the quantum cochain complex $(CQ^*(-\epsilon f), \delta^Q)$ by changing the downward Novikov ring to the upward one. In other words, we define

$$CQ^*(-\epsilon f) := CM_{2n-*}(\epsilon f) \otimes \Lambda^\uparrow, \quad \delta^Q := \partial_{(\epsilon f, g_{J_0})} \otimes \Lambda^\uparrow_\omega.$$

Again we would like to emphasize that $CQ^*(-\epsilon f)$ is *not* isomorphic to the dual space of $CQ_*(-\epsilon f)$ as a $\mathbb{Q}$-vector space.

To emphasize the role of the Morse function in the level of complex, we denote the corresponding homology by $HQ^*(-\epsilon f) \cong QH^*(M)$. With these definitions, we have the obvious nondegenerate pairing

$$CQ^*(-\epsilon f) \otimes CQ_*(-\epsilon f) \to \mathbb{Q}$$

induced by the duality pairing (not the Poincaré pairing!)

$$CM_{2n-*}(\epsilon f) \otimes CM_*(-\epsilon f) \to \mathbb{Q}$$

which also induces the pairing above in homology.

We now choose a generic Morse function f and an almost complex structure J_0 as before. Then for any given homotopy $(\mathcal{H}, j)$ with $\mathcal{H} = \{H^s\}_{s\in[0,1]}$ with $H^0 = \epsilon f$ and $H^1 = H$, we denote by

$$h_{(\mathcal{H},j)} : CQ_*(-\epsilon f) = CF_{*-n}(\epsilon f) \to CF_{*-n}(H) \tag{7.4}$$

the standard Floer chain map from ϵf to H via the homotopy $\mathcal{H}$. This induces a homomorphism

$$h_{(\mathcal{H},j)} : HQ_*(-\epsilon f) \cong HF_{*-n}(\epsilon f, J_0) \to HF_{*-n}(H, J). \tag{7.5}$$

Although (7.4) depends on the choice $(\mathcal{H}, j)$, (7.5) is canonical, i.e, does not depend on the homotopy $(\mathcal{H}, j)$. One confusing point in this isomorphism is the issue of grading. See the next section for a review of the construction of this chain map and the issue of grading of $HF_*(H, J)$.

7.2. **Grading convention.** We set up our grading convention of the Floer homology. We denote by $\mu_H([z, w])$ the Conley-Zehnder index of $[z, w]$ for the Hamiltonian H. The convention of the grading of $CF_*(H)$ from [Oh9] is

$$\deg([z, w]) = \mu_H([z, w]) \tag{7.6}$$

for $[z, w] \in \mathrm{Crit}\mathcal{A}_H$. This convention is the analog to the one we use in [Oh4] in the context of Lagrangian submanifolds.

We next compare this grading and the Morse grading of the Morse complex of the *negative* gradient flow equation of $-f$, (i.e., of the *positive* gradient flow of f

$$\dot\chi - \mathrm{grad}\, f(\chi) = 0.$$

This corresponds to the *negative* gradient flow of the action functional $\mathcal{A}_{\epsilon f}$). This gives rise to the relation between the Morse indices $\mu^{Morse}_{-\epsilon f}(p)$ and the Conley-Zehnder indices $\mu_{\epsilon f}([p, \widehat{p}])$ in our convention (See Lemma 7.2 [SZ] but with some care about the different convention of the Hamiltonian vector field. Their definition of X_H is $-X_H$ in our convention.):

$$\mu_{\epsilon f}([p, \widehat{p}]) = \mu^{Morse}_{-\epsilon f}(p) - n$$

or

$$\mu^{Morse}_{-\epsilon f}(p) = \mu_{\epsilon f}([p, \widehat{p}]) + n$$

Recalling that we chose the Morse complex

$$CM_*(-\epsilon f) \otimes \Lambda^{\downarrow}$$

for the quantum chain complex $CQ_*(-\epsilon f)$, we have the following grading preserving isomorphism

$$QH^{n-k}(M) \to QH_{n+k}(M) \cong HQ_{n+k}(-\epsilon f) \to HF_k(\epsilon f, J_0) \to HF_k(H, J).$$

We will also show in subsection 7.3 that this grading convention makes the pants product, denoted by $*$, has the degree $-n$

$$*: HF_k(H) \otimes HF_\ell(K) \to HF_{(k+\ell)-n}(H\#K) \tag{7.7}$$

which will be compatible with the degree preserving quantum product

$$\cdot : QH^a(M) \otimes QH^b(M) \to QH^{a+b}(M)$$

under the ring isomorphism between QH^* and HF_* [PSS], [LT2].

Finally we state an important identity relating the Conley-Zehnder index and the first Chern number $c_1(A)$ under the action by 'gluing a sphere' $[z,w] \mapsto [z, A\#w]$. We like to emphasize that in our convention, the sign in front of the first Chern number term in the formula is '-'. The difference of the sign from the formula in [HS] is due to the different convention of the canonical symplectic form on $\mathbb{C}^n$: when we identify $\mathbb{R}^{2n} \cong T^*\mathbb{R}^n$ and denote by $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ the corresponding canonical coordinates, then the canonical symplectic form is given by

$$\omega_0 = \sum dq_i \wedge dp_i$$

in our convention, while it is given by

$$\omega_0' = -\omega_0 = \sum dp_i \wedge dq_i.$$

according to the convention of [HS], [SZ], or [Po3]. We will provide a complete self-contained proof starting from the definition of the Conley-Zehnder index from [SZ].

Theorem 7.1. *Let $z : S^1 = \mathbb{R}/\mathbb{Z} \to M$ be a given one-periodic solution of $\dot{x} = X_H(x)$ and w, w' two given bounding discs. Then we have the identity*

$$\mu_H([z,w']) = \mu_H([z,w]) - 2c_1([w'\#\overline{w}]). \tag{7.8}$$

In particular we have

$$\mu_H([z,A\#w]) = \mu_H([z,w]) - 2c_1(A). \tag{7.9}$$

7.3. **Hamiltonian fibrations and the pants product.** To start with the proof of the triangle inequality, we need to recall the definition of the "pants product"

$$HF_*(H,J^1) \otimes HF_*(F,J^2) \to HF_*(H\#F,J^3).$$

For the purpose of studying the effect on the filtration under the product, we need to define this product in the chain level in an optimal way as in [Oh4], [Sc]. For this purpose, we will mostly follow the description provided by Entov [En1] with few notational changes and differences in the grading. Except the grading convention, the conventions in [En1], [En2] on the definition of Hamiltonian vector field and the action functional coincide with our conventions in [Oh3]-[Oh11] and also here.

Let Σ be the compact Riemann surface of genus 0 with three punctures. We fix a holomorphic identification of a neighborhood of each puncture with either $[0,\infty) \times S^1$ or $(-\infty, 0] \times S^1$ with the standard complex structure on the cylinder. We call punctures of the first type *negative* and the second type *positive*. In terms of the "pair-of-pants" $\Sigma \setminus \cup_i D_i$, the positive puncture corresponds to the *outgoing ends* and the negative corresponds to the *incoming ends*. We denote the neighborhoods of the three punctures by D_i, $i = 1,2,3$ and the identification by

$$\varphi_i^+ : D_i \to (-\infty, 0] \times S^1 \quad \text{for } i = 1,\, 2$$

for positive punctures and

$$\varphi_3^- : D_3 \to [0,\infty) \times S^1$$

for negative punctures. We denote by (τ, t) the standard cylindrical coordinates on the cylinders.

We fix a cut-off function $\rho^+ : (-\infty, 0] \to [0,1]$ defined by

$$\rho = \begin{cases} 1 & \tau \leq -2 \\ 0 & \tau \geq -1 \end{cases}$$

and $\rho^- : [0,\infty) \to [0,1]$ by $\rho^-(\tau) = \rho^+(-\tau)$. We will just denote by ρ these cut-off functions for both cases when there is no danger of confusion.

We now consider the (topologically) trivial bundle $P \to \Sigma$ with fiber isomorphic to (M,ω) and fix a trivialization

$$\Phi_i : P_i := P|_{D_i} \to D_i \times M$$

on each D_i. On each P_i, we consider the closed two form of the type

$$\omega_{P_i} := \Phi_i^*(\omega + d(\rho H_t dt))$$

for a time periodic Hamiltonian $H : [0,1] \times M \to \mathbb{R}$. The following is an important lemma whose proof we omit (see [En1]).

Lemma 7.2. *Consider three normalized Hamiltonians H_i, $i = 1,2,3$. Then there exists a closed 2-form ω_P such that*

(1) $\omega_P|_{P_i} = \omega_{P_i}$
(2) ω_P *restricts to* ω *in each fiber*
(3) $\omega_P^{n+1} = 0$

Such ω_P induces a canonical symplectic connection $\nabla = \nabla_{\omega_P}$ [GLS], [En1]. In addition it also fixes a natural deformation class of symplectic forms on P obtained by those

$$\Omega_{P,\lambda} := \omega_P + \lambda\omega_\Sigma$$

where ω_Σ is an area form and $\lambda > 0$ is a sufficiently large constant. We will always normalize ω_Σ so that $\int_\Sigma \omega_\Sigma = 1$.

Next let $\widetilde{J}$ be an almost complex structure on P such that

(1) $\widetilde{J}$ is ω_P-compatible on each fiber and so preserves the vertical tangent space
(2) the projection $\pi : P \to \Sigma$ is pseudo-holomorphic, i.e, $d\pi \circ \widetilde{J} = j \circ d\pi$.

When we are given three t-periodic Hamiltonian $H = (H_1, H_2; H_3)$, we say that $\widetilde{J}$ is (H,J)-compatible, if $\widetilde{J}$ additionally satisfies

(3) For each i, $(\Phi_i)_*\widetilde{J} = j \oplus J_{H_i}$ where

$$J_{H_i}(\tau, t, x) = (\phi_{H_i}^t)^* J$$

for some t-periodic family of almost complex structure $J = \{J_t\}_{0 \leq t \leq 1}$ on M over a disc $D_i' \subset D_i$ in terms of the cylindrical coordinates. Here $D_i' = \varphi_i^{-1}((-\infty, -K_i] \times S^1)$, $i = 1$, 2 and $\varphi_3^{-1}([K_3,\infty) \times S^1)$ for some $K_i > 0$. See [Oh9] for a more detailed discussion on $\widetilde{J}$. The condition (3) implies that the $\widetilde{J}$-holomorphic sections v over D_i' are precisely the solutions of the equation

$$\frac{\partial u}{\partial \tau} + J_t\Big(\frac{\partial u}{\partial t} - X_{H_i}(u)\Big) = 0 \tag{7.10}$$

if we write $v(\tau,t) = (\tau,t,u(\tau,t))$ in the trivialization with respect to the cylindrical coordinates (τ,t) on D_i' induced by $\phi_i^{\pm}$ above.

Now we are ready to define the moduli space which will be relevant to the definition of the pants product that we need to use. To simplify the notations, we denote

$$\widehat{z} = [z,w]$$

in general and $\widehat{z} = (\widehat{z}_1, \widehat{z}_2, \widehat{z}_3)$ where $\widehat{z}_i = [z_i, w_i] \in \mathrm{Crit}\mathcal{A}_{H_i}$ for $i = 1,2,3$.

Definition 7.1. Consider the Hamiltonians $H = \{H_i\}_{1\leq i\leq 3}$ with $H_3 = H_1 \# H_2$, and let $\widetilde{J}$ be a H-compatible almost complex structure. We denote by $\mathcal{M}(H,\widetilde{J};\widehat{z})$ the space of all $\widetilde{J}$-holomorphic sections $u : \Sigma \to P$ that satisfy

1. The maps $u_i := u \circ (\varphi_i^{-1}) : (-\infty, K_i] \times S^1 \to M$ which are solutions of (7.10), satisfy
$$\lim_{\tau\to-\infty} u_i(\tau,\cdot) = z_i, \quad i = 1,2$$
and similarly for $i = 3$ changing $-\infty$ to $+\infty$.
2. The closed surface obtained by capping off $pr_M \circ u(\Sigma)$ with the discs w_i taken with the same orientation for $i = 1,2$ and the opposite one for $i = 3$ represents zero in $\pi_2(M)$.

Note that $\mathcal{M}(H,\widetilde{J};\widehat{z})$ depends only on the equivalence class of $\widehat{z}$'s: we say that $\widehat{z}' \sim \widehat{z}$ if they satisfy

$$z_i' = z_i, \quad w_i' = w_i \# A_i$$

for $A_i \in \pi_2(M)$ and $\sum_{i=1}^3 A_i$ represents zero (mod) Γ. The (virtual) dimension of $\mathcal{M}(H,\widetilde{J};\widehat{z})$ is given by

$$\begin{aligned}
\dim \mathcal{M}(H,\widetilde{J};\widehat{z}) &= 2n - (-\mu_{H_1}(z_1) + n) - (-\mu_{H_2}(z_2) + n) - (\mu_{H_3}(z_3) + n) \\
&= -n + (\mu_{H_1}(z_1) + \mu_{H_2}(z_2) - \mu_{H_3}(z_3)).
\end{aligned}$$

Note that when $\dim \mathcal{M}(H,\widetilde{J};\widehat{z}) = 0$, we have

$$n = -\mu_{H_3}(\widehat{z}_3) + \mu_{H_1}(\widehat{z}_1) + \mu_{H_2}(\widehat{z}_2)$$

which is equivalent to

$$\mu_{H_3}(\widehat{z}_3) = (\mu_{H_1}(\widehat{z}_1) + \mu_{H_2}(\widehat{z}_2)) - n$$

which provides the degree of the pants product (7.7) in our convention of the grading of the Floer complex we adopt in the present paper. Now the pair-of-pants product $*$ for the chains is defined by

$$\widehat{z}_1 * \widehat{z}_2 = \sum_{\widehat{z}_3} \#(\mathcal{M}(H,\widetilde{J};\widehat{z}))\widehat{z}_3 \tag{7.11}$$

for the generators $\widehat{z}_i$ and then by linearly extending over the chains in $CF_*(H_1) \otimes CF_*(H_2)$. Our grading convention makes this product is of degree $-n$. Now with this preparation, we are ready to prove the triangle inequality.

7.4. **Proof of the triangle inequality.** Let $\alpha \in CF_*(H)$ and $\beta \in CF_*(F)$ be Floer cycles with $[\alpha] = [\beta] = a^\flat$ and consider their pants product cycle $\alpha * \beta := \gamma \in CF_*(H\#F)$. Then we have

$$[\alpha * \beta] = (a \cdot b)^\flat$$

and so

$$\rho(H\#F; a \cdot b) \leq \lambda_{H\#F}(\alpha * \beta). \tag{7.12}$$

Let $\delta > 0$ be any given number and choose $\alpha \in CF_*(H)$ and $\beta \in CF_*(F)$ so that

$$\lambda_H(\alpha) \leq \rho(H; a) + \frac{\delta}{2} \tag{7.13}$$

$$\lambda_F(\beta) \leq \rho(F; b) + \frac{\delta}{2}. \tag{7.14}$$

Then we have the expressions

$$\alpha = \sum_i a_i [z_i, w_i] \text{ with } \mathcal{A}_H([z_i, w_i]) \leq \rho(H; a) + \frac{\delta}{2}$$

and

$$\beta = \sum_j a_j [z_j, w_j] \text{ with } \mathcal{A}_F([z_j, w_j]) \leq \rho(F; b) + \frac{\delta}{2}.$$

Now using the pants product (7.11), we would like to estimate the level of the chain $\alpha\#\beta \in CF_*(H\#F)$. The following is a crucial lemma whose proof we omit but refer to Section 4.1 [Sc] or Section 5 [En1].

Lemma 7.3. *Suppose that $\mathcal{M}(H, \widetilde{J}; \widehat{z})$ is non-empty. Then we have the identity*

$$\int v^*\omega_P = -\mathcal{A}_{H_1\#H_2}([z_3, w_3]) + \mathcal{A}_{H_1}([z_1, w_1]) + \mathcal{A}_{H_2}([z_2, w_2])$$

for any $\in \mathcal{M}(H, \widetilde{J}; \widehat{z})$

Now since $\widetilde{J}$-holomorphic and $\widetilde{J}$ is $\Omega_{P,\lambda}$-compatible, we have

$$0 \leq \int v^*\Omega_{P,\lambda} = \int v^*\omega_P + \lambda \int v^*\omega_\Sigma = \int v^*\omega_P + \lambda.$$

Lemma 7.4. [Theorem 3.6.1 & 3.7.4, [En1]**]** *Let H_i's be as in Definition 7.1. Then for any given $\delta > 0$, we can choose a closed 2-form ω_P so that $\Omega_{P,\lambda} = \omega_P + \lambda\omega_\Sigma$ becomes a symplectic form for all $\lambda \geq \delta$.*

We recall that from the definition of $*$ that for any $[z_3, w_3] \in \alpha * \beta$ there exist $[z_1, w_1] \in \alpha$ and $[z_2, w_2] \in \beta$ such that $\mathcal{M}(\widetilde{J}, H; \widehat{z})$ is non-empty with the asymptotic condition

$$\widehat{z} = ([z_1, w_1], [z_2, w_2]; [z_3, w_3]).$$

Applying this and the above two lemmata to H and F for λ arbitrarily close to 0, and also applying (7.12)-(7.14), we immediately derive

$$\begin{aligned} \mathcal{A}_{H\#F}([z_3, w_3]) &\leq \mathcal{A}_H([z_1, w_1]) + \mathcal{A}_F([z_2, w_2]) + \delta \\ &\leq \lambda_H(\alpha) + \lambda_F(\beta) + \delta \\ &\leq \rho(H; a) + \rho(F; b) + 2\delta \end{aligned} \tag{7.15}$$

for any $[z_3, w_3] \in \alpha * \beta$. Combining (7.12), (7.13)-(7.15), we derive

$$\rho(H\#F; a \cdot b) \leq \rho(H; a) + \rho(F; b) + 2\delta$$

Since this holds for any δ, we have proven

$$\rho(H\#F; a\cdot b) \leq \rho(H;a) + \rho(F;b).$$

This finishes the proof.

8. Spectral norm of Hamiltonian diffeomorphisms

In this section, we will explain our construction of an invariant norm of Hamiltonian diffeomorphisms following [Oh9], which we call the *spectral norm.* This involves a careful usage of the spectral invariant $\rho(H;1)$ corresponding to the quantum cohomology class $1 \in QH^*(M)$.

8.1. Construction of the spectral norm.

Using $\rho(H;1)$, we define a function

$$\gamma : C_m^\infty([0,1]\times M) \to \mathbb{R}$$

by

$$\gamma(H) = \rho(H;1) + \rho(\overline{H};1), \tag{8.1}$$

on $C_m^\infty([0,1]\times M)$. Obviously we have $\gamma(H) = \gamma(\overline{H})$ for any H. The general triangle inequality

$$\rho(H;a) + \rho(F;b) \geq \rho(H\#F; a\cdot b)$$

for the spectral invariants restricted to $a = b = 1$, and the normalization axiom $\rho(id;1) = 0$ imply

$$\gamma(H) = \rho(H;1) + \rho(\overline{H};1) \geq \rho(\underline{0};1) = 0. \tag{8.2}$$

Here $a\cdot b$ is the quantum product of the quantum cohomology classes a, $b \in QH^*(M)$ and $\underline{0}$ is the zero function.

The following theorem generalizes the inequality (8.4) proven in [Oh3], [Oh4] for the exact case to the general quantum cohomology classes on non-exact symplectic manifolds.

Theorem 8.1. *For any H and $0 \neq a \in QH^*(M)$, we have*

$$-E^+(H) + v(a) \leq \rho(H;a) \leq E^-(H) + v(a). \tag{8.3}$$

In particular for any classical cohomology class $b \in H^*(M) \hookrightarrow QH^*(M)$, we have

$$-E^+(H) \leq \rho(H;b) \leq E^-(H) \tag{8.4}$$

for any Hamiltonian H.

Proof. We first recall the following general inequality

$$\int_0^1 -\max(H-K)\,dt \leq \rho(H,a) - \rho(K,a) \leq \int_0^1 -\min(H-K)\,dt.$$

proven in [Oh8], which can be rewritten as

$$\rho(K;a) + \int_0^1 -\max(H-K)\,dt \leq \rho(H;a) \leq \rho(K;a) + \int_0^1 -\min(H-K)\,dt.$$

Now let $K \to 0$ which results in

$$\rho(0;a) + \int_0^1 -\max(H)\,dt \leq \rho(H;a) \leq \rho(0;a) + \int_0^1 -\min(H)\,dt. \tag{8.5}$$

By the normalization axiom, we have $\rho(0;a) = v(a)$ which turns (8.5) to

$$v(a) - E^+(H) \leq \rho(H;a) \leq v(a) + E^-(H)$$

for any H. (8.4) immediately follow from the definitions and the identity $v(b) = 0$ for a classical cohomology class b. This finishes the proof. $\square$

Applying the right hand side of (8.4) to $b = 1$, we derive $\rho(H;1) \leq E^-(H)$ and $\rho(\overline{H};1) \leq E^-(\overline{H})$ for arbitrary H. On the other hand, we also have $E^-(\overline{H}) = E^+(H)$ for arbitrary H's and hence

$$\gamma(H) \leq \|H\|.$$

The nonnegativity (8.2) leads us to the following definition.

Definition 8.1. We define $\gamma : \mathcal{H}am(M,\omega) \to \mathbb{R}_+$ by

$$\gamma(\phi) := \inf_{H \mapsto \phi} (\rho(H;1) + \rho(\overline{H};1)).$$

Theorem 8.2. *Let γ be as above. Then $\gamma : Ham(M,\omega) \to \mathbb{R}_+$ defines an invariant norm i.e., it enjoys the following properties.*

1. $\phi = id$ *if and only if* $\gamma(\phi) = 0$
2. $\gamma(\eta^{-1}\phi\eta) = \gamma(\phi)$ *for any symplectic diffeomorphism* η
3. $\gamma(\phi\psi) \leq \gamma(\phi) + \gamma(\psi)$
4. $\gamma(\phi^{-1}) = \gamma(\phi)$
5. $\gamma(\phi) \leq \|\phi\|$

In the remaining subsection, we will give the proofs of these statements postponing the most non-trivial statement, nondegeneracy, to the next subsection modulo the Fundamental Existence Theorem whose proof we refer either to [Oh8] or [Oh9]. We split the proof of this theorem item by item.

Proof of (2). We recall the symplectic invariance of spectral invariants $\rho(H;a) = \rho(\eta^*H;a)$. Applying this to $a = 1$, we derive the identity

$$\begin{aligned}\gamma(\phi) &= \inf_{H \mapsto \phi} \Big(\rho(H;1) + \rho(\overline{H};1)\Big) \\ &= \inf_{H \mapsto \phi} \Big(\rho(\eta^*H;1) + \rho(\overline{\eta^*H};1)\Big) = \gamma(\eta^{-1}\phi\eta),\end{aligned}$$

which finishes the proof.

Proof of (3). We first recall the triangle inequality

$$\rho(H\#K;1) \leq \rho(H;1) + \rho(K;1) \tag{8.6}$$

and

$$\rho(\overline{K}\#\overline{H};1) \leq \rho(\overline{K};1) + \rho(\overline{H};1). \tag{8.7}$$

Adding up (8.6) and (8.7), we have

$$\begin{aligned}\rho(H\#K;1) \quad &+ \quad \rho(\overline{H\#K};1) = \rho(H\#K;1) + \rho(\overline{K}\#\overline{H};1) \\ &\leq \quad \Big(\rho(H;1) + \rho(\overline{H};1)\Big) + \Big(\rho(K;1) + \rho(\overline{K};1)\Big).\end{aligned} \tag{8.8}$$

Now let $H \mapsto \phi$ and $K \mapsto \psi$. Because $H\#K$ generates $\phi\psi$, we have

$$\gamma(\phi\psi) \leq \rho(H\#K;1) + \rho(\overline{H\#K};1)$$

and hence

$$\gamma(\phi\psi) \leq \Big(\rho(H;1) + \rho(\overline{H};1)\Big) + \Big(\rho(K;1) + \rho(\overline{K};1)\Big)$$

from (8.8). By taking the infimum of the right hand side over all $H \mapsto \phi$ and $K \mapsto \psi$, (3) is proved.

Proof of (4). The proof immediately follows from the observation that the definition of γ is symmetric over the map $\phi \mapsto \phi^{-1}$.

Proof of (5). By taking the infimum of

$$\gamma(H) \leq \|H\|$$

over $H \mapsto \phi$, we have proved $\gamma(\phi) \leq \|\phi\|$. □

It now remains to prove nondegeneracy of γ, which we will do in the next two sections. We like to mention that our proof of nondegeneracy of γ provides another proof of nondegeneracy of the Hofer norm via the inequality $\gamma(\phi) \leq \|\phi\|$.

8.2. **The ϵ-regularity theorem and its consequences.** The entirety of this and the next subsections will be occupied with the proof of nondegeneracy of the semi-norm

$$\gamma : \mathcal{H}am(M, \omega) \to \mathbb{R}_+$$

defined in section 8. First we note that the null set

$$\text{null}(\gamma) := \{\phi \in \mathcal{H}am(M, \omega) \mid \gamma(\phi) = 0\}$$

is a normal subgroup of $\mathcal{H}am(M, \omega)$ by the symplectic invariance of γ. Therefore by Banyaga's theorem [Ba], it is enough to exhibit one ϕ such that $\gamma(\phi) \neq 0$. We will prove that $\gamma(\phi) > 0$ for any *nondegenerate* Hamiltonian diffeomorphism and so for all $\phi \neq id$.

Suppose ϕ is a nondegenerate Hamiltonian diffeomorphism. Denote by J_0 a compatible almost complex structure on (M, ω). For given such a pair (ϕ, J_0), we consider the set of paths J'

$$j_{(\phi, J_0)} = \{J' : [0, 1] \to \mathcal{J}_\omega \mid J'(0) = J_0, \quad J'(1) = \phi^* J_0\}.$$

We extend J' to $\mathbb{R}$ so that

$$J'(t + 1) = \phi^* J'(t).$$

For each given $J' \in j_{(\phi, J_0)}$, we define the constant

$$A_S(\phi, J_0; J') = \inf \Big\{ \omega([u]) \quad | \quad u : S^2 \to M \text{ non-constant and satisfying } \overline{\partial}_{J'_t} u = 0 \text{ for some } t \in [0, 1] \Big\}.$$

A priori it is not obvious whether $A_S(\phi, J_0; J')$ is not zero. This is an easy consequence of the so called *ϵ-regularity theorem*, first introduced by Sacks-Uhlenbeck [SU] in the context of harmonic maps. We state a parameterized version of this theorem in the context of pseudo-holomorphic curves from [Oh1].

Lemma 8.3. [**ϵ-Regularity Theorem**] *Let g be any given background almost Kähler metric of (M, ω). We denote by $D = D^2(1)$ the unit open disc . Let J_0 be any almost complex structure and let $u : D \to M$ be a J_0-holomorphic map. Then there exists some $\epsilon = \epsilon(g, J_0) > 0$ such that if $\int_D |Du|^2 < \epsilon$, then for any smaller disc $D' = D^2(r)$ with $\overline{D}' \subset D$, we have*

$$\|Du\|_{\infty, D'} := \max_{z \in D'} |Du(z)| \leq C$$

where $C > 0$ depends only on g, ϵ, J_0 and D', not on u. Furthermore, the same C^1-bound holds for any compact family K of compatible almost complex structures with $\epsilon = \epsilon(g, K)$ and $C = C(g, \epsilon, K, D')$ depending on K.

An immediate corollary of this ϵ-regularity theorem is the following uniform C^1-estimate of pseudo-holomorphic curves whose derivation is the standard covering method in the geometric analysis. We refer to [Oh9] for its complete proof especially in the parametric form.

Corollary 8.4. *Let $J' \in j_{(\phi,J_0)}$. Then there exists an $\epsilon = \epsilon(J') > 0$ such that if $\omega(u) < \epsilon$, then we have*

$$\|Du\|_\infty := \max_{z \in S^2} |Du(z)| \le C$$

for any J'_t-holomorphic sphere $u : S^2 \to M$ and for any $t \in [0,1]$ where $C = C(\epsilon, J')$ does not depend on u.

The following positivity is an important consequence of the above uniform C^1-estimate for a pseudo-holomorphic map with small energy. To illustrate the usage of this C^1-estimate, we provide a complete proof borrowed from [Oh9].

Proposition 8.5. *Let ϕ, J_0 and J' be as above. Then we have*

$$A_S(\phi, J_0; J') > 0.$$

Proof. Suppose $A_S(\phi, J_0; J') = 0$. Then there exists a sequence $t_j \in [0,1]$ and a sequence of non-constant maps $u_j : S^2 \to M$ such that u_j is J_{t_j}-holomorphic and

$$\omega(u_j) = E_{J_{t_j}}(u_j) \to 0$$

as $j \to \infty$. By choosing a subsequence of t_j, again denoted by t_j, we may assume that $t_j \to t_\infty \in [0,1]$ and so J_{t_j} converges to J_{t_∞} in the C^∞-topology. We choose sufficiently large $N \in \mathbb{Z}_+$ so that

$$E_{J_{t_j}}(u_j) = \omega(u_j) < \epsilon(J')$$

for all $j \ge N$, where $\epsilon(J')$ is the constant ϵ provided in Corollary 8.4. Then we have the uniform C^1-bound

$$0 < \|Du_j\|_\infty \le C(\epsilon, J').$$

The Ascoli-Arzela theorem then implies that there exists a subsequence, again denoted by u_j, such that u_j converges uniformly to a continuous map $u_\infty : S^2 \to M$. Recalling that all the u_j are J_{t_j}-holomorphic and J_{t_j} converges to J_{t_∞} in the C^∞-topology, the standard boot-strap argument implies that $\{u_j\}$ converges to u_∞ in the C^1 topology (and so in the C^∞-topology). However we have

$$E_{J_{t_\infty}}(u_\infty) = \lim_{j \to \infty} E_{J_{t_j}}(u_j) = 0$$

and hence u_∞ must be a constant map, say $u_\infty \equiv x \in M$. Therefore $\{u_j\}$ converges to the point x in the C^∞-topology. In particular, if j is sufficiently large, then the image of u_j is contained in a (*contractible*) Darboux neighborhood of x. Therefore we must have $\omega([u_j]) = 0$ and in turn

$$E_{J_{t_j}}(u_j) = 0$$

for all sufficiently large j, because $E_{J_{t_j}}(u) = \omega(u)$ holds for any J_{t_j}-holomorphic curve u. This contradicts the assumption that u_j is non-constant. This finishes the proof. $\square$

Next for each given $J' \in j_{(\phi,J_0)}$, we consider the equation of $v : \mathbb{R}\times\mathbb{R}\to M$

$$\begin{cases} \frac{\partial v}{\partial \tau} + J'_t \frac{\partial v}{\partial t} = 0 \\ \phi(v(\tau, t+1)) = v(\tau,t), \quad \int_{\mathbb{R}\times[0,1]} |\frac{\partial v}{\partial \tau}|^2_{J'_t} < \infty. \end{cases} \tag{8.9}$$

Now it is a crucial matter to produce a non-constant solution of (8.9). For this purpose, using the fact that $\phi \neq id$, we choose a symplectic ball $B(\lambda)$ such that

$$\phi(B(\lambda)) \cap B(\lambda) = \emptyset \tag{8.10}$$

where $B(\lambda)$ is the image of a symplectic embedding $g : B^{2n}(r) \to B(\lambda) \subset M$ of the standard Euclidean ball $B^{2n}(r) \subset \mathbb{C}^n$ of radius r with $\lambda = \pi r^2$. We then study (8.9) together with

$$v(0,0) \in B(\lambda). \tag{8.11}$$

Because of (8.10) and (8.11), it follows

$$v(\pm\infty) \in \mathrm{Fix}\ \phi \subset M \setminus B(\lambda).$$

Therefore any such solution cannot be constant.

We now define the constant

$$A_D(\phi, J_0; J') := \inf_v \Big\{ \int_{\mathbb{R}\times[0,1]} v^*\omega \,\Big|\, v \text{ non-constant solution of (8.9)} \Big\}$$

for each $J' \in j_{(\phi,J_0)}$ as in subsection 2.6. Obviously we have $A_D(\phi, J_0; J') \geq 0$. We will prove $A_D(\phi, J_0; J') \neq 0$. We first derive the following simple lemma.

Lemma 8.6. *Let H be nondegenerate. Suppose that $u : \mathbb{R}\times S^1 \to M$ is any finite energy solution of*

$$\begin{cases} \frac{\partial u}{\partial \tau} + J_t\Big(\frac{\partial u}{\partial t} - X_H(u)\Big) = 0 \\ \int |\frac{\partial u}{\partial \tau}|^2_{J_t} < \infty. \end{cases} \tag{8.12}$$

that satisfies

$$u(-\infty, t) = u(\infty, t). \tag{8.13}$$

Then $\int_{\mathbb{R}\times S^1} u^*\omega$ converges, and we have

$$E_J(u) = \int_{\mathbb{R}\times S^1} u^*\omega. \tag{8.14}$$

Proof. First note that when H is nondegenerate, any finite energy solution has well-defined asymptotic limits $z^\pm = u(\pm\infty)$. Then we pick any bounding discs $w^\pm$ of $z^\pm$ such that $w^+ \sim w^- \# u$. Now (8.14) is an immediate consequence of (4.1) applied to $\mathcal{H} \equiv H$, since we assume (8.13), i.e., $z^+ = z^-$. □

With this proposition, we are ready to prove positivity of $A_D(\phi, J_0; J')$.

Proposition 8.7. *Suppose that ϕ is nondegenerate, and J_0 and $J' \in j_{(\phi,J_0)}$ as above. Then we have*

$$A_D(\phi, J_0; J') > 0.$$

Proof. We prove this by contradiction. Suppose $A_D(\phi, J_0; J') = 0$ so that there exists a sequence of non-constant maps $v_j : \mathbb{R}\times[0,1] \to M$ that satisfy (8.9) and

$$E_{J'}(v_j) \to 0 \quad \text{as } j \to \infty.$$

Therefore we have

$$E_{J'}(v_j) < \epsilon(J')$$

for all sufficiently large j', where $\epsilon(J')$ is the constant in Lemma 8.3 and Corollary 8.4. In particular, the sequence v_j cannot bubble off. This implies that v_j locally uniformly converge, and in turn that v_j must (globally) uniformly converge to a constant map because $E_{J'}(v_j) \to 0$. Since there are only finitely many fixed points of ϕ by the nondegeneracy hypothesis, by choosing a subsequence if necessary, we conclude

$$v_j(-\infty) = v_j(\infty) = p \tag{8.15}$$

for all j's for some $p \in \text{Fix}\ \phi$. Now we fix any Hamiltonian $H : [0,1] \times M \to \mathbb{R}$ that is zero near $t = 0,\ 1$ and with $H \mapsto \phi$, and consider the following maps

$$u_j : \mathbb{R} \times S^1 \to M, \quad u_j(\tau, t) := (\phi_H^t)(v_j(\tau, t)).$$

It follows from (8.15) that

$$u_j(-\infty, t) = u_j(\infty, t).$$

Furthermore for the family $J = \{J_t\}_{0 \leq t \leq 1}$ with

$$J_t := (\phi_H^t)_*(J'_t),$$

the u_j's satisfy the perturbed Cauchy-Riemann equation (8.12).

We note that (8.13) and the exponential convergence of $u_j(\tau)$ to $u_j(\pm\infty)$, as $\tau \to \pm\infty$ respectively, allows us to compactify the maps u_j and consider each of them as a cycle defined over a torus T^2. Therefore the integral $\int u_j^*\omega$ depends only on the homology class of the compactified cycles.

Now, because $v_j : \mathbb{R} \times [0,1] \to M$ uniformly converges to the constant map $p \in \text{Fix}\ \phi$, the image of u_j will be contained in a tubular neighborhood of the closed orbit $z_H^p : S^1 \to M$ of $\dot{x} = X_H(x)$ given by

$$z_H^p(t) = \phi_H^t(p).$$

In particular, $\int u_j^*\omega = 0$ because the cycle $[u_j]$ is homologous to the one dimensional cycle $[z_H^p]$. Then Lemma 8.6 implies the energy $E_J(u_j) = 0$. But by the choice of J above, Lemma 2.8 implies $E_{J'}(v_j) = 0$, a contradiction to the hypothesis that v_j are non-constant. This finishes the proof of Proposition 8.7. □

We then define

$$A(\phi, J_0; J') = \min\{A_S(\phi, J_0; J'), A_D(\phi, J_0; J')\}.$$

Proposition 8.5 and 8.7 imply

$$A(\phi, J_0; J') > 0.$$

The finiteness

$$A(\phi, J_0; J') < \infty$$

is a consequence of the Fundamental Existence Theorem, Theorem 8.13 in the next section. Finally we define

$$A(\phi, J_0) := \sup_{J' \in j_{(\phi, J_0)}} A(\phi, J_0; J')$$

and

$$A(\phi) = \sup_{J_0} A(\phi, J_0).$$

By definition, we have $A(\phi, J_0) > 0$ and so we have $A(\phi) > 0$. However a priori it is not obvious whether they are finite, which will be again a consequence of the Fundamental Existence Theorem.

8.3. **Proof of nondegeneracy.** With the definitions and preliminary studies of the invariants of $A(\phi, J_0; J')$, the following is the main theorem we will prove in this section, modulo the proof of Theorem 8.13 which we refer to [Oh9] and omit here.

Theorem 8.8. *Suppose that ϕ is nondegenerate. Then for any J_0 and $J' \in j_{(\phi,J_0)}$, we have*

$$\gamma(\phi) \geq A(\phi, J_0; J') \tag{8.16}$$

and hence

$$\gamma(\phi) \geq A(\phi).$$

In particular, $A(\phi)$ is finite.

We have the following two immediate corollaries. The first one proves nondegeneracy of γ and the second provides a new lower bound for the Hofer norm itself.

Corollary 8.9. *The pseudo-norm is nondegenerate, i.e., $\gamma(\phi) = 0$ if and only if $\phi = id$.*

Corollary 8.10. *Let ϕ be as in Theorem 8.8. Then we have*

$$\|\phi\| \geq A(\phi).$$

Remark 8.2. The function $\phi \mapsto A(\phi)$ is *not* C^0-continuous. However there is another geometric invariant $A(\phi; 1)$ introduced in [Oh9] which enjoys better C^0-continuity property than $A(\phi)$ and which we call the *homological area* of ϕ. This invariant $A(\phi; 1)$ satisfies $A(\phi; 1) \geq A(\phi)$ and is more computable than $A(\phi)$. Furthermore in [Oh9] we proved results stronger than those of Theorem 8.8 and Corollary 8.9 by replacing $A(\phi)$ by $A(\phi; 1)$. We expect that $A(\phi; 1)$ is C^0-continuous. We refer to [Oh9] for further discussions on $A(\phi; 1)$ in relation to the *optimal energy capacity inequality.*

The rest of this section will be occupied by the proof of Theorem 8.8.

Let ϕ be a nondegenerate Hamiltonian diffeomorphism with $\phi \neq id$. In particular, we can choose a small symplectic ball $B(\lambda)$ with $\lambda = \pi r^2$ such that

$$B(\lambda) \cap \phi(B(\lambda)) = \emptyset.$$

By the definition of γ, for any given $\delta > 0$, we can find $H \mapsto \phi$ such that

$$\rho(H; 1) + \rho(\overline{H}; 1) \leq \gamma(\phi) + \delta. \tag{8.17}$$

For any Hamiltonian $H \mapsto \phi$, we know that $\overline{H} \mapsto \phi^{-1}$. However we will use another Hamiltonian

$$\widetilde{H}(t, x) := -H(1 - t, x)$$

generating ϕ^{-1}, which is more useful than $\overline{H}$, at least in the study of duality and pants product. We refer to [Oh9] for the proof of the following lemma.

Lemma 8.11. *Let H be a Hamiltonian generating ϕ. Then $\widetilde{H} \mapsto \phi^{-1}$ and $\overline{H} \sim \widetilde{H}$, i.e.,*

$$[\phi^{-1}, \overline{H}] = [\phi^{-1}, \widetilde{H}].$$

In particular, we have $\rho(\overline{H}; a) = \rho(\widetilde{H}; a)$.

One advantage of using the representative $\widetilde{H}$ over $\overline{H}$ is that the time reversal

$$t \mapsto 1 - t$$

acting on the loops $z : S^1 \to M$ induces a natural one-one correspondence between $\mathrm{Crit}(H)$ and $\mathrm{Crit}(\widetilde{H})$. Furthermore the space-time reversal

$$(\tau, t) \mapsto (-\tau, 1 - t)$$

acting on the maps $u : \mathbb{R} \times S^1 \to M$ induces a bijection between the moduli spaces $\mathcal{M}(H, J)$ and $\mathcal{M}(\widetilde{H}, \widetilde{J})$ of the perturbed Cauchy-Riemann equations corresponding to (H, J) and $(\widetilde{H}, \widetilde{J})$ respectively, where $\widetilde{J}_t = J_{1-t}$. This correspondence reverses the direction of the Cauchy-Riemann flow and the corresponding actions satisfy

$$\mathcal{A}_{\widetilde{H}}([\widetilde{z}, \widetilde{w}]) = -\mathcal{A}_H([z, w]). \tag{8.18}$$

Here $[\widetilde{z}, \widetilde{w}]$ is the class corresponding to $\widetilde{z}(t) := z(1 - t)$ and $\widetilde{w} = w \circ c$ where $c : D^2 \to D^2$ is the complex conjugation of $D^2 \subset \mathbb{C}$.

The following estimate of the action difference is an important ingredient in our proof of nondegeneracy. The proof here is similar to the analogous non-triviality proof for the Lagrangian submanifolds studied in sections 6-7 [Oh2].

Proposition 8.12. *Let J_0 be any compatible almost complex structure, $J' \in j_{(\phi, J_0)}$ and J be the one-periodic family $J_t = (\phi_H^t)_* J'_t$. Let H be any Hamiltonian with $H \mapsto \phi$. Consider the equation*

$$\begin{cases} \frac{\partial u}{\partial \tau} + J_t\Big(\frac{\partial u}{\partial t} - X_H(u)\Big) = 0 \\ u(-\infty) = [z_-, w_-],\ u(\infty) = [z_+, w_+] \\ w_- \# u \sim w_+, \quad u(0,0) = q \in B(\lambda) \end{cases} \tag{8.19}$$

for a map $u : \mathbb{R} \times S^1 \to M$. If (8.19) has a broken trajectory solution (without sphere bubbles attached)

$$u_1 \# u_2 \# \cdots\cdots \# u_N$$

which is a connected union of solutions of (8.19) for H that satisfies the asymptotic condition

$$\begin{aligned} u_N(\infty) &= [z', w'],\ u_1(-\infty) = [z, w] \\ u_j(0,0) &= q \text{ for some } j. \end{aligned} \tag{8.20}$$

For some $[z, w] \in \mathrm{Crit}\, \mathcal{A}_H$ and $[\widetilde{z}', \widetilde{w}'] \in \mathrm{Crit}\, \mathcal{A}_{\widetilde{H}}$, then we have

$$\mathcal{A}_H(u(-\infty)) - \mathcal{A}_H(u(\infty)) \geq A_D(\phi, J_0; J').$$

Proof. Suppose u is such a solution. Opening up u along $t = 0$, we define a map $v : \mathbb{R} \times [0, 1] \to M$ by

$$v(\tau, t) = (\phi_H^t)^{-1}(u(\tau, t)).$$

It is straightforward to check that v satisfies (8.9). Moreover we have

$$\int \Big|\frac{\partial v}{\partial \tau}\Big|_{J'_t}^2 = \int \Big|\frac{\partial u}{\partial \tau}\Big|_{J_t}^2 < \infty \tag{8.21}$$

from Lemma 2.8. Since $\phi(B(\lambda)) \cap B(\lambda) = \emptyset$, we have

$$v(\pm\infty) \in \mathrm{Fix}\, \phi \subset M \setminus B(\lambda).$$

On the other hand since $v(0,0) = u(0,0) \in B(\lambda)$, v cannot be a constant map. In particular, we have

$$\int \Big|\frac{\partial v}{\partial \tau}\Big|^2_{J'_t} = \int v^*\omega \geq A_D(\phi, J_0; J').$$

Combining this and (8.21), we have proven

$$\mathcal{A}_H(u(-\infty)) - \mathcal{A}_H(u(\infty)) = \int \Big|\frac{\partial u}{\partial \tau}\Big|^2_{J_t} \geq A_D(\phi, J_0; J').$$

This finishes the proof. □

This proposition demonstrates relevance of the existence result of the equation (8.19) to Theorem 8.8. However we still need to control the asymptotic condition (8.20) and to establish some relevance of the asymptotic condition to the inequality (8.16). For this, we will use (8.18) and impose the condition

$$u(-\infty) = [z,w],\ u(\infty) = [z', w']$$

in (8.20) so that

$$[z,w] \in \alpha_H, \quad [\widetilde{z}', \widetilde{w}'] \in \beta_{\widetilde{H}} \tag{8.22}$$

for the *suitably chosen* fundamental Floer cycles α_H of H and $\beta_{\widetilde{H}}$ of $\widetilde{H}$.

We recall from (8.17) that we have

$$\rho(H;1) + \rho(\widetilde{H};1) \leq \gamma(\phi) + \delta.$$

We choose $H \mapsto \phi$ so that

$$\rho(H;1) + \rho(\widetilde{H};1) \leq \gamma(\phi) + \delta.$$

By the definition of ρ and from (2.23), there exist $\alpha_H \in CF_n(H)$ and $\beta_{\widetilde{H}} \in CF_n(\widetilde{H})$ representing $1^\flat = [M]$ such that

$$\rho(H;1) \quad \leq \quad \lambda_H(\alpha_H) \leq \rho(H;1) + \frac{\delta}{2} \tag{8.23}$$

$$\rho(\widetilde{H};1) \quad \leq \quad \lambda_{\widetilde{H}}(\beta_{\widetilde{H}}) \leq \rho(\widetilde{H};1) + \frac{\delta}{2}. \tag{8.24}$$

Once we have these, by adding (8.23) and (8.24), we obtain

$$\begin{aligned} 0 \leq \rho(H;1) + \rho(\widetilde{H};1) \quad &\leq \quad \lambda_H(\alpha_H) + \lambda_{\widetilde{H}}(\beta_{\widetilde{H}}) \\ &\leq \quad \rho(H;1) + \rho(\widetilde{H};1) + \delta. \end{aligned}$$

The fundamental cycles α_H and $\beta_{\widetilde{H}}$ that satisfy (8.23) and (8.24) respectively will be used as the asymptotic boundary condition required in (8.22).

The following is the fundamental existence theorem of the Floer trajectory *with its asymptotic limits lying near the 'top' of the given Floer fundamental cycles* which will make the difference

$$\mathcal{A}_H([z,w]) - \mathcal{A}_H([z',w']) = \mathcal{A}_H([z,w]) + \mathcal{A}_{\widetilde{H}}([\widetilde{z}', \widetilde{w}'])$$

as close to $\rho(H;1) + \rho(\widetilde{H};1)$ as possible. We refer readers to [Oh9] for other interesting consequences of this theorem besides the proof of nondegeneracy of γ.

Theorem 8.13. [Fundamental Existence Theorem] *Let ϕ, H, J_0 and $J' \in j_{(\phi,J_0)}$, be as in Proposition 8.12. and let $q \in M \setminus \mathrm{Fix}(\phi)$ be given. Choose any δ such that*

$$0 < \delta < A_D(\phi, J_0; J').$$

Then there exist some fundamental cycles α_H of (H, J) for $J_t = (\phi_H^t)_* J_t'$, and $\beta_{\widetilde{H}}$ of $(\widetilde{H}, \widetilde{J})$ such that

$$\begin{aligned}\lambda_H(\alpha_H) &\leq \rho(H; 1) + \frac{\delta}{2}\\ \lambda_{\widetilde{H}}(\beta_{\widetilde{H}}) &\leq \rho(\widetilde{H}; 1) + \frac{\delta}{2}\end{aligned}$$

and we can find some generators $[z, w] \in \alpha_H$ and $[\widetilde{z}', \widetilde{w}'] \in \beta_{\widetilde{H}}$ that satisfy the following alternative:

(1) *(8.19) has a broken-trajectory solution (without sphere bubbles attached)*

$$u_1 \# u_2 \# \cdots \cdots \# u_N$$

which is a connected union of Floer trajectories for H that satisfies the asymptotic condition

$$u_N(\infty) = [z', w'],\ u_1(-\infty) = [z, w], \quad u_j(0, 0) = q \in B(\lambda)$$

for some $1 \leq j \leq N$, (and hence

$$\mathcal{A}_H([z, w]) - \mathcal{A}_H([z', w']) \geq A_D(\phi, J_0; J')$$

from Proposition 8.12) or,

(2) *there exists a J_t'-holomorphic sphere $v_\infty : S^2 \to M$ for some $t \in [0, 1]$ passing through the point $q \in B(\lambda)$, and hence*

$$\mathcal{A}_H([z, w]) - \mathcal{A}_H([z', w']) \geq A_S(\phi, J_0; J') - \delta.$$

This in particular implies

$$A(\phi, J_0; J') < \rho(H; 1) + \rho(\widetilde{H}; 1) + \delta < \infty \tag{8.25}$$

for any ϕ and J_0.

Finish-up of the proof of nondegeneracy. Let ϕ be a nondegenerate Hamiltonian diffeomorphism. From the definition of $\gamma(\phi)$, since δ and H are arbitrary as long as $H \mapsto \phi$, we immediately derive, from (8.25),

$$A(\phi, J_0; J') \leq \gamma(\phi) \tag{8.26}$$

for all J_0 and $J' \in j_{(\phi,J_0)}$. Next by taking the supremum of $A(\phi, J_0; J')$ over all J_0 and $J' \in j_{(\phi,J_0)}$ in (8.26), we also derive

$$A(\phi) \leq \gamma(\phi).$$

This finishes the proof of Theorem 8.8 and so the proof of nondegeneracy.

9. Applications to Hofer geometry of $Ham(M,\omega)$

9.1. **Quasi-autonomous Hamiltonians and the minimality conjecture.** In this section, we drop the one-periodicity of the Hamiltonian function H, unless otherwise stated. The norm of H

$$\|H\| = \int_0^1 (\max H_t - \min H_t)\, dt$$

can be identified with the Finsler length

$$\mathrm{leng}(\phi_H) = \int_0^1 \Big(\max_x H(t, (\phi_H^t)(x)) - \min_x H(t, (\phi_H^t)(x)\Big)\, dt$$

of the path $\phi_H : t \mapsto \phi_H^t$ where the Banach norm on $T_{id}Ham(M,\omega) \cong C^\infty(M)/\mathbb{R}$ defined by

$$\|h\| = \mathrm{osc}(h) = \max h - \min h$$

for a normalized function $h : M \to \mathbb{R}$.

Definition 9.1. [The Hofer topology] Consider the metric

$$d : \mathcal{P}(Ham(M,\omega), id) \to \mathbb{R}_+$$

defined by

$$d(\lambda, \mu) := \mathrm{leng}(\lambda^{-1} \circ \mu)$$

where $\lambda^{-1} \circ \mu$ is the Hamiltonian path $t \in [0,1] \mapsto \lambda(t)^{-1}\mu(t)$. We call the induced topology on $\mathcal{P}(Ham(M,\omega), id)$ the *Hofer topology*. The Hofer topology on $Ham(M,\omega)$ is the strongest topology for which the evaluation map $\lambda \mapsto \lambda(1)$ is continuous.

It is easy to see that this definition of the Hofer topology of $Ham(M,\omega)$ coincides with the usual one induced by the Hofer norm function given in Definition 4.4, which also shows that the Hofer topology is metrizable. Of course nontriviality of the topology is not a trivial matter which was proven by Hofer [Ho1] for $\mathbb{C}^n$ and by Lalonde and McDuff [LM1] in its complete generality. It is also immediate to check that the Hofer topology of $Ham(M,\omega)$ is *locally path-connected* (see the proof of Theorem 3.15 [Oh10] for the relevant argument).

Hofer [Ho2] also proved that the path of any compactly supported *autonomous* Hamiltonian on $\mathbb{C}^n$ is length minimizing as long as the corresponding Hamilton's equation has no non-constant periodic orbit *of period less than or equal to one.* This result has been generalized in [En1], [MSl] and [Oh5]-[Oh7] under the additional hypothesis that the linearized flow at each fixed point is not over-twisted i.e., has no closed trajectory of period less than one. In [BP] and [LM2], Bialy-Polterovich and Lalonde-McDuff proved that any length minimizing (respectively, locally length minimizing) Hamiltonian path is generated by *quasi-autonomous* (respectively, *locally quasi-autonomous*) Hamiltonian paths.

Definition 9.2. A Hamiltonian H is called *quasi-autonomous* if there exists two points x_-, $x_+ \in M$ such that

$$H(t, x_-) = \min_x H(t,x), \quad H(t,x_+) = \max_x H(t,x)$$

for all $t \in [0,1]$.

We now recall the Ustilovsky-Lalonde-McDuff's necessary condition on the stability of geodesics. Ustilovsky [U] and Lalonde-McDuff [LM2] proved that for a generic ϕ in the sense that all its fixed points are isolated, any stable geodesic ϕ_t, $0 \leq t \leq 1$ from the identity to ϕ must have at least two fixed points which are under-twisted.

Definition 9.3. Let $H : [0,1] \times M \to \mathbb{R}$ be a Hamiltonian which is not necessarily time-periodic and ϕ_H^t be its Hamiltonian flow.

(1) We call a point $p \in M$ a *time T periodic point* if $\phi_H^T(p) = p$. We call $t \in [0,T] \mapsto \phi_H^t(p)$ a *contractible time T-periodic orbit* if it is contractible.
(2) When H has a fixed critical point p over $t \in [0,T]$, we call p *over-twisted* as a time T-periodic orbit if its linearized flow $d\phi_H^t(p)$; $t \in [0,T]$ on T_pM has a closed trajectory of period less than or equal to T. Otherwise we call it *under-twisted*. If in addition the linearized flow has only the origin as the fixed point, then we call the fixed point *generically under-twisted*.

Here we follow the terminology used by Kerman and Lalonde in [KL] for the "generically under-twisted". Note that under this definition of the under-twistedness, under-twistedness is C^2-stable property of the Hamiltonian H.

The following conjecture was raised by Polterovich, Conjecture 12.6.D [Po2]. (See also [Po3], [LM2] and [MSl].)

[Minimality Conjecture]. *Any autonomous Hamiltonian path that has no contractible periodic orbits of period less than equal to one is Hofer-length minimizing in its path-homotopy class relative to the boundary.*

9.2. **Length minimizing criterion via $\rho(H;1)$.** In this subsection, we describe a simple criterion of the length minimizing property of Hamiltonian paths in terms of the spectral invariant $\rho(H;1)$, which was given in [Oh7]. The criterion is similar to the one used in [Ho2] and in [BP] for the case of $\mathbb{C}^n$. In fact, Bialy and Polterovich [BP] predicted existence of such a criterion via the Floer homology on general symplectic manifolds, and this criterion indeed confirms their prediction.

To describe this criterion, we recall

$$\|H\| = E^-(H) + E^+(H)$$

where

$$\begin{aligned} E^-(H) &= \int_0^1 -\min H\, dt \\ E^+(H) &= \int_0^1 \max H\, dt. \end{aligned}$$

These are called the *negative Hofer-length* and the *positive Hofer-length* of H respectively. We will consider them separately. First note

$$E^+(H) = E^-(\overline{H}).$$

Theorem 9.1. *Let $G : [0,1] \times M \to \mathbb{R}$ be any Hamiltonian that satisfies*

$$\rho(G;1) = E^-(G) \tag{9.1}$$

Then H is negative Hofer-length minimizing in its homotopy class with fixed ends. In particular, G must be quasi-autonomous.

Proof. Let F be any Hamiltonian with $F \sim G$. Then we have a string of equalities and inequality

$$E^-(G) = \rho(G;1) = \rho(F;1) \leq E^-(F)$$

from (9.1), (6.2) for $a = 1$, (8.4) respectively. The last statement follows from Bialy-Polterovich, Ustilovsky and Lalonde-McDuff's criterion for the minimality. This finishes the proof. □

On the other hand, *if G is one-periodic*, we can consider the associated action functional $\mathcal{A}_G$. Then $\mathcal{A}_G$ has two obvious critical values of $\mathcal{A}_G$ for a quasi-autonomous Hamiltonian G given by

$$\begin{aligned}
\mathcal{A}_G([x^-, \widehat{x}^-]) &= \int_0^1 -G(t, x^-)\, dt \\
\mathcal{A}_G([x^+, \widehat{x}^+]) &= \int_0^1 -G(t, x^+)\, dt
\end{aligned}$$

which coincide with

$$\begin{aligned}
E^-(G) &= \int_0^1 -\min G_t\, dt \\
E^+(G) &= \int_0^1 \max G_t\, dt
\end{aligned}$$

respectively. We note that when G is one-periodic and quasi-autonomous having x^- and x^+ its uniform minimum and maximum points, then $\widetilde{G}$ given by

$$\widetilde{G}(t, x) = -G(1-t, x)$$

is also one-periodic and quasi-autonomous and has x^+ and x^- as a uniform minimum and a maximum point respectively. We also know (Lemma 8.11) that $\widetilde{G} \sim \overline{G}$.

Now we explain how to dispose the periodicity and extend the definition of $\rho(H;a)$ for arbitrary time dependent Hamiltonians $H : [0,1] \times M \to \mathbb{R}$. Note that it is obvious that the semi-norms $E^\pm(H)$ and $\|H\|$ are defined without assuming the periodicity. For this purpose, the following lemma from [Oh5] is important. We leave its proof to readers or to [Oh5].

Lemma 9.2. *Let H be a given Hamiltonian $H : [0,1] \times M \to \mathbb{R}$ and $\phi = \phi_H^1$ be its time-one map. Then we can re-parameterize ϕ_H^t in time so that the re-parameterized Hamiltonian H' satisfies the following properties:*

1. $\phi_{H'}^1 = \phi_H^1$
2. $H' \equiv 0$ *near* $t = 0, 1$ *and in particular H' is time periodic*
3. *Both $E^\pm(H' - H)$ can be made as small as we want*
4. *If H is quasi-autonomous, then so is H'*
5. *For the Hamiltonians H', H'' generating any two such re-parameterizations of ϕ_H^t, there is canonical one-one correspondences between $Per(H')$ and $Per(H'')$, and Crit $\mathcal{A}_{H'}$ and Crit $\mathcal{A}_{H''}$ with their actions fixed .*

Furthermore this re-parametrization is canonical with the "smallness" in (3) can be chosen uniformly over H depending only on the C^0-norm of H.

Using this lemma, we can now define $\rho(H;a)$ for arbitrary H by

$$\rho(H;a) := \rho(H';a)$$

where H' is the Hamiltonian generating the canonical re-parametrization of ϕ_H^t in time provided in Lemma 9.2. It follows that this definition is well-defined because any such re-parameterizations are homotopic to each other with fixed ends. This being said, we will always assume that our Hamiltonians are time one-periodic without mentioning further in the rest of the paper.

9.3. **Canonical fundamental Floer cycles.** Now we are ready to introduce the following concept of homological essentialness in the chain level theory, which is the heart of matter in the chain level Floer theory.

Definition 9.4. We call a Floer cycle $\alpha \in CF(H)$ *tight* if it satisfies the following non-pushing down property under the Cauchy-Riemann flow (3.4): for any Floer cycle $\alpha' \in CF(H)$ homologous to α (in the sense of Definition 3.8 (2)), it satisfies

$$\lambda_H(\alpha') \geq \lambda_H(\alpha).$$

In terms of the length minimizing criterion in Theorem 9.1, we would like to construct a *tight* fundamental Floer cycle of G whose level is precisely $E^-(G)$ for a quasi-autonomous Hamiltonian G.

As often done in [Oh5], one natural way of constructing a Floer fundamental cycle of general Hamiltonian H is to transfer a Morse cycle using Floer's chain map. More precisely, we consider a Morse function f and the fundamental Morse cycle α of $-\epsilon f$ *for a sufficiently small* $\epsilon > 0$ *such that Theorem 5.1 holds.* Then α also becomes a Floer cycle of ϵf. We then transfer α and define a fundamental Floer cycle of H as

$$\alpha_H := h_{\mathcal{L}}(\alpha) \in CF(H)$$

where $h_{\mathcal{L}}$ is the Floer chain map over the *canonically given* linear path

$$\mathcal{L}:\ s \mapsto (1-s)\epsilon f + sH.$$

We call any of such transferred cycle a *canonical fundamental Floer cycle* of H as in [Oh9]. We however note that this cycle depends on the choice of the Morse function f. In general, we do not expect this cycle will be tight even when H is quasi-autonomous.

Now we apply this construction to a quasi-autonomous Hamiltonian G *that has the unique nondegenerate global minimum* x^- *that is undertwisted for all* $t \in [0,1]$, which was studied by Kerman and Lalonde [KL]. In this case, they made the following particular choice of the Morse function f in the above linear path $\mathcal{L}$ so that

(1) f has a global minimum point at x^-
(2) f satisfies
$$f(x^-) = 0, \quad f(x^-) < f(x_j)$$
for all other critical points x_j.

Having f adapted to the given G this way, Kerman and Lalonde [KL] proved the following basic result on the transferred cycle α_G for the aspherical manifold. Their proof was then generalized by the author [Oh7] for general symplectic manifolds. We refer readers to [Oh7] for the details of the proof.

Proposition 9.3. *Suppose that* G *is a generic one-periodic Hamiltonian such that* G_t *has the unique nondegenerate global minimum* x^- *which is fixed and undertwisted for all* $t \in [0,1]$. *Suppose that* $f: M \to \mathbb{R}$ *is a Morse function such that*

f has the unique global minimum point x^- and $f(x^-) = 0$. Then the canonical fundamental cycle has the expression

$$\alpha_G = [x^-, \widehat{x}^-] + \beta \in CF(G) \tag{9.2}$$

for some Floer Novikov chain $\beta \in CF(G)$ with the inequality

$$\lambda_G(\beta) < \lambda_G([x^-, \widehat{x}^-]) = \int_0^1 -G(t, x^-)\, dt. \tag{9.3}$$

In particular its level satisfies

$$\begin{aligned} \lambda_G(\alpha_G) &= \lambda_G([x^-, \widehat{x}^-]) \\ &= \int_0^1 -G(t, x^-)\, dt = \int_0^1 - \min G\, dt. \end{aligned} \tag{9.4}$$

9.4. **The case of autonomous Hamiltonians.** In this section, we will restrict to the case of autonomous Hamiltonians G. The following result was proven in [Oh7].

Theorem 9.4. *Let (M, ω) be an arbitrary closed symplectic manifold. Suppose that G is an autonomous Hamiltonian such that*

(1) *it has no non-constant contractible periodic orbits "of period one"*
(2) *it has a maximum and a minimum that are generically under-twisted*
(3) *all of its critical points are nondegenerate in the Floer theoretic sense (i.e., the linearized flow of X_G at each critical point has only the zero as a periodic orbit).*

Then the one parameter group ϕ_G^t is length minimizing in its homotopy class with fixed ends for $0 \leq t \leq 1$.

And the same result with the condition (1) is replaced by the one in which the phrase "of period one" replaced by "of period less than equal to one" was proven in [Oh5] earlier. There are also similar results proven in [MSl], and [En1] (for the strongly semi-positive case) with slightly different hypotheses. The improvement of the phrase "of period less than equal to one" being replaced by "of period one" is due to Kerman and Lalonde [KL] in the case of symplectically aspherical (M, ω).

To prove Theorem 9.4, according to the criterion Theorem 9.1, it will be enough to prove that the value $\mathcal{A}_G([x^-, \widehat{x}^-]) = E^-(G)$ coincides with the mini-max value $\rho(G; 1)$. This latter fact is an immediate consequence of the following theorem, which is a special case of the main theorem in [Oh7] restricted to the strongly semi-positive case. Here we provide details of the proof for the strongly semi-positive case.

Theorem 9.5. *Suppose that G is an autonomous Hamiltonian satisfying the hypotheses in Theorem 9.4. Then the canonical fundamental cycle α_G constructed in Proposition 9.3 is tight, i.e.,*

$$\rho(G; 1) = \lambda_G(\alpha_G)\, (= -G(x^-) = E^-(G)).$$

Proof. Note that the conditions in Theorem 9.4 in particular imply that G is non-degenerate. We fix a *time-independent* J_0 which is G-regular.

Suppose that α is homologous to the canonical fundamental Floer cycle α_G, i.e.,

$$\alpha = \alpha_G + \partial_G(\gamma) \tag{9.5}$$

for some Floer Novikov chain $\gamma \in CF_*(G)$. When G is autonomous and $J \equiv J_0$ is t-independent, there is no non-stationary t-independent trajectory of $\mathcal{A}_G$ landing at $[x^-,\widehat{x}^-]$ because any such trajectory comes from the negative Morse gradient flow of G but x^- is the minimum point of G. Therefore any non-stationary Floer trajectory u landing at $[x^-,\widehat{x}^-]$ must be t-dependent. Because of the assumption that G has no non-constant contractible periodic orbits of period one, any critical points of $\mathcal{A}_G$ has the form

$$[x,w] \quad \text{with } x \in \text{Crit } G.$$

Let u be a trajectory starting at $[x,w]$, $x \in \text{Crit } G$ with

$$\mu([x,w]) - \mu([x^-,\widehat{x}^-]) = 1, \tag{9.6}$$

and denote by $\mathcal{M}_{(G,J_0)}([x,w],[x^-,\widehat{x}^-])$ the corresponding Floer moduli space of connecting trajectories. The general index formula shows

$$\mu([x,w]) = \mu([x,w_x]) - 2c_1([w]). \tag{9.7}$$

We consider two cases separately: the cases of $c_1([w]) = 0$ or $c_1([w]) \neq 0$. If $c_1([w]) \neq 0$, we derive from (5.4), (5.5) that $x \neq x^-$. This implies that any such trajectory must come with (locally) free S^1-action, i.e., the moduli space

$$\widehat{\mathcal{M}}_{(G,J_0)}([x,w],[x^-,\widehat{x}^-]) = \mathcal{M}_{(G,J_0)}([x,w],[x^-,\widehat{x}^-])/\mathbb{R}$$

and its stable map compactification have a locally free S^1-action *without fixed points*. Then it follows from the S^1-equivariant transversality theorem from [FHS] that $\widehat{\mathcal{M}}_{(G,J_0)}([x,w],[x^-,\widehat{x}^-])$ becomes empty for a suitable choice of an autonomous J_0. This is because the quotient has the virtual dimension -1 by the assumption (9.6). We refer to [FHS] for more explanation on this S^1-invariant regularization process. Now consider the case $c_1([w]) = 0$. First note that (9.6) and (9.7) imply that $x \neq x^-$. On the other hand, if $x \neq x^-$, the same argument as above shows that the perturbed moduli space becomes empty.

It now follows that there is no trajectory of index 1 that land at $[x^-,\widehat{x}^-]$. Therefore $\partial_G(\gamma)$ cannot kill the term $[x^-,\widehat{x}^-]$ in (9.5) away from the cycle

$$\alpha_G = [x^-,\widehat{x}^-] + \beta$$

and hence we have

$$\lambda_G(\alpha) \geq \lambda_G([x^-,\widehat{x}^-])$$

by the definition of the level λ_G. Combined with (9.4), this finishes the proof. □

10. Remarks on the transversality for general (M,ω)

Our construction of various maps in the Floer homology works as they are in the previous section for the strongly semi-positive case [Se], [En1] by the standard transversality argument. On the other hand in the general case where constructions of operations in the Floer homology theory requires the machinery of virtual fundamental chains through multi-valued abstract perturbation, we need to explain how this general machinery can be incorporated in our construction. The full details will be provided elsewhere. We will use the terminology 'Kuranishi structure' adopted by Fukaya and Ono [FOn] for the rest of the discussion.

One essential point in our proofs is that various numerical estimates concerning the critical values of the action functional and the levels of relevant Novikov cycles do *not* require transversality of the solutions of the relevant pseudo-holomorphic sections, but *depends only on the non-emptiness of the moduli space*

$$\mathcal{M}(H,\widetilde{J};\widehat{z})$$

which can be studied for *any*, not necessarily generic, Hamiltonian H. Since we always have suitable a priori energy bound which requires some necessary homotopy assumption on the pseudo-holomorphic sections, we can compactify the corresponding moduli space into a compact Hausdorff space, using a variation of the notion of stable maps in the case of nondegenerate Hamiltonians H. We denote this compactification again by

$$\mathcal{M}(H, \widetilde{J}; \widehat{z}).$$

This space could be pathological in general. But because we assume that the Hamiltonians H are nondegenerate, i.e, all the periodic orbits are nondegenerate, the moduli space is not completely pathological but at least carries a Kuranishi structure in the sense of Fukaya-Ono [FOn] for any H-compatible $\widetilde{J}$. This enables us to apply the abstract multi-valued perturbation theory and to perturb the compactified moduli space by a Kuranishi map Ξ so that the perturbed moduli space

$$\mathcal{M}(H, \widetilde{J}; \widehat{z}, \Xi)$$

is transversal in that the linearized equation of the perturbed equation

$$\overline{\partial}_{\widetilde{J}}(v) + \Xi(v) = 0$$

is surjective and so its solution set carries a smooth (orbifold) structure. Furthermore the perturbation Ξ can be chosen so that as $\|\Xi\| \to 0$, the perturbed moduli space $\mathcal{M}(H, \widetilde{J}; \widehat{z}, \Xi)$ converges to $\mathcal{M}(H, \widetilde{J}; \widehat{z})$ in a suitable sense (see [FOn] for the precise description of this convergence).

Now the crucial point is that non-emptiness of the perturbed moduli space will be guaranteed as long as certain topological conditions are met. For example, the followings are the prototypes that we have used in this paper:

(1) $h_{\mathcal{H}} : CF_0(\epsilon f) \to CF_0(H)$ is an isomorphism in homology and so $[h_{\mathcal{H}}(1^\flat)] \neq 0$. This is immediately translated as an existence result of solutions of the perturbed Cauchy-Riemann equation.

(2) The definition of the pants product $*$ and the identity

$$[\alpha * \beta] = (a \cdot b)^\flat$$

in homology guarantee non-emptiness of the relevant perturbed moduli space $\mathcal{M}(H, \widetilde{J}; \widehat{z}, \Xi)$ for $\alpha \in CF_*(H_1)$, $\beta \in CF_*(H_2)$ with $[\alpha] = a^\flat$ and $[\beta] = b^\flat$ respectively.

Once we prove non-emptiness of $\mathcal{M}(H, \widetilde{J}; \widehat{z}, \Xi)$ and an a priori energy bound for the non-empty perturbed moduli space and *if the asymptotic conditions $\widehat{z}$ are fixed*, we can study the convergence of a sequence $v_j \in \mathcal{M}(H, \widetilde{J}; \widehat{z}, \Xi_j)$ as $\Xi_j \to 0$ by the Gromov-Floer compactness theorem. However a priori there are infinite possibility of asymptotic conditions for the pseudo-holomorphic sections that we are studying, because we typically impose that the asymptotic limit lie in certain Novikov cycles like

$$\widehat{z}_1 \in \alpha,\ \widehat{z}_2 \in \beta,\ \widehat{z}_3 \in \alpha * \beta$$

Because the Novikov Floer cycles are generated by an infinite number of critical points $[z, w]$ in general, one needs to control the asymptotic behavior to carry out compactness argument. For this purpose, we need to establish a *lower bound* for the actions which will enable us to consider only a finite number of possibilities for the asymptotic conditions because of the finiteness condition in the definition of Novikov chains. We would like to emphasize that obtaining a lower bound is the heart of matter in the classical mini-max theory of the *indefinite* action functional which requires a linking property of semi-infinite cycles. On the other hand, obtaining an *upper bound* is usually an immediate consequence of the identity like (4.1).

With such a lower bound for the actions, we may then assume, by taking a subsequence if necessary, that the asymptotic conditions are fixed when we take the limit and so we can safely apply the Gromov-Floer compactness theorem to produce a (cusp)-limit lying in the compactified moduli space $\mathcal{M}(H, \widetilde{J}; \widehat{z})$. This would then justify all the statements and proofs in this paper for the complete generality, without assuming the strong semi-positivity assumption.

Appendix A. Proof of the index formula

In this appendix, we give the proof of the index formula (1.1), Theorem 7.1. The only thing that enters in the definition of the Conley-Zehnder index is a periodic solution of the Hamilton's equation

$$\dot{x} = X_H(x)$$

on a symplectic manifold (M, ω) for a one-periodic Hamiltonian function $H : S^1 \times M \to \mathbb{R}$. We will give the proof of the index formula in several steps.

0. **(Other convention)** There is another package of conventions that have been consistently used by Salamon-Zehnder [SZ], Polterovich [Po3] and others. In that convention, there are two things to watch out in relation to the index formula, when compared to our convention. The first thing is that their definition of the Hamiltonian vector field, also called as the symplectic gradient and denoted by $\mathrm{sgrad}\, H$, is given by

$$\mathrm{sgrad}\, H \rfloor \omega = -dH. \tag{A.1}$$

Therefore we have $X_H = -\mathrm{sgrad} H$. The second thing is that their definition of the canonical symplectic form on $T^*\mathbb{R}^n = \mathbb{R}^{2n} \cong \mathbb{C}^n$ in the coordinates $z_j = q_j + ip_j$ is given by

$$\omega_0' = \sum_{j=1} dp_j \wedge dq^j = -\omega_0 \tag{A.2}$$

Cancelling out two negatives, the definition of the Hamiltonian vector field of a function H on $\mathbb{R}^{2n}$ in this package becomes the same vector field as ours that is given by

$$J_0 \nabla H$$

where ∇H is the usual gradient vector field of H with respect to the standard Euclidean inner product on $\mathbb{R}^{2n}$.

1. **(Canonical symplectic form)** Our convention of the canonical symplectic form of on $T^*\mathbb{R}^n = \mathbb{R}^{2n} \cong \mathbb{C}^n$ in the coordinates $z_j = q_j + ip_j$ is given by

$$\omega_0 = \sum_{j=1} dq^j \wedge dp_j. \tag{A.3}$$

This means that on $\mathbb{R}^{2n}$ J_0 is the standard complex structure on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ obtained by multiplication by the complex number i.

2. **(Canonical complex structure)** In our convention of the canonical symplectic form ω_0 $\mathbb{C}^n$, the associated Hermitian structure

$$\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$$

becomes *complex linear in the first argument, but anti-linear in the second argument.* In other words, the Hermitian inner product is given by

$$\langle u, v\rangle = g(u, v) - i\omega_0(u, v). \tag{A.4}$$

We like to note that this Hermitian structure on $\mathbb{C}^n$ is the conjugate to that of [HS], [SZ], [Po3], which corresponds to

$$\langle u, v\rangle = g(u, v) + i\omega_0(u, v). \tag{A.5}$$

(See the remark right before Lemma 5.1 [SZ].) Equivalently, the latter Hermitian structure is associated to the almost Kähler structure

$$(g, \omega_0', J_0')$$

where J_0' is the almost complex structure conjugate to J_0. *This change of complex structure on $\mathbb{C}^n$ affects the sign of the first Chern number of general complex vector bundles E :* we recall the following general formula for the Chern classes of the complex vector bundle E

$$c_k(\overline{E}) = (-1)^k c_k(E).$$

3. **(The Conley Zehnder index on $\mathcal{S}P^*(1)$)** We follow the definition from [SZ] of the Conley-Zehnder index, denoted by $\mathrm{ind}_{CZ}(\alpha)$ as in [FH], for a paths α lying in $\mathcal{S}P^*(1)$: we denote

$$\mathcal{S}P^*(1) = \{\alpha : [0,1] \to Sp(2n,\mathbb{R}) \mid \alpha(0) = id,\ \det(\alpha(1) - id) \neq 0\} \tag{A.6}$$

following the notation from [SZ]. Note that the definition of $Sp(2n,\mathbb{R})$ are the same in both of the above conventions. We will define the Conley-Zehnder index function $\mathrm{ind}_{CZ} : \mathcal{S}P^*(1) \to \mathbb{Z}$ to be the same as that of [SZ]. This index is then characterized by Proposition 5 [FH].

4. **(Symplectic trivialization)** A given pair $[\gamma, w] \in \widetilde{\Omega}_0(M)$ determines a preferred homotopy class of trivialization of the symplectic vector bundle γ^*TM on $S^1 = \partial D^2$ that extends to a trivialization

$$\Phi_w : w^*TM \to D^2 \times (\mathbb{R}^{2n}, \omega_0)$$

over D^2 of where $D^2 \subset \mathbb{C}$ is the unit disc with the standard orientation. A symplectic trivialization $\overline{\Phi}_w : w^*TM \to D^2 \times (\mathbb{R}^{2n}, \omega_0')$ of w^*TM in terms of $(\mathbb{R}^{2n}, \omega_0')$ is then obtained by the composition

$$\overline{\Phi}_w = c \circ \Phi_w; \quad \overline{\Phi}_w(z, v) := \overline{\Phi_w(z, v)}, \quad (z, v) \in w^*TM \tag{A.7}$$

where c is the complex conjugation on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ in the obvious sense.

5. **(The Conley-Zehnder index, $\mu_H([z, w])$)** Let $z : \mathbb{R}/\mathbb{Z} \times M$ be a one-periodic solution of $\dot{x} = X_H(x)$. Any such one-periodic solution has the form $z(t) = \phi_H^t(p)$ for a fixed point $p = z(0) \in \mathrm{Fix}(\phi_H^1)$. When we are given a one-periodic solution z and its bounding disc $w : D^2 \to M$, we consider the one-parameter family of the symplectic maps

$$d\phi_H^t(z(0)) : T_{z(0)}M \to T_{z(t)}M$$

and define a map $\alpha_{[z,w]} : [0,1] \to Sp(2n,\mathbb{R})$ by

$$\alpha_{[z,w]}(t) = \Phi_w(z(t)) \circ d\phi_H^t(z(0)) \circ \Phi_w(z(0))^{-1}. \tag{A.8}$$

Obviously we have $\alpha_{[z,w]}(0) = id$, and nondegeneracy of H implies that

$$\det(\alpha_{[z,w]}(1) - id) \neq 0$$

and hence

$$\alpha_{[z,w]} \in \mathcal{S}P^*(1).$$

Then according to the definition of [SZ], [HS] the Conley-Zehnder index of $[z, w]$ is defined by

$$\mu_H([z, w]) := \mathrm{ind}_{CZ}(\overline{\alpha}_{[z,w]}) \tag{A.9}$$

where $\overline{\alpha}_{[z,w]} = c \circ \alpha_{[z,w]}$.

6. When we are given two maps

$$w,\, w' : D^2 \to M$$

with $w|_{\partial D^2} = w'|_{\partial D^2}$, we define the glued map $u = w\#\overline{w}' : S^2 \to M$ in the following way:

$$u(z) = \begin{cases} w(z) & z \in D^+ \\ w'(1/\overline{z}) & z \in D^-. \end{cases}$$

Here D^+ is D^2 with the same orientation, and D^- with the opposite orientation. This is a priori only continuous but we can deform to a smooth one without changing its homotopy class by 'flattening' the maps near the boundary: In other words, we may assume

$$w(z) = w(z/|z|) \quad \text{for } |z| \geq 1 - \epsilon$$

for sufficiently small $\epsilon > 0$. We will always assume that the bounding disc will be assumed to be flat in this sense. With this adjustment, u defines a smooth map from S^2.

7. **(The marking condition)** For the given $[z, w]$, $[z, w']$ with a periodic solution $z(t) = \phi_H^t(z(0))$, we impose the additional *marking* condition

$$\Phi_w(z(0)) = \Phi_{w'}(z(0)) \tag{A.10}$$

as a map from $T_{z(0)}M$ to $\mathbb{R}^{2n}$ for the trivialization

$$\Phi_w, \Phi_{w'} : w^*TM \to D^2 \times (\mathbb{R}^{2n}, \omega_0)$$

which is always possible. *With this additional condition*, we can write

$$\alpha_{[z,w']}(t) = S_{w'w}(t) \cdot \alpha_{[z,w]}(t) \tag{A.11}$$

where $S_{w'w} : S^1 = \mathbb{R}/\mathbb{Z} \to Sp(2n, \mathbb{R})$ is the *loop* defined by the relation (A.11). Note that this really defines a loop because we have

$$\alpha_{[z,w']}(0) = \alpha_{[z,w]}(0)\,(= id) \tag{A.12}$$

$$\alpha_{[z,w']}(1) = \alpha_{[z,w]}(1) \tag{A.13}$$

where (A.13) follows from the marking condition (A.10). In fact, it follows from the definition of (A.11) and (A.10) that we have the identity

$$\begin{aligned} S_{w'w}(t) &= \Big(\Phi_{w'}(z(t)) \circ d\phi_H^t(z(0)) \circ \Phi_{w'}(z(0))^{-1}\Big) \\ &\quad \circ \Big(\Phi_w(z(t)) \circ d\phi_H^t(z(0)) \circ \Phi_w(z(0))^{-1}\Big)^{-1} \\ &= \Phi_{w'}(z(t)) \circ \\ &\quad \Big(d\phi_H^t(z(0)) \circ \Phi_{w'}(z(0))^{-1} \circ \Phi_w(z(0)) \circ (d\phi_H^t)^{-1}(z(0))\Big) \\ &\quad \circ (\Phi_w(z(t)))^{-1}. \end{aligned} \tag{A.14}$$

Then the marking condition (A.10) implies the middle terms in (A.14) are cancelled away and hence we have proved

$$S_{w'w}(t) = \Phi_{w'}(z(t)) \circ \Phi_w(z(t))^{-1} \tag{A.15}$$

Then we derive the following formula, from the definition μ_{CZ} in [CZ] and from (A.15),

$$\mathrm{ind}_{CZ}(\overline{\alpha}_{[z,w']}) = 2\ \mathrm{wind}(\overline{\widehat{S}}_{w'w}) + \mathrm{ind}_{CZ}(\overline{\alpha}_{[z,w]}) \tag{A.16}$$

where $\widehat{S}_{w'w} : S^1 \to U(n)$ is a loop in $U(n)$ that is homotopic to $S_{w'w}$ inside $Sp(2n, \mathbb{R})$. (See Proposition 5 [FH] for this formula.) Such a homotopy always exists and is unique upto homotopy because $U(n)$ is a deformation retract to $Sp(2n, \mathbb{R})$. And $\mathrm{wind}(\overline{\widehat{S}}_{w'w})$ is the degree of the obvious determinant map

$$\det_{\mathbb{C}}(\overline{\widehat{S}}_{w'w}) : S^1 \to S^1.$$

8. **(Normailization of c_1)** Finally, we recall the definition of the first Chern class c_1 of the symplectic vector bundle $E \to S^2$. We normalize the Chern class so that the tangent bundle of $S^2 \cong \mathbb{C}P^1$ has the first Chern number 2, which also coincides with the standard convention in the literature. *We like to note that this normalization is compatible with the Hermitian structure on $\mathbb{C}^n$ given by (A.4) in our convention.* (See p 167 [MSt].)

We decompose $S^2 = D^+ \cup D^-$ and consider the symplectic trivializations $\Phi_+ : E|_{D^+} \to D^2 \times (\mathbb{R}^{2n}, \omega_0)$ and $\Phi_- : E|_{D^-} \to D^2 \times (\mathbb{R}^{2n}, \omega_0)$. Note that under the Hermitian structure on $\mathbb{C}^n$ in our convention, these are homotopic to a unitary trivialization, while in other convention they are homotopic to a conjugate unitary trivialization.

Denote by the transition matrix loop

$$\phi_{+-} : S^1 \to Sp(2n, \mathbb{R})$$

which is the loop determined by the equation

$$\Phi_+|_{S^1} \circ (\Phi_-|_{S^1})^{-1}(t, \xi) = (t, \phi_{+-}(t)\xi)$$

for $(t, \xi) \in E|_{S^1}$, where $S^1 = \partial D^+ = \partial D^-$. Then, by definition, we have

$$c_1(E) = \mathrm{wind}(\widehat{\phi}_{+-}) \tag{A.17}$$

in our convention. Equivalently, we have

$$c_1(E) = -\mathrm{wind}(\overline{\widehat{\phi}}_{+-}). \tag{A.18}$$

Now we apply this to $u^*(TM)$ where $u = w' \# \overline{w}$ and Φ_w and $\Phi_{w'}$ are the trivializations given in 4. It follows from (A.15) that $S_{w'w}$ is the transition matrix loop between Φ_w and $\Phi_{w'}$. Then by definition, the first Chern number $c_1(u^*TM)$ is provided by the winding number $\mathrm{wind}(\widehat{S}_{w'w})$ of the loop of unitary matrices

$$\widehat{S}_{w'w} : t \mapsto \widehat{S}_{w'w}(t); \quad S^1 \to U(n)$$

in the Hermitian structure of $\mathbb{C}^n$ in our convention. One can easily check that this winding number is indeed 2 when applied to the tangent bundle of S^2 and so consistent with the convention of the Chern class that we are adopting.

9. **(Wrap-up of the proof)** These steps, in particular, Step 2 and Step 7 and 8 combined, (A.9) (A.16) and (A.18) turn into the index formula we want to prove. We restate this in the following theorem.

Theorem A.1. *Let (M, ω) be a symplectic manifold and X_H a Hamiltonian vector field defined by*

$$X_H \rfloor \omega = dH$$

of any contractible one-periodic Hamiltonian function $H : [0,1] \times M \to \mathbb{R}$. For a given one-periodic solution $z : S^1 = \mathbb{R}/\mathbb{Z} \to M$ of $\dot{x} = X_H(x)$ and two given bounding discs w, w', we have the identity

$$\mu_H([z, w']) = \mu_H([z, w]) - 2c_1([w' \# \overline{w}]).$$

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