Relative Floer and quantum cohomology and the symplectic topology of Lagrangian submanifolds

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Dedicated to the memory of Andreas Floer

Abstract: Floer (co)homology of the symplectic manifold which was originally introduced by Floer in relation to the Arnol'd conjecture has recently attracted much attention from both mathematicians and mathematical physicists either from the algebraic point of view or in relation to the quantum cohomology and the mirror symmetry.

Very recent progress in its relative version, the Floer (co)homology of Lagrangian submanifolds, has revealed quite different mathematical perspective: The Floer (co)homology theory of Lagrangian submanifolds is a powerful tool in the study of *symplectic topology of Lagrangian submanifolds*, just as the classical (co)homology theory in topology has been so in the study of *differential topology of submanifolds* on differentiable manifolds.

In this survey, we will review the Floer theory of Lagrangian submanifolds and explain the recent progress made by Chekanov and by the present author in this relative Floer theory, which have found several applications of the Floer theory to the symplectic topology of Lagrangian submanifolds which demonstrates the above perspective: They include a Floer theoretic proof of Gromov's non-exactness theorem, an optimal lower bound for the symplectic disjunction energy, construction of new symplectic invariants, proofs of the non-degeneracy of Hofer's distance on the space of Lagrangian submanifolds in the cotangent bundles and new results on the Maslov class obstruction to the Lagrangian embedding in $\mathbb{C}^n$. Each of these applications accompanies new development in the Floer theory itself: localizations, semi-infinite cycles and minimax theory in the Floer theory and a spectral sequence as quantum corrections and others. We will also define the relative version of the quantum cohomology, and explain its relation to the Floer cohomology of Lagrangian submanifolds and its applications.

§1. Introduction

It is a classical fact that a gradient vector field on a compact smooth manifold tend to have more zeros than a generic vector field. This is due to the fact that to each such a gradient vector field, by definition, is associated a smooth function whose critical points correspond to zeros of the gradient vector field. In particular for a nondegenrate smooth function (i.e., Morse function), one can apply the classical Morse theory to derive information on the zeros of the associated gradient vector field. It is by now well-known that Morse theory is much more closely tied to the topology of underlying manifold than the degree theory of generic vector fields. For example, the degree theory only captures the Euler characteristic while the Morse theory recovers the whole cohomology (or homology) of the underlying manifold. One should recall that although the zeros of the gradient vector field (or the topology of underlying manifold) are independent of the choice of the Riemannian metric that is used to define the gradient, having a metric is an essential ingredient in doing Morse theory.

Now by counting the number of zeros of Morse functions on M, one can define a *diffeomorphism invariant* of M,

$$CRN(M) := \inf_f \#\{\text{Crit } (f) \mid f \in C^\infty(M) \text{ is Morse } \} \tag{1.1}$$

From the classical Morse theory, this $CRN(M)$ is bounded below by a *homological invariant*

$$SB(M) := \sum_{i=0}^{n} \dim H^i(M, \mathbb{Z}). \tag{1.2}$$

By Smale's h-cobordism theorem, we have

$$CRN(M) = SB(M)$$

provided M is simply connected and dim $M \geq 5$. However when dim $M \leq 4$ or M is not simply connected, the relation between these two invariants is not well understood.

One can also define another diffeomorphism invariant, allowing *degenerate* functions,

$$CR(M) := \inf_f \#\{ \text{ Crit } (f) \mid f \in C^\infty(M)\} \tag{1.3}$$

From the Lusternik-Schinirelmann theory, it is known that $CR(M)$ is bounded below by a *homeomorphism invariant*

$$\begin{aligned} \text{cat}(M) = \inf_{\mathcal{U}} \{r \mid \mathcal{U} = \{U_1, \cdots, U_r\} &\text{ is an open covering of } M \\ &\text{such that each } U_i \text{ is contractible}\} \end{aligned} \tag{1.4}$$

It is also known that $\mathrm{cat}(M)$ is bounded by the *cohomological invariant*, $CL(M)+1$ where

$$CL(M) := \min\{k \mid \text{ there exist } \beta_1, \cdots, \beta_k \in H^*(M) - H^0(M), \\ \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k \neq 0\} \tag{1.5}$$

Geometrically the zeros of a general vector field on M can be identified with the intersections of two *middle* dimensional submanifolds (that are homotopic to each other) of the tangent bundle TM, one the zero section and the other the graph of the vector field in TM. The degree theory on M can be translated into the Lefschetz intersection theory in this setting. However for the gradient vector field, it is more natural to consider its dual version: *the graph of df in the contangent bundle T^*M* instead. This is because this dual description does not involve the extra structure, the *Riemannian metric*. Now the Morse theory (or the Lusternik-Schnirelmann theory) can be translated into the *Lagrangian intersection theory on T^*M*: both the zero section and the Graph df in T^*M are *Lagrangian submanifolds* on T^*M with respect to the canonical symplectic structure on T^*M. Furthermore the two Lagrangian submanifolds are isotopic through Hamiltonian diffeomorphisms (i.e., *Hamiltonian isotopic*). Although due to the birth of *caustics* the general Hamiltonian isotopy does not preserve the stucture of *being a graph*, Arnold [Ar1] posed the following celebrated conjecture, which is known as the intersection version of the *Arnold conjecture* and which first predicted the existence of the Lagrangian intersection theory

Arnold's Conjecture. *Let M be a compact n-manifold and*

$$L_0 = \text{ the zero section of } T^*M, \quad L_1 = \phi(L_0)$$

where ϕ is a Hamiltonian diffeomorphism. Then

$$\begin{aligned} \#(L_0 \cap L_1) &\geq CRN(M) \quad \text{for the transversal case} \\ &\geq CR(M) \quad \text{for the general case.} \end{aligned}$$

Because of the lack of understanding of the invariants $CRN(M)$ or $CR(M)$, this conjecture is widely open. However its cohomological version was proven by Hofer [H1] using a version of classical critical point theory, which was inspired by Conley-Zehnder's proof of Arnold's conjecture on the fixed points of Hamiltonian symplectic diffeomorphisms [CZ]. A much simpler proof using *gererating functions* was given by Laudenbach and Sikorav [LS]. It is also a corollary of Floer's more general theorem [F2, F4], which we will describe in detail later.

Theorem I [Hofer, Laudenbach-Sikorav]. *Under the same hypothesis as in Arnol'd's Conjecture, we have*

$$\begin{aligned}\#(L_0 \cap L_1) &\geq SB(M) \quad \textit{for the transversal case} \\ &\geq CL(M) + 1 \quad \textit{for the general case.}\end{aligned}$$

One might tempt to generalize the conjecture further by considering general symplectic manifolds (P, ω) and Lagrangian submanifolds $L \subset P$. But it is easy to see that this general version is obviously false: consider a small torus in (P, ω) which always exists by Darboux theorem and which can be easily disjuncted by Hamiltonian isotopies. In this respect, Floer [F2,4] and Hofer [H2] for (1.7) proved the following theorem for a special class of Lagrangian submanifolds,

Theorem II [Floer]. *Let (P, ω) be a symplectic manifold that is tame and $L \subset P$ be a compact Lagrangian submanifold with $\pi_2(P, L) = \{e\}$ (or more generally with $\omega|_{\pi_2(P,L)} \equiv 0$). Then for $L_0 = L$, $L_1 = \phi(L)$, we have*

$$\#(L_0 \cap L_1) \geq SB(L : \mathbb{Z}_2) \quad \textit{for the transversal case} \tag{1.6}$$

$$\geq CL(L : \mathbb{Z}_2) + 1 \quad \textit{for the general case.} \tag{1.7}$$

Since T^*M is tame for all compact M and $\pi_2(T^*M, L) = \{e\}$ for the zero section L, the above cotangent bundle theorem is a corollary of Floer's (at least in $\mathbb{Z}_2$-coefficients as it stands. In fact, one can prove in the case (T^*M, o_M) as in Theorem I that the Floer complex can be given a coherent orientation to define the $\mathbb{Z}$-valued Floer (co)homology and so Floer's proof of Theorem II works for the integer coefficient. See [O10] for more details). The proof of this theorem involves by now the well-known *Floer (co)homology* for the pair (L, P). However there had been some evidence that the restrction $\omega|_{\pi_2(P,L)} \equiv 0$ would not be an essential restriction (see [CJ] or [Gi2] for the case $(\mathbb{R}P^n, \mathbb{C}P^n)$). Eventually, the present author proved the following theorem in [O5] which relies on our generalization of the Floer cohomology for the *monotone pair* (P, L) with the *minimal Maslov number* greater than equal to 2 (see [O2,8] or Section 6 below).

Theorem III [Oh]. *Let (P, ω) be a compact Hermitian symmetric space and $L \subset P$ be an orientable real form, or (P, ω) be irreducible and $L \subset (P, \omega)$ be arbitrary real forms. Then (1.6) holds.*

Although the statement of Theorem II and III look the same, the geometry behind their proofs is completely different. One could roughly state that in Therem I and II, *the quantized picture is the same as the classical picture*, while in Theorem III *the quantized picture is different from the classical one but the quantum effect does not contribute.* We will make this comparison more precise later. Similar phenomenon occurs for the case of the Floer cohomology for the (semi-positive) symplectic manifolds. However unlikely from the case of Lagrangian submanifolds, the absence of the quantum effect on the *module structure* of the cohomology of the symplectic manifolds is omnipresent, i.e, *the Floer cohomology and the ordinary cohomology are always isomorphic as modules (over the Novikov ring) for the case of symplectic manifolds.* (See [F5], [HS], [On], [PSS] and [RT2].) For the general estimates (1.7), it has been proven for the case $\pi_2(P, L) = \{e\}$ by Floer himself [F6], which involves a product structure on the Floer cohomology (See also [H2] for a proof which does not directly involve the product structure.) It is not clear at this moment whether (1.7) holds everytime (1.6) holds, which seems unlikely to the author. We would like to call the Floer cohomology for the symplectic manifold the *nonrelative* version of the Floer cohomology and that of Lagrangian submanifolds in a symplectic manifold the *relative* version of the Floer cohomology with respect to the ambient symplectic manifold, and will do so from now on.

Although the nonrelative Floer cohomology has recently attracted much attention from both mathematicians and mathematical physists motivated partly by its relation to the quantum cohomology and mirror symmetry, the current status of the subject is somewhat unsatisfactory as far as applications to the symplectic topology are concerned: There have been few applications (see [FH2] [CFH] for some applications of idea relevant to this) of the big machinery beyond the original motivation, the Arnold conjecture. Most of the current research has been focused on the effort of unveiling the hidden algebraic structure of the Floer cohomology, e.g., product structure and its relation to quantum cohomology (see e.g., [BzRd], [PSS] or [RT2]). On the other hand, the relative version, the Floer cohomology of Lagrangian submanifolds has received little attention since Floer's pioneering work [F2] and the author's generalization [O3,4,5] to the monotone Lagrangian submanifolds, which were also developed in the attmept to prove the relative version of Arnol'd's conjecture. As far as the author perceives, this has happened partly because the relative case does not seem to have much interaction to other areas of mathematics (except to the Atiyah's conjecture in the Floer homology for 3-manifolds [At], [F7]) and also because until very recently applications of the relative

Floer cohomology have been none beyond the Arnold conjecture. Furthermore there has been no survey article on the relative version of the Floer (co)homology other than the original articles by Floer [F2] and the author [O3,4,5], while there have appeared several good literature that survey the nonrelative version of the Floer cohomology (See e.g. [McD] and the books [MS], [HZ]).

One of the main purposes of this survey is to remedy this situation by reviewing the construction of the Floer cohomology and clarifying the differences between the relative and the nonrelative cases. Another purpose is to advertize recent exciting new developments made by Chekanov [C2] and by the author [O9,10] in the relative Floer theory and its applications to the symplectic topology of Lagrangian submanifolds. Combining the works in [C2] and [O9,10], it appears that most of the results on the Lagrangian submanifolds (see e.g., [Po1,2,3] [V3]) that have been proven since Gromov's paper [Gr] appeared can be not only reproved but also strengthend further in a systematic way using the machinery of the Floer (co)homology. One most notable exception so far is the result by Viterbo [V2] on the *general* Lagragian tori in $\mathbb{C}^n$. We strongly believe that this development is only the beginning of the new era of the Floer theory, which is waiting to be unfold: *the Floer theory in the symplectic topology of Lagrangian submanifolds is a powerful tool in understanding symplectic topology of Lagrangian submanifolds, just as the classical (co)homology theory in the topology has been so in understanding differential topology of submanifolds on differentiable manifolds.* The third purpose is to briefly explain joint works of Fukaya and the author on the relation between the relative versions of the Floer and the quantum cohomology. However the main emphasis of this survey is not on the structural study of the Floer cohomology or the quantum cohomology but their applications to the study of Lagrangian submanifolds. Finally we would like to mention a connection of the relative Floer theory to the open string theory by Witten (See [W] and [Fu2] for detailed discussions).

The organization of the paper is as follows. In Section 2, we briefly review the known invariants of compact Lagrangian embeddings. In Section 3, we review the construction of the Floer cohomology given in [F2] and [O3] and give an axiomatic definition of it. After then the survey will consist of two parts, one from Section 4 to Section 7 which deals with the case *without quantum effects* (i.e., without bubbling phenomena) and the other from Section 8 to Section 10 which deals with the case *with quantum effects* (i.e, with bubbling). Section 10 is of the nature of research announcements of works still in progress.

Here is a more detailed description section by section. In Section 4, we verify the compactness properties required in the definition of Floer cohomology given in Section 3 emphasizing details which are less known in the literature. The general scheme of proving these compactness properties are by now well-known and explained well in the literature, for example, in [McD]. In Section 5, we define the local version of the Floer cohomology that was introduced in [O9] a version of which had been used already in [F5] in the context of non-relative theory. As an application of this local Floer cohomology, we give a Floer theorectic proof of Gromov's celebrated non-exactness theorem [Gr] of compact Lagrangian embeddings in $\mathbb{C}^n$. In Section 6, we outline the proof of Chekanov's result [C2] on the symplectic disjunction energy which strengthens Polterovich's theorem [Po4]. In Section 7, we explain the author's recent work [O10] on the Floer theoretic construction of the invariants of Viterbo type [V3] and an application to Hofer's geometry on Lagrangian submanifolds in the contangent bundle providing two different proofs of the fact that Hofer's pseudo-distance on the space of Lagrangian submanifolds is nondegenerate. In the course of doing this, we discover semi-infinite cycles which are closely tied to the Floer theory, and an application to the structure of wave fronts of Lagrangian submanifolds.

In Section 8, we review the construction of the Floer cohomology for monotone Lagrangian submanifolds again emphasizing the details less known in the literature. In Section 9, we explain the author's recent work [O9] on a new symplectic invariant of the Lagrangian embedding, a spectral sequence which computes the Floer cohomology of Lagrangian embeddings. This spectral sequence measures the extrincic geometry versus the intrincic geometry of Lagrangian embeddings. As applications, we state several theorems on monotone Lagrangian embeddings in $\mathbb{C}^n$, an optimal upper bound for the minimal Maslov number, a result on the Audin's question [Au] on the Maslov class of Lagrangian tori in $\mathbb{C}^n$ and a new obstruction to compact Lagrangian embeddings in $\mathbb{C}^n$. In Section 10, we define the relative version of the quantum cohomology and give an outline of the proof of the equivalence between the Floer cohomology and the quantum cohomology of the Lagrangian embedding.

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§2. Invariants of Lagrangian submanifolds

2.1. Classical invariants

For Lagrangian submanifolds $L \subset (\mathbb{C}^n, \omega_0)$ with the standard symplectic structure $\omega_0 = -d\theta$ where θ is the canonical one form, so called the *Liouville form*, there are two well-known symplectic invariants of them, one the *Maslov class* denoted by $\mu_L \in H^1(L, \mathbb{Z})$ and the other the *symplectic period* or the *Liouville class* $[\theta|_L] \in H^1(L, \mathbb{R})$. One of the fundamental results in the symplectic topology proven by Gromov [Gr] is the following non-triviality theorem of the Liouville class.

Theorem 2.1 [Gromov]. *For any compact Lagrangian embedding $L \subset \mathbb{C}^n$, $[\theta|_L] \neq 0$ in $H^1(L, \mathbb{R})$.*

But it is still an open question in general whether the corresponding result holds for the Maslov class μ_L.

Conjecture 2.2. For any compact Lagrangian embedding $L \subset \mathbb{C}^n$, $\mu_L \neq 0$ in $H^1(L, \mathbb{Z})$.

Some partial results are known: the conjecture holds for any Lagrangian tori [V2] and for two dimensional surfaces in $\mathbb{C}^2$ [Po1]. It holds for monotone Lagranigan embeddings for any dimension [Po1,2].

On the general symplectic manifolds (P, ω), these classes are not defined as they are but can be generalized as homomorphisms from $\pi_2(P, L)$ to $\mathbb{Z}$ and $\mathbb{R}$ respectively. We first briefly recall how we define the *Maslov index* homomorphism $I_{\mu,L}$ on $\pi_2(P, L)$ for a general Lagrangian submanifold L in P, which we will often just denote by μ for the notational convenience. If $w : (D^2, \partial D^2) \to (P, L)$ is a smooth map of pairs, we can find a unique trivialization (up to homotopy) of the pull-back bundle $w^*TP \simeq D^2 \times \mathbb{C}^n$ as a symplectic vector bundle. This trivialization defines a map $\alpha_w : S^1 \to \Lambda(\mathbb{C}^n)$ from $S^1 = \partial D^2$ to $\Lambda(\mathbb{C}^n) =$ *the set of Lagrangian planes* in $\mathbb{C}^n$ where there is a well-known

Maslov class $\mu \in H^1(\Lambda(\mathbb{C}^n), \mathbb{Z})$ (see [Ar2]). We define

$$I_{\mu,L}(w) := \mu(\alpha_w) \in \mathbb{Z} \ .$$

It is easy to show that $I_{\mu,L}$ defines a homomorphism on $\pi_2(P, L)$. This Maslov index is invariant under Hamiltonian isotopy ϕ_t of P and indeed under any symplectic isotopy of P. For this isotopy gives the same trivialization of w_t^*TP, $D^2 \times \mathbb{C}^n$ (up to homotopy) and α_{w_t} defines a homotopy between α_{w_0} and α_{w_1} in $\Lambda(\mathbb{C}^n)$, which proves that α_{w_0} and α_{w_1} have the same Maslov indices. Here we denoted $w_t := \phi_t(w_0)$.

The homomorphism $I_{\omega,L}$ on $\pi_2(P, L)$ is much easier to define and is given by

$$I_{\omega,L}(w) = \int_{D^2} w^*\omega \ .$$

This $I_{\omega,L}$ is not invariant under the general symplectic isotopy but invariant *only under the Hamiltonian isotopy.*

2.2. Quantum invariants

More modern invariants involving pseudo-holomorphic curves were first introduced and used by Gromov [Gr] to prove, among other things, the celebrated non-exactness theorem, Theorem 2.1, of compact Lagrangian embedding in $\mathbb{C}^n$, which then gives rise to the existence of exotic symplectic structure on $\mathbb{C}^n$ for $n \geq 2$. However these invariants can be defined so far, as invariants under Hamiltonian isotopies of Lagranigan submanifolds, only for a restricted class of Lagrangian submanifolds like *monotone Lagrangian submanifolds*:

Definition 2.1. A Lagrangian submanifold L on P is called *monotone* if two homomorphisms $I_{\mu,L}$ and $I_{\omega,L}$ satisfy

$$I_{\omega,L} = \lambda\, I_{\mu,L} \ , \qquad \text{for some} \quad \lambda \geq 0 \ .$$

Remark 2.2.

i) The monotonicity is preserved under the Hamiltonian deformations of L.
ii) One can easily show that if (P, ω) admits a monotone Lagrangian submanifold, then P itself must be a monotone symplectic manifold in the sense of Floer [F5]. Moreover the monotonicity constant λ in Definition 2.1 does not depend on L but depends only on (P, ω), if $I_{\omega,L}|_{\pi_2(P)} \neq 0$ (See [O3]).

For the monotone pair (L, P), one can define a whole variety of invariants by counting the number of (pseudo)-holomorphic discs (see Section 8 or [O8] for a brief description of these invariants), which are similar to the *Gromov-Witten invariants* for the (semi-positive) symplectic manifolds (see [Ru] or [MS]). These invariants are preserved under Hamiltonian isotopies of monotone Lagrangian submanifolds and can be used to distinguish certain monotone Lagrangian tori that have the same classical invariants (see [C1] and [EP2] for relevant examples). More sophistigated invariants which are the main subject of this survey are the celebrated *Floer (co)homology* $HF^*(L : P)$ which has already shown its power in the proof of Arnold's conjecture by Floer, or the *relative quantum cohomology* $QH^*(L : P)$ which we will define in Section 10. As we will explain in Section 9 and 10, it turns out that these latter invariants are particularly useful for studying the Maslov class of Lagrangian submanifolds in $\mathbb{C}^n$, *exactly because the invariants must vanish therein as soon as they are well-defined.* We refer the precise definition of these invariants and their applications to the Maslov class obstruction to Lagrangian embeddings to later sections.

2.3. Invariants for pairs (L_1, L_2) and the Hofer's distance

In the above, we have described various invariants of one Lagrangian that are preserved under Hamiltonian isotopies. These invariants distinguish only the orbits of actions by the group of Hamiltonian diffeomorphisms and so can be considered invariants defined on the space of orbits. When we consider a pair (L_1, L_2) in the same orbit, one can associate an interseting invariant of the pair (L_1, L_2), the *Hofer's (pseudo)-distance* between them as in the case of Hamiltonian diffeomorphisms [H3] (see Section 7).

As in the case of diffeomorphism [H3], we take the point of view of the Finsler geometry. Let L_0 be a fixed compact Lagrangian submanifold in (P, ω) and denote by $\Lambda_\omega(L_0 : P)$ the set of Lagranigan submanifolds that are Hamiltonian isotopic to L_0. The tangent space of $\Lambda_\omega(L_0 : P)$ at $L \in \Lambda_\omega(L_0 : P)$ can be canonically identified with

$$\widetilde{C}^\infty(L) := C^\infty(L)/\{\text{constant functions}\}.$$

We now define a norm on this set by the oscillation

$$\|f\| = \max_{x \in L} f(x) - \min_{x \in L} f(x) \tag{2.1}$$

for $f \in \widetilde{C}^\infty(L)$ and a length of the Hamiltonian isotopy $\overline{L} = \{L_s\}$ between them by

$$\|\overline{L}\| = \int_0^1 \|f_s\| ds$$

where $f_s \in \widetilde{C}^\infty(L_s)$ is the element corresponding to the "tangent vector field" at L_s

$$\frac{d}{ds}(L_s), \quad s \in [0,1]$$

For given two $L_1, L_2 \in \Lambda_\omega(L_0 : P)$, we define the (pseudo)-distance between them by

$$d(L_1, L_2) := \inf_{\overline{L}} \|\overline{L}\| \tag{2.2}$$

where the infimum is taken over all smooth path $\overline{L}$ between L_0, L_1. It is easy to check that

$$d(L_1, L_2) = \inf_{\phi} \|\phi\| \tag{2.3}$$

where the infimum is taken over all $\phi \in \mathcal{D}^c_\omega(P)$ with $\phi(L_1) = L_2$. One can easily prove that this definition satisfies the triangle inequality. The main nontrivial question is whether this pseudo-distance is indeed a distance i.e, whether it holds that

$$d(L_1, L_2) = 0 \quad \text{ if and only if } \; L_1 = L_2. \tag{2.4}$$

It is still not known in general whether this (pseudo)-distance is non-degenerate but we will show in Section 7 (and see [O10] for details) that this is the case for the orbit of the zero section of the cotangent bundle T^*M of any compact manifold M. In fact in this case, we can use a version of Floer theory to construct a family of invariants parametrized by $H^*(M, \mathbb{R})$ which are similar to the invariants that Viterbo constructed in [V3] using *generating functions*. See Section 7 below for more detailed account. The complete detalis will appear in [O10].

§3. Definition of the Floer cohomology.

Since the Floer theory should be considered as a version of (infinite dimensional) Morse theory, we should at least say why. For two given compact Lagrangian submanifolds L_0, L_1 in (P, ω), we consider the space

$$\Omega = \Omega(L_0, L_1) = \{z : [0,1] \to P \mid z(0) \in L_0, z(1) \in L_1\}.$$

On this space, we define a *closed* one form (the *action one-from*) α by

$$\alpha(z)(v) = \int_0^1 \omega(\dot{z}, v)dt \tag{3.1}$$

for each $v \in T_z\Omega$. Locally one can express this form as

$$\alpha = d\mathcal{A} \tag{3.2}$$

where $\mathcal{A}$ is the *action functional* on Ω which is not globally defined in general. As in the finite dimensional Morse theory, we study the gradient flow of $\mathcal{A}$ in terms of a given "Riemannian metric" on Ω. We choose a L^2-metric on Ω with respect to some Riemannian metric on P that is compatible to the given symplectic structure ω: let $S_\omega = S_\omega(P)$ be the bundle of all almost complex structures $J \in \text{End}\ (TP)$ whose fiber is given by

$$S_x = \{J \in \text{End}\ (TP) \mid J^2 = -id \text{ and } \omega(\cdot, J\cdot) \text{ is positive definite}\}.$$

Then any smooth section of S_ω is just an almost complex structure *compatible* (or *calibrated*) to (P, ω). The set of compatible almost complex structure is known to be contractible [Gr]. In terms of the metric

$$g_J := \omega(\cdot, J\cdot), \tag{3.3}$$

(3.1) can be re-written as

$$\alpha(z)(v) = \int_0^1 g_J(J\dot{z}, v)dt$$

and so the L^2-gradient of the action functional $\mathcal{A}$ can be written as

$$\text{grad}_{g_J}\mathcal{A}(z) = J\dot{z}.$$

Hence the flow $u(\tau)$ on Ω of the vector field

$$-\text{grad}\ _{g_J}\mathcal{A}$$

satisfies the equation (as a map $u : \mathbb{R} \times [0, 1] \to P$)

$$\begin{cases} \overline{\partial}_J u := \frac{\partial u}{\partial \tau} + J(u)\frac{\partial u}{\partial t} = 0 \\ u(\tau, 0) \subset L_0, u(\tau, 1) \subset L_1 \\ u(\tau, t) := u(\tau)(t). \end{cases}$$

This equation is an example of the elliptic boundary value problem but on the non-compact space $\mathbb{R} \times [0, 1]$. As usual, we need to study the space of solutions of this equation in terms of the Fredholm theory. The appropriate Fredholm set-up goes as follows. To introduce a suitable topology on Ω, we fix a metric on P of the type in (3.3) and give the topology on Ω induced by the Sobolev norms on P with respect to this fixed metric. All the estimates implicit in the later discussions are in terms of this fixed metric. We now consider the space

$$\mathcal{P}^p_{k:loc}(L_0, L_1 : P) = \{u \in L^p_{k:loc}(\Theta, P) \mid u(\mathbb{R} \times \{0\}) \subset L_0, u(\mathbb{R} \times \{1\} \subset L_1\}$$

and

$$\begin{aligned} j_\omega &= j_\omega(P) = C^\infty(S_\omega) \\ &= \text{the space of compatible almost complex structures.} \\ \mathcal{J}_\omega &= \mathcal{J}_\omega(P) := C^\infty([0,1] \times S_\omega) \\ &= \text{the space of parametrized compatible almost complex structures.} \end{aligned} \tag{3.4}$$

For each given $J \in \mathcal{J}_\omega$, we study

$$\begin{cases} \frac{\partial u}{\partial \tau} + J(u,t)\frac{\partial u}{\partial t} = 0 \\ u(\tau, 0) \subset L_0, u(\tau, 1) \subset L_1 \\ u(\tau, t) := u(\tau)(t). \end{cases} \tag{3.5}$$

We denote

$$\begin{aligned} \mathcal{M}_J &= \mathcal{M}_J(L_0, L_1) \\ &= \big\{u \in \mathcal{P}^p_{k:loc} \mid u \ \text{ satisfies (3.5) and } \int_\Theta \big|\frac{\partial u}{\partial t}\big|^2 < \infty\big\} \\ &= \big\{u \in \mathcal{P}^p_{k:loc} \mid u \ \text{ satisfies (3.5) and } \int_\Theta u^*\omega < \infty\big\}. \end{aligned}$$

and for each give pair $x, y \in L_0 \cap L_1$, we define

$$\begin{aligned} \mathcal{M}_J(x,y) &= \big\{u \in \mathcal{M}_J \mid \lim_{\tau \to \infty} u = x, \lim_{\tau \to -\infty} u = y\big\} \\ \widehat{\mathcal{M}}_J(x,y) &= \mathcal{M}_J(x,y)/\mathbb{R}. \end{aligned}$$

This $\mathcal{M}_J(x,y)$ plays a role of the space of *connecting orbits*, i.e, the trajectories connecting two critical points of a smooth function in the finite dimensional Morse theory. The main analytic properties of $\mathcal{M}_J$ are summarized in the following propositions, which were essentially proven in [F1] for the case $x \neq y$, however with some unsettling points in the proof of the transversality. These points are carefully addressed in [FHS] and [O6].

Proposition 3.1 [Proposition 2.1, F1]. *For $x, y \in L_0 \cap L_1$, we have*

$$\mathcal{M}_J = \bigcup_{x,y \in L_0 \cap L_1} \mathcal{M}_J(x,y) \ .$$

Moreover, if L_0 intersects L_1 transversally, then for each $x, y \in L_0 \cap L_1$, there exist smooth Banach manifolds

$$\mathcal{P}(x,y) = \mathcal{P}^p_k(x,y) \subset \mathcal{P}^p_{k:loc}$$

so that (3.1) *defines a smooth section $\overline{\partial}_J : u \mapsto \overline{\partial}_J u$ of a smooth Banach space bundle $\mathcal{L}$ over $\mathcal{P}(x,y)$ with fibers $\mathcal{L}_u = L^p_{k-1}(u^*TP)$, and so that $\mathcal{M}_J(x,y)$ is the zero set of $\overline{\partial}_J$. The tangent space $T_u\mathcal{P} = T_u\mathcal{P}(x,y)$ consists of all elements ξ of $L^p_k(u^*TP)$ so that $\xi(\tau,0) \in TL_0$ and $\xi(\tau,1) \in TL_1$ for all $\tau \in \mathbb{R}$. The linearizations*

$$E_u := D\bar{\partial}(u) = T_u\mathcal{P} \to \mathcal{L}_u$$

are Fredholm operators for $u \in \mathcal{M}_J(x,y)$. There is a dense set $\mathcal{J}_{reg}(L_0, L_1) \subset \mathcal{J}_\omega$ so that if $J \in \mathcal{J}_{reg}(L_0, L_1)$, then E_u is surjective for all $u \in \mathcal{M}_J(x,y)$.

Proposition 3.2 [Theorem 1, F1]. *If $x, y \in L_0 \cap L$ are transverse intersections, then the Fredholm index of the linearization*

$$E_u := D\overline{\partial}_J(u) : T_u\mathcal{P} \to \mathcal{L}_u$$

is the same as the Maslov-Viterbo index $\mu_u(x,y)$ (See [V1] for its definition). *In particular $\mathcal{M}_J(x,y)$ becomes a smooth manifold with dimension equal to $\mu_u(x,y)$ for $J \in \mathcal{J}_{reg}(L_0, L_1)$.*

Definition 3.3. We call $u \in \mathcal{M}_J(L_0, L_1)$ *regular* if Coker $E_u = 0$ and we call (J, L_0, L_1) *regular* if u is regular for all $u \in \mathcal{M}_J(L_0, L_1)$.

By Proposition 3.1, if x and y are transversal intersections of L_0 and L_1, and if u is regular, then $\mathcal{M}_J(x,y)$ is a smooth manifold near u whose tangent space at u is isomorphic to KerE_u. Now we are ready to give the definition of the *Floer cohomology* of the quadruple $(L_0, L_1, J; P)$. Since we are mostly ignore the grading problem in this survey, we will not distinguish the homology and cohomology but exclusively use the cohomology to naturally relate to the quantum cohomology in Section 10. The only exception is Section 7 where it is more fuctorial to use the homology and so we use the homology there. For those who are interested in the more functorial treatment of the Floer (co)homology, we recommend readers to read [BzRd] or [PSS], where a careful functorial treatment is given in the context of non-relative theory.

Definition 3.4. We define

$$\begin{aligned} I(L_0, L_1) &= \text{the set of intersection points} \\ \mathcal{D}^* &= \text{the } \mathbb{Z}_2\text{-free module over } I(L_0, L_1). \end{aligned}$$

Assume that

(1) The triple (J, L_0, L_1) is regular,
(2) The number of zero dimensional components of $\widehat{\mathcal{M}}_J(x, y)$, denoted by $n(x, y)$, is finite,
(3) the integers $n(x, y)$ satisfy

$$\sum_{y \in I(L_0, L_1)} n(x, y) n(y, z) = 0 \mod 2$$

for any $x, z \in I(L_0, L_1)$.

We then define the Floer cohomology of the quadruple $(L_0, L_1, J : P)$ on P as

$$HF^*_J(L_0, L_1; P) := \text{Ker } \delta / \text{Im } \delta$$

where $\delta : \mathcal{D}^* \to \mathcal{D}^*$ is the operator defined by the matrix elements $n(x, y)$, i.e.,

$$\delta y = \sum_{x \in I(L_0, L_1)} n(x, y) x.$$

The main hypotheses in this definition concern certain compactness properties of the zero dimensional (for (2)) and one dimensional (for (3)) components of $\widehat{\mathcal{M}}_J(L_0, L_1)$. Such compactness properties in this generality of arbitrary pairs (L_0, L_1) were proven either for the case when $\omega|_{\pi_2(P, L_i)} \equiv 0$ [F2] or for the case when L_i's are *monotone* and one of L_i's satisfies the property that Im $(\pi_1(L_i)) \subset \pi_1(P)$ is a torsion subgroup for at least one of $i = 0, 1$ [O3].

We now restrict to the case when L_1 is Hamiltonian isotopic to $L_0 = L$. In this case one can order the set of connected components of Ω by the elements in $\pi_1(P, L)$:

Let a Hamiltonian isotopy $\Phi = \{\phi_t\}_{0 \le t \le 1}$ be given so that $L_0 = L$ and $L_1 = \phi_1(L)$. Then for each path $z : [0, 1] \to P$ with $z(0) \in L_0, z(1) \in L_1$, the map

$$t \mapsto \phi_t^{-1}(z(t))$$

defines an element in $\pi_1(P, L)$. As in [F2], [O3], we fix the component corresponding to the case when this element is trivial in $\pi_1(P, L)$ and denote

$$\begin{aligned} \Omega_\Phi = \Omega(L, \Phi) = \{z : [0, 1] \to P \mid z(0) \in L, z(1) \in \phi_1(L) \text{ and} \\ [t \mapsto \phi_t^{-1} z(t)] = 0 \text{ in } \ \pi_1(P, L)\} \end{aligned}$$

This is the analogoue in this relative Floer theory to the set of contractible loops in the nonrelative Floer homology theory [F5]. Note that the dependence of Ω_Φ on Φ is not strong. In fact, it depends only on the homotopy class of the path $\Phi = \{\phi_t\}$ from id to ϕ_1. Main reasons why we restrict to this component are similar to the nonrelative theory (see [F5], [HS]).

We now denote

$$\begin{aligned} \mathcal{D}_\omega(P) &= \{\Phi_H \mid H \text{ has compact support }\} \\ \mathcal{D}_\omega(L : P) &= \{\Phi \in \mathcal{D}_\omega(P) \mid \phi_1(L) \text{ meets } L \text{ transverely }\} \end{aligned}$$

and define the restricted moduli space of trajectories

$$\begin{aligned} \mathcal{M}_{J,\Phi} &= \mathcal{M}_{J,\Phi}(L : P) \\ &= \big\{u : \mathbb{R} \to \Omega_\Phi \mid u \text{ satisfies (3.5) and } \int_\Theta \Big|\frac{\partial u}{\partial t}\Big|^2 < \infty\big\} \\ &= \big\{u : \mathbb{R} \to \Omega_\Phi \mid u \ \text{ satisfies (3.5) and } \int_\Theta u^*\omega < \infty\big\}. \end{aligned}$$

and for each $x, y \in L \cap \phi_1(L)$

$$\begin{aligned} \mathcal{M}_{J,\Phi}(x, y) &= \big\{u \in \mathcal{M}_{J,\Phi} \mid \lim_{\tau \to \infty} u = x, \lim_{\tau \to -\infty} u = y\big\} \\ \widehat{\mathcal{M}}_{J,\Phi}(x, y) &= \mathcal{M}_{J,\Phi}(x, y)/\mathbb{R}. \end{aligned}$$

Using this restricted moduli space, we now modify Definition 3.4 slightly for this case as follows

Definition 3.5 We define for each $\Phi \in \mathcal{D}_\omega(L : P)$

$$\begin{aligned} CF(L, \Phi) &= \{x \in L \cap \phi_1(L) \mid [t \mapsto \phi_t^{-1} x] = 0 \text{ in } \pi_1(P, L)|\} \\ CF^*(L, \Phi) &= \text{the } \mathbb{Z}_2\text{-free module over } CF(L, \Phi). \end{aligned}$$

Assume that

(1) The triple $(J, L, \phi_1(L))$ is regular,

(2) The number of zero dimensional components, denoted by $n(x, y)$, is finite,

(3) the integers $n(x, y)$ satisfy

$$\sum_{y \in CF(L,\Phi)} n(x, y)n(y, z) = 0 \mod 2$$

for any $x, z \in CF(L, \Phi)$. We then define the Floer cohomology of the quadruple $(L, \Phi, J : P)$ on P as

$$HF_J^*(L, \Phi : P) := \text{Ker } \delta / \text{Im } \delta$$

where $\delta : CF^*(L, \Phi) \to CF^*(L, \Phi)$ is the operator defined by the matrix elements $n(x, y)$, i.e.,

$$\delta y = \sum_{x \in CF(L,\Phi)} n(x, y)x.$$

So far there have been two distinguished classes of Lagrangian submanifolds in the literature for which the conditions in Definition 3.5 can be achieved: one for Floer's case [F2] of $\omega|_{\pi_2(P,L)} \equiv 0$ and the other for the author's *monotone* case [O3]. In fact, we have slightly changed the definition of monotone Lagrangian submanifolds from [O3] in Section 2 as in [O8] which makes Floer's case a special case of monotone Lagrangian submanifolds. As in the non-relative case, it is still an open problem to generalize a definition of the Floer cohomology to arbitrary Lagrangian submanifolds.

PART I: FLOER COHOMOLOGY WITHOUT QUANTUM EFFECTS

§4. The construction and the invariance.

The first essential ingredient in proving the compactness properties required in Definition 3.5 is getting an a priori bound for the L^2-norm $\int |Du|^2$ of the derivative of maps $u : \Theta \to P$ *under the hypothesis that they are in* $\mathcal{M}_J(x,y)$ *for a fixed pair* (x,y) *and of the same* Index u, *which is the same as the topological index* $\mu_u(x,y)$ defined by Viterbo [V1]. Under the assumption $\omega|_{\pi_2(P,L)} \equiv 0$, this a priori bound immediately follows from the fact that the action functional $\mathcal{A}$ is well-defined on Ω_Φ and from the following general lemma

Lemma 4.1. *If* $\omega|_{\pi_2(P,L)} \equiv 0$, *then there is a globally defined action functional* $\mathcal{A} : \Omega_\Phi \to \mathbb{R}$. *Moreover for any map* $u : [0,1] \to \Omega_\Phi$ *with* $u(0) = y, u(1) = x \in CF(L,\Phi)$, *we have*

$$\mathcal{A}(y) - \mathcal{A}(x) = \int u^*\omega. \tag{4.1}$$

In particular, for any two $u, v \in \mathcal{M}_{J,\Phi}(x,y)$, *we have*

$$\int |Du|_J^2 = \int |Dv|_J^2. \tag{4.2}$$

We would like to emphasize that $\mathcal{A}$ is well-defined only on Ω_Φ not on the whole Ω. Because the condition $\omega|_{\pi_2(P,L)} \equiv 0$ also rules out the phenomenon of bubbling off both spheres and disks, it follows that for regular (J,L,Φ), the zero dimensional component of $\widehat{\mathcal{M}}_J(x,y)$ is a compact zero dimensional manifold and so the number $n(x,y)$ is finite, which verifies (1) of Definition 3.5. Now we consider the one dimensional component of $\widehat{\mathcal{M}}_J(x,y)$. Since bubbling does not occur, the only source of the failure of compactness is "splitting" into a pair $(u_1,u_2) \in \widehat{\mathcal{M}}_J(x,z) \times \widehat{\mathcal{M}}_J(z,y)$ with Index $u_i = 1$ for $i = 1,2$. This together with the gluing construction implies that one dimensional component of $\widehat{\mathcal{M}}_J(x,y)$ can be compactified into a compact one dimensional manifold by adding all such pairs (u_1,u_2) to $\widehat{\mathcal{M}}_J(x,y)$ where $u_1 \in \widehat{\mathcal{M}}_J(x,z)$ and $u_2 \in \widehat{\mathcal{M}}_J(z,y)$ for all z. Therefore the number of such pairs must be even because the number of boundaries of any compact one dimensional manifold is even. On the other hand the left hand side of the equation in (3) of Definition 3.5 is exactly this number, which proves that (3) holds for the case $\omega|_{\pi_2(P,L)} \equiv 0$. Hence

Proposition 4.2. *Under the condition* $\omega|_{\pi_2(P,L)} \equiv 0$, *for any given Hamiltonian diffeomorphism* $\phi = \phi_1$, *the hypotheses in Definition 3.5 hold for regular* (J,L,Φ) *and* *therefore $HF^*_J(L, \Phi : P)$ is well-defined as $\mathbb{Z}_2$-modules which depends only on ϕ and the homotopy class $[\Phi]$.*

Remark 4.3. Under the stronger hypothesis $\pi_2(P, L) = \{e\}$ as in [F2], there is a well-defined map

$$\mu : CF(L, \Phi) \to \mathbb{Z}$$

which defines a $\mathbb{Z}$-grading on $HF^*_J(L, \Phi : P)$. In this paper, we will not consider this grading problem.

Now we prove the invariance of the module $HF^*_J(L, \Phi : P)$ under the change of the pair (J, Φ). More precisely we will prove that there exists a natural homomorphism

$$h^{\alpha\beta} : HF^*_{J^\alpha}(L, \Phi^\alpha : P) \to HF^*_{J^\beta}(L, \Phi^\beta : P) \tag{4.3}$$

for each given regular (J^α, Φ^α) and (J^β, Φ^β). We then prove that these homomorphisms also satify the composition rule

$$h^{\alpha\beta} \circ h^{\beta\gamma} = h^{\alpha\gamma}. \tag{4.4}$$

(4.4) will also imply, by considering the case $\alpha = \gamma$, that the homomorphism $h^{\alpha\beta}$ is indeed an isomorphism.

We define $h^{\alpha\beta}$ in (4.3) by studying a variation of the equation (3.5). For each given (J^α, Φ^α) and (J^β, Φ^β), we choose a smooth one parameter family of $(\overline{J}, \overline{\Phi}) = \{(J^s, \Phi^s)\}$ that is constant in s outside $[0, 1]$ and extends smoothly to $\mathbb{R}$ so that

$$(J^s, \Phi^s) = \begin{cases} (J^\alpha, \Phi^\alpha) & \text{for } s \leq 0 \\ (J^\beta, \Phi^\beta) & \text{for } s \geq 1. \end{cases} \tag{4.5}$$

We now choose a monotone funtion $\rho : \mathbb{R} \to [0, 1]$ such that

$$\begin{aligned} \rho(\tau) &= 0 \quad \text{if } \tau < -K \\ &= 1 \quad \text{if } \ \tau > K \end{aligned}$$

for $K > 0$ sufficiently large. For each pair $(x^\alpha, x^\beta) \in CF(L, \Phi^\alpha) \times CF(L, \Phi^\beta)$, we consider the solution set, denoted by $\mathcal{M}_\rho(x^\alpha, x^\beta)$ of the equation

$$\begin{cases} \overline{\partial}_\rho u(\tau, t) := \frac{\partial u}{\partial \tau}(\tau, t) + J_t^{\rho(\tau)} \frac{\partial u}{\partial t}(\tau, t) = 0 \\ u(\tau, 0) \in L, \quad u(\tau, 1) \in \phi_1^{\rho(\tau)}(L) \\ \lim_{\tau \to -\infty} u = x^\alpha, \quad \lim_{\tau \to \infty} u = x^\beta. \end{cases} \tag{4.6}$$

Because of the assumption that (J^s, Φ^s) is constant outside $[0,1]$, the relevant analytic properties of $\mathcal{M}_\rho(x^\alpha, x^\beta)$ will not change from $\mathcal{M}_J(x,y)$ except that (4.6) is not invariant under the translation of τ. We prove that the number of solutions whose index is zero, is finite and denote the number by $n^{\alpha\beta}(x^\alpha, x^\beta)$. Then we define the homomorphism

$$h^{\alpha\beta} : CF^*(L, \Phi^\alpha) \to CF^*(L, \Phi^\beta)$$

by

$$h^{\alpha\beta}(x^\alpha) = \sum n^{\alpha\beta}(x^\alpha, x^\beta) x^\beta. \tag{4.7}$$

By studying the one dimensional component of $\mathcal{M}_\rho(x^\alpha, x^\beta)$, we prove that this $h^{\alpha\beta} : CF^*(L^\alpha, \Phi^\alpha) \to CF^*(L^\beta, \Phi^\beta)$ will satisfy

$$h^{\alpha\beta} \circ \delta^\alpha = \delta^\beta \circ h^{\alpha\beta}$$

which proves that this map descends to the cohomology. We again denote this map by

$$h^{\alpha\beta} : HF^*_{J^\alpha}(L, \Phi^\alpha) \to HF^*_{J^\beta}(L, \Phi^\beta).$$

One can also prove by considering a parametrized version of (4.6) that this map does not depend on the choice of the one parameter family (J^s, Φ^s) we used in the equation (4.6) (See [McD] for a detailed account of this kind of argument for the nonrelative case. The argument for our case will be the same.)

Again in the above proofs, appropriate compactness properties of $\mathcal{M}_\rho(x^\alpha, x^\beta)$ will be crucial. Hence so is the following a priori bound for $|[\omega](u)|$ for $u \in \mathcal{M}_\rho(x^\alpha, x^\beta)$.

Propostion 4.4. *Let u_1, u_2 be two maps from $[0,1]^2$ to P such that*

$$\begin{aligned} &u_j(\tau, 0) \in L, \quad u_j(\tau, 1) \in \phi_1^\tau(L) \\ &u_j(0, t) \equiv x^\alpha, \quad u_j(1, t) \equiv x^\beta \quad j = 1, 2 \end{aligned}$$

Then we have

$$|[\omega](u_1) - [\omega](u_2)| \leq C(\overline{\Phi}) \tag{4.8}$$

where $C = C(\overline{\Phi})$ depends only on $\overline{\Phi}$.

We leave the proof of this to readers or refer to [Lemma 5.2, O3] where we proved a version of this for the monotone case, which is harder to prove. This finally establishes the construction of the Floer cohomology for the pair (L, P) for the case $\omega|_{\pi_2(P,L)} \equiv 0$

by defining $HF^*(L:P)$ to be $HF^*_J(L,\Phi:P)$ for the regular (J,L,Φ). By defintion, it follows that

$$\#(L\cap\phi(L))\geq \text{rank } HF^*(L:P) \tag{4.9}$$

for any nondegenerate Hamiltonian diffeomorphism ϕ. Therefore the proof of a version of Arnold conjecture as in Theorem II [Floer] in the introduction will follow from the following theorem

Theorem 4.5. [Floer, F4] *Under the assumption $\omega|_{\pi_2(P,L)}\equiv 0$, we have*

$$HF^*(L:P)\cong H^*(L,\mathbb{Z}_2)$$

as $\mathbb{Z}_2$-modules.

Floer [F4] finished his proof of this theorem only for the case when $P=T^*M$ and $L\subset T^*M$ is the zero section and took it for granted for the general case $\pi_2(P,L)=\{e\}$. By the Weinstein-Darboux theorem that any Lagrangian submanifold in (P,ω) has a neighborhood which is symplectomorphic to a neighborhood of the zero section in T^*M with respect to the canonical symplectic structure, his proof a priori implies only that a *local* Floer cohomology group $HF^*(L,U:P)$, which we will define in the next section, is isomorphic to $H^*(L,\mathbb{Z}_2)$ for *any* pair (L,P). What is omitted in Floer's proof of the Arnold's conjecture is the proof of the results that the above localization is justified and that the *global* Floer cohomology $HF^*(L:P)$ is isomorphic to the *local* Floer cohomology $HF^*(L,U:P)$ *provided* $\omega|_{\pi_2(P,L)}\equiv 0$. Although the proof of these facts is not hard to come by, which might be the reason why Floer did not spell it out in his proof, the present author's understanding of this dichotomy is precisely what made possible his discovery [O10] of the spectral sequence that bridges the local and global Floer cohomology for the case of *monotone* Lagrangian submanifolds.

§5. Local Floer cohomology

In this section, we essentially quote the materials in Section 2 and 3 of [O9]. We take any pair (L,P), not necessarily monotone, where (P,ω) is a symplectic manifold and L is a compact Lagrangian submanifold therein. We fix a pair $V\subset\overline{V}\subset U$ of Darboux neighborhoods of L. We also denote

$$\begin{aligned}\mathcal{D}^V_\omega(P)&=\{\Phi_H\in\mathcal{D}_\omega(P)\mid \text{supp } H\subset[0,1]\times V\}\\ \mathcal{D}^V_\omega(L:P)&:=\mathcal{D}_\omega(L:P)\cap\mathcal{D}^V_\omega(P)\end{aligned}\tag{5.1}$$

and for each time-independent J_0 we define

$$\mathcal{J}_\omega^{J_0}(V:P) = \{J \in \mathcal{J}_\omega(P) \mid J_t \equiv (\phi_t)_* J_0 \text{ for all } t \in [0,1] \\ \text{where } \Phi = \{\phi_t\} \in \mathcal{D}_\omega^V(L:P)\} \tag{5.2}$$

We denote, for each (time independent) J_0,

$$A_{J_0} := \inf\{\int w^*\omega > 0 \mid w : (D^2, \partial D^2) \to (P, L), \overline{\partial}_{J_0} w = 0\} \ . \tag{5.3}$$

It is well-known (see e.g, [Corollary 3.5, O2]) that $A_{J_0} > 0$ for *any*, not necessarily monotone, pair (P, L) as long as L is comapct. (It is easy to see that

$$A_{J_0} \geqq A_{(P,L)}$$

for any J_0 compatible to ω, where $A_{(P,L)}$ is defined by

$$A_{(P,L)} := \inf\{\int w^*\omega > 0 \mid w : (D^2, \partial D^2) \to (P, L)\}.$$

But $A_{(P,L)}$ could be zero in general.)

The following proposition will be crucial for the localization of the Floer cohomology, which was proven in [O9].

Proposition 5.1. *Let U be a Darboux neighborhood of L in (P, ω). Then for any given $\alpha > 0$ and for any fixed* time-independent J_0*, there exists a constant $\epsilon_1 > 0$ such that if $|\phi - id|_{C^1, P\times[0,1]} < \epsilon_1$ and $|J - J_0|_{P\times[0,1]} < \epsilon_1$, we have*

$$\text{Image } u \subset U \tag{5.4}$$

for all $u \in \mathcal{M}_J(L, \Phi : P)$, provided

$$\frac{1}{2}\int |\nabla u|_J^2 = \int u^*\omega < A_{J_0} - \alpha \ .$$

Moreover all the other $u \in \mathcal{M}_J(L, \Phi : P)$ which are not contained in U satisfy

$$\frac{1}{2}\int |\nabla u|_J^2 = \int u^*\omega > A_{J_0} - \epsilon_2 \tag{5.5}$$

for sufficiently small $\epsilon_2 = \epsilon_2(\epsilon_1) > 0$ which does not depend on α.

Now we are ready to define the local Floer cohomology group of the triple $(L, U : P)$

Definition 5.2. For any $(J, \Phi) \in \mathcal{J}_\omega^{J_0}(V : P) \times \mathcal{D}_\omega^V(P)$, we define

$$\mathcal{M}_J(L, \Phi, U : P) := \{u \in \mathcal{M}_J(L, \Phi : P) \mid u(\tau)(t) \in U \text{ for all } t, \tau\}.$$

Theorem 5.3. *Under the same hypotheses as in Proposition 5.1, there are only finitely many isolated trajectories in $\widehat{\mathcal{M}}_J(L, \Phi, U : P)$. And if we denote*

$$n(x, y : U) := \# \text{ of isolated trajectories in } \widehat{\mathcal{M}}_J(x, y : U)$$

then the homomorphism $\delta_U : CF^* \to CF^*$ defined by

$$\delta_U y = \sum_{y \in CF(L,\Phi)} n(x, y : U) x$$

satisfies $\delta_U \circ \delta_U = 0$. Furthermore the quotients

$$HF_J^*(L, \Phi, U : P) := \text{Ker } \delta_U / \text{Im } \delta_U$$

are isomorphic under the change of $(J, \Phi) \in \mathcal{J}_\omega^{J_0}(V : P) \times \mathcal{D}_\omega^V(P)$.

Once this localization process has been justified, it is not difficult to prove that the local Floer cohomology group $HF^*(L, U : P)$ is always isomorphic to $HF^*(L : T^*L)$ (See [O9] for details). Now Floer [F4] has proved that $HF^*(L : T^*L)$ is isomophic to $H^*(L, \mathbb{Z}_2)$ and so we conclude

Theorem 5.4. **[Theorem 3.7, O9]** *Let $L \subset (P, \omega)$ be any compact Lagrangian submanifold and U be a Darboux neighborhood of L. Then we have*

$$HF^*(L, U : P) \cong H^*(L, \mathbb{Z}_2).$$

Remark 5.5. When one could prove

$$\mathcal{M}_J(L, \Phi : P) = \mathcal{M}_J(L, \Phi, U : P) \tag{5.6}$$

for a regular $(J, \Phi) \in \mathcal{J}_\omega^{J_0}(V : P) \times \mathcal{D}_\omega^V(P)$, then it would follow from the definition that the local Floer cohomology $HF^*(L, U : P)$ is isomorphic to $HF^*(L : P)$, as long as $HF^*(L : P)$ is well-defined. However in case (5.6) does not hold, $HF^*(L : P)$ is not necessarily isomorphic to the $HF^*(L, U : P)$ even when it is well-defined. This can be easily seen from *monotone* Lagrangian embeddings in $\mathbb{C}^n$ (see Section 5).

As an easy application of Theorem 5.4 and Remark 5.5, we now give a Floer theoretic proof of Gromov's celebrated nonexactness theorem [Gr] of comapct Lagrangian embeddings in $\mathbb{C}^n$.

Theorem 5.6 [Gromov]. *For any comapct Lagrangian embedding $L \subset \mathbb{C}^n$, the Lioville class $[\theta|_L] \in H^*(L, \mathbb{R})$ is not zero.*

Proof. We prove this by contradiction. Suppose that a compact Lagrangian embedding $L \subset \mathbb{C}^n$ is exact. Then the action functional is well-defined on the path space Ω_Φ and so we have the apriori bound on the area of $u \in \mathcal{M}_{J,\Phi}(x, y)$ for each given pair $x, y \in CF(L, \Phi : \mathbb{C}^n)$. Once we have this, it is now a standard argument to prove that the Floer cohomology $HF_J^*(L, \Phi : \mathbb{C}^n)$ is well-defined and invariant under the change of (J, Φ). Furthermore since the exactness assumption also implies that there cannot be any J-holomorphic disc with boundary on L, it also implies by standard compactness argument that all J-holomorphic trajectories will be contained in a Darboux neighborhood U of L when Φ is sufficiently C^1 close to id. Therefore the Floer cohomology $HF^*(L : \mathbb{C}^n)$ must be isomorphic to the local Floer cohomology $HF^*(L, U : \mathbb{C}^n)$ by Remark 5.5, which in turn is isomorphic to $H^*(L, \mathbb{Z}_2)$ by Therem 5.4. On the other hand, since one can disjunct L from itself by linear translations on $\mathbb{C}^n$, $HF^*(L : \mathbb{C}^n)$ must also be trivial by the invariance property and so $H^*(L, \mathbb{Z}_2)$ must be trivial which would be obsurd. This proves that L cannot be exact.

Q.E.D.

§6. Chekanov's theorem

In [H3], Hofer introduced a norm on the space of $\mathcal{H}$ of compactly supported functions on $[0, 1] \times P$,

$$\|H\| = \int_0^1 (\max_{x \in P} H(s, x) - \min_{x \in P} H(s, x)) ds \tag{6.1}$$

and define the *energy* of a symplectic diffeomorphism $\phi : P \to P$ by

$$E(\phi) = \inf\{\|H\| \mid H \mapsto \phi, \; H \in \mathcal{H} \}.$$

He also introduced the notion of *disjunction energy* (or also called the *displacement energy*) of subsets in P by the infimum of $E(\phi)$ for which ϕ disjuncts the given subset. Polterovich [Po4] showed that the disjunction energy for compact Lagrangian submanifolds are nonzero (at least) for *rational Lagrangian submanifolds*:

Definition 6.1 A Lagrangian submanifold $L \subset (P, \omega)$ is called *rational* if the subgroup of $\mathbb{R}$

$$\Sigma_{\omega,L} := \{\int_{D^2} w^*\omega \mid w : (D^2, \partial D^2) \to (P, L)\}$$

is discrete. We will denote by $\sigma = \sigma_{\omega,L}$ the positive generator of $\Sigma_{\omega,L}$, i.e.,

$$\Sigma_{\omega,L} = \sigma_{\omega,L} \cdot \mathbb{Z}.$$

Theorem 6.2 [Polterovich]. *Suppose that (P, ω) is tame and $L \subset (P, \omega)$ is a compact rational Lagrangian submanifold. Then if $E(\phi) < \frac{\sigma_{\omega,L}}{2}$, then*

$$L \cap \phi(L) \neq \emptyset.$$

In other words, the disjunction energy of L is greater than equal to $\frac{\sigma_{\omega,L}}{2}$.

Polterovich's proof in [Po4] relies on Gromov's figure 8 trick and a refinement of Gromov's existence scheme of J-holomorphic discs that was used in Gromov's proof of the non-exactness theorem [Gr], Theorem 5.6 in Section 5. Polterovich used the figure 8 trick to reduce the estimation of the disjunction energy to that of the cylindrical capacity which was previously estimated by Sikorav [Si] in $\mathbb{C}^n$. In fact, Sikorav [Si] proved that the lower bound for the disjunction energy for the standard tori in $\mathbb{C}^n$ is $\sigma_{\omega,L}$, in which sense the lower bound in Theorem 6.2 had known to be not optimal in general. Recently, Chekanov [C2] was able to use a version of Floer cohomology techniques in the study of the disjunction energy of rational Lagrangian submanifolds and proved a theorem which strengthens Polterovich's in two different ways: He proved an optimal lower bound in Theorem 6.2 and obtained a lower bound for the number of elements in $L \cap \phi(L)$

Theorem 6.3 [Chekanov]. *Under the same hypotheses as in Theorem 6.1, if $E(\phi) < \sigma_{\omega,L}$, then*

$$\#(L \cap \phi(L)) \geq \dim H_*(L, \mathbb{Z}_2)$$

provided L intersects $\phi(L)$ transversely.

In [C2], Chekanov introduced several interesting new ideas in the Floer theory. In the rest of this section, we will give an outline of Chekanov's proof of this theorem. We will mostly follow his presentation and notations in [C2] with minior modifications.

We first note that when the Hamiltonian $H : P \times [0, 1] \to \mathbb{R}$ is given, it defines not only ϕ_1 but the isotopy $\Phi = \{\phi_s\}_{0 \leq s \leq 1}$ and so a path $\overline{L} = \{L_s\}, L_s = \phi_s(L)$ of

Lagrangian submanifolds. One can assume that for any $s \in [0,1]$, $H(s,\cdot)$ assumes the value zero. Denote

$$\Omega_s := \{\gamma \in C^\infty([0,1],P) \mid \gamma(0) \in L, \gamma(1) \in L_s\}$$
$$\Omega = \cup_{s\in[0,1]}\Omega_s \subset [0,1] \times C^\infty([0,1],P).$$

The first ingeneous idea of Chekanov is to consider the family of $\mathbb{R}/\sigma\mathbb{Z}$-valued functionals (which is the action functional) i.e., anti-derivatives $F_s : \Omega_s \to \mathbb{R}$ of the action one-form (2.1) in a uniform way: Recall that F_s is defined up to additive constant *for each* s and so if we choose the anti-derivatives in an arbitrary way, it will involve one parameter family of constants, which makes difficult to compare for two different s's. Chekanov chooses the constant once and for all at $s = 0$ in a way that the rest of constants will be determined automatically, by introducing the one-form $\alpha - \beta$ on Ω where

$$\alpha(s,\gamma) = \alpha_s(\gamma) = \text{the action one-form in (3.1)}$$
$$\beta(s,\gamma) = H(s,\gamma(1))\ ds$$

A computation shows that the one-form $\alpha - \beta$ is closed on Ω. Note that α restricts to the closed one-form α_s on Ω_s for each s. Therefore one may choose the anti-derivative of $\alpha - \beta$, $F : \Omega \to \mathbb{R}/\sigma\mathbb{Z}$ so that $dF = \alpha - \beta$. We denote

$$F_s := F(s,\cdot) : \Omega_s \to \mathbb{R}/\sigma\mathbb{Z}.$$

One can choose F so that F_0 *has critical value 0.*

Now we denote

$$a_+ = \int_0^1 \max_{x\in P} H(s,x)\ ds, \quad a_- = \int_0^1 \min_{x\in P} H(s,x)\ ds,$$

and then we have $a_- \le 0 \le a_+$ because we assume that $H(s,\cdot)$ assumes the value zero for all $s \in [0,1]$. From now on, we assume as in Theorem 6.3 that $E(\phi) < \sigma$ and so choose $H \mapsto \phi$ such that

$$a_+ - a_- = \|H\| < \sigma \tag{6.2}$$

and choose an interval $[b_-, b_+)$ such that

$$[a_-, a_+] \subset (b_-, b_+) \quad \text{and} \quad b_+ - b_- = \sigma.$$

Denote $f_s = p|_{[b_-,b_+)} \circ F_s$ where $p : \mathbb{R}/\sigma\mathbb{Z} \to \mathbb{R}$ is the canonical projection. With this f_s, we now consider the space of solutions of (2.5) with $L_0 = L$ and $L_1 = L_s$ that have distinguished "length"

$$\ell(u) := \int u^*\omega = f_s(x_-) - f_s(x_+).$$

We denote

$$\begin{aligned}\mathcal{M}_s^d(x_+, x_-) := \{u \in \mathcal{M}(L, L_s) \mid \lim_{\tau\to-\infty} = x_-,\ \lim_{\tau\to\infty} = x_+, \\ \text{and}\ \ \ell(u) = f_s(x_-) - f_s(x_+)\}\end{aligned}$$

and then we have

Proposition 6.4. (1) *The zero dimensional components of $\widehat{\mathcal{M}}_s^d(x,y)$ is compact for a dense set of $s \in (0,1]$. We define for these s*

$$n_s^d(x,y) = \#(\textit{zero dimensional components of}\ \ \widehat{\mathcal{M}}_s^d(x_+,x_-)). \tag{6.3}$$

(2) *The numbers $n_s^d(x,y)$ satisfy*

$$\sum_{y\in L\cap L_s} n_s^d(x,y)n_s^d(y,z) = 0 \quad \textit{mod}\ 2. \tag{6.4}$$

The compactness properties of $\mathcal{M}_s^d(x,y)$ required in the proof of (6.3) and (6.4) hold essentially because the bubbling cannot occur under the assumption that

$$\ell(u) = f_s(y) - f_s(x) < \sigma \tag{6.5}$$

This proposition makes the corresponding Floer cohomology group denoted by $HF_d^*(L, L_s : P)$ well-defined (see Definition 2.4).

Then Chekanov [C2] compares these groups for two values of s, one $s = \epsilon$ close to 0 and the other $s = 1$. Since $f_s(x_-) - f_s(x_+) \to 0$ as $s \to 0$, it follows that

$$\mathcal{M}_\epsilon^d(x,y) = \mathcal{M}(L, L_\epsilon, U : P)$$

for some Darboux neighborhood U of L and so that

$$HF_d^*(L, L_\epsilon : P) \cong HF^*(L, U : P) \cong H^*(L, \mathbb{Z}_2) \tag{6.6}$$

from Theorem 5.4. As in Section 4, we pick a function $\rho : \mathbb{R} \to [0,1]$ such that

$$\begin{aligned} \rho(\tau) &= s_- \quad \text{when } \ \tau < -K \\ &= s_+ \quad \text{when } \ \tau > -K \end{aligned}$$

for sufficiently large $K > 0$ and define

$$\begin{aligned} \mathcal{M}^d_\rho(x_-, x_+) &= \{u \in \mathcal{M}_\rho(x_-, x_+) \mid \ell(u) = f_{s_-}(x_-) - f_{s_+}(x_+)\} \\ n_\rho(x_+, x_-) &= \#(\text{ zero dimensional components of } \mathcal{M}^d_\rho(x_+, x_-)) \end{aligned}$$

Chekanov calls such ρ a (s_-, s_+)-*continuation function.* Now we define a linear map $Q^{s_+}_{s_-} : C(s_-) \to C(s_+)$ by

$$Q^{s_+}_{s_-}(y) = \sum_{y \in C(s_-)} n_\rho(x, y) x. \tag{6.7}$$

Proposition 6.5. *For $0 < s_- < \epsilon, s_+ = 1$ or $s_- = 1, 0 < s_+ < \epsilon$, $n_\rho(x_+, x_-)$ are finite and so (6.7) is well-defined. Furthermore we have*

$$Q^{s_+}_{s_-} \circ \delta_{s_+} = \delta_{s_-} \circ Q^{s_+}_{s_-}. \tag{6.8}$$

Again the compactness properties required in the proof of this proposition follows essentially from the non-existence of bubbling property and the following crucial estimate. A similar estimate of this sort was previoulsy derived in Lemma 5.2 [O3] (Also see the remark after the proof of Lemma 5.2 [O3]).

Lemma 6.6. *If $u \in \mathcal{M}^d_\rho(x_-, x_+)$, then*

$$\int u^*\omega < \sigma.$$

Proposition 6.5 enables one to conclude that Q^ϵ_1, Q^1_ϵ induce the homomorphisms

$$\begin{aligned} V^\epsilon_1 &: HF^*_d(L, L_\epsilon : P) \to HF^*_d(L, L_1 : P) \\ V^1_\epsilon &: HF^*_d(L, L_1 : P) \to HF^*_d(L, L_\epsilon : P) \end{aligned}$$

Now Chekanov proves the following proposition which will finish the proof of Theorem 6.3.

Proposition 6.7. *The composition map*

$$V_1^\epsilon \circ V_\epsilon^1 : HF_d^*(L, L_\epsilon : P) \to HF_d^*(L, L_\epsilon : P)$$

is the identity, i.e, $V_1^\epsilon \circ V_\epsilon^1 = id$*. In particular, the map*

$$V_\epsilon^1 : HF_d^*(L, L_s : P) \to HF_d^*(L, L_1 : P)$$

is injective.

In the proof of this proposition [Lemma 7, C2], Chekanov uses another interesting idea of using a special parametrized moduli space: Construct a family of (ϵ, ϵ)-continuation function π_c where $c \in [0, \infty]$ satisfying the conditions

(1) $\pi_0(\tau) \equiv \epsilon$,

(2) $c \mapsto \pi_c(0)$ is a monotone smooth function onto $[\epsilon, 1]$,

(3) $\frac{d\pi_c}{d\tau}(\tau) \geq 0$ for $\tau > 0$, and $\frac{d\pi_c}{d\tau}(\tau) \leq 0$ for $\tau < 0$,

(4) For c large enough, π_c satisfies

$$\pi_c(\tau) = \begin{cases} \rho^-(\tau + c) & \text{where } \tau < 0 \\ \rho^+(\tau - c) & \text{where } \tau \geq 0 \end{cases}$$

Figure 1.

Then consider the space

$$\mathcal{M}_\pi(x_-, x_+) = \{(c, u) \mid u \in \mathcal{M}_{\pi_c}(x_-, x_+)\}$$

where $x_\pm \in C(\epsilon)$ and the distinguished part of $\mathcal{M}_\pi$

$$\mathcal{M}_\pi^d(x_-, x_+) = \{(c, u) \mid u \in \mathcal{M}_{\pi_c}^d(x_-, x_+)\}.$$

Then Chekanov uses some crucial calculation which we refer to [C2] and (6.2) to prove the following estimate, from which Proposition 6.7 can be proven by a standard argument in the Floer theory.

Lemma 6.8. *If $x_-, x_+ \in C(\epsilon)$ and $(c, u) \in \mathcal{M}_\pi^d(x_-, x_+)$, then we have*

$$\int u^*\omega < \sigma.$$

Combining Proposition 6.7 with (6.6), Theorem 6.3 immediately follows. We would like to emphasize that the homomorphism $V_\epsilon^1 : HF_d^*(L, L_s : P) \to HF_d^*(L, L_1 : P)$ in Proposition 6.7 may not be surjective in general, which raises an interesting question about how this happens.

§7. Floer theory, symplectic invariants and Hofer's geometry

In this section, we will outline the results proven in our paper [O10], which employes the idea of the Floer (co)homology to study symplectic invariants of Lagrangian submanifolds. In [H3], Hofer introduced a remarkable bi-invariant norm on the group $\mathcal{D}_\omega^c(P)$, which gives a structure of Finsler manifold on it. Recall that the Lie algebra of the group $\mathcal{D}_\omega^c(P)$ is the set of compactly supported Hamiltonian H's. The Hofer's norm for H is defined by

$$\|H\| = \int_0^1 (\max_x H_t - \min_x H_t)\ dt \tag{7.1}$$

and the norm $\|\phi\|$, which corresponds to the distance from the identity to ϕ, is defined by

$$\|\phi\| = \inf_{H \mapsto \phi} \|H\| \tag{7.2}$$

Then the (pseudo)-distance, denoted by $d(\phi, \psi)$, between ϕ and ψ in $\mathcal{D}_\omega^c(P)$ is defined to be

$$d(\phi, \psi) = \|\phi^{-1}\psi\|. \tag{7.3}$$

It is a highly nontrivial theorem (see [H3] and [LM]) that the (pseudo)-norm in (7.2) is nondegenerate or equivalently the d in (7.3) indeed defines a *distance.*

Theorem 7.1 [Hofer ($\mathbb{R}^n$), Lalonde-McDuff (in general)]. *The norm defined as in (7.2) is nondegenerate, i.e, $\|\phi\| = 0$ if and only if ϕ is the identity.*

Hofer's proof [H3] uses a variational theory of the action functional

$$\mathcal{A}_H(\gamma) = -\int_\gamma pdq + \int H(\gamma(t), t)dt$$

on the Sobolev space $H^{1/2}(S^1, \mathbb{C}^n)$, while Lalonde and McDuff [LM] use the Gromov theory of pseudo-holomorphic discs together with some ingenious method of constructing optimal symplectic embedding of balls. Lalonde and McDuff in fact show that the nondegenracy question is a consequence of the general nonsqueezing theorem which they prove in [LM] in its full generality : "For any symplectic manifold (P, ω), the standard ball $B^{2n}(R) \subset \mathbb{C}^n$ can be symplectically embedded into $B^2(r) \times P$ only if $R \leq r$."

In this section, we will develop the analogoue of Hofer's geometry on the space of Lagrangian submanifolds. We recall the definition of Hofer's pseudo-distance from (2.3) on the space $\Lambda_\omega(L_0 : P)$ of Lagrangian submanifolds and the fundamental non-degeneracy question (2.4) of the pseudo-distance. Whether this non-nondegeracy holds for Lagrangian submanifolds on general symplectic manifolds is still an open question and so we will focus on the case

$$P = T^*M \quad \text{ and } L_0 = o_M = \text{ the zero section.}$$

To motivate our constructions of symplectic invariants, we first briefly review Viterbo's techniques of generating functions (see [V3] for details).

7.1. Viterbo's invariants

For a given Lagrangian submanifold $L \subset T^*M$, we call a function $S : E \to \mathbb{R}$ is called a *generating function* for L if L can be expressed as

$$L = \{(x, \frac{\partial S}{\partial x}(e)) \mid \frac{\partial S}{\partial \xi}(e) = 0\}$$

where the map $\pi : E \to M$ is a submersion (typically a vector bundle). We denote

$$\Sigma_S = \{e \in E \mid \frac{\partial S}{\partial \xi}(e) = 0\}$$

and by $i_S : \Sigma_S \to T^*M$ the map

$$i_S(e) = (x, \frac{\partial S}{\partial x}(e)) \quad \text{for } e \in \Sigma_S.$$

One of the essential facts for the generating function is that $i_S : \Sigma_S \to T^*M$ is a Lagrangian immersion and that the identity

$$i_S^*\theta = d(S|_{\Sigma_S}) \quad \text{on} \ \ \Sigma_S \tag{7.4}$$

holds. When E is a (finite dimensional) vector bundle over M, one can introduce a special generating function called *a generating function quadratic at infinity* (abbreviated as GFQI): A generating function $S : E \to \mathbb{R}$ is called a GFQI if $S(x,\xi) - Q(x,\xi)$ has compact support where $Q(x,\xi)$ is a fiberwise nondegerate quadratic form in ξ.

Theorem 7.2 [Laudenbach-Sikorav [LS], Viterbo [V3]]. *If $L = \phi(o_M)$, then L has a GFQI. Moreover it is essentially unique up to the stabilization and the gauge invariance.*

This has the consequence that the cohomology group $H^*(S^b, S^a)$ is independent of the choice of S but depends only on L *if one normalizes S appropriately.* Note that for $c > 0$ sufficiently large, we have

$$(S^c, S^{-c}) = (Q^c, Q^{-c}) \cong (D(E^-), S(E^-))$$

and so

$$H^*(S^c, S^{-c}) \cong H^{*-k}(M) \quad k = \dim D(E^-)$$

which is independent of S as long as c is sufficiently big. Here we denote by E^- the negative bundle of the quadratic form Q and by $D(E^-)$ and $S(E^-)$ the disc and the sphere bundles associated to E^-. One denotes

$$E^\infty = Q^c, \quad E^{-\infty} = Q^{-c}$$

for any such c. Now the Thom isomorphism provides the isomorphism

$$H^*(M) \to H^*(D(E^-), S(E^-)) \cong H^*(E^\infty, E^{-\infty}), \quad u \mapsto Tu := \pi^*u \cup \mathcal{T}_{E^-}$$

where $\mathcal{T}_{E^-}$ is the Thom class of the vector bundle E^-.

Definition 7.3. Let S be a GFQI for $L = \phi(o_M) \subset T^*M$. For each $u \in H^*(M, \mathbb{R})$, we assign the number

$$c(S,u) := \inf_\lambda \{\lambda \mid j_\lambda^* Tu \neq 0 \quad \text{in} \ \ H^*(E^\lambda, E^{-\infty})\}. \tag{7.5}$$

Note that it is easy to prove from (7.4) that the collection of differences $c(S,u) - c(S,1)$ depends only on L but not on S as long as S generates L. Since this does not depend on the choice of S, one could define $c(L,u)$, as invariants of L, by the common number

$$c(L,u) := c(S,u)$$

for *suitably normalized* S's. However it is not completely clear what would be the best normalization in general. To define invariants of compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$, Viterbo used a compactification of $\mathbb{R}^{2n}$ which provides a natural normalization in this case (see [V3]).

One of the main theorms in [V3] that is related to the nondegeracy question is the following

Theorem 7.4. [Viterbo, V3] *Set $\gamma(L) = c(L,1) - c(L,\mu_M)$ where $1 \in H^0(M,\mathbb{R})$, $\mu_M \in H^n(M,\mathbb{R})$ are the canonical generators respectively. Then we have $\gamma(L) \geq 0$ and*

$$\gamma(L) = 0 \quad \textit{if and only if} \quad L = o_M.$$

7.2. Action functional: the universal generating function

Since the work [LS] by Laudenbach and Sikorav, it has been a forklore that GFQI is a version of finite dimensional approximation to a *more canonical* generating function of the given Lagrangian submanifold $L = \phi(o_M)$ that exists when a generating Hamiltonian $H \mapsto \phi$ is given. For a notational convenience, we will also adopt the notation

$$H \mapsto L \quad \text{if} \quad L = \phi_H(o_M).$$

More precisely, consider the classical action functional

$$\mathcal{A}_H(\gamma) = -\int \gamma^*\theta + \int_0^1 H(\gamma(t),t)\, dt$$

defined on the space of paths that are *free at the final time*,

$$\Omega = \{\gamma : I \to T^*M \mid \gamma(0) \in o_M\}.$$

The space Ω has the natural fibration

$$p : \Omega \to M, \quad p(\gamma) := \pi(\gamma(1)).$$

We denote its fiber at q by Ω_q, i.e.,

$$\begin{aligned}\Omega_q &:= \{\gamma \in \Omega \mid \gamma(1) \in T^*_q M\} \\ &= \{\gamma : I \to T^*M \mid \gamma(0) \in o_M,\ \gamma(1) \in T^*_q M\ \}.\end{aligned}$$

A straightforward computation shows that for each $\xi \in T_\gamma \Omega$,

$$d\mathcal{A}_H(\gamma)(\xi) = \int_0^1 \big(-\omega(\dot\gamma, \xi) + dH_t(\gamma)\xi \big)\ dt - \langle \xi(1), \theta(\gamma(1))\rangle. \tag{7.6}$$

From this, we get

$$\frac{d\mathcal{A}_H}{d\xi}(\gamma) = 0 \quad \text{ if and only if } \ \dot\gamma \rfloor \omega - dH_t = 0, \quad \text{i.e., } \ \dot\gamma = X_{H_t}(\gamma)$$

where $\frac{d\mathcal{A}_H}{d\xi}$ stands for the vertical derivative of the fibration $p : \Omega \to M$. In other words, we have

$$\Sigma_{\mathcal{A}_H} = \{\gamma \in \Omega \mid \dot\gamma = X_{H_t}(\gamma)\ \}. \tag{7.7}$$

Furthermore, it follows from (7.6) that the map $i_{\mathcal{A}_H} : \Sigma_{\mathcal{A}_H} \to T^*M$ becomes

$$i_{\mathcal{A}_H}(\gamma) = \gamma(1) = \phi_H(\gamma(0)) \tag{7.8}$$

Now (7.7) and (7.8) imply that $\mathcal{A}_H : \Omega \to \mathbb{R}$ *is a generating function of* $L = \phi_H(o_M)$. By the same reason as in (7.4), this discussion leads to

Proposition 7.5. *The set of differences of the critical values of $\mathcal{A}_H$ depends only on $L = \phi(o_M)$, i.e is independent of the Hamiltonian H as long as $H \mapsto L$.*

This universal generating function has an advantage over its finite dimensional approximation GFQI in that it is canonical and does not involve any choice. Recall from [V3] that the proof of the uniqueness of GFQI up to the stabilization and the gauge invariance forms one of the essential ingredients and requires some nontrivial topological machinery in [V3]. On the other hand, the action functional is defined on the *infinite dimensional* path space Ω, for which it is well-known that the classical critical point theory does not work well in the general case other than when $M = \mathbb{R}^n$, i.e $T^*M = \mathbb{R}^{2n}$. Still, however, it is very natural to attempt to develop analogoues to Viterbo's work [V3] directly working with the actional functional $\mathcal{A}_H : \Omega \to \mathbb{R}$. There are two major difficulties in this attempt:

(1) *The standard direct approcach to the functional* $\mathcal{A}_H : \Omega to \mathbb{R}$ *for general* M *does not work by various reasons (e.g, lack of the global coordinates, the failure of Palais-Smale conditions and etc.)* –Analytical aspect–

(2) *There does not exist the Thom isomorphism on the fibration* $p : \Omega \to M$ *in the classical algebraic topological sense because the fiber* Ω_q *is infinite dimensional. More geometrically saying, it is not a priori obvious which mini-maxing sets one should choose to have the linking properties and so to pick out certain critical values of* $\mathcal{A}_H$ *or restrictions to the cycles chosen.* –Topological aspect–

In the rest of this section, we will outline how to overcome both of these two difficulties using the *Floer theory* which will substitute the traditional mini-max theory. The complete details have appeared in [O10].

7.3. Semi-infinite cycles

The difficulties we mentioned in the end of Section 7.2 turn out to be inter-related. In the classical critical point theory, the *mini-maxing sets* are the ones that define nontrivial *cycles in terms of the gradient flow of the given functional.* In the literature like [Rb], [BnRb], [Bn] and so on, the choice of such cycles depend on the type of the given functional. Indeed in the literature related to the periodic orbit problem of the Hamiltonian system on $\mathbb{R}^{2n} \cong \mathbb{C}^n$, the notion of *semi-infinite cycles* has been implicitly used. This cycle must be *nontrivial* in terms of the given action functional on the loop space on $\mathbb{C}^n$ with various *universal* hypotheses at infinity on the Hamiltonian (see e.g., [Rb], [Bn]). In the above literature, the global gradient flow of the action functional on the Sobolev space $H^{1/2}(S^1, \mathbb{C}^n)$ is well-defined in the standard sense and so one can still apply the classical variational theory using the mountain-pass type theorem. With these experiences in hand, we will try to choose our *semi-infinite cycles* with respect to which the *Floer theory* on Ω works well.

We start with the formula (7.6)

$$d\mathcal{A}_H(\gamma)(\xi) = \int_0^1 \big(-\omega(\dot\gamma, \xi) + dH_t(\gamma)\xi \big)\, dt - \langle \xi(1), \theta(\gamma(1)) \rangle.$$

As usual by now, we choose an almost complex structure J on T^*M that is compatible with the symplectic structure ω. Then one can re-write (7.6) as

$$d\mathcal{A}_H(\gamma)(\xi) = \int_0^1 \langle J\dot\gamma + \text{grad } H, \xi \rangle\, dt - \langle \xi(1), \theta(\gamma(1)) \rangle. \tag{7.9}$$

If one tries to write down the equation corresponding to the gradient flow of $-\mathcal{A}_H$ as in the Floer theory, one immediately encounters a difficulty due to the boundary term $\langle \xi(1), \theta(\gamma(1)\rangle$ in (7.9). To do the Floer theory correctly *in the analytical point of view*, one should try to get rid of this difficulty by *choosing certain subsets of* Ω *so that if we restrict the functional* $\mathcal{A}_H$ *to the subset, the boundary term drops out for the gradient flow of the restricted functional.* It is a remarkable fact that this attempt of ours to overcome the analytical difficulty gives rise to the way to associate a semi-infinite cycle to *each* compact submanifold of M and hence solves the topological difficulty mentioned above as well.

From the definition of the canonical one-form θ on T^*M, we can re-write the boundary term

$$\langle \xi(1), \theta(\gamma(1))\rangle = \langle T\pi\xi(1), \gamma(1)\rangle. \tag{7.10}$$

Main Observation. The term $\langle T\pi\xi(1), \gamma(1)\rangle$ vanishes if we impose the condition that $\gamma(1)$ *lies in the co-normal bundle* $N^*S \subset T^*M$ *of any submanifold* $S \subset M$ *and* $\xi(1)$ *is tangent to* N^*S *because* $\theta|_{N^*S} \equiv 0$ *for any submanifold.*

Now we are ready to assign to each $u \in H^*(M, \mathbb{R})$ a semi-infinite cycle whose topology is non-trivial in terms of the gradient flow of the action functional, i.e, *which cannot be pushed away to infinity by the gradient flow* of $\mathcal{A}_H$. This linking property of the cycles will be detected by the Floer homology theory. We choose a submanifold $S \subset M$ and associate a subset of Ω

$$\Omega(S) = \{\gamma \in \Omega \mid \gamma(1) \in N^*S\} \tag{7.11}$$

which will play a role as the cycles that have the required linking property. We emphasize that this choice of the above semi-infinite cycles does not involve the Hamiltonian H at all, which will be a crucial ingredient in defining invariants of Lagrangian submanifolds later.

Example 7.6. (1) When $S = M$, we have $N^*S = o_M$ and so the corresponding semi-infinite cycle becomes

$$\begin{aligned} \Omega(M) &= \{\gamma \in \Omega \mid \gamma(1) \in o_M\} \\ &= \{\gamma : I \to T^*M \mid \gamma(0),\ \gamma(1) \in o_M\} \end{aligned}$$

We call this cycle the *basic cycle.*

(2) When $S = \{pt\}$, then we assign to each $q \in M$

$$\begin{aligned} \Omega(q) &= \{\gamma \in \Omega \mid \gamma(1) \in T_q^*M\} \\ &= \{\gamma : I \to T^*M \mid \gamma(0) \in o_M,\ \gamma(1) \in T_q^*M = \Omega_q\} \end{aligned}$$

7.4. Floer homology of submanifolds

The L^2-gradient flow on $\Omega(S)$ of $\mathcal{A}_H|_{\Omega(S)}$ is the solution of the following perturbed Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\Big(\frac{\partial u}{\partial t} - X_H(u)\Big) = 0 \\ u(\tau, 0) \in o_M \\ u(\tau, 1) \in N^*S. \end{cases} \tag{7.12}$$

We will denote by $\mathcal{M}_J(H,S)$ the set of bounded solutions of (7.13), i.e ones u with

$$\int_{\mathbb{R}\times[0,1]} |\frac{\partial u}{\partial \tau}|_J^2 + |\frac{\partial u}{\partial t} - X_H(u)|_J^2 \, d\tau dt < \infty.$$

Note that the stationary points for this flow are the solutions of Hamilton's equaiton of H

$$\dot z = X_H(z), \quad z(0) \in o_M,\, z(1) \in N^*S. \tag{7.13}$$

We denote

$$CF(H,S) := \text{ the set of solutions of (7.13)}$$

which will be a set of finite elements for *generic* choice of S, if we fix a *generic* H (see [O10] the precise meaning of this genericity of the choice of H and S). Although N^*S is a noncompact Lagrangian submanifold, one can prove the necessary C^0-estimates to prove that all the solutions of (7.12) with finite energy with a uniform bound will be contained in a compact subset of T^*M and each has the limits as $\tau \to \pm\infty$ (see [FH2] for relevant estimates for the periodic orbit cases). This uses the fact that X_H has compact support and so the perturbed equation (7.12) becomes a genuine Cauchy-Riemann equation outside the compact set and then one can use the fact that T^*M is tame at infinity in the Gromov's sense to prove the C^0-estimates. The necessary analytic details will be carried out elsewhere [O10]. Once these are done, it is standard to construct a Floer-type homology for (7.12) (see Definition 3.5 in Section 3). In this section, we will exclusively use the homology instead of the cohomology. We now form the free $\mathbb{Z}$-module (or G-module with arbitrary abelian group), denoted by $CF_*(H,S)$ over $CF(H,S)$. We also prove in [O10] that $CF_*(H,S)$ carries a canonical grading and the moduli space $\mathcal{M}_J(H,S)$ has coherent orientations. This enables us to define the $\mathbb{Z}$-graded Floer homology group $HF_*(H,S,J)$ with *arbitrary coefficients* as

$$HF_*(H,S,J) = \text{Ker } \partial_J \backslash \text{Im } \partial_J$$

where we define the boundary operator

$$\partial : CF_*(H,S) \to CF_*(H,S)$$

as before. Note that unlikely from the case of the non-relative Floer theory where it is always possible to give a coherent orientation on the Floer complex (see [FH1] for detailed discussions on this), it is not possible to give a coherent orientation in the general relative Floer theory studied in [F2] or [O3]. By the standard compactness and corbodism argument, one can prove that these groups will be isomorphic as graded groups under the generic change of H. The following theorem implies that this homology is nontrivial and so the semi-infinite cycle $\Omega(S)$ cannot be pushed away to infinity.

Theorem 7.7. *For given (H,S,Ji), there exists a natural isomorphism*

$$F_{(H,S,J)} : H_*(S,\mathbb{Z}) \to HF_*(H,S,J). \tag{7.14}$$

Example 7.8. (1) When $S = M$, we have

$$HF_*(H,M,J) \cong H_*(M,\mathbb{Z}). \tag{7.15}$$

(2) When $S = \{q\}$, we have

$$HF_*(H,\{q\},J) \cong \mathbb{Z}.$$

7.5. A semi-infinite version of the Thom isomorphism

Recall the fibration $p : \Omega \to M$ and the basic cycle

$$\begin{aligned}\Omega(M) &= \{\gamma \in \Omega \mid \gamma(1) \in o_M\} \\ &= \{\gamma : [0,1] \to T^*M \mid \gamma(0),\, \gamma(1) \in o_M\}.\end{aligned}$$

We denote

$$CF^k(H:M) := \mathrm{Hom}(CF_k(H,M),\mathbb{Z})$$
$$\delta = \mathrm{Hom}(\partial) : CF^k(H:M) \to CF^{k+1}(H:M)$$

Then one has the natural isomorphism (with a shift of the grading)

$$F_H : H^*(M,\mathbb{Z}) \to HF^*(H:M) \tag{7.16}$$

We regard this isomorphism as *a semi-infinite version of the Thom isomorphism* for the fibration $p:\Omega \to M$.

Now for the later purpose, it is useful to realize a version of the Poincare duality. Note that $\Omega([1]) = \Omega(M)$ has a natural $\mathbb{Z}_2$-action which is induced from the time reversal: $t \to 1-t$ and which is not shared by other $\Omega(S)$'s. We denote

$$\widetilde{z}(t) := z(1-t), \quad \widetilde{H}(t,x) = -H(1-t,x)$$

and then one can easily check that this induces a natural isomorphism

$$\sigma_H : CF_{n-k}(\widetilde{H}:M) \to CF^k(H:M) \tag{7.17}$$

and the following identity holds

$$\mathcal{A}_{\widetilde{H}}(z) = -\mathcal{A}_H(\widetilde{z}). \tag{7.18}$$

In particular, σ_H induces a natural isomorphism

$$\sigma_H : HF_{n-k}(\widetilde{H}:M) \to HF^k(H:M) \tag{7.19}$$

Combining this with the isomorphism between $HF_*(H:M)$ and $HF_*(\widetilde{H}:M)$, one can also obtain the Poincare duality isomorphism

$$PD_H : HF^k(H:M) \to HF_{n-k}(H:M).$$

7.6. Construction of invariants, wave fronts and Hofer's distance

In this section, we outline a construction of symplectic invaritants using the machinery developed in the previous sections. Furthermore all the properties of our invariants can be directly proved without referring to Viterbo's and our invariants can be directly related to the Hamiltonian H's, not to the generating functions as in [V3]. Our construction enables us to give two different proof of the nondegenracy of the distance defined in (2.3), one using properties of the *wave fronts* of the Lagrangian submanifolds (see [E] for some explanation of properties of the wave front) and the other using the pants product on the Floer cohomology (See [F6] for its definition and also [BzRd], [LO], [PSS] for the non-relative context). We refer to [O10] for complete details.

We first note that the flow (7.12) preserves the filtration given by the values of the action functional $\mathcal{A}_H$ and so as in [FH2] one can define the relative homology $HF_*^{[a,b)}(H,S)$ for any real values $b > a$ by considering the complex induced by the boundary maps

$$\partial : CF^{[a,b)}(H,S) \to CF^{[a,b)}(H,S)$$

where $CF^{[a,b)}(H,S) := CF^b(H,S)/CF^a(H,S)$ and

$$C^a(H,S) := \{c \in \text{Crit } \mathcal{A}_H|_{\Omega(S)} \mid \mathcal{A}_H(c) < a\}.$$

Definition 7.9. For each H and $S \subset M$, we define

$$\rho(H,S) := \inf_\lambda \{\lambda \mid HF_*^{(-\infty,\lambda)}(H,S) \to HF_*(H,S) \text{ is surjective } \} \tag{7.20}$$

Lemma 7.10. *$\rho(H,S)$ is a (finite) critical value of $\mathcal{A}_H|_{\Omega(S)}$ and a continuous function with respect to S in the C^1-topology. The corresponding critical points are solutions of (7.13).*

Now the following theorem summarizes the basic properties of $\rho(H,S)$.

Theorem 7.11. *Let S and $\rho(H,S)$ be the ones in (7.20). Then*
(1) *The collection of the differences of $\rho(H,S)$ among $S \subset M$ depends only on $L = \phi_H(o_M)$. In other words if $\phi_H(o_M) = \phi_{H'}(o_M)$, then*

$$\rho(H,S) - \rho(H',S) = c(H,H') \tag{7.21}$$

where $c(H,H')$ does not depend on $S \subset M$.
(2) *When $H \equiv 0$, $\rho(H,S) = 0$ for any $S \subset M$.*
(3) *For any $S \subset M$,*

$$\int_0^1 \min_x \ (H_2 - H_1) \ dt \le \rho(H_2,S) - \rho(H_1,S) \le \int_0^1 \max_x \ (H_2 - H_1) \ dt. \tag{7.22}$$

In particular, when combined with (2)*, we have*

$$\int_0^1 \min_x H \, dt \le \rho(H,S) \le \int_0^1 \max_x H \, dt \tag{7.23}$$

Furthermore (7.22) also implies the inequality

$$|\rho(H_1,S) - \rho(H_2,S)| \le \|H_2 - H_1\|$$

which in particular implies that for each $S \subset M$, the function $H \mapsto \rho(H,S)$ is continuous with respect to the topology induced by Hofer's norm $\|H\|$. Combining Lemma 7.10, we can now extend the definition of ρ to all C^0-Hamiltonians H's and C^1-submanifolds S's.

Each of the statements looks quite standard in the point of view of the critical point theory, but the proof of any of them turn out to require certain forms of "geometrization" of the arguments in the traditional mini-max theory (e.g., ones in [BnRb] or [V3]). These arguments heavily rely on some invariance property of the Floer homology and on some important computations which we refer to [O10]. For example, the finiteness of the $\rho(H,S)$ is a consequence of Theorem 7.7 which proves the linking property of the semi-infinite cycles under the gradient flow of the action functional $\mathcal{A}_H$. This is the analogoue to the linking property of the mini-maxing sets in the traditional mini-max theory (e.g., [BnRb], [Bn]). We refer to [O10] for the complete details of the proof.

The special case $S = \{\text{point}\}$ is particularly interesting because it is closely related to the wave front of the Lagrangian submanifold L. In this case the map $q \mapsto \rho(H,\{q\})$ defines a continuous function on M. We denote this function by

$$f_H(q) := \rho(H,\{q\})$$

and call it the *basic phase fuction* of H or of the Lagrangian submanifold L. We prove that this function is independent of the choice of $H \mapsto L$ up to addition of a constant. It has the following remarkable property.

Proposition 7.12. *The basic phase function f_H is smooth away from a set of codimension at least one, and at smooth points q it satisfies*

$$(q, df_H(q)) \in L = \phi_H(o_M).$$

In other words, the graph $G_{f_H} \subset M \times \mathbb{R}$ of the function f_H is a subset of the wave front of L.

Now we are ready to give the first proof of the nondegeracy of the Hofer's distance defined in (2.3).

Theorem 7.13. *The distance in (2.3) is nondegenerate,i.e,*

$$d(L_1, L_2) = 0 \quad \textit{if and only if} \;\; L_1 = L_2.$$

Proof. When $L_1 = o_M$ and $L_2 = \phi(o_M)$, we have from definition

$$d(L_1, L_2) = \inf_{H \mapsto L_2} \|H\| \tag{7.24}$$

where $H \mapsto L_2$ means that $L_2 = \phi_H(o_M)$. On the other hand, from (7.19), it follows

$$\text{osc }(f_H) := \max f_H - \min f_H \leq \int_0^1 \max_x H \; dt - \int_0^1 \min_x H \; dt = \|H\|.$$

Suppose now that $d(L_2, L_1) = 0$. Then by taking the infimum over all $H \mapsto \phi$ and using the fact that the wave front set of L_2 is a compact subset of $M \times \mathbb{R}$, we conclude that the graph of a *constant function* is contained in the wave front of L_2. This immediately implies that L_2 must be the zero section, which proves $L_2 = L_1$ in the case $L_1 = o_M$. For the general case of $L_1 = \phi(o_M)$ and $L_2 = \psi(o_M)$, we first note that

$$d(L_1, L_2) = d(\eta(L_1), \eta(L_2))$$

for any $\eta \in \mathcal{D}^c_\omega(P)$. Therefore one can reduce the general case to the special case when $L_1 = o_M$, which will finish the proof.

Q.E.D.

Example 7. 14. Let us consider the case $M = S^1$ and the Lagrangian submanifold L as in the following picture.

Figure 2.

Here we denoted by z's the intersections of L with the zero section, by x's the caustics and by y the point at which the two shaded regions in the picture have the same area. The corresponding wave front can be easily drawn as

Figure 3.

Note that the points z's correspond to critical points, x's to the points where the wave front bifurcates and y to the point where two different branches of the wave front cross. Now by the continuity of the basic phase function f_H where $H \mapsto L$, one can easily see that the graph of the function f_H is the one bold-lined in Figure 3. From this, it follows that the value $\min_{q \in M} f_H(q)$ is *not* a critical value of $\mathcal{A}_H$ but a value coresponding to an intersection of two different branches of the wave front. One the other hand, all the Viterbo's invariants in [V3] correspond to critical values of $\mathcal{A}_H$ which are associated to intersections of L with the zero section.

Now we describe a construction of cohomological invariants, which uses the Thom isomorphism in Section 7.5. This way of constructing the invariants was observed by D. Milinkovic during our discussion with him. We expect that these invariants coincide with Viterbo's under suitable nomalizations. For each cohomology class $u \in H^*(M, \mathbb{Z})$, consider the Floer cohomology class $F_H(u) \in HF^*(H : M)$ given by the isomorphism (7.16). Using the filtration above, we can define the relative cohomolgy group $HF^k_{[a,b)}(H)$ for any real values $b > a$ by considering the complex induced by the coboundary maps

$$\delta : CF^k_{[a,b)}(H : M) \to CF^{k+1}_{[a,b)}(H : M)$$

where $CF^k_{[a,b)} := CF^k_a / CF^k_b$ and

$$CF^k_a(H : M) := \text{Hom}\ (CF^k_a(H, M), \mathbb{Z}).$$

As in [FH2], there exists canonical homomorphism $j^* : HF^k_{[c,d)}(H) \to HF^k_{[a,b)}(H)$ whenever $c \geq a$, $d \geq b$ In particular, there exists a natural homomorphism

$$j_\lambda^* : HF^k(H : M) \to HF^k_{(\lambda,\infty)}(H : M).$$

Definition 7.15. For each $u \in H^*(M, \mathbb{Z})$, we define

$$\rho(H, u) := \sup\{\lambda \mid j_\lambda^* F_H(u) \neq 0 \quad \text{in } HF^*_{(\lambda,\infty)}(H : M)\ \} \tag{7.25}$$

The following theorem summarizes the basic properties of $\rho(H, u)$.

Theorem 7.16. *Let $u \in H^*(M, \mathbb{Z})$ and $\rho(H, u)$ as above. Then*
(1) *All of these $\rho(H, u)$ are critical values of $\mathcal{A}_H$ on Ω which are independent of the choice of $H \mapsto L$, up to addition of a constant that is independent of u.*
(2) *When $H \equiv 0$, $\rho(H, u) = 0$ for any $u \in H^*(M, \mathbb{Z})$.*
(3) *For any $u \in H^*(M, \mathbb{Z})$,*

$$-\int_0^1 \max_x\ (H_2 - H_1)\ dt \leq \rho(H_2, u) - \rho(H_1, u) \leq -\int_0^1 \min_x\ (H_2 - H_1)\ dt. \tag{7.26}$$

In particular, when combined with (2), we have

$$-\int_0^1 \max_x H\ dt \leq \rho(H, u) \leq -\int_0^1 \min_x H\ dt \tag{7.27}$$

Furthermore (7.26) also implies the inequality

$$|\rho(H_1, u) - \rho(H_2, u)| \leq \|H_2 - H_1\|_{C^0}$$

which in particular implies that for each $u \in H^*(M, \mathbb{Z})$, the function $H \mapsto \rho(H, u)$ is continuous with respect to the topology induced by Hofer's norm $\|H\|$.
(4) *For any $u, v \in H^*(M, \mathbb{Z})$, we have*

$$\rho(H, u) \geq \rho(H, u \cup v). \tag{7.28}$$

Now as in [V3], we define a capacity of L as follows.

Definition 7.17 [ρ-capacity]. For any L that is Hamiltonian isotopic to o_M i.e., $L = \phi(o_M)$, we define the ρ-capacity of L by

$$\gamma(L) := \rho(H, \mu_M) - \rho(H, 1), \tag{7.29}$$

for any $H \mapsto L$.

The following is the analogoue to Theroem 7.4, which is Corollary 2.3 in [V3].

Theroem 7.18. *We have*

$$\gamma(L) = 0 \quad \textit{if and only if} \;\; L = o_M.$$

The proof of this relies on our construction of the pants product and the study of the change of the filtration under the pants product. We refer to [O10] for details of the proof of this theorem.

One can also easily derive from Theorem 7.18 and Theorem 7.11 that the Hofer's distance in (2.3) is nondegenerate. In fact, we have the inequality

$$\gamma(L) \leq d(L, o_M). \tag{7.30}$$

Remark 7.19. (1) It turns out that there is a very natural universal normalization of the action fuctional $\mathcal{A}_H$ for those H's such that $H \mapsto L$ in our story. This enables us to define invariants $\widetilde{\rho}(L, u)$ of L associated to $u \in H^*(M, \mathbb{R})$ that depends only on L but not on H as long as $H \mapsto L$. Recall that Viterbo's invariant $c(L, u)$ are also defined up to certain normalizations of GFQI (see the remark in the beginning of Section 2 in [V3]). So far this normalization problem in general has not been carefully addressed in the literature. Up to a suitable normalization of Viterbo's invariants, we conjecture that our invariants $\rho(L, u)$ coincide with Viterbo's $c(L, u)$.

(2) Strictly speaking, in the discussions of this Section 7, we always have to first look at some "generic" choices of the Hamiltonian H's and the cycle S's, which we intentionally suppress in this survey for the clarity of the exposition. It is very important to have the continuity property of $\rho(H, S)$ or $\rho(H, u)$ stated in Lemma 7.10, in Theorem 7.11 and in Theorem 7.16 when we pass to the non-generic cases by limiting arguments. The rigorous details of this limiting argument can be found in [O10].

(3) In [O10], we investigate further properties of our invariants ρ, where we refer to more foundational works on the relative Floer theory on the cotangent bundle.

PART II: FLOER COHOMOLOGY WITH QUANTUM EFFECTS

§8. Monotone Lagrangian submanifolds

We first look at some examples of the monotone Lagrangian submanifold.

Examples 8.1.

(i) Any Lagrangian submanifold L with $I_{\omega,L} \equiv 0$. In this case, $\lambda = 0$.

(ii) *Symmetric Lagrangian submanifolds.*

Let (P, ω) be a monotone symplectic manifold defined as in [F5], i.e.

$$I_c = \alpha \, I_\omega$$

on $\pi_2(P)$ for some $\alpha > 0$, and let $\sigma : P \to P$ be an anti-symplectic involution i.e., $\sigma^*\omega = -\omega$ and $\sigma^2 = \mathrm{id}$. The fixed pont set of σ, $L = \mathrm{Fix}\ \sigma$ is a Lagrangian submanifold if it is nonempty. Any such Lagrangian submanifold is indeed a monotone Lagrangian submanifold. One interesting example is the totally geodesic $\mathbb{R}P^n$ in $\mathbb{C}P^n$ or more generally any real forms of Hermitian symmetric spaces with the standard Kähler form (see [O1]).

(iii) *The Clifford torus T^n in $\mathbb{C}P^n$.*

We follow the description of the Clifford torus given in [O1]. Consider the isometric embedding

$$T^{n+1} := \underbrace{S^1(\frac{1}{\sqrt{n+1}}) \times \cdots \times S(\frac{1}{\sqrt{n+1}})}_{n+1 \text{ times}} \hookrightarrow S^{2n+1}(1) \subset \mathbb{C}^{n+1} \ .$$

This embedding is Lagrangian in $\mathbb{C}^{n+1}$ and the standard action by S^1 on $\mathbb{C}^{n+1}$ restricts to both the above torus and $S^{2n+1}(1)$. By taking the quotients by this action, the torus $T^n := T^{n+1}/S^1$ in $\mathbb{C}P^n = S^{2n+1}(1)/S^1$ is Lagrangian by the general push forward operation of Lagrangian submanifolds. This torus is a minimal submanifold in Riemannian geometry, which is called the *Clifford torus* in $\mathbb{C}P^n$. This Clifford torus $T^n \subset \mathbb{C}P^n$ is monotone with respect to the standard symplectic structure on $\mathbb{C}P^n$ with the monotonicity constant given by $\lambda = \frac{2(n+1)}{\pi}$ (Compare [O3]).

(iv) *Chekanov tori in $\mathbb{R}^{2n}$*

The general construction Chekanov used in [C1] is a kind of suspensions of Lagrangian submanifolds. For any embedded Lagrangian submanifold $L \subset \mathbb{R}^{2n}$ and $a \in \mathbb{R}$, we define an embedded Lagrangian submanifold $\Theta_a(L) \subset \mathbb{R}^{2n+2}$ as follows: Identifying $S^1 = \mathbb{R}/\mathbb{Z}$, let us consider the map

$$i_n : S^1 \times \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}, (t, x_1, x_2, \cdots, x_n) \mapsto (e^{x_1} \cos 2\pi t, e^{x_1} \sin 2\pi t, \cdots, x_n). \tag{8.1}$$

Denote by N_a the Lagranigian submanifold $\{p_t = a\} \subset T^*S^1$ where (t, p_t) is the canonical coordinates on T^*S^1. Define

$$\Theta_a(L) := I_n(N_a \times L), \quad I_n := (i_n^*)^{-1} : T^*(S^1 \times \mathbb{R}^n) \hookrightarrow T^*\mathbb{R}^{n+1} \cong \mathbb{R}^{2n}.$$

It is easy to see that if $L \subset \mathbb{R}^{2n}$ is monotone, then $\Theta_0(L)$ is monotone in $\mathbb{R}^{2n+2}$. For example, one can start with the standard circle $S^1 \subset R^2$ to produce a monotone Lagrangian torus $\Theta_0(S^1) \subset \mathbb{R}^4$. It is a very interesting theorem by Chekanov [C1] that this torus is not symplectically equivalent to the standard product torus $S^1 \times S^1 \subset \mathbb{R}^4$, which the classical invariants cannot detect.

Theorem 8.2 [Chekanov, C1]. *The monotone Lagrangian torus $\Theta_0(S^1)$ is not symplectically equivalent to the product torus $S^1 \times S^1 \subset \mathbb{R}^4$.*

Chekanov uses an argument using the symplectic capacity function in his proof. Later Eliashberg and Polterovich [EP] gave a different proof of this theorem using the quantum invariants mentioned in Section 2.

Since we have already looked at the case when $\lambda = 0$ in Section 4, we will assume that $\lambda > 0$ in this section. The essential difference between the two cases is the existence of bubbling phenomena. As we have seen from Section 4, the picture for the case $\lambda = 0$ is almost the same as in the finite dimensional Morse theory on L. In fact, as far as the structure of moduli spaces of the Floer complex and the Morse complex are concerned, they turn out to be diffeomorphic to each other when Φ is C^1-close to identity (see [F4] and Section 10 for more discussions on this picture). Theorem 4.5 is just a consequence of this diffeomorhpism. However when $\lambda > 0$, the bubbling phenomenon is unavoidable in general and the whole picture diverges from the finite dimensional Morse theory: we have to take into account "quantum contributions" given by J-holomorphic discs or spheres. However as we see from Definition 3.5, the construction of the Floer cohomology involves only one or two dimensional components of the moduli space $\mathcal{M}_J(L, \Phi : P)$. Imitating Floer's argument [F5] for the monotone symplectic manifolds, the author was able to give a definition of the Floer cohomology for the monotone case with $\Sigma_L \geq 3$

in [O3] by analyzing the structure of low dimensional components of the moduli space, and then has been recently able to generalize to the case $\Sigma_L = 2$ [O8] with a finer argument involving the relative version of Gromov-Witten invariant (see [Ru] or [MS] for an account for the nonrelative case). However unlikely from the non-relative case where Floer [F5] (and others [HS], [On], [PSS] and [RT2]) proved that $HF^*_{\mathbb{Z}}(P)$ is always isomorphic to $H^*(P,\mathbb{Z})$ as modules, it is not the case that $HF^*_{\mathbb{Z}_2}(L:P)$ is isomorphic to $H^*(L,\mathbb{Z}_2)$ in general. For example, for the monotone Lagrangian embeddings in $\mathbb{C}^n$, where $HF^*(L:\mathbb{C}^n)$ must be obviously trivial, once it is well-defined and invariant under Hamiltonian isotopy of L. On the other hand, the author [O5] proved that they are isomorphic in the case of the real forms of compact Hermitian symmetric spaces. It had not been clear what would be the general relation between these two until the author [O9] discoverd a spectral sequence starting from $H^*(L,\mathbb{Z}_2)$ which converges to $HF^*(L:P)$. We will explain more about this spectral sequence in Section 9.

Now we go back to the discussion of the construction of the Floer cohomology of monotone Lagrangian submanifolds. The following proposition will substitute Lemma 4.1 in the case $\lambda = 0$ for the monotone case, which provides the a priori bound for the L^2-norm $\int |Du|^2$ for maps $u : \Theta \to P$ under the hypothesis that they are in $\mathcal{M}_J(x,y)$ for a fixed pair (x,y) and of the same Index u.

Proposition 8.3 [Proposition 2.7 & 2.10, O3]. *Let L be a monotone Lagrangian submanifold and Φ be a Hamitonian isotopy. Let u and v be two maps from $[0,1]\times[0,1]$ to Ω_Φ such that*

$$u(\tau,0), v(\tau,0) \in L \ , \quad u(\tau,1), v(\tau,1) \in \phi_1(L) \ \text{and}$$
$$u(0,t) = v(0,t) \equiv x \ , \quad u(1,t) = v(1,t) \equiv y$$

where $x, y \in CF(L,\Phi)$ and the path $u(\tau)(t) := u(\tau,t)$ (respectively $v(\tau)(t) := v(\tau,t)$) defines for each $\tau \in [0,1]$ an element of Ω_Φ. Then

$$[\omega](u) = [\omega](v) \text{ if and only if } \mu_u(x,y) = u_v(x,y)$$

where μ_u is the Maslov-Viterbo index (see [F3] or [V1] for the definition). In particular, if $u, v \in \mathcal{M}_J(x,y)$ for $J \in \mathcal{J}_\omega(P)$, then

$$\int \|\nabla u\|^2_J = \int \|\nabla v\|^2_J \text{ if and only if } \mu_u(x,y) = \mu(x,y) \ , \tag{8.2}$$

With this a priori bound for the L^2-norm of $|Du|$, we proceed the study of compactness properties of zero and one dimensional components of $\widehat{\mathcal{M}}_J(x,y)$. To analyze these compactness properties, we need to study the structure of the set of J-holomorphic disks and spheres and to understand how they intersect the space $\mathcal{M}_J(L, \Phi : P)$. In [Gr], Gromov studied the convergence properties of pseudo-holomorphic maps from a Riemann surface Σ (resp. $(\Sigma, \partial\Sigma)$) to P (resp. (P, L)) and introduced the notion of the *cusp-curve* which occurs as a limit of a sequence of holomorphic curves on P or (P, L). The failure of compactness as a standard holomorpic curve is due to either bubbling off a sphere or bubbling off a disk out of L (See [PW] for a nice detailed account on this convergence in the context of J-holomorphic spheres). Similarly as in [F5], we now introduce the notion of *k-cusp trajectories* $\widehat{\mathcal{MC}}^k_J(x,y)$.

Definition 8.4. Let $J = \{J_t\}_{0\leq t\leq 1}$. A k-cusp trajectory $(\underline{\mathrm{u}}, \underline{\mathrm{v}}, \underline{\mathrm{w}}) \in \widehat{\mathcal{MC}}^k_J(x,y)$ is a k-trajectory $\underline{\mathrm{u}} \in \widehat{\mathcal{M}}^k_J(x,y)$ together with a finite collection $\underline{\mathrm{v}}$, $\underline{\mathrm{w}}$ of maps

$$v : \mathbb{C} \to P$$
$$w : (\mathbb{C}_+, \partial\mathbb{C}_+) \to (P, L) \text{ or } (P, \phi_1(L))$$

with finite areas labeled by elements ξ_v or $\xi_w \in \underline{\mathrm{u}} \cup \underline{\mathrm{v}} \cup \underline{\mathrm{w}} \cup \{x, y\}$ and points θ_v and θ_w in $\mathbb{R} \times S^1$, S^2 or D^2 such that

$$v_\infty := \lim_{|z|\to\infty} v(z) = \xi_v(\theta_v)$$

or

$$w_\infty := \lim_{|z|\to\infty} w(z) = \xi_w(\theta_w)\ .$$

Moreover we assume that each v or w are *J_t-holomorphic* for some $t \in [0,1]$. Here $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \ \mathrm{Im}\ z \geq 0\}$.

By removing singularities (see [PW] for the interior singularities, and [O2] for a unified proof of the theorem of removing singularities for both interior and boundary cases), each $v \in \underline{\mathrm{v}}$ or $w \in \underline{\mathrm{w}}$ is obtained from J_t-holomorphic maps $\tilde{v} : S^2 \to P$ or $\tilde{w} : (D^2, \partial D^2) \to (P, L_0)$ or (P, L_1) by a conformal equivalence. By introducing some topology around a point in $\widehat{\mathcal{MC}}_J(x,y)$ (see Definition 3a.3 in [F5] or [PW]) which is geometrically quite clear, we have the following proposition..

Proposition 8.5. *Let $(J^\alpha, L^\alpha, \Phi^\alpha) \to (J, L, \Phi)$ be a convergent sequence with L^α_0 transversally intersecting $\phi^\alpha_1(L)$ and let $(x_\alpha, y_\alpha) \in CF(L^\alpha, \Phi^\alpha)$ converges to $(x,y) \in$*

$CF(L,\Phi)$. Then for any sequence $u_\alpha \in \mathcal{M}_{J_\alpha}(x_\alpha, y_\alpha)$ with constant index I and with $\int |Du_\alpha|^2 < M < \infty$, there exists a subsequence converging to some $(\underline{\mathrm{u}}, \underline{\mathrm{v}}, \underline{\mathrm{w}}) \in \mathcal{MC}^k_J(x,y)$ for some $k \geq 0$. Moreover, we have

$$I = \sum_{i=1}^{k} \mathrm{Index}(u_i) + 2\sum_j c_1(v_j) + \sum_\ell \mu(w_\ell)\ . \tag{8.3}$$

$$\overline{\lim}_{\alpha\to\infty} \ell^2(u_\alpha) = \sum_i \ell^2(u_i) + \frac{1}{2}\sum_j \int v_j^*\omega + \frac{1}{2}\sum_\ell \int w_\ell^*\omega \tag{8.4}$$

where

$$\ell^2(u) := \int_\Theta u^* w = \int_{\mathbb{R}} \int_I |\frac{\partial u}{\partial \tau}|^2_J \, dt d\tau.$$

8.1. The zero dimensional component

In this section, we in fact prove *for a general pair (L_0, L_1) that without any assumption on Σ besides the monotonicity of L_0 and L_1*, the zero dimensional component of $\widehat{\mathcal{M}}_J(x,y) = \mathcal{M}_J(x,y)/\mathbb{R}$ is compact for a subset of $\mathcal{J}_{reg}(L_0, L_1)$, which is again a dense set of $\mathcal{J}_\omega$. The corresponding statement for the pair $(L, \phi_1(L))$ can be proven in the same way. Throughout this section, we always implicitly assume the fact that constant index implies a bound for the energy, which is provided by Proposition 8.2 for the case of our main interest $(L, \phi_1(L))$.

Proposition 8.6. *There is a subset $\mathcal{J}_1(L_0, L_1) \subset \mathcal{J}_{reg}(L_0, L_1)$, which is dense in $\mathcal{J}_\omega$ and for which the zero dimensional component of $\widehat{\mathcal{M}}_J(x,y)$ is compact. For the case of $(L, \phi_1(L))$, there is a dense subset $\mathcal{J}_1(L, \phi_1) \subset \mathcal{J}_\omega$ such that for any $J \in \mathcal{J}_1(L, \phi_1)$ the zero-dimensional component is compact.*

Proof: We apply Proposition 8.5 to the case $J_\alpha = J$, $L_0^\alpha = L_0$, $L_1^\alpha = L$ and $y_\alpha = y$, $x_\alpha = x$. First, note that by the monotonicity of L_0, L_1, the second and third sums in (8.3) become positive if $\underline{\mathrm{v}}$ and $\underline{\mathrm{w}}$ are non-empty respectively. Moreover the second sum is greater than or equal to 2 if $\underline{\mathrm{v}} \neq \emptyset$.

We now consider two cases separately: The first is the case when $x \neq y$ and the second when $x = y$. When $x \neq y$, $J \in \mathcal{J}_{reg}(L_0, L_1)$ and $I = 1$, then $\underline{\mathrm{u}} \neq \emptyset$ and so $\underline{\mathrm{v}}$ and $\underline{\mathrm{w}}$ must be empty by the above. Therefore, the zero dimensional component of $\widehat{\mathcal{M}}_J(x,y)$ cannot bubble off and so is compact in this case.

Now consider the case $x = y$. This case is new which we did not have to consider in the case $\omega|_{\pi_2(P,L)} \equiv 0$. However in the general (monotone) case, it is possible that there is a *homoclinic* connecting orbit starting from and ending at the same point. In this case, we have to consider the possibility in (8.4) that $\underline{\mathrm{u}} = \emptyset$ and there occurs some bubbling off either a sphere or a disk out of the point x. The possibility of a bubbling-off-a-sphere can be easily ruled out by the same reasoning as above. It makes RHS in (8.3) greater than one. Ruling out a bubbling-off-a-disk is more subtle since we cannot rule it out by the above dimension counting. However note that such a holomorphic disc must have the Maslov index 1. We first need the following standard lemma (see [O3] for its proof).

Lemma 8.7. *There exists a dense set $\mathcal{J}_D(L_0, L_1)$ of $\mathcal{J} = C^\infty([0,1] \times S_\omega)$ such that for any $J \in \mathcal{J}_D(L_0, L_1)$, the linearizations*

$$D\bar\partial_{J_i}(v) : L^p_k(v^*TP, L_i) \to L^p_{k-1}(v^*TP)$$

are surjective for any J_i-holomorphic $v : (D, \partial D^2) \to (P, L_i)$ satisfying the condition that there is $z \in D^2$ such that $dv(z) \neq 0$ and $v^{-1}(v(z)) = \{z\}$. Here we denote $J_i := J(i), i = 0, 1$ and define

$$L^p_k(v^*TP, L_i) := \{\xi \in L^p_k(v^*TP) \mid \xi(\partial D) \subset TL_i\} \ .$$

We continue the proof of Proposition 8.6. First note that although we consider a time dependent almost complex structure J and the equation $\bar\partial_J u = 0$, if a bubbling off a disk occurs, the disk must be holomorphic with respect to either J_0 or J_1 depending on whether the boundary of the disk is on L_0 or L_1. This is because the bubbling is *localized* and it can be easily seen by carefully looking at the bubbling argument (see [F1], [H2] or [O2]). Therefore, we assume without loss of any generalities that the disk v is J_0-holomorphic and regular. Since $\mu_{L_0}(w)$ is must be 1, the dimension of $M_p(J_0, L_0 : [w])$, the set of parametrized holomorphic disks with boundary on L_0 in the class $[w]$, is $n + 1$. We consider the evaluation map

$$\begin{aligned}
&\mathrm{ev} : M_p(J_0, L_0 : [w]) \times_G S^1 \to L_0 \\
&\mathrm{ev}(f, \theta) = f(\theta) \qquad f \in M_p(J_0, L_0 : [w]) \quad \theta \in S^1
\end{aligned}$$

where G is the automorphism group of D^2 whose dimension is 3. Therefore

$$\dim(M_p(J_0, L_0 : [v]) \times_G S^1) = n + 1 + 1 - 3 = n - 1 \ .$$

On the other hand we assume that $L_0 \cap L_1$ is finite and so we can avoid those intersections for a dense set of j_ω. Apply this to all possible classes $[w]$ and take for j_0, the intersection of all those dense sets. Then since the number of possible classes is countable, j_0 is again a dense set of j_ω. The same argument goes for $i = 1$. Now we define $\overline{\mathcal{J}}_D(L_0, L_1)$ to be the set of all smooth paths connecting $\overline{j}_0$ and $\overline{j}_1$. This is again a dense set of $\mathcal{J}$. Now we can rule out the bubbling off disks with its Maslov index 1 for $J \in \overline{\mathcal{J}}_D(L_0, L_1)$. Therefore the zero dimensional component of $\widehat{\mathcal{M}}_J(x, y)$ is compact if $J \in \mathcal{J}_1(L_0, L_1) := \mathcal{J}_{reg}(L_0, L_1) \cap \overline{\mathcal{J}}_D(L_0, L_1)$. Now it is easy to show that $\mathcal{J}_1(L_0, L_1)$ is dense in $\mathcal{J}$

Q.E.D.

8.2. The one-dimensional component

In this section, we assume that L is monotone with $\Sigma_L \geq 2$ and study the compactness properties of one dimensional components of $\widehat{\mathcal{M}}_J(L, \Phi : P)$.

Proposition 8.8. *Under the assumption $\Sigma_L \geq 3$, there is a dense set $\mathcal{J}_2(L, \phi_1) \subset \mathcal{J}_{reg}(L, \phi_1)$ of $\mathcal{J}_\omega$ such that the one dimensional component of $\widehat{\mathcal{M}}_J(x, y)$ is compact up to the splitting of two isolated trajectories for $J \in \mathcal{J}_2(L, \phi_1)$.*

Proof: Let u_α be a sequence in $\mathcal{M}_J(x, y)$ with Index $u_\alpha = 2$. Again, we will apply Proposition 8.5. As before, we consider the two cases separately.

First we consider the case when $x \neq y$. If $x \neq y$, $\underline{\mathrm{u}}$ cannot be empty for $J \in \mathcal{J}_{reg}(L_0, L_1)$. Therefore by the assumption $\Sigma_L \geq 3 > 2$, RHS in (8.3) becomes greater than two unless $\underline{\mathrm{v}} = \emptyset$ and $\underline{\mathrm{w}} = \emptyset$. Therefore one dimensional components of $\widehat{\mathcal{M}}_J(x, y)$ are compact up to splitting for $J \in \mathcal{J}_{reg}(L, \phi_1)$.

Now consider the case of $x = y$. A priori it is possible in the limit that

$$\underline{\mathrm{u}} = \emptyset\ , \quad \underline{\mathrm{w}} = \emptyset\ , \quad \text{but} \quad \underline{\mathrm{v}} = \{v\} \quad \text{with} \quad c_1(v) = 1.$$

Of course this possibility is ruled out if we assume that $\Sigma \geq 3$ for all $J \in \mathcal{J}(L, \phi_1)$. However for the later purposes, we analyze the case $\Sigma_L = 2$ further. To take care of the case for $\Sigma_L = 2$, we proceed as follows. Consider the parametrized family

$$M_p(J, S^2, [v]) := \bigcup_{t \in [0,1]} M_p(J_t, S^2, [v]).$$

By a now standard argument, one can prove that there exists a dense set $\mathcal{J}_S(L, \phi_1)$ such that if $J \in \mathcal{J}_S(L_0, L_1)$, $M_p(J, S^2, [v])$ is a smooth manifold with dimension $2(n+1)+1$.

Consider the parametrized evaluation map

$$\begin{aligned} &\mathrm{Ev} : [0,1] \times M_p(J, S^2, [v]) \times_G S^2 \to [0,1] \times P \\ &\mathrm{Ev}(t, v, z) = (t, v(z)) \ . \end{aligned}$$

Again this map Ev can be made transversal to the one dimensional submanifold $[0,1] \times L_0 \cap L_1 \subset [0,1] \times P$ for $J \in \mathcal{J}_T(L, \phi_1)$ where $\mathcal{J}_T(L, \phi_1)$ is a dense set of $\mathcal{J}_\omega$. However, we have

$$\dim([0,1] \times M_p(J, S^2, [v]) \times_G S^2 = 2(n+1) + 1 + 2 - 6 = 2n - 1$$

and

$$\dim([0,1] \times P) = 2n + 1 \ .$$

Therefore, for $J \in \mathcal{J}_T(L, \phi_1)$, we can avoid the intersections of $\mathrm{Im}(\mathrm{ev}(M_p(J_t, S^2, [v]))$ and $L_0 \cap L_1$ for all $t \in [0,1]$ and so there cannot occur a bubbling off a sphere at any point in $L_0 \cap L_1$.

Now it remains to take care of the possibility that

$$\underline{\mathrm{u}} = \emptyset \ , \quad \underline{\mathrm{v}} = \emptyset \ , \quad \text{but} \quad \underline{\mathrm{w}} = \{w\} \quad \text{with} \quad \mu(w) = 2.$$

We cannot rule out this possibility by the above generic argument when $\Sigma_L = 2$ and indeed it occurs in reality. For example, we can easily produce such limits for the case of area bisecting curves on S^2, or more generally for the case of the Clifford torus $T^n \subset \mathbb{C}P^n$. This is why we require the hypothesis $\Sigma_L \geq 3$ in this proposition and in our first paper [O3] in the Floer theory. By taking $\mathcal{J}_2(L, \phi_1) = \mathcal{J}_T(L, \phi_1) \cap \mathcal{J}_{reg}(L, \phi_1)$, we have finished the proof of Proposition 8.8.

Q.E.D.

Now we present the argument carried out in [O4] and [O8] which makes it possible to get around this difficulty for the case $\Sigma_L = 2$. First we need to define some "quantum" invariant of the pair (L, P), which is the relative version of a Gromov-Witten invariant (see [Ru] for the precise definition for the nonrelative case).

Definition 8.9. We define by $\Phi_J(x, L : P)$ is the number (mod 2) of J-holomorphic discs with Maslov index 2 that pass through the point x in $L \subset P$.

Then the following general lemma can be proven by the discussions above and by a version of gluing argument. (See [O8] for a detailed discussion on this gluing.)

Lemma 8.10. *There is a dense subset $\mathcal{J}_2(L_0, L_1) \subset \mathcal{J}_1(L_0, L_1)$ of Baire type such that if $J \in \mathcal{J}_2(L_0, L_1)$, for each $x \in I(L_0, L_1)$, $\delta : I(L_0, L_1) \to I(L_0, L_1)$ (which can be defined for $J \in \mathcal{J}_1(L_0, L_1)$ by Proposition 8.6), satisfies*

$$\langle \delta \circ \delta x, x \rangle = \Phi_J(x, L_0 : P) + \Phi_J(x, L_1 : P) \pmod 2 \tag{8.5}$$

for any $x \in I(L_0, L_1)$.

The next important fact is that the number $\Phi_J(x, L : P)$ does not depend on the choice of J nor the choice of $x \in [x]$, but depends only on the connected component $[x] \in \pi_0(L)$. We denote this common number by $\Phi_L([x] : P)$. This can be proven by now the standard compactness and cobordism argument (see e.g, [Chapter 7, MS]), since each homotopy class in $\pi_2(M, L)$ of J-holomorphic discs with Maslov index 2 is *simple* due to the assumption of monotonicity and $\Sigma_L \geqq 2$. This common number, which we denote by $\Phi_L([x] : P)$, is the sum of $\Phi_B(pt : L_{[x]}, P)$'s over all $B \in \pi_2(P, L_{[x]})$ with $\mu_L(B) = 2$ which is the relative analogoue for the pair $(P, L_{[x]})$ to the Gromov-Witten invariant $\Phi_A(pt : P)$ for the symplectic manifold P (see [Ru], [Chapter 7, MS] for the definition of Gromov-Witten invariants and their basic propeties). Moreover, again from the cobordism argument, it follows that the "quantum" number $\Phi_L([x] : P)$ is preserved under Hamiltonian isotopies of L, i.e, we have

$$\Phi_L([x] : P) = \Phi_{\phi_1(L)}([\phi_1(x)] : P). \tag{8.6}$$

This shows that for a general monotone pair (L_0, L_1) with minimal Maslov number greater than equal to 2, $\delta : \mathcal{D}^* \to \mathcal{D}^*$ satisfies

$$\langle \delta \circ \delta x, x \rangle = (\Phi_{L_0}([x] : P) + \Phi_{L_1}([x] : P)). \tag{8.7}$$

Now we go back to the case where L_1 is Hamiltonian isotopic to L_0, say $L_0 = L$ and $L_1 = \phi_1(L)$. We recall that the complex $(CF^*(L, \Phi), \delta)$ was defined by

$$\begin{aligned} CF^*(L, \Phi) &= \text{the free } \mathbb{Z}_2\text{-module over} \quad CF(L, \Phi) \\ CF(L, \Phi) &:= \{x \in L \cap \phi_1(L) \mid [t \mapsto \phi_t^{-1}(x)] = 0 \in \pi_1(P, L)\}. \end{aligned}$$

Therefore if $x \in CF(L, \Phi)$, we have

$$\Phi_{L_0}([x] : P) = \Phi_{\phi_1(L_0)}([\phi_1(x)] : P) = \Phi_{L_1}([x] : P) \tag{8.8}$$

from (8.6) and because by the definion of $CF(L, \Phi)$ x and $\phi_1^{-1}(x)$ are in the same component of L. Combining Lemma 8.10 and (8.8), we have obtained

$$\langle \delta \circ \delta x, x \rangle \equiv 0 \pmod 2$$

for all $x \in CF(L, \Phi)$ and hence $\delta \circ \delta = 0$ on $CF^*(L, \Phi)$, which in turn finishes the proof of the well-definedness of $HF^*(L : P)$ for any L (whether L is connected or not) even for the case $\Sigma_L = 2$. This finally finishes the proof of the following main theorem in this section by now a standard construction of the coboundary operator.

Theroem 8.11. *Let L be a monotone Lagrangian submanifold in (P, ω) and $\Phi = \{\phi_t\}_{0 \le t \le 1}$ be a Hamiltonian isotopy such that L transversely meets $\phi_1(L)$. Assume $\Sigma_L \ge 2$. Then there exists a homomorphism*

$$\delta : CF^*(L, \Phi) \to CF^*(L, \Phi)$$

with $\delta \circ \delta = 0$, provided $J \in \mathcal{J}_0(L, \phi_1) := \mathcal{J}_1(L, \Phi) \cap \mathcal{J}_2(L, \Phi)$. We define the Floer cohomology for the quadruple $(L, \Phi, J : P)$ as

$$HF^*_J(L, \Phi : P) := \mathrm{Ker}\ \delta / \mathrm{Im}\ \delta$$

as $\mathbb{Z}_2$-modules.

One can easily see that we can give a $\mathbb{Z}/\Sigma_L$-grading to $HF^*(L, \Phi : P)$.

8.3. Invariance.

The possibility of non-invariance of the Floer cohomology under Hamiltonian deformations even when it is defined for individual (L, Φ), is illustrated by the example of small circles in S^2. (Here note that a simple closed curve is monotone if and only if it divides S^2 into two equal area pieces.) Although the small circles are not monotone, we can still define $HF^*_{J_0}(L, \Phi)$ for each nondegenerate Hamiltonian isotopy Φ and for the standard complex structure J_0 but $HF^*_{J_0}(L, \Phi)$ will not be invariant under the change of Φ. For example, if we separate a small circle $= L$ from itself by a rotation which is always possible unless L is area-bisecting, $HF^*_{J_0}(L, \phi)$ is changed from $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ to $\{0\}$. The reason behind this is that there are two holomorphic trajectories connecting two intersectins with the same index 1 but with different actions (one approaches 0 and the other approaches $4\pi - 2A$ where A is the area of the smaller pieces of $S^2 \setminus L$). This phenomenon never happens in the monotone case. The main theorem we want to consider in this section is that $HF^*_J(L, \Phi : P)$ defined in the previous section are invariant under the change of (L, Φ, J).

Remark 8.12. The most interesting invariance property is the one under the change of Φ. We need some caution in stating the invariance property when we compare the Floer cohomologies for two Hamiltonian isotopies Φ^α and Φ^β with $\phi^\alpha_1(L) = \phi^\beta_1(L)$. In this case, we will always assume that the Hamiltonian isotopies of Lagrangian submanifolds

$$t \mapsto \phi^\alpha_t(L) \quad \text{and } t \mapsto \phi^\beta_t(L)$$

are homotopic.

Theorem 8.13. *Let P, L satisfy the hypotheses as in Theorem 8.11. Suppose $\Phi^\alpha = \{\phi^\alpha_t\}$ and $\Phi^\beta = \{\phi^\beta_t\}$ are Hamiltonian isotopies such that $\phi^\alpha_1(L)$, $\phi^\beta_1(L)$ meet L transversally. For $J^\alpha \in \mathcal{J}_0(L, \phi^\alpha_1)$ and $J^\beta \in \mathcal{J}_0(L, \phi^\beta_1)$, there is an isomorphism*

$$h^{\alpha\beta} : HF^*_{J^\alpha}(L, \Phi^\alpha) \to HF^*_{J^\beta}(L, \Phi^\beta)$$

(which preserves grading).

Proof: We introduce a one parameter family

$$(\overline{J}, \overline{\Phi}) = \{(J^\lambda, \Phi^\lambda)\}_{\lambda \in \mathbb{R}}$$

which is constant in λ outside $[0, 1]$ and

$$\begin{aligned} \Phi^0 &= \Phi^\alpha \ , \qquad & \Phi^1 &= \Phi^\beta \quad \text{and} \\ J^0 &= J^\alpha \ , \qquad & J^1 &= J^\beta \end{aligned}$$

We also assume that ϕ^λ_t is also Hamiltonian under the change of λ. Recall that J and Φ already involve one parameter t. Moreover we assume that $\phi_0 = id$ and so ϕ^λ_t is two parameter family of Hamiltonian isotopies contractible to the identity. Such $\overline{\Phi}$ connecting Φ and Φ' certainly exists when $\phi_1(L) \neq \phi'_1(L)$. As mentioned in Remark 8.12, when $\phi_1(L) = \phi'_1(L)$, we assume that Φ and Φ' satisfy the hypothesis in Remark 8.12 and so such isotopies exist by the hypothesis. Now consider the same equation as (4.6) and the moduli space $\mathcal{M}_\rho(x, x')$ for $x \in CF(L, \Phi)$, $x' \in CF(L, \Phi')$. To carry out the analysis, we need the following lemma which is the analogue of Proposition 4.4 for the general monotone case. We refer readers to the proof of Lemma 5.2 [O3].

Lemma 8.14. *Let u_1, u_2 be two maps from $[0,1]^2 \to P$ such that*

$$u_i(\tau, 0) \in L \ , \qquad u_i(\tau, 1) \in \phi_1^\tau(L) \quad \text{and}$$
$$u_i(0, t) \equiv x \ , \qquad u_i(1, t) \equiv x' \ , \qquad i = 1, 2 \ .$$

and u_i define a path in Ω_{ϕ^τ}. Suppose $\mu_{u_1}(x, x') = \mu_{u_2}(x, x')$. Then

$$|[\omega](u_1) - [\omega](u_2)| \leq C(\overline{\Phi})$$

for some constant $C(\overline{\Phi})$ which depends only on $\overline{\Phi}$.

The rest of the proof of Theorem 8.13 will be the same as outlined in Section 4.1. This finally establishes the construction of $HF^*(L : P)$ which will be invariant under the Hamiltonian isotopies of L.

§9. Spectral sequence

In the previous section, we have established the definition and the invariance properties of $HF^*(L : P)$ for the monotone Lagrangian submanifolds with $\Sigma_L \geq 2$. Next thing important in applications is to develop a method of computing this group. As Floer [F4] did in the case $\pi_2(P, L) = \{e\}$ or more generally in the case of $\omega|_{\pi_2(P,L)} \equiv 0$, we will analyze the structure of the moduli space $\mathcal{M}_J(L, \Phi : P)$ for the isotopy Φ C^1-close to the identity and for an appropriately chosen J. In the Floer's case, this leads to the isomorphism between $H^*(L, \mathbb{Z}_2)$ and $HF^*(L : P)$ which gives rise to the proof of Arnold's conjecture

$$\#(L \cap \phi(L)) \geq \text{SB } (L : \mathbb{Z}_2)$$

for any Hamiltonian diffeomorphism ϕ such that L meets $\phi(L)$ transversely. The way how this ismorphism was established is through a much stronger statement that the moduli space of the Floer complex is diffeomorphic to the corresponding one of the Morse complex, *when ϕ is sufficiently C^1-close and when $\omega|_{\pi_2(P,L)} \equiv 0$*. It turns out that this kind of equivalence between the Morse theory and its *quantization* holds in a much more general context (see [Fu1,2] and [FO1,2] for details), which leads to an isomorphism between these two in the sense of A^∞-category of Fukaya [Fu1,2].

However in the general monotone case, this equivalence fails due to the presence of holomorphic discs or spheres: Even when the isotopy Φ is C^1-close to identity, the images of Floer's trajectories may not be "thin". In general, as Φ approaches to the constant isotopy, each Floer's trajectory will be decomposed into the "thin" parts and "thick" parts. What is really going on in this limit as Φ converges to identity, had not been clear until the author organized this degeneration and discovered the spectral sequence relating $H^*(L, \mathbb{Z}_2)$ and $HF^*(L : P)$. This also allows us to be able to define what should be the relative version of the quantum cohomology of the pair (L, P) (see Section 10 for some remarks on this), and to prove the equivalence between the Floer cohomology and the quantum cohomology in this relative context [FO2]. Rigorous mathematical proofs of a similar equivalence problem for the non-relative case have been announced by Piunikhin-Salamon-Schwarz [PSS] and Ruan-Tian [RT2] independently. Previously this latter equivalence proof was outlined by Pinuikhin [P] in a heuristic level. We would like to note that before the author's work [O9] (and Fukaya's work [Fu2] in a certain level), the question what would be the analogy of this equivalence problem in the relative case of the pairs (L, P), has not been addressed in the literature at all. For example, there has not been even the definition of the relative version of the quantum cohomology of (L, P) until the author indicated the existence of such a definition in [O9], which we will explain in Section 10.

The main goal in this section is to describe the spectral sequence in the Floer cohomology which was introduced in [O9] and to explain several consequences of this spectral sequence.

Theorem 9.1. *Suppose that $L \subset (P, \omega)$ is monotone with $\Sigma_L \geqq 2$ and suppose that $\phi = \{\phi_t\}_{0 \leqq t \leqq 1}$ is C^1-close to id. Then there exists a differential d_F on $H_L := H^*(L, \mathbb{Z}_2)$ which preserves the filtration*

$$H_L = F^0 H_L \supset F^1 H_L \supset \cdots \supset F^n H_L \supset \{0\}$$
$$F^p H_L := \bigoplus_{0 \leqq j \leqq n-p} H^j(L, \mathbb{Z}_2), \quad 0 \leqq p \leqq n$$

and the spectral sequence associated to the filtered complex (H_L, d_F) collapses at the n-th term and converges to the Floer cohomology $HF_L := HF^*(L : P)$. Furthermore if we denote by $P(t : H_L)$ and $P(t : FH_L)$ the Poincaré series of H_L and FH_L (as filtered modules) respectively, they satisfy the identity

$$P(t : H_L) = \sum_{l=1}^{[\frac{n+1}{\Sigma_L}]} (1 + t^{l\Sigma_L - 1}) Q_l(t) + P(t : FH_L) \tag{9.1}$$

where Q_l are polynomials of non-negative coefficients (and so of degree less than equal to $n - l\Sigma_L + 1$).

Furthermore, we will prove in the next section that if we write the polynomials

$$Q_l(t) = \sum_{j=1}^{n-l\Sigma_L+1} a_j^\ell t^j$$

then we have

$$a_j^\ell = a_{n-\ell\Sigma_L+1-j}^\ell. \tag{9.2}$$

By applying (9.1) to $L \subset \mathbb{C}^n$, we immediately obtain the following result on the obstruction to monotone Lagrangian embeddings into $\mathbb{C}^n$ in terms of the Poincare series of the underlying manifold L. This is the first result that relates the underlying topology of the manifold of L to the obstruction to the existence of Lagrangian embeddings of L in a *systematic* way.

Theorem 9.2. *Suppose that L allows a monotone Lagrangian embedding into $\mathbb{C}^n$. Then its Poincare polynomial must satisfy*

$$P(t : H_L) = \sum_{l=1}^{[\frac{n+1}{\Sigma_L}]} (1 + t^{l\Sigma_L - 1}) Q_l(t). \tag{9.3}$$

Now we state several corollaries of (9.2) and (9.3) which were proven in [O9] by different arguments in the final step of the proofs. We refer to [O9] for their proofs.

Corollary 9.3. [O9] *For any compact monotone Lagrangian embedding $L \subset \mathbb{C}^n$, we have*

$$1 \le \Sigma_L \le n.$$

This corollary strengthens an earlier result by Polterovich [Po2] and the inequality here is optimal (see [O9] for a further discussion).

Corollary 9.4. [O9] *Let $L \subset \mathbb{C}^n$ be a compact embedded Lagrangian torus that is monotone. Then we have*

$$\Sigma_L = 2$$

provided $n \le 24$.

The proof of Theorem 9.1 will occupy the rest of this section. Throughout this section, we will fix Darboux neighborhoods

$$V \subset \overline{V} \subset U$$

of L as in the previous section. We also denote by J_g the canonical almost complex structure on T^*L induced from a Riemannian metric g on L. We define

$$j_g := \{J \in j \mid J \equiv J_g \quad \text{on } U\}.$$

We also fix a Morse function f and define in U

$$L_k := \text{Graph } (\frac{1}{k} df) \subset V, \quad k \in \mathbb{N},$$

where we identify U as an open neighborhood of L in T^*L. As in Section 5, for each k we choose a Hamiltonian isotopy $\Phi_k = \{\phi_{k,t}\}$ so that

$$|\Phi_k - id| < \epsilon_3, \quad \phi_{k,1}(L) = L_k \tag{9.4}$$

where Φ_k is the Hamiltonian isotopy corresponding to $\frac{1}{k} f$ which is supported in V as in Section 5. Note that (9.4) can be always achieved by choosing f so that $|f''|$ is sufficiently small. Now we choose $J_k = \{J_{k,t}\}$ which are defined by

$$J_{k,t} := (\phi_{k,t})_* J_0 (\phi_{k,t})_*^{-1}. \tag{9.5}$$

Note that

$$\phi_k \to id, \quad J_k \to J_0$$

as $k \to \infty$ in C^∞-topology. By Theorem 5.4 and Remark 5.5, if all the isolated trajectories in $\mathcal{M}_J(L, \Phi : P)$ were contained in U for (J_k, Φ_k) for sufficiently large k, then we would have obtained

$$HF^*(L : P) \cong H^*(L, \mathbb{Z}_2)$$

in which case, the spectral sequence will be trivial. Suppose the contrary. Then there exists a sequence, by re-numbering k's if necessary,

$$u_k \in \mathcal{M}_{J_k}(L, \Phi_k : P) \quad \text{with Index } u_k = 1 \quad \text{and Image } u_k \not\subset U.$$

We may assume by choosing a subsequence that $u_k \in \mathcal{M}_{J_k}(L, \Phi_k : P)$ connect the same pair of points $x, y \in \text{Crit } (f)$, i.e.,

$$u_k \in \mathcal{M}_{J_k}(x, y, \Phi_k : P).$$

Since Image $u_k \not\subset U$, there exists some $\delta > 0$ and a sequence (τ_k, t_k) such that

$$\mathrm{dist}(L, u_k(\tau_k, t_k)) \geqq \delta > 0$$

for all k. Therefore by Proposition 5.1 (5.4), we have

$$\int u_k^*\omega \geqq \frac{1}{2}A. \tag{9.6}$$

To continue our proof, the following proposition will be crucial.

Proposition 9.5. *Let L_0, L_1 be two arbitrary Lagrangian submanifolds in P such that L_0 intersects L_1 transversely. And let u, u' be maps from $[0,1] \times [0,1]$ to P with*

$$\begin{aligned} &u(\tau, 0) \subset L_0\ ,\ \ u(\tau, 1) \subset L_1 \\ &u'(\tau, 0) \subset L_1\ ,\quad u'(\tau, 1) \subset L_0 \quad \textit{and} \\ &u(0, t) \equiv u'(0, t) \equiv y\ ,\quad u(1, t) \equiv u'(1, t) \equiv x \end{aligned}$$

for $x, y \in L_0 \cap L_1$. Assume that

$$u(\tau, 1) \equiv u'(\tau, 0)$$

and let $w : [0,1] \times [0,2] \to (P, L)$ be the map obtained by gluing u and u' along $u(\tau, 1) \equiv u'(\tau, 0)$. Then we have

$$\mu_L(w) = \mu_u(x, y) - \mu_{u'}(x, y) \tag{9.7}$$

Proof: See the proof of Proposition 4.9 in [O9].

Q.E.D

Now note that for sufficiently large k we can obtain a disk $w : (D^2, \partial D^2) \to (P, L)$ with *positive* symplectic area by gluing u_k to a thin strip between L_k and L_0 connecting y and x. Therefore by applying the monotonicity of L and Proposition 9.5, we get

$$0 < \mu_L(w) = \mu_{u_k}(x, y) - (\text{ index } d^2 f(x) - \text{ index } d^2 f(y)\,) \leq 1 + n \tag{9.8}$$

because $\mu_{u_k}(x, y) = 1$ and

$$-n \leq \text{index } d^2 f(x) - \text{ index } d^2 f(y) \leq n$$

where $n =$ dim L (See the proof of Proposition 4.9 in [O9]).

From this analysis of thick trajectories, it is easy to see that the coboundary operator $\delta : \mathcal{C}^* \to \mathcal{C}^*$ has the form

$$\delta = \delta_0 + \delta'$$

where δ_0 comes from the coboundary operator associated to the local Floer complex defined as in Theorem 5.4 and δ' is the contribution of big trajectories. To analyse δ' further, we start with (9.8) which becomes

$$0 < \mu_L(w) = 1 - (\text{index } d^2 f(x) - \text{index } d^2 f(y)) \leqq n + 1. \tag{9.9}$$

It follows from this that

$$\text{index } d^2 f(x) - \text{index } d^2 f(y) = 1 - \mu_L(w)$$

and

$$\mu_L(w) = l\Sigma \quad \text{for} \quad 1 \leqq l \leqq [\frac{n+1}{\Sigma}].$$

Therefore each big trajectory maps C^* to $C^{*-l\Sigma+1}$ for some $1 \leqq l \leqq [\frac{n+1}{\Sigma}]$ and so δ' is decomposed into

$$\delta' = \partial_1 + \cdots + \partial_{[\frac{n+1}{\Sigma}]} \tag{9.10}$$

where $\partial_l : C^* \to C^{*-l\Sigma+1}$ is the map induced from the trajectories connecting critical points of f with indices $*$ and $* - l\Sigma + 1$ (for $l\Sigma - 1 \leqq * \leqq n$). Recall that δ_0 maps C^* to C^{*+1}. In other words, with respect to the grading on C^* given by the Morse index of f, δ_0 has degree $+1$ and ∂_l has degree $-\ell\Sigma + 1$. Since we assume $\Sigma \geqq 2$, ∂_l has degree less than or equal to -1. By comparing degrees of each summand of the equation

$$0 = \delta \circ \delta = (\delta_0 + \delta')^2,$$

we immediately obtain $\delta_0 \circ \delta_0 = 0$ and

$$\delta_0 \circ \delta' + \delta' \circ \delta_0 = 0 \tag{9.11}$$

$$\delta' \circ \delta' = 0. \tag{9.12}$$

Now (9.11) implies that δ' descends to $H(\mathcal{C}^*, \delta_0) = H^*(L, \mathbb{Z}_2)$ and (9.12) implies that it defines a differential there. We denote this differential by $d_F : H_L \to H_L$. Since each component ∂_l of δ' in (9.10) has negative degree, it obviously preserves the decreasing filtration

$$H_L = F^0 H \supset F^1 H \supset \cdots \supset F^n H \supset \{0\}$$

where

$$F^pH := \bigoplus_{0 \leqq j \leqq n-p} H^j(L, \mathbb{Z}_2), \quad 0 \leqq p \leqq n.$$

At this stage, we now apply the general theorem on spectral sequences (see e.g., [Theorem 2.1, McC]), which we briefly recall below. We recall the general construction of the spectral sequence associated to the filtered complex (A, d) (see [§2.2.2, McC]). We should, however, completely ignore the grading in our construction. We define

$$\begin{aligned}
Z_r^p &= \text{elements in } F^pA \text{ which are boundaries in } F^{p+r}A \\
&= F^pA \cap d^{-1}(F^{p+r}A) \\
B_r^p &= \text{elements in } F^pA \text{ form the image of } d \text{ from } F^{p-r}A \\
&= F^pA \cap d(F^{p-r}A) \\
Z_\infty^p &= \ker d \cap F^pA \\
B_\infty^p &= \mathrm{Im}\ d \cap F^pA.
\end{aligned}$$

These induce a tower of submodules of A,

$$B_0^p \subset B_1^p \subset \cdots \subset B_l^p \subset \cdots \subset B_\infty^p \subset Z_\infty^p \subset \cdots \subset Z_l^p \subset \cdots \subset Z_1^p \subset Z_0^p.$$

We define

$$E_r^p = Z_r^p/(Z_{r+1}^{p+1} + B_{r-1}^p)$$

and define $\eta_r^p : Z_r^p \to E_r^p$ to be the canonical projection with $\ker \eta_r^p = Z_{r-1}^{p+1} + B_r^p$. Then the differential, as a mapping $d : Z_r^p \to Z_r^{p+r}$ induces a homomorphism, d_r, so that the following diagram commutes

Figure 4.

and so $d_r \circ d_r = 0$. Then the sequence (E_r, d_r) is the desired spectral sequence, which has the following basic properies:

1. $H(E_r, d_r) \cong E_{r+1}$

2. $E^p_0 = F^p A/F^{p+1} A$
3. $E^p_\infty \cong F^p H(A, d)/F^{p+1} H(A, d)$.

Now we specialize this general construction to our case (H_L, d_F). Since $\partial_l : \mathcal{C}^* \to \mathcal{C}^{*-l\Sigma+1}$ has degree $-\ell\Sigma + 1 \leqq -1$, it follows from (9.11) that each ∂_l commutes δ_0 and so descends to a map $[\partial_l] : H_L \to H_L$ which has degree $-\Sigma + 1$ and hence d_F can be written as in the form

$$d_F = [\partial_1] + [\partial_2] + \cdots + [\partial_{[\frac{n+1}{\Sigma}]}]. \tag{9.13}$$

From this, it is easy to check that

$$\begin{aligned} &Z_0 = Z_1 = \cdots = Z_{\Sigma-1} \supset Z_\Sigma = Z_{\Sigma+1} = \cdots Z_{2\Sigma-1} \supset \cdots \supset Z_\infty \\ &B_0 = B_1 = \cdots = B_{\Sigma-1} \subset B_\Sigma = B_{\Sigma+1} \cdots = B_{2\Sigma-1} \subset \cdots \subset B_\infty. \end{aligned}$$

Therefore we have

$$\begin{aligned} E_0 = E_1 = \cdots = E_{\Sigma-1}, &\quad d_0 = d_1 = \cdots = d_{\Sigma-1} = 0 \\ E_\Sigma = E_{\Sigma+1} = \cdots = E_{2\Sigma-1}, &\quad d_{\Sigma+1} = \cdots = d_{2\Sigma-1} = 0 \\ \vdots \qquad &\qquad \vdots \\ E_{[\frac{n+1}{\Sigma}]\Sigma} = E_{[\frac{n+1}{\Sigma}]+1} = \cdots = E_n, &\quad d_{[\frac{n+1}{\Sigma}]+1} = \cdots = d_n. \end{aligned}$$

We note that each (E_r, d_r) inherits a filtration induced from H_L. We denote its associated Poincaré polynomials by $P(t : E_r)$, i.e,

$$P(t : E_r) = \sum_{k=0}^{n} \dim\, (F^k E_r/F^{k+1} E_r) t^{n-k}. \tag{9.14}$$

Now we are ready to prove (9.1) by induction applied to $(E_{l\Sigma}, d_{l\Sigma})$ over $0 \leqq l \leqq [\frac{n+1}{\Sigma}]$. For $l = 0$, we have $d_0 = 0$ and so $E_0 = E_\Sigma \cong H_L$. Recall that $d_\Sigma : E_\Sigma \to E_\Sigma$ has (filtration) degree $-\Sigma + 1$. Therefore, there exists a polynomial $Q_l(t)$ with $0 \leqq Q_l(t) \leqq P(t : E_\Sigma) = P(t : H_L)$ such that

$$P(t : H_L) = (1 + t^{\Sigma-1}) Q_1(t) + P(t : H(E_\Sigma)).$$

We recall that $E_{2\Sigma} \cong H(E_\Sigma)$ and that $d_{2\Sigma} : E_{2\Sigma} \to E_{2\Sigma}$ has degree $-2\Sigma + 1$. Now we repeat the same reasoning as above to $(E_{2\Sigma}, d_{2\Sigma})$ and to $(E_{l\Sigma}, d_{l\Sigma})$ for other l's and derive the identity

$$P(t : H_L) = \sum_{l=1}^{[\frac{n+1}{\Sigma}]} (1 + t^{l\Sigma-1}) Q_l(t) + P(t : E_{n+1}) \tag{9.15}$$

for some $Q_l \geq 0$. However, since we have already proven in the first part of proof that

$$E_{n+1} \cong E_\infty \cong HF_L = HF^*(L:P),$$

(9.15) now finishes the proof of Theorem 9.1.

§10. Relative quantum cohomology

We have defined the Floer cohomology $HF^*(L:P)$ as an invariant of the pair (L,P) in the previous sections. The definition involves choices of Hamiltonian isotopies Φ and we have defined $HF^*(L:P)$ to be any $HF^*_J(L,\Phi:P)$ for a generic (J,Φ) which however are isomorphic through canonical isomorphisms between them. Although this way of defining the invariant from its definition gives rise to a scheme of proving the celebrated Arnold's conjecture, it would have been much nicer for defining the invariant itself, if one could define it *directly working with the pair* (P,L) *without involving perturbations of* L. Note that the definition of the quantum cohomology $QH^*(P)$ turned out to be exactly such one in the nonrelative case. It turns out that the analysis in Section 9 of the Floer complex $\mathcal{M}_J(L,\Phi:P)$ when Φ is close to the identity suggests a way of giving such a construction of the invariant by studying the degeneration of the complex $\mathcal{M}_J(L,\Phi:P)$ as $\Phi \to id$. Floer's proof of Theorem 4.5 proves that the Floer complex $\mathcal{M}_J(L,\Phi:P)$ degenerates to the Morse complex on L if one choose Φ's converging to the identity and J's in an appropriate way as described in Section 5. In this sense, the *Morse cohomology* (see [Sc] for a detailed exposition) will be such a construction in the case when $\omega|_{\pi_2(P,L)} \equiv 0$. The reason why such a degeneration works is that all the elements in $\mathcal{M}_J(L,\Phi:P)$ will become "thin" i.e., the images of them will be contained in a Darboux neighborhood of L as $\Phi \to id$, provided $\omega|_{\pi_2(P,L)} \equiv 0$ or more precisely provided that there exist no J-holomorphic discs with boundary on L or no J-holomorhphic spheres.

However when there exist J-holomorphic spheres or J-holomorphic discs, it is possible that there exist "thick" trajectories, i.e, ones that are not contained in any Darboux neighborhood of L even when Φ is close to the identity. Then the limit configurations of elements in $\mathcal{M}_J(L,\Phi:P)$, as $\Phi \to id$, will be a union of lines intersecting with J-holomorphic discs and spheres. This heuristic argument indeed gives rise to a way of defining the above invariant directly from the pair (L,P), which leads us to what should be *the relative version of the quantum cohomology* which we will denote by $QH^*(L:P)$.

In this section, we will give the definition of the quantum cohomology $QH^*(L:P)$ and also state a theorem of the equivalence between $QH^*(L:P)$ and the Floer cohomoloyg $HF^*(L : P)$: One can prove that the Floer complex $\mathcal{M}_J(L, \Phi_\epsilon : P)$ is strata-wise diffeomorphic to the *quantum complex* $\mathcal{Q}_J(L, f : P)$ when Φ_ϵ is appropriately associated to the function f as in Section 5. We will describe the quantum complex $\mathcal{Q}_J(L, f : P)$ below in more detail. The full analytic details of this equivalence will appear in [FO2] where we will prove a theorem of more general equivalence between the *Floer A^∞-category* and the *quantum A^∞-category* in the sense of Fukaya [Fu1,2]. We would like to emphasize that these cohomologies of the pair (P, L) are not isomorphic to the *Morse cohomology* of L in general even in terms of their module structures, while for the nonrelative case these are isomorphic to the Morse cohomology of L (and so to the singular cohomology of L), as far as the module structure is concerned.

We first consider the space of configurations, denoted by $T(k, \vec{\ell})$ for $\vec{\ell} = (\ell_1, \cdots, \ell_k)$, which is pictured as follows,

Figure 5.

i.e, it consists of two outer edges, one left and the other right, $k-1$ inner edges and k unit discs attached to the edges at $\pm 1 \in \partial D^2$ as in the figure. The left outer edge is parametrized by $(-\infty, 0]$, the right is by $[0, \infty)$ and the inner edges are by $[0, \ell_j]$ for $j = 1, \cdots, k$. For each given $T(k, \ell)$, we consider the space $\mathcal{Q}_{k,\vec{\ell}:J_0}(L : P)$ of maps $u : T(k, \vec{\ell}) \to P$ such that

(1) Each edge solves the equation $\dot{x} = -\text{grad } f$,

(2) The $-\infty$ in the left outer edge maps to a point in Crit f and the ∞ in the rigth outer edge maps to a point in Crit f,

(3) On the disc parts, they satisfy the equaiton

$$\begin{cases} \overline{\partial}_{J_0} w = g, \qquad g \in \mathcal{G} \\ w(\partial D^2) \subset L \end{cases}$$

where $\mathcal{G}$ is the set of non-homogeneous terms introduced as in [Gr] or [Ru], to which we refer for its precise definition. We denote

$$\mathcal{Q}_{k:(J_0,g)}(L:P) = \bigcup_{\vec{\ell}} \mathcal{Q}_{k,\vec{\ell}:(J_0,g)}(L:P) \quad \text{and} \quad \mathcal{Q}_{(J_0,g)}(L:P) = \bigcup_{k} \mathcal{Q}_{k:(J_0,g)}(L:P). \tag{10.3}$$

For each given pair $(x_+, x_-) \subset \mathrm{Crit}\, f$ and an element $A \in \pi_2(P, L)$, we now consider the subset of $\mathcal{Q}_{(J_0,g)}(L:P)$

$$\begin{aligned} \mathcal{Q}^A_{(J_0,g)}(x_+, x_-) = \{u \in & \mathcal{Q}_{(J_0,g)}(L:P) \mid u(-\infty) = x_-, u(\infty) = x_+ \\ & \text{and } [w_1] + [w_2] + \cdots + [w_k] = A \ \text{ in } \pi_2(P, L)\} \end{aligned} \tag{10.4}$$

and $\mathcal{Q}_{(J_0,g)}(x_+, x_-) = \cup_A \mathcal{Q}^A_{(J_0,g)}(x_+, x_-)$. Furthermore note that there is a natural $\mathbb{R}$-action on each disc part of the map and so we mod out $\mathcal{Q}^A_{(J_0,g)}(x_+, x_-)$ by these $\mathbb{R}$-actions and denote the quotient by $\widehat{\mathcal{Q}}^A_{(J_0,g)}(x_+, x_-)$.

$\widehat{\mathcal{Q}}_{(J_0,g)}(x_+, x_-)$ will be the analogoue to $\widehat{\mathcal{M}}_J(x_+, x_-)$ in the Floer complex. We form the union

$$\widehat{\mathcal{Q}}_{(J_0,g)}(x_+, x_-) = \bigcup_A \widehat{\mathcal{Q}}^A_{(J_0,g)}(x_+, x_-).$$

and call $\widehat{\mathcal{Q}}_{(J_0,g)}(L:P)$ the *quantum complex* associated to $(L, J_0 : P)$. The following is the main structure theorem of this quantum complex.

Proposition 10.1. *For each $f \in C^\infty(L)$, a Morse fuction, there exist dense subsets of $j_0 \subset j_\omega$ and of $\mathcal{G}$ respectively such that $\widehat{\mathcal{Q}}^A_{(J_0,g)}(x_+, x_-)$ becomes a smooth manifold of dimension*

$$\mu_f(x_-) - \mu_f(x_+) - 1 + \mu_L(A), \tag{10.5}$$

where μ_f is the Morse index and μ_L is the Maslov index.

We note that when certain positivity assumptions (like monotonicity) for the Maslov index of J_0-holomorphic discs are assumed, elements with $A = 0$ are exactly the connecting trajectories of $-\mathrm{grad}\, f$ from x_- to x_+. Now we define the number

$$q(x_+, x_-) := \#(\text{zero dimensional components of } \mathcal{Q}_{(J_0,g)}(x_+, x_-)). \tag{10.6}$$

Definition 10.2. We define for each $f \in C^\infty_M(L)$, the set of Morse functions

$$C^*(L, f) = \text{the } \mathbb{Z}_2\text{-free module over Crit } (f).$$

Assume that

(1) The triple (L, J_0, g) satisfies the properties stated in Proposition 10.1,

(2) The number $q(x, y)$ is finite for each $(x, y) \in \text{Crit } (f)$,

(3) The integers $q(x, y)$ satisfy

$$\sum_{y \in \text{ Crit } (f)} q(x, y)q(y, z) = 0 \mod 2$$

for all $x, z \in \text{Crit } (f)$. We then define the *quantum cohomology* of the quadruple $(L, f, J_0 : P)$ by

$$QH^*_{(J_0,g)}(L, f : P) := \text{Ker } \delta / \text{Im } \delta$$

where $\delta : C^*(L, f) \to C^*(L, f)$ is the operator defined by

$$\delta y = \sum_{x \in \text{Crit } (f)} q(x, y)x.$$

It turns out that the hypotheses in this defintion can be verified if (L, P) is a monotone pair with $\Sigma_L \geq 2$ as in the case of the Floer cohomology. Furthermore one can also prove the invariance property of $QH^*_{(J_0,g)}(L, f : P)$ under the generic change of (J_0, g, f) by a similar cobordism argument as in the case of Floer cohomology. The complete details of these properties will appear elsewhere. Now we are ready to define the quantum cohomolgy $QH^*(L : P)$

Definition 10.3 We define $QH^*(L : P)$ to be $QH^*_{(J_0,g)}(L, f : P)$ for any generic (J_0, g, f).

One of the nice things about the quantum cohomology is that some form of the *Poincare duality* is quite obvious to see and hence *a forsteri* so that of the Floer cohomology by the equivalence theorem below, Theorem 10.5. Before mentioning what we mean by the Poincare duality, we can again introduce a spectral sequence in the quantum cohomolgy using the quantum complex above. The defintion is exactly parallel to the one in the case of the Floer cohomology explained in Section 9, which is in fact more straghtforward than the latter case: From (10.5), it follows as in Section 9 that the coboundary operator $\delta : C^*(L, f) \to C^*(L, f)$ can be decomposed into

$$\delta = \delta_0 + \delta'$$

where δ_0 comes from the coboundary operator associated to the Morse-Witten complex of f on L and δ' is the contribution from ones with $A \neq 0$. After then exactly the same construction as in Section 9 gives rise to a spectral sequence similar to that in Section 9. This will also give a decomposition of the Poincare polynomial $P(t : H_L)$

$$P(t : H_L) = \sum_{\ell=1}^{[\frac{n+1}{\Sigma_L}]} (1 + t^{\ell\Sigma_L - 1}) R_\ell(t) + P(t : QH_L) \tag{10.7}$$

where R_ℓ are polynomials of nonnegative coefficients

$$R_\ell(t) = \sum_{j=1}^{n-\ell\Sigma_L+1} b_j^\ell t^j.$$

The following version of Poincare duality is easy to prove.

Proposition 10.4. *The above coefficients b_j^ℓ satisfy the identity*

$$b_j^\ell = b_{n-\ell\Sigma_L+1-j}^\ell \tag{10.8}$$

for all j and ℓ.

Proof. We prove this by induction on ℓ. We will just give the proof for $\ell = 1$ and leave the rest for readers. First note that

$$b_j^1 = \text{rank } \partial_1|_{H^j(L,f)} : H^j(L, f) \to H^{j-\Sigma_L+1}(L, f).$$

Furthermore there exist canonical isomorphisms between $H^j(L, f)$'s for different f's and the space of J-holomorphic discs are independent of the choice of f. Therefore if we replace f by $-f$, it is easy to see from the geometry of the configurations

$$\begin{aligned} b_j^1 &= \text{rank } \ \partial_1|_{H^j(L,f)} : H^j(L, f) \to H^{j-\Sigma_L+1}(L, f). \\ &= \text{rank } \ \partial_1|_{H^{n-\Sigma_L+1-j}(L,f)} = b_{n-\Sigma_L+1-j}^1 \end{aligned}$$

which finishes the proof.

Q.E.D.

As we indicated in the beginning of this section, one can prove that there exists a natural isomorphism between $QH^*(L : P)$ and $HF^*(L : P)$. In fact, a much stronger equivalence theorem can be proved in the level of Fukaya's A^∞-category but we restrict to the present cohomology level in this survey. The above isomorphism is an immediate consequence of the following analytical theorem [FO2] which is a version of gluing theorem that has been also used in other contexts (see [F2], [RT1], [MS] and [FO1]).

Theorem 10.5. **[FO2]** *For each given $J_0 \in j_0$ and $f \in C^\infty_M(L)$, there exists $J \in \mathcal{J}_\omega(L : P), \Phi^\epsilon$ and $g \in \mathcal{G}$ such that there exists a natural gluing diffeomorphism between $\mathcal{Q}^A_{(J_0,g)}(x,y)$ and $\mathcal{M}^A_J(x,y) \subset \mathcal{M}_J(L, \Phi^\epsilon : P)$, which induces an isomorphism between the Floer complex and the quantum complex in the chain level.*

This theorem immediately gives an isomorphism between the two spectral sequences we introduced for the Floer and the quantum cohomology and in particular identifies the coefficients a^ℓ_j's and b^ℓ_j's. Therefore we have proved from (10.8) as we promised in Section 9. A complete proof of this theorem will be given in [FO2] in a stronger form of the A^∞-category of Fukaya.

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