THIRD ANSWER.

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Contents

This is an answer to question 4.

In the questions posted on March 14, question 4 concerns only the case of moduli space of pseudo-holomorphic curve of genus 0 with one marked point and the homology class is primitive. So there is no bubble. If the question is only on this particular case it seems to us that there is nothing more to reply than what we wrote on March 21. (Surjectivity, injectivity, smoothness etc. that is mentioned in March 23's post is an immediate consequence of implicit function theorem, that is certainly a standard result in this case.) On the other hand, in the post on March 23, 'gluing' is mentioned. (Line 7 of the paragraph starting Q4.) This is contradictory. So we gave up replying the question word by word but explain the construction of Kuranishi structure on the moduli space of pseudo-holomorphic curve in general.

Our construction of Kuranishi charts does not use Fredholm theory at infinity.

We do not understand what means 'slicing', the word that appeared in the post on March 23.

There is a well-established technique to find the moduli space as a manifold with boundary in certain situation. It was used by Donaldson in gauge theory (in his first paper [D1] to show that 1 instanton moduli of ASD connection on 4 manifold M with $b_+(2) = 0$ has M as a boundary.) In this method we take some parameter (that is the degree of concentration of the curvature in the case of ASD equation and the parameter T in the situation of section 1 below). We consider the submanifold where that parameter T is large, say T_0. We throw away everything where $T > T_0$. Then the part $T = T_0$ becomes the boundary of the 'moduli space' we obtain. It was more detailed in a book by Freed and Uhlenbeck [FU] in the gauge theory case. Abouzaid used this techniqe in his paper [Ab] about exotic sphere in T^*S^n, including the case of corners. At least as far as the results in [FOn1] are concerned we can use this technique since we need to study moduli space of virtual dimension 0 and 1 only to prove all the results in [FOn1]. In other words we can use something like Theorem 1.10 for large and fixed T, but does not need to estimate the T derivative or study the bahavior of the moduli space at $T = \infty$. The reason is as follows. In case we consider codimension 2 or higher corner, then since the virtual dimension of the moduli space is 1 or 0, the restriction to that corner has negative virtual dimension. So after generic multivalued perturbation the zero set on the corner becomes empty. So all we need is to extend multivalued perturbation. (The C^0 extention is enough for this purpose.) For codimension 1 boundary and the case of moduli space of virtual dimension 1, after generic perturbation we have isolated zero of the perturbed moduli space. So, for large T_0, Theorem 1.10 or its analogue implies that the zero on the 'boundary $T = T_0$' corresponds one to one to the zero at the actual boundary ($T = \infty$). So we do not need to see carefully what happens in a neighborhood of the set $T = \infty$. (All we need is to extend this given perturbation at $T = T_0$ to the inside.) This argument is good enough to establish all the results in [FOn1].

As we mentioned explicitly in [FOn1, page 978 line 13] our argument there, in analytic points, is basically the same as [MS]. (Let us remark however the proof of 'surjectivity' that is written in [FOn1, Section 14] is slightly different from one in [MS].) So the novelity of [FOn1] does *not* lie in the analytic point but in the general strategy, that is

(1) To define some general notion of 'spaces' that contain various moduli spaces of pseudo-holomorphic curves as examples and work out transversality issue in that abstract setting,
(2) Use multivalued abstract perturbation, that we call multisection.

When we go beyond that and prove results such as those we had proved in [FOOO], we need to study the moduli spaces of higher virtual dimension and study chain level intersection theory. In that case we are not sure whether the above mentioned technique is enough. (It may work. But we did not think enough about it.) It is not the way we had taken in [FOOO].

Our method in [FOOO] was using exponential decay estimate ([FOOO, Lemma A1.59]) and use $s = 1/T$ as the coordinate on the normal direction to the stratum to define smooth coordinate of the Kuranishi structure. We refer [FOOO, subsection A1.4] and [FOOO, subection 7.1.2] where this construction is written.

Below, we provide more details of the way how to use alternating method to construct smooth chart at infinity following the argument in [FOOO, subsection A1.4].

1. A simple case

1.1. **Setting.** We will describe the general case in Section 2. To simplify the notation and clarify the main analytic point of the proof we prove the case where we glue holomorphic maps from two stable bordered Riemann surfaces to (X, L) in this section.

Let Σ_i be a bordered Riemann surface with one end. ($i = 1, 2$.) We identify their ends as follows.

$$\begin{aligned} \Sigma_1 &= K_1 \cup ((-5T, \infty) \times [0,1]), \\ \Sigma_2 &= ((-\infty, 5T) \times [0,1]) \cup K_2. \end{aligned} \tag{1.1}$$

Here K_i are compact and $\pm\infty$ are the ends. We put

$$\Sigma_T = K_1 \cup ((-5T, 5T) \times [0,1]) \cup K_2. \tag{1.2}$$

We use τ for the coordinate of the factors $(-5T, \infty)$, $(-\infty, 5T)$, or $(-5T, 5T)$ and t for the coordinate of the second factor $[0, 1]$.

Let X be a symplectic manifold with compatible (or tame) almost complex structure and L be its Lagrangian submanifold.

Let

$$u_i : (\Sigma_i, \partial\Sigma_i) \to (X, L), \qquad i = 1, 2$$

be pseudo-holomorphic maps of finite energy. Then, by the removable singularity theorem that is now standard, we have asymptotic value

$$\lim_{\tau\to\infty} u_1(\tau, t) \in L \tag{1.3}$$

and

$$\lim_{\tau\to-\infty} u_2(\tau, t) \in L. \tag{1.4}$$

The limits (1.3) and (1.4) are independent of t.

We assume that the limit (1.3) coincides with (1.4) and denote it by $p_0 \in L$.

We fix a coordinate of X and of L in a neighborhood of p_0. So a trivialization of the tangent bundle TX and TL in a neighborhood of p_0 is fixed. Hereafter we assume the following:

$$\mathrm{Diam}(u_1([-5T, \infty) \times [0, 1])) \le \epsilon_1, \qquad \mathrm{Diam}(u_2((-\infty, 5T] \times [0, 1])) \le \epsilon_1. \tag{1.5}$$

The maps u_i determine homology classes $\beta_i = [u_i] \in H_2(X, L)$.

We take $K_i^{\rm obst}$ a compact subset of the interior of K_i and take

$$E_i \subset \Gamma(K_i^{\rm obst}; u_i^*TX \otimes \Lambda^{0,1}) \tag{1.6}$$

a finite dimensional linear subspace consisting of smooth sections supported in $K_i^{\rm obst}$.

For simplicity we also fix a complex structure of the source Σ_i. The version where it can move will be discussed later. We also assume that Σ_i equipped with marked points $\vec z_i$ is stable. The process to add marked points to stabilize it will be discussed later also. Let

$$D_{u_i}\overline{\partial} : L^2_{m+1,\delta}((\Sigma_i, \partial\Sigma_i); u_i^*TX, u_i^*TL) \to L^2_{m,\delta}(\Sigma_i; u_i^*TX \otimes \Lambda^{01}) \tag{1.7}$$

be the linearization of the Cauchy-Riemann equation. Here we define the weighted Sobolev space we use as follows.

Definition 1.1. ([FOOO, Section 7.1.3])[1] Let $L^2_{m+1,loc}((\Sigma_i, \partial\Sigma_i); u_i^*TX; u_i^*TL)$ be the set of the sections s of u_i^*TX which is locally of L^2_{m+1}-class, (Namely its differential up to order $m + 1$ is of L^2 class. Here m is sufficiently large, say larger than 10.) We also assume $s(z) \in u_i^*TL$ for $z \in \partial\Sigma_i$.

[1] In [FOOO] L^p_1 space is used in stead of L^2_m space.

The weighted Sobovel space $L^2_{m+1,\delta}((\Sigma_i,\partial\Sigma_i);u_i^*TX,u_i^*TL)$ is the set of all pairs (s,v) of elements s of $L^2_{m+1,loc}((\Sigma_i,\partial\Sigma_i);u_i^*TX;u_i^*TL)$ and $v\in T_{p_0}L$, (here $p_0\in L$ is the point (1.3) or (1.4)) such that

$$\sum_{k=0}^{m+1}\int_{\Sigma_i\setminus K_i} e^{\delta|\tau\pm 5T|}|\nabla^k(s-\mathrm{Pal}(v))|^2<\infty, \tag{1.8}$$

where $\mathrm{Pal}:T_{p_0}X\to T_{u_i(\tau,t)}X$ is defined by the trivialization we fixed right after (1.4). (Here $\pm$ is $+$ for $i=1$ and $-$ for $i=2$.) The norm is defined as the sum of (1.8), the norm of v and the L^2_{m+1} norm of s on K_i. (See (1.26).)

$L^2_{m,\delta}(\Sigma_i;u_i^*TX\otimes\Lambda^{01})$ is defined similarly without boundary condition and with out v. (See (1.28).)

When we define $D_{u_i}\overline{\partial}$ we forget v component and use s only.

Remark 1.2. The positive number δ is chosen as follows. (1.3) and a standard estimate implies that there exists $\delta_1>0$ such that

$$\left|\frac{d}{d\tau}u_i\right|_{C^k}(\tau,t)<C_ke^{-\delta_1|\tau|}, \tag{1.9}$$

for any k. We choose δ smaller than $\delta_1/10$.

(1.9) implies

$$(D_{u_i}\overline{\partial})(\mathrm{Pal}(v))<C_ke^{-\delta_1|\tau|/10}.$$

Therefore (1.7) is defined and bounded.

It is a standard fact that (1.7) is Fredholm.

We work under the following assumption.

Assumption 1.3.

$$D_{u_i}\overline{\partial}:L^2_{m+1,\delta}((\Sigma_i,\partial\Sigma_i);u_i^*TX,u_i^*TL)\to L^2_{m,\delta}(\Sigma_i;u_i^*TX\otimes\Lambda^{01})/E_i \tag{1.10}$$

is surjective. Moreover the following (1.12) holds. Let $(D_{u_i}\overline{\partial})^{-1}(E_i)$ be the kernel of (1.10). We define

$$\mathrm{Dev}_{i,\infty}:L^2_{m+1,\delta}((\Sigma_i,\partial\Sigma_i);u_i^*TX,u_i^*TL)\to T_{p_0}L \tag{1.11}$$

by

$$\mathrm{Dev}_{i,\infty}(s,v)=v.$$

Then

$$\mathrm{Dev}_{1,\infty}-\mathrm{Dev}_{2,\infty}:(D_{u_1}\overline{\partial})^{-1}(E_1)\oplus(D_{u_2}\overline{\partial})^{-1}(E_2)\to T_{p_0}L \tag{1.12}$$

is surjective.

Let us start stating the result. Let

$$u':(\Sigma_T,\partial\Sigma_T)\to(X,L) \tag{1.13}$$

be a smooth map. We consider the following condition depending $\epsilon>0$.

Condition 1.4. (1) $u'|_{K_i}$ is ϵ-close to $u_i|_{K_i}$ in C^1 sense.
(2) The diameter of $u'([-5T,5T]\times[0,1])$ is smaller than ϵ.

We take ϵ_2 sufficiently small compared to the 'injectivity radius' of X so that the next definition makes sense. [2] For u' satisfying Condition 1.4 for $\epsilon < \epsilon_2$:

$$I_{u'} : E_i \to \Gamma(\Sigma_T; (u')^*TX \otimes \Lambda^{01})$$

is the complex linear part of the parallel translation along the short geodesic (between $u_i(z)$ and $u'(z)$. Here $z \in K_i^{\rm obst}$). We put

$$E_i(u') = I_{u'}(E_i). \tag{1.14}$$

The equation we study is

$$\overline{\partial} u' \equiv 0, \qquad \text{mod } E_1(u') \oplus E_2(u'). \tag{1.15}$$

Remark 1.5. In the actual construction of Kuranishi structure, we take several u_i's and take E_i's for each of them. Then in place of $E_1(u') \oplus E_2(u')$ we take sum of finitely many of them. Here we simplify the notation. There is no difference between the proof of Theorem 1.10 and the corresonding result in case we take several such u_i's and E_i's. See [Fu2, pages 4-5] and Section 2.

Theorem 1.10 describes all the solutions of (1.15). To state this precisely we need a bit more notations.

We consider the following condition for $u'_i : (\Sigma_i, \partial\Sigma_i) \to (X, L)$.

Condition 1.6.
(1) $u'_i|_{K_i}$ is ϵ-close to $u_i|_{K_i}$ in C^1 sense.
(2) The diameter of $u'_1([-5T, \infty) \times [0, 1])$, (resp. $u'_2((-\infty, 5T]) \times [0, 1])$) is smaller than ϵ.

Then we define

$$I_{u'_i} : E_i \to \Gamma(\Sigma_i; (u'_i)^*TX \otimes \Lambda^{01})$$

by using parallel transport in the same was as $I_{u'_T}$. (This makes sense if u'_i satisfies Condtion 1.6 for $\epsilon < \epsilon_2$.) We put

$$E_i(u'_i) = I_{u'_i}(E_i). \tag{1.16}$$

So we can define an equation

$$\overline{\partial} u'_i \equiv 0, \qquad \text{mod } E_i(u'_i). \tag{1.17}$$

Definition 1.7. The set of solutions of equation (1.17) with finite energy and satisfying Condition 1.6 for $\epsilon = \epsilon_2$ is denoted by $\mathcal{M}^{E_i}((\Sigma_i, \vec z_i); \beta_i)_{\epsilon_2}$. Here β_i is the homology class of u_i.

Remark 1.8. In the usual story of pseudo-holomorphic curve, we identify u_i and u'_i if there exists a biholomorphic map $v : (\Sigma_i, \vec z_i) \to (\Sigma_i, \vec z_i)$ such that $u'_i = u_i \circ v$. In our situation where Σ_i has no sphere or disk bubble and has nontrivial boundary with at least one boundary marked points (that is $\tau = \pm\infty$), such v is necessary the identity map. Namely Σ_i has no nontrivial automorphism.

[2] More precisely we assume that

$$\{(x, y) \in X \times X \mid d(x, y) < \epsilon_2\} \subset \mathrm{E}(\{(x, v) \in TX \mid |v| < \epsilon\}),$$

where $\mathrm{E} : \{(x, v) \in TX \mid |v| < \epsilon\} \to X$ is induced by an exponential map of certain connection of TX. See (1.30).

The surjectivity of (1.11), (1.12) and the implicit function theorem imply that if ϵ_2 is small then there exists a finite dimensional vector space $\tilde V_i$ and its neighborhood V_i of 0 such that

$$\mathcal{M}^{E_i}((\Sigma_i,\vec z_i);\beta_i)_{\epsilon_2} \cong V_i.$$

Since we assume that Σ_i is nonsingular the group $\mathrm{Aut}((\Sigma_i,\vec z_i),u_i)$ is trivial. (In the case when there is a sphere bubble, the automorphism group can be nontrivial. That case will be discussed later.)

For any $\rho_i \in V_i$ we denote by $u_i^{\rho_i} : (\Sigma_i,\partial\Sigma_i) \to (X,L)$ the corresponding solution of (1.17).

We have an evaluation map

$$\mathrm{ev}_{i,\infty} : \mathcal{M}^{E_i}((\Sigma_i,\vec z_i);\beta_i)_{\epsilon_2} \to L$$

that is smooth. Namely

$$\mathrm{ev}_{i,\infty}(u'_i) = \lim_{\tau\to\pm\infty} u'_i(\tau,t).$$

(Here $\pm = +$ for $i=1$ and $-$ for $i=2$.)[3] We consider the fiber product:

$$\mathcal{M}^{E_1}((\Sigma_1,\vec z_1);\beta_1)_{\epsilon_2} \times_L \mathcal{M}^{E_2}((\Sigma_2,\vec z_2);\beta_2)_{\epsilon_2}. \tag{1.18}$$

The surjectivity of (1.12) implies that this fiber product is transversal so is

$$V_1 \times_L V_2.$$

And an element of $V_1 \times_L V_2$ is written as $\rho = (\rho_1,\rho_2)$.

Definition 1.9. Let $\beta = \beta_1 + \beta_2$. We denote by $\mathcal{M}^{E_1+E_2}((\Sigma_T,\vec z);\beta)_\epsilon$ the set of solutions of (1.15) satisfying the Condition 1.4 with $\epsilon_2 = \epsilon$.

Theorem 1.10. *For each sufficiently small ϵ_3 and sufficiently large T, there exist ϵ_1,ϵ_2 and a map*

$$\mathrm{Glu}_T : \mathcal{M}^{E_1}((\Sigma_1,\vec z_1);\beta_1)_{\epsilon_2} \times_L \mathcal{M}^{E_2}((\Sigma_2,\vec z_2);\beta_2)_{\epsilon_2} \to \mathcal{M}^{E_1+E_2}((\Sigma_T,\vec z);\beta)_{\epsilon_1}$$

that is a diffeomorphism to its image. The image contains $\mathcal{M}^{E_1+E_2}((\Sigma_T,\vec z);\beta)_{\epsilon_3}$.

The result about exponential decay estimate of this map is in Subsection 1.4. (Theorem 1.34.)

1.2. **Proof of Theorem 1.10 : 1 - Bump function and weighted Sobolev norm.**

The proof of Theorem 1.10 was given in [FOOO, Section 7.1.3]. The exponential decay estimate of the solution was proved in [FOOO, Section A1.4] together with a slightly modified version of the proof of Theorem 1.10. Here we follow the proof of [FOOO, Section A1.4] and give its more detail. As mentioned there the origin of the proof is Donaldson's paper [D2], and its Bott-Morse version in [Fu1].

We first introduce certain bump functions. First let $\mathcal{A}_T \subset \Sigma_T$ and $\mathcal{B}_T \subset \Sigma_T$ be the domains defined by

$$\mathcal{A}_T = [-T-1,-T+1]\times[0,1], \qquad \mathcal{B}_T = [T-1,T+1]\times[0,1].$$

We may regard $\mathcal{A}_T,\mathcal{B}_T \subset \Sigma_i$. The third domain is

$$\mathcal{X} = [-1,1]\times[0,1] \subset \Sigma_T.$$

We may also regard $\mathcal{X} \subset \Sigma_i$.

[3]This is a consequence of the fact that u_i is pseudo-holomorphic outside a compact set and has finite energy.

Let $\chi_{\mathcal{A}}^{\leftarrow}$, $\chi_{\mathcal{A}}^{\rightarrow}$ be functions on $[-5T,5T]\times[0,1]$ such that

$$\chi_{\mathcal{A}}^{\leftarrow}(\tau,t)=\begin{cases}1 & \tau<-T-1\\ 0 & \tau>-T+1.\end{cases} \tag{1.19}$$

$$\chi_{\mathcal{A}}^{\rightarrow}=1-\chi_{\mathcal{A}}^{\leftarrow}.$$

We define

$$\chi_{\mathcal{B}}^{\leftarrow}(\tau,t)=\begin{cases}1 & \tau<T-1\\ 0 & \tau>T+1.\end{cases} \tag{1.20}$$

$$\chi_{\mathcal{B}}^{\rightarrow}=1-\chi_{\mathcal{B}}^{\leftarrow}.$$

We define

$$\chi_{\mathcal{X}}^{\leftarrow}(\tau,t)=\begin{cases}1 & \tau<-1\\ 0 & \tau>1.\end{cases} \tag{1.21}$$

$$\chi_{\mathcal{X}}^{\rightarrow}=1-\chi_{\mathcal{X}}^{\leftarrow}.$$

We extend these functions to Σ_T and Σ_i $(i=1,2)$ so that it is locally constant outside $[-5T,5T]\times[0,1]$. We denote them by the same symbol.

We next introduce weighted Sobolev norm and its local version for sections on Σ_T or Σ_i as follows.

We define $e_{i,\delta}:\Sigma_i\to[1,\infty)$ of C^∞ class as follows.

$$e_{1,\delta}(\tau,t)\begin{cases}=e^{\delta|\tau+5T|} & \text{if } \tau>1-5T\\ =1 & \text{on } K_1\\ \in[1,10] & \text{if } \tau<1-5T\end{cases} \tag{1.22}$$

$$e_{2,\delta}(\tau,t)\begin{cases}=e^{\delta|\tau-5T|} & \text{if } \tau<5T-1\\ =1 & \text{on } K_2\\ \in[1,10] & \text{if } \tau>5T-1\end{cases} \tag{1.23}$$

We also define $e_{T,\delta}:\Sigma_T\to[1,\infty)$ as follows:

$$e_{T,\delta}(\tau,t)\begin{cases}=e^{\delta|\tau-5T|} & \text{if } 1<\tau<5T-1\\ =e^{\delta|\tau+5T|} & \text{if } -1>\tau>1-5T\\ =1 & \text{on } K_1\cup K_2\\ \in[1,10] & \text{if } |\tau-5T|<1 \text{ or } |\tau+5T|<1\\ \in[e^{5T\delta}/10,e^{5T\delta}] & \text{if } |\tau|<1.\end{cases} \tag{1.24}$$

We remark that the weighted Sobolev norm we use for $L^2_{m,\delta}(\Sigma_i;u_i^*TX\otimes\Lambda^{01})$ is

$$\|s\|^2_{L^2_{m,\delta}}=\sum_{k=0}^{m}\int_{\Sigma_i}e_{i,\delta}|\nabla^k s|^2\mathrm{vol}_{\Sigma_i}. \tag{1.25}$$

For $(s,v)\in L^2_{m+1,\delta}((\Sigma_i,\partial\Sigma_i);u_i^*TX,u_i^*TL)$ we define

$$\begin{aligned}\|(s,v)\|^2_{L^2_{m+1,\delta}}=&\sum_{k=0}^{m+1}\int_{K_i}|\nabla^k s|^2\mathrm{vol}_{\Sigma_i}\\ &+\sum_{k=0}^{m+1}\int_{\Sigma_i\setminus K_i}e_{i,\delta}|\nabla^k(s-\mathrm{Pal}(v))|^2\mathrm{vol}_{\Sigma_i}+\|v\|^2.\end{aligned} \tag{1.26}$$

We next define weighted Sobolev norm for the sections on Σ_T. Let

$$s \in L^2_{m+1}((\Sigma_T, \partial\Sigma_T); u^*TX, u^*TL).$$

Since we take m large s is continuous. So $s(0,1/2) \in T_{u(0,1/2)}X \otimes \Lambda^{01}$ is well defined. There is a canonical trivialization of TX in a neighborhood of p_0 that we fixed right after (1.4). We use it to define Pal below. We put

$$\begin{aligned}\|s\|^2_{L^2_{m+1,\delta}} = &\sum_{k=0}^{m+1}\int_{K_1} |\nabla^k s|^2 \mathrm{vol}_{\Sigma_1} + \sum_{k=0}^{m+1}\int_{K_2} |\nabla^k s|^2 \mathrm{vol}_{\Sigma_2}\\ &+ \sum_{k=0}^{m+1}\int_{[-5T,5T]\times[0,1]} e_{T,\delta}|\nabla^k(s - \mathrm{Pal}(s(0,1/2)))|^2 \mathrm{vol}_{\Sigma_i}\\ &+ \|s(0,1/2)\|^2.\end{aligned} \tag{1.27}$$

For

$$s \in L^2_m((\Sigma_T, \partial\Sigma_T); u^*TX \otimes \Lambda^{01})$$

we define

$$\|s\|^2_{L^2_{m,\delta}} = \sum_{k=0}^{m}\int_{\Sigma_T} e_{T,\delta}|\nabla^k s|^2 \mathrm{vol}_{\Sigma_1}. \tag{1.28}$$

These norms were used in [FOOO, Section 7.1.3].

For a subset W of Σ_i or Σ_T we define $\|s\|_{L^2_{m,\delta}(W\subset\Sigma_i)}$, $\|s\|_{L^2_{m,\delta}(W\subset\Sigma_T)}$ by restricting the domain of the integration (1.28) or (1.27) to W.

Let $(s_j, v_j) \in L^2_{m+1,\delta}((\Sigma_i, \partial\Sigma_i); u_i^*TX, u_i^*TL)$ for $j = 1, 2$. We define the inner product among them by:

$$\begin{aligned}\langle\!\langle (s_1, v_1), (s_2, v_2)\rangle\!\rangle_{L^2_\delta} = &\int_{\Sigma_i\setminus K_i} \langle\!\langle (s_1 - \mathrm{Pal}v_1, s_2 - \mathrm{Pal}v_2\rangle\!\rangle\\ &+ \int_{K_i} \langle\!\langle s_1, s_2\rangle\!\rangle + \langle\!\langle v_1, v_2\rangle\!\rangle.\end{aligned} \tag{1.29}$$

We also use an exponential map. (The same map was used in [FOOO, pages 410-411].) We take a diffeomorphism

$$\mathrm{E} = (\mathrm{E}_1, \mathrm{E}_2) : \{(x, v) \in TX \mid |v| < \epsilon\} \to X \times X \tag{1.30}$$

to its image such that

$$\mathrm{E}_1(x, v) = x, \qquad \left.\frac{d\mathrm{E}_2(x, tv)}{dt}\right|_{t=0} = v$$

and

$$\mathrm{E}(x, v) \in L \times L, \qquad \text{for } x \in L,\ v \in T_xL.$$

Furthermore we may take it so that

$$\mathrm{E}(x, v) = (x, x + v) \tag{1.31}$$

on a neighborhood of p_0.

To find such E, we take linear connection ∇ (that may not be a Levi-Civita conneciton of a Riemannian metric) of TX such that TL is parallel with respect to ∇. We then use geodesic with respect to ∇ to define an exponential map. We then define E such that $t \mapsto \mathrm{E}_2(x, tv)$ is a geodesic with initial direction v.

Note we may take ∇ so that in a neighborhood of p_0 it coincides with the standard trivial connection with respect the coordinate we fixed. (1.31) follows.

1.3. **Proof of Theorem 1.10 : 2 - Gluing by alternating method.** Let us start with $u^\rho = (u_1^{\rho_1}, u_2^{\rho_2}) \in \mathcal{M}^{E_1}((\Sigma_1,\vec z_1);\beta_1)_{\epsilon_2} \times_L \mathcal{M}^{E_2}((\Sigma_2,\vec z_2);\beta_2)_{\epsilon_2}$. Here $\rho_i \in V_i$ and corresponding map $(\Sigma_i, \partial\Sigma_i) \to (X, L)$ is denoted by $u_i^{\rho_i}$. Let $\rho = (\rho_1, \rho_2)$. We put

$$p^\rho = \lim_{\tau\to\infty} u_1^{\rho_1}(\tau,t) = \lim_{\tau\to-\infty} u_2^{\rho_2}(\tau,t).$$

Preglueing:

Definition 1.11. We define

$$u^\rho_{T,(0)} = \begin{cases} \chi_{\mathcal{B}}^{\leftarrow}(u_1^{\rho_1} - p^\rho) + \chi_{\mathcal{A}}^{\rightarrow}(u_2^{\rho_2} - p^\rho) + p^\rho & \text{on } [-5T,5T]\times[0,1] \\ u_1^{\rho_1} & \text{on } K_1 \\ u_2^{\rho_2} & \text{on } K_2. \end{cases} \tag{1.32}$$

Note we use the coordinate of the neighborhood of p_0 to define the sum in the first line.

Step 0-3:

Lemma 1.12. *If $\delta < \delta_1/10$ then there exists $\mathfrak{e}^\rho_{i,T,(0)} \in E_i$ such that*

$$\|\overline\partial u^\rho_{T,(0)} - \mathfrak{e}^\rho_{1,T,(0)} - \mathfrak{e}^\rho_{2,T,(0)}\|_{L^2_{m,\delta}} < C_{1,m} e^{-\delta T} \tag{1.33}$$

Moreover

$$\|\mathfrak{e}^\rho_{i,T,(0)}\|_{L^2_m(K_i)} < \epsilon_{4,m}. \tag{1.34}$$

Here $\epsilon_{4,m}$ is a positive number which we may choose arbitraly small by taking V_i to be a sufficiently small neighborhood of zero in $\tilde V_i$.

Moreover $\mathfrak{e}^\rho_{i,T,(0)}$ is independent of T.

Proof. We put

$$\mathfrak{e}_{i,T,(0)} = \overline\partial u_i^\rho \in E_i.$$

Then by definition the support of $\overline\partial u^\rho_{T,(0)} - \mathfrak{e}^\rho_{1,T,(0)} - \mathfrak{e}^\rho_{2,T,(0)}$ is on $[-5T,5T]\times[0,1]$. Moreover it is estimated as (1.33). □

Step 0-4:

Definition 1.13. We put

$$\begin{aligned}\mathrm{Err}^\rho_{1,T,(0)} &= \chi_{\mathcal{X}}^{\leftarrow}(\overline\partial u^\rho_{T,(0)} - \mathfrak{e}^\rho_{1,T,(0)}),\\ \mathrm{Err}^\rho_{2,T,(0)} &= \chi_{\mathcal{X}}^{\rightarrow}(\overline\partial u^\rho_{T,(0)} - \mathfrak{e}^\rho_{2,T,(0)}).\end{aligned}$$

We regard them as elements of the weighted Sobolev spaces $L^2_{m,\delta}((\Sigma_1,\partial\Sigma_1);(u_1^\rho)^*TX\otimes\Lambda^{01})$ and $L^2_{m,\delta}((\Sigma_2,\partial\Sigma_2);(u_2^\rho)^*TX\otimes\Lambda^{01})$ respectively. (We extend them by 0 outside compact set.)

Step 1-1: We first cut $u^{\rho}_{T,(0)}$ and extend to obtain maps $\hat u^{\rho}_{i,T,(0)} : (\Sigma_i,\partial\Sigma_i) \to (X,L)$ $(i = 1,2)$ as follows. (This map is used to set the linearized operator (1.36).)

$$\hat u^{\rho}_{1,T,(0)}(z) = \begin{cases} \chi_{\mathcal B}^{\leftarrow}(\tau - T,t)u^{\rho}_{T,(0)}(\tau,t) + \chi_{\mathcal B}^{\rightarrow}(\tau - T,t)p^{\rho} & \text{if } z = (\tau,t) \in [-5T,5T]\times[0,1] \\ u^{\rho}_{T,(0)}(z) & \text{if } z \in K_1 \\ p^{\rho} & \text{if } z \in [5T,\infty)\times[0,1]. \end{cases}$$

$$\hat u^{\rho}_{2,T,(0)}(z) = \begin{cases} \chi_{\mathcal A}^{\rightarrow}(\tau + T,t)u^{\rho}_{T,(0)}(\tau,t) + \chi_{\mathcal A}^{\leftarrow}(\tau + T,t)p^{\rho} & \text{if } z = (\tau,t) \in [-5T,5T]\times[0,1] \\ u^{\rho}_{T,(0)}(z) & \text{if } z \in K_2 \\ p^{\rho} & \text{if } z \in (-\infty,-5T]\times[0,1]. \end{cases} \tag{1.35}$$

Let

$$\begin{aligned} D_{\hat u^{\rho}_{i,T,(0)}}\overline\partial : L^2_{m+1,\delta}((\Sigma_i,\partial\Sigma_i);(\hat u^{\rho}_{i,T,(0)})^*TX,(\hat u^{\rho}_{i,T,(0)})^*TL) \\ \to L^2_{m,\delta}(\Sigma_i;(\hat u^{\rho}_{i,T,(0)})^*TX\otimes\Lambda^{01}) \end{aligned} \tag{1.36}$$

be the linearization of Cauchy-Riemann equation.

Lemma 1.14. *We put $E_i = E_i(\hat u^{\rho}_{i,T,(0)})$. We have*

$$\mathrm{Im}(D_{\hat u^{\rho}_{i,T,(0)}}\overline\partial) + E_i = L^2_{m,\delta}(\Sigma_i;(\hat u^{\rho}_{i,T,(0)})^*TX\otimes\Lambda^{01}). \tag{1.37}$$

Moreover

$$Dev_{1,\infty} - Dev_{2,\infty} : (D_{\hat u^{\rho}_{1,T,(0)}}\overline\partial)^{-1}(E_1)\oplus(D_{\hat u^{\rho}_{2,T,(0)}}\overline\partial)^{-1}(E_2) \to T_{p^{\rho}}L \tag{1.38}$$

is surjective.

Proof. Since $\hat u^{\rho}_{i,T,(0)}$ is close to u_i in exponential order this is a consequence of Assumption 1.3. □

Note $E_i(u'_i)$ actually depends on u'_i. So to obtain a linearized equation of (1.15) we need to take into account of that effect. Let $\Pi_{E_i(u'_i)}$ be the projection to $E_i(u'_i)$ with respect to the L^2 norm. Namely we put

$$\Pi_{E_i(u'_i)}(A) = \sum_{a=1}^{\dim E_i} \langle\langle A, \mathbf e_{i,a}(u'_i)\rangle\rangle_{L^2(K_1)}\mathbf e_{i,a}(u'_i), \tag{1.39}$$

where $\mathbf e_{i,a}$, $a = 1,\dots,\dim E_i(u'_i)$ is an orthonormal basis of $E_i(u'_i)$ which are supported in K_i.

We put

$$(D_{u'_i}E_i)(A,v) = \frac{d}{ds}(\Pi_{E_i(\mathrm E(u'_i,sv))}(A))|_{s=0} \tag{1.40}$$

Here $v \in \Gamma((\Sigma_i,\partial\Sigma_i),(u'_i)^*TX,(u'_i)^*TL)$. (Then $\mathrm E(u'_i,sv)$ is a map $(\Sigma_i,\partial\Sigma_i) \to (X,L)$ defined in (1.30).)

Remark 1.15. We use an isomorphism

$$\Gamma(\Sigma_i;\mathrm E(u'_i,sv)^*TX\otimes\Lambda^{01}) \cong \Gamma(\Sigma_i;(u'_i)^*TX\otimes\Lambda^{01}) \tag{1.41}$$

to define the right hand side of (1.40). The map (1.41) is defined as follows. Let $z \in \Sigma_i$. We have a path $r \mapsto \mathrm{E}(u'_i(z), rsv(z))$ joining $u'_i(z)$ to $\mathrm{E}(u'_i, sv)(z)$. We use a connection ∇ such that TL is parallel to define a parallel transport along this path. Its complex linear part defines an isomorphism (1.41).

We remark the same isomorphism (1.41) is used also to define $D_{u'_i}\overline{\partial}$. Namely

$$(D_{u'_i}\overline{\partial})(v) = \frac{d}{ds}(\overline{\partial}\mathrm{E}(u'_i, sv))|_{s=0}$$

where the right hand side is defined by using (1.41).

We put

$$\Pi^{\perp}_{E_i(u'_i)}(A) = A - \Pi_{E_i(u'_i)}(A).$$

The equation (1.17) is equivalent to the following.

$$\Pi^{\perp}_{E_i(u'_i)}\overline{\partial}u'_i = 0 \tag{1.42}$$

We calculate the linearization

$$\left.\frac{\partial}{\partial s}\Pi^{\perp}_{E_i(\mathrm{E}(u'_i,sV))}\overline{\partial}\mathrm{E}(u'_i, sV))\right|_{s=0},$$

to obtain the linearized equation:

$$D_{u'_i}\overline{\partial}(V) - (D_{u'_i}E_i)(\overline{\partial}u'_i, V) \equiv 0 \mod E_i(u'_i). \tag{1.43}$$

We remark that

$$\overline{\partial}\hat{u}^{\rho}_{i,T,(0)} - \mathfrak{e}^{\rho}_{i,T,(0)}$$

is exponentially small. So we use the operator

$$V \mapsto D_{\hat{u}^{\rho}_{i,T,(0)}}\overline{\partial}(V) - (D_{\hat{u}^{\rho}_{i,T,(0)}}E_i)(\mathfrak{e}^{\rho}_{i,T,(0)}, V), \tag{1.44}$$

as an approximation of the linearlization of (1.42).

Lemma 1.16. *We put $E_i = E_i(\hat{u}^{\rho}_{i,T,(0)})$. We have*

$$\mathrm{Im}(D_{\hat{u}^{\rho}_{i,T,(0)}}\overline{\partial}-(D_{\hat{u}^{\rho}_{i,T,(0)}}E_i)(\mathfrak{e}^{\rho}_{i,T,(0)},\cdot))+E_i = L^2_{m,\delta}(\Sigma_i;(\hat{u}^{\rho}_{i,T,(0)})^*TX\otimes\Lambda^{01}). \tag{1.45}$$

Moreover

$$\begin{aligned}D\mathrm{ev}_{1,\infty}-D\mathrm{ev}_{2,\infty} : (D_{\hat{u}^{\rho}_{1,T,(0)}}\overline{\partial} - (D_{\hat{u}^{\rho}_{1,T,(0)}}E_1)(\mathfrak{e}^{\rho}_{1,T,(0)},\cdot))^{-1}(E_1)\\ \oplus (D_{\hat{u}^{\rho}_{2,T,(0)}}\overline{\partial} - (D_{\hat{u}^{\rho}_{2,T,(0)}}E_2)(\mathfrak{e}^{\rho}_{2,T,(0)},\cdot))^{-1}(E_2) \to T_{p^{\rho}}L\end{aligned} \tag{1.46}$$

is surjective.

Proof. (1.34) implies that $(D_{\hat{u}^{\rho}_{1,T,(0)}}E_1)(\mathfrak{e}^{\rho}_{1,T,(0)},\cdot)$ is small in operator norm. The lemma follows from Lemma 1.14. □

Remark 1.17. Note (1.34) is proved by taking V_i in a small neighborhood of 0 (in $\tilde{V}_i$) with respect to the C^m norm. (Note $V_i \subset \mathcal{M}^{E_i}((\Sigma_i,\vec{z}_i);\beta_i)_{\epsilon_2}$ and V_i consists of smooth maps.) However we can take V_i that is independent of m and the conclusion of Lemma 1.16 holds for m. In fact the elliptic regularity implies that if the conclusion of Lemma 1.16 holds for some m then it holds for all $m' > m$. (The inequality (1.34) holds for that particular m only. However this inequality is used to show Lemma 1.16 only.)

We consider

$$\begin{aligned}&\mathrm{Ker}(D\mathrm{ev}_{1,\infty} - D\mathrm{ev}_{2,\infty})\\ &\cap \Big((D_{\hat u^{\rho}_{1,T,(0)}}\overline{\partial} - (D_{\hat u^{\rho}_{1,T,(0)}}E_1)(\mathfrak{e}^{\rho}_{1,T,(0)},\cdot)))^{-1}(E_1)\\ &\quad\oplus (D_{\hat u^{\rho}_{2,T,(0)}}\overline{\partial} - (D_{\hat u^{\rho}_{2,T,(0)}}E_2)(\mathfrak{e}^{\rho}_{2,T,(0)},\cdot))^{-1}(E_2)\Big).\end{aligned} \tag{1.47}$$

This is a finite dimensional subspace of

$$\mathrm{Ker}(D\mathrm{ev}_{1,\infty} - D\mathrm{ev}_{2,\infty}) \cap \bigoplus_{i=1}^{2} L^2_{m+1,\delta}((\Sigma_i,\partial\Sigma_i);(\hat u^{\rho}_{i,T,(0)})^*TX,(\hat u^{\rho}_{i,T,(0)})^*TL) \tag{1.48}$$

consisting of smooth sections.

Definition 1.18. We denote by $\mathfrak{H}(E_1,E_2)$ the intersection of the L^2 orthonormal complement of (1.47) with (1.48). Here L^2 inner product is defined by (1.29).

Definition 1.19. We define $(V^{\rho}_{T,1,(1)},V^{\rho}_{T,2,(1)},\Delta p^{\rho}_{T,(1)})$ as follows.

$$\begin{aligned}(D_{\hat u^{\rho}_{i,T,(0)}}\overline{\partial})(V^{\rho}_{T,i,(1)}) - (D_{\hat u^{\rho}_{i,T,(0)}}E_i)(\mathfrak{e}^{\rho}_{i,T,(0)},V^{\rho}_{T,i,(1)})&\\ + \mathrm{Err}^{\rho}_{i,T,(0)} \in E_i(\hat u^{\rho}_{i,T,(0)}).&\end{aligned} \tag{1.49}$$

$$D\mathrm{ev}_{\infty}(V^{\rho}_{T,1,(1)}) = D\mathrm{ev}_{-\infty}(V^{\rho}_{T,2,(1)}) = \Delta p^{\rho}_{T,(1)}. \tag{1.50}$$

Moreover

$$((V^{\rho}_{T,1,(1)},\Delta p^{\rho}_{T,(1)}),(V^{\rho}_{T,2,(1)},\Delta p^{\rho}_{T,(1)})) \in \mathfrak{H}(E_1,E_2).$$

Lemma 1.16 implies that such $(V^{\rho}_{T,1,(1)},V^{\rho}_{T,2,(1)},\Delta p^{\rho}_{T,(1)})$ exists and is unique.

Lemma 1.20. *If $\delta < \delta_1/10$, then*

$$\|(V^{\rho}_{T,i,(1)},\Delta p^{\rho}_{T,(1)})\|_{L^2_{m+1,\delta}(\Sigma_i)} \le C_{2,m}e^{-\delta T}, \qquad |\Delta p^{\rho}_{T,(1)}| \le C_{2,m}e^{-\delta T}. \tag{1.51}$$

This is immediate from construction and the uniform boundedness of the right inverse of $D_{\hat u^{\rho}_{i,T,(0)}}\overline{\partial} - (D_{\hat u^{\rho}_{i,T,(0)}}E_i)(\mathfrak{e}^{\rho}_{i,T,(0)},\cdot)$.

Step 1-2: We use $(V^{\rho}_{T,1,(1)},V^{\rho}_{T,2,(1)},\Delta p^{\rho}_{T,(1)})$ to find an approximate solution $u^{\rho}_{T,(1)}$ of the next level.

Definition 1.21. We define $u^{\rho}_{T,(1)}(z)$ as follows. (Here E is as in (1.30).)

(1) If $z \in K_1$ we put

$$u^{\rho}_{T,(1)}(z) = \mathrm{E}(\hat u^{\rho}_{1,T,(0)}(z),V^{\rho}_{T,1,(1)}(z)) \tag{1.52}$$

(2) If $z \in K_2$ we put

$$u^{\rho}_{T,(1)}(z) = \mathrm{E}(\hat u^{\rho}_{2,T,(0)}(z),V^{\rho}_{T,2,(1)}(z)) \tag{1.53}$$

(3) If $z = (\tau,t) \in [-5T,5T]\times[0,1]$ we put

$$\begin{aligned}u^{\rho}_{T,(1)}(\tau,t) =& \chi_{\mathcal{B}}^{\leftarrow}(\tau,t)(V^{\rho}_{T,1,(1)}(\tau,t) - \Delta p^{\rho}_{T,(1)})\\ &+ \chi_{\mathcal{A}}^{\rightarrow}(\tau,t)(V^{\rho}_{T,2,(1)}(\tau,t) - \Delta p^{\rho}_{T,(1)}) + u^{\rho}_{T,(0)}(\tau,t) + \Delta p^{\rho}_{T,(1)}.\end{aligned} \tag{1.54}$$

We recall that on K_1 we have $\hat u^{\rho}_{1,T,(0)}(z) = u^{\rho}_{T,(0)}(z)$ and on K_2 we have $\hat u^{\rho}_{2,T,(0)}(z) = u^{\rho}_{T,(0)}(z)$.

Step 1-3: Let $0 < \mu < 1$. We fix it throughout the proof.

Lemma 1.22. *There exists δ_2 such that for any $\delta < \delta_2$, $T > T(\delta, m, \epsilon_{5,m})$ there exists $\mathfrak{e}^\rho_{i,T,(1)} \in E_i$ with the following properties.*

$$\|\overline{\partial} u^\rho_{T,(1)} - (\mathfrak{e}^\rho_{1,T,(0)} + \mathfrak{e}^\rho_{1,T,(1)}) - (\mathfrak{e}^\rho_{2,T,(0)} + \mathfrak{e}^\rho_{2,T,(1)})\|_{L^2_{m,\delta}} < C_{1,m}\mu\epsilon_{5,m}e^{-\delta T}$$

(Here $C_{1,m}$ is the constant given in Lemma 1.12.) Moreover

$$\|\mathfrak{e}^\rho_{i,T,(1)}\|_{L^2_m(K_i)} < C_{3,m}e^{-\delta T}. \tag{1.55}$$

Proof. The existence of $\mathfrak{e}^\rho_{i,T,(1)}$ satisfying

$$\|\overline{\partial} u^\rho_{T,(1)} - (\mathfrak{e}^\rho_{1,T,(0)} + \mathfrak{e}^\rho_{1,T,(1)}) - (\mathfrak{e}^\rho_{2,T,(0)} + \mathfrak{e}^\rho_{2,T,(1)})\|_{L^2_{m,\delta}(K_1\cup K_2 \subset \Sigma_T)} < C_{1,m}\mu\epsilon_{5,m}e^{-\delta T}/10$$

is a consequene of the fact that (1.43) is the linearized equation of (1.42) and the estimate (1.51). More explicitly we can prove it by a routine calculation as follows. We first estimate on K_1. We have:

$$\begin{aligned}
&\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(0)}, V^\rho_{T,1,(1)})) \\
&\quad = \overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(0)}, 0)) + \int_0^1 \frac{\partial}{\partial s}\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(0)}, sV^\rho_{T,1,(1)}))ds \\
&\quad = \overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(0)}, 0)) + (D_{\hat u^\rho_{1,T,(0)}}\overline{\partial})(V^\rho_{T,1,(1)}) \\
&\qquad\qquad + \int_0^1 ds \int_0^s \frac{\partial^2}{\partial r^2}\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(0)}, rV^\rho_{T,1,(1)}))dr.
\end{aligned} \tag{1.56}$$

We remark

$$\begin{aligned}
&\left\|\int_0^1 ds \int_0^s \frac{\partial^2}{\partial r^2}\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(0)}, rV^\rho_{T,1,(1)}))dr\right\|_{L^2_m(K_1)} \\
&\le C_{3,m}\|V^\rho_{T,1,(1)}\|^2_{L^2_{m+1,\delta}} \le C_{4,m}e^{-2\delta T}.
\end{aligned} \tag{1.57}$$

We have

$$\begin{aligned}
&\Pi^\perp_{E_1(\mathrm{E}(\hat u^\rho_{1,T,(0)}, V^\rho_{T,1,(1)}))} \\
&\quad = \Pi^\perp_{E_1(\hat u^\rho_{1,T,(0)})} + \int_0^1 \frac{\partial}{\partial s}\Pi^\perp_{E_i(\mathrm{E}(\hat u^\rho_{1,T,(0)}, sV^\rho_{T,1,(1)}))}ds \\
&\quad = \Pi^\perp_{E_1(\hat u^\rho_{1,T,(0)})} - (D_{\hat u^\rho_{1,T,(0)}}E_1)(\cdot, V^\rho_{T,1,(1)}) \\
&\qquad\qquad + \int_0^1 ds \int_0^s \frac{\partial^2}{\partial r^2}\Pi^\perp_{E_1(\mathrm{E}(\hat u^\rho_{1,T,(0)}, rV^\rho_{T,1,(1)}))}dr
\end{aligned} \tag{1.58}$$

We can estimate the third term of the right hand side of (1.58) in the same way as (1.57).

On the other hand, (1.56) implies that

$$\left\|\overline{\partial}(\mathrm{E}(\hat u^\rho_{1,T,(0)}, V^\rho_{T,1,(1)})) - \mathfrak{e}^\rho_{1,T,(0)}\right\|_{L^2_m(K_1)} \le C_{6,m}e^{-\delta T}. \tag{1.59}$$

Therefore, using (1.58) and (1.51), we have

$$\begin{aligned}\Big\|&\Pi^{\perp}_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))}\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))\\ &-\Pi^{\perp}_{E_1(\hat u^{\rho}_{1,T,(0)},0)}\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))\\ &-\Pi^{\perp}_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))}(\mathfrak{e}^{\rho}_{1,T,(0)})+\Pi^{\perp}_{E_1(\hat u^{\rho}_{1,T,(0)},0)}(\mathfrak{e}^{\rho}_{1,T,(0)})\Big\|_{L^2_m(K_1)}\le C_{7,m}e^{-2\delta T}.\end{aligned} \tag{1.60}$$

Therefore using (1.58) we have:

$$\begin{aligned}&\|\Pi^{\perp}_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))}\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))\\ &-\Pi^{\perp}_{E_1(\hat u^{\rho}_{1,T,(0)},0)}\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))\\ &+(D_{\hat u^{\rho}_{1,T,(0)}}E_1)(\mathfrak{e}^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)})\|_{L^2_m(K_1)}\le C_{8,m}e^{-2\delta T}\end{aligned} \tag{1.61}$$

By (1.49) and Definition 1.13, we have:

$$\begin{aligned}&\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},0))+(D_{\hat u^{\rho}_{1,T,(0)}}\overline{\partial})(V^{\rho}_{T,1,(1)})\\ &\qquad-(D_{\hat u^{\rho}_{1,T,(0)}}E_1)(\mathfrak{e}^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)})\in E_1(\hat u^{\rho}_{1,T,(0)})\end{aligned} \tag{1.62}$$

on K_1.

(1.61) and (1.62) imply

$$\begin{aligned}&\|\Pi^{\perp}_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))}\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))\\ &-\Pi^{\perp}_{E_1(\hat u^{\rho}_{1,T,(0)},0)}\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))\\ &+\Pi^{\perp}_{E_1(\hat u^{\rho}_{1,T,(0)},0)}\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},0))\\ &+\Pi^{\perp}_{E_1(\hat u^{\rho}_{1,T,(0)},0)}(D_{\hat u^{\rho}_{1,T,(0)}}\overline{\partial})(V^{\rho}_{T,1,(1)})\|_{L^2_m(K_1)}\le C_{9,m}e^{-2\delta T}\end{aligned} \tag{1.63}$$

Combined with (1.56) and (1.57), we have

$$\begin{aligned}&\|\Pi^{\perp}_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))}(\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)})))\|_{L^2_m(K_1)}\\ &\le C_{10,m}e^{-2\delta T}\le C_{1,m}e^{-\delta T}\epsilon_{5,m}\mu/10,\end{aligned} \tag{1.64}$$

for $T>T_m$ if we choose T_m so that $C_{10,m}e^{-\delta T_m}<C_{1,m}\epsilon_{5,m}\mu/10$.

It follows from (1.59) and (1.64) that

$$\|\Pi_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))}(\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))-\mathfrak{e}^{\rho}_{1,T,(0)}\|_{L^2_m(K_1)}\le C_{11,m}e^{-\delta T}.$$

(1.55) then follows, by selecting

$$\mathfrak{e}^{\rho}_{1,T,(1)}=\Pi_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)}))}(\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(0)},V^{\rho}_{T,1,(1)})-\mathfrak{e}^{\rho}_{1,T,(0)}).$$

The estimate on K_2 is the same.

Let us estimate $\overline{\partial}u^{\rho}_{T,(1)}$ on $[-T+1,T-1]\times[0,1]$. The inequality

$$\|\overline{\partial}u^{\rho}_{T,(1)}\|_{L^2_{m,\delta}([-T+1,T-1]\times[0,1]\subset\Sigma_T)}<C_{1,m}\mu\epsilon_{5,m}e^{-\delta T}/10$$

is also a consequece of the fact that (1.43) is the linearized equation of (1.42) and the estimate (1.51). (Note the bump functions $\chi^{\leftarrow}_{\mathcal{B}}$ and $\chi^{\rightarrow}_{\mathcal{A}}$ are $\equiv 1$ there.) On $\mathcal{A}_T$ we have

$$\overline{\partial}u^{\rho}_{T,(1)}=\overline{\partial}(\chi^{\rightarrow}_{\mathcal{A}}(V^{\rho}_{T,2,(1)}-\Delta p^{\rho}_{T,(1)})+V^{\rho}_{T,1,(1)}+u^{\rho}_{T,(0)}) \tag{1.65}$$

Note

$$\|\overline{\partial}(\chi_{\mathcal{A}}^{\rightarrow}(V^{\rho}_{T,2,(1)} - \Delta p^{\rho}_{T,(1)})\|_{L^2_m(\mathcal{A}_T)} \le C_{3,m}e^{-6T\delta}\|V^{\rho}_{T,2,(1)} - \Delta p^{\rho}_{T,(1)}\|_{L^2_{m+1,\delta}(\mathcal{A}_T\subset\Sigma_2)} \le C_{12,m}e^{-7T\delta}.$$

The first inequality follows from the fact the weight function $e_{2,\delta}$ is around $e^{6T\delta}$ on $\mathcal{A}_T$. The second inequality follows from (1.51). On the other hand the weight function $e_{T,\delta}$ is around $e^{4T\delta}$ at $\mathcal{A}_T$.[4] Therefore

$$\|\overline{\partial}(\chi_{\mathcal{A}}^{\rightarrow}(V^{\rho}_{T,2,(1)} - \Delta p^{\rho}_{T,(1)}))\|_{L^2_{m,\delta}(\mathcal{A}_T\subset\Sigma_T)} \le C_{13,m}e^{-3T\delta}. \tag{1.66}$$

Note

$$\mathrm{Err}^{\rho}_{2,T,(0)} = 0$$

on $\mathcal{A}_T$. Using this in the same way as we did on K_1 we can show

$$\|\overline{\partial}(V^{\rho}_{T,1,(1)} + u^{\rho}_{T,(0)})\|_{L^2_{m,\delta}(\mathcal{A}_T\subset\Sigma_T)} \le C_{1,m}e^{-\delta T}\epsilon_{5,m}\mu/20 \tag{1.67}$$

for $T > T_m$. Therefore by taking T large we have

$$\|\overline{\partial}u^{\rho}_{T,(1)}\|_{L^2_{m,\delta}(\mathcal{A}_T\subset\Sigma_T)} < C_{1,m}\mu\epsilon_{5,m}e^{-\delta T}/10. \tag{1.68}$$

(Note the almost complex structure may not be integrable. So the almost complex structure may not be constant with respect to the flat metric we are taking in the neighborhood of p_0. However we can still deduce (1.68) from (1.67) and (1.66).)

The estimate on $\mathcal{B}_T$ and on $([-5T, -T-1]\cup[T+1, 5T])\times[0,1]$ are similar. The proof of Lemma 1.22 is complete. □

Step 1-4:

Definition 1.23. We put

$$\mathrm{Err}^{\rho}_{1,T,(1)} = \chi_{\mathcal{X}}^{\leftarrow}(\overline{\partial}u^{\rho}_{T,(1)} - (\mathfrak{e}^{\rho}_{1,T,(0)} + \mathfrak{e}^{\rho}_{1,T,(1)})),$$
$$\mathrm{Err}^{\rho}_{2,T,(1)} = \chi_{\mathcal{X}}^{\rightarrow}(\overline{\partial}u^{\rho}_{T,(1)} - (\mathfrak{e}^{\rho}_{2,T,(0)} + \mathfrak{e}^{\rho}_{2,T,(1)})).$$

We regard them as elements of the weighted Sobolev spaces $L^2_{m,\delta}(\Sigma_1;(u^{\rho}_1)^*TX\otimes\Lambda^{01})$ and $L^2_{m,\delta}(\Sigma_2;(u^{\rho}_2)^*TX\otimes\Lambda^{01})$ respectively. (We extend them by 0 outside compact set.)

We put $p^{\rho}_{(1)} = p^{\rho} + \Delta p^{\rho}_{T,(1)}$.

We now come back to the Step 2-1 and continue. In other words, we will prove the following by induction on κ.

$$\left\|(V^{\rho}_{T,i,(\kappa)}, \Delta p^{\rho}_{T,(\kappa)})\right\|_{L^2_{m+1,\delta}(\Sigma_i)} < C_{2,m}\mu^{\kappa-1}e^{-\delta T}, \tag{1.69}$$

$$\left\|\Delta p^{\rho}_{T,(\kappa)}\right\| < C_{2,m}\mu^{\kappa-1}e^{-\delta T}, \tag{1.70}$$

$$\left\|u^{\rho}_{T,(\kappa)} - u^{\rho}_{T,(0)}\right\|_{L^2_{m+1,\delta}(\Sigma_T)} < C_{14,m}e^{-\delta T}, \tag{1.71}$$

$$\left\|\mathrm{Err}^{\rho}_{i,T,(\kappa)}\right\|_{L^2_{m,\delta}(\Sigma_i)} < C_{1,m}\epsilon_{5,m}\mu^{\kappa}e^{-\delta T}, \tag{1.72}$$

$$\left\|\mathfrak{e}^{\rho}_{i,T,(\kappa)}\right\|_{L^2_m(K_i^{\mathrm{obst}})} < C_{15,m}\mu^{\kappa-1}e^{-\delta T}, \quad \text{for } \kappa \ge 1. \tag{1.73}$$

[4] This drop of the weight is the main part of the idea. It was used in [FOOO, page 414]. See [FOOO, Figure 7.1.6].

Remark 1.24. The left hand side of (1.71) is defined as follows. We define $\mathfrak{u}^{\rho}_{T,(\kappa)}$ by $u^{\rho}_{T,(\kappa)} = \mathrm{E}(u^{\rho}_{T,(\kappa-1)}, \mathfrak{u}^{\rho}_{T,(\kappa)})$. Then the left hand side of (1.71) is

$$\|\mathfrak{u}^{\rho}_{T,(\kappa)}\|_{L^2_{m+1,\delta}((\Sigma_T,\partial\Sigma_T);(u^{\rho}_{T,(\kappa-1)})^*TX,(u^{\rho}_{T,(\kappa-1)})^*TL)}.$$

More precisely the claim we will prove is: for any $\epsilon_{5,m}$ we can choose T_m so that (1.69) and (1.70) imply (1.72) and (1.73) for given $T > T_m$, and we can choose $\epsilon_{5,m}$ so that (1.72) and (1.73) for κ implies (1.69) and (1.70) for $\kappa + 1$. (It is easy to see that (1.69) and (1.70) imply (1.71).)

Below we describe Steps κ-1,…,κ-4.

Step κ-1:

We first cut $u^{\rho}_{T,(\kappa-1)}$ and extend to obtain maps $\hat u^{\rho}_{i,T,(\kappa-1)} : (\Sigma_i, \partial\Sigma_i) \to (X, L)$ $(i = 1, 2)$ as follows.

$$\hat u^{\rho}_{1,T,(\kappa-1)}(z) = \begin{cases} \chi^{\leftarrow}_{\mathcal B}(\tau - T, t)u^{\rho}_{T,(\kappa-1)}(\tau, t) + \chi^{\rightarrow}_{\mathcal B}(\tau - T, t)p^{\rho}_{(\kappa-1)} & \text{if } z = (\tau, t) \in [-5T, 5T] \times [0, 1] \\ u^{\rho}_{T,(\kappa-1)}(z) & \text{if } z \in K_1 \\ p^{\rho}_{T,(\kappa-1)} & \text{if } z \in [5T, \infty) \times [0, 1]. \end{cases}$$

$$\hat u^{\rho}_{2,T,(\kappa-1)}(z) = \begin{cases} \chi^{\rightarrow}_{\mathcal A}(\tau + T, t)u^{\rho}_{T,(\kappa-1)}(\tau, t) + \chi^{\leftarrow}_{\mathcal A}(\tau + T, t)p^{\rho}_{(\kappa-1)} & \text{if } z = (\tau, t) \in [-5T, 5T] \times [0, 1] \\ u^{\rho}_{T,(\kappa-1)}(z) & \text{if } z \in K_2 \\ p^{\rho}_{T,(\kappa-1)} & \text{if } z \in (-\infty, -5T] \times [0, 1]. \end{cases} \tag{1.74}$$

Let

$$\begin{aligned} D_{\hat u^{\rho}_{i,T,(\kappa-1)}}\overline{\partial} : L^2_{m+1,\delta}((\Sigma_i, \partial\Sigma_i);(\hat u^{\rho}_{i,T,(\kappa-1)})^*TX, (\hat u^{\rho}_{i,T,(\kappa-1)})^*TL) \\ \to L^2_{m,\delta}(\Sigma_i; (\hat u^{\rho}_{i,T,(\kappa-1)})^*TX \otimes \Lambda^{01}). \end{aligned} \tag{1.75}$$

Lemma 1.25. *We have*

$$\mathrm{Im}(D_{\hat u^{\rho}_{i,T,(\kappa-1)}}\overline{\partial}) + E_i = L^2_{m,\delta}(\Sigma_i; (\hat u^{\rho}_{i,T,(\kappa-1)})^*TX \otimes \Lambda^{01}). \tag{1.76}$$

Moreover

$$D\mathrm{ev}_{1,\infty} - D\mathrm{ev}_{2,\infty} : (D_{\hat u^{\rho}_{1,T,(0)}}\overline{\partial})^{-1}(E_1) \oplus (D_{\hat u^{\rho}_{2,T,(0)}}\overline{\partial})^{-1}(E_2) \to T_{p^{\rho}_{T,(\kappa-1)}}L \tag{1.77}$$

is surjective.

Proof. Since $\hat u^{\rho}_{i,T,(\kappa-1)}$ is close to u_i in exponential order this is a consequence of Assumption 1.3. □

We denote

$$(\mathfrak{se})^{\rho}_{i,T,(\kappa-1)} = \sum_{a=0}^{\kappa-1} \mathfrak{e}^{\rho}_{i,T,(a)}. \tag{1.78}$$

Lemma 1.26. *We have*

$$\begin{aligned} &\mathrm{Im}(D_{\hat u^{\rho}_{i,T,(\kappa-1)}}\overline{\partial} - (D_{\hat u^{\rho}_{i,T,(\kappa-1)}}E_i)((\mathfrak{se})^{\rho}_{i,T,(\kappa-1)}, \cdot)) + E_i \\ &= L^2_{m,\delta}(\Sigma_i; (\hat u^{\rho}_{i,T,(\kappa-1)})^*TX \otimes \Lambda^{01}). \end{aligned} \tag{1.79}$$

Moreover

$$\begin{aligned}&\mathrm{Dev}_{1,\infty} - \mathrm{Dev}_{2,\infty}\\ &:(D_{\hat u^\rho_{1,T,(\kappa-1)}}\overline{\partial} - (D_{\hat u^\rho_{1,T,(\kappa-1)}}E_1)((\mathfrak{se})^\rho_{1,T,(\kappa-1)},\cdot)))^{-1}(E_1)\\ &\oplus (D_{\hat u^\rho_{2,T,(\kappa-1)}}\overline{\partial} - (D_{\hat u^\rho_{2,T,(\kappa-1)}}E_2)((\mathfrak{se})^\rho_{2,T,(\kappa-1)},\cdot))^{-1}(E_2) \to T_{p^\rho_{T,(\kappa-1)}}L\end{aligned} \tag{1.80}$$

is surjective.

Proof.

$$\left\|\sum_{a=0}^{\kappa-1}\mathfrak{e}^\rho_{i,T,(a)}\right\|_{L^2_m(K_i)} < \epsilon_{4,m} + C_{15,m}\frac{e^{-\delta T}}{1-\mu}. \tag{1.81}$$

imply that $(D_{\hat u^\rho_{1,T,(0)}}E_1)(\mathfrak{e}^\rho_{1,T,(0)},\cdot)$ is small in operator norm. The lemma follows from Lemma 1.25. □

Note Remark 1.17 still applies to Lemma 1.26.

Definition 1.27. We define $(V^\rho_{T,1,(\kappa)}, V^\rho_{T,2,(\kappa)}, \Delta p^\rho_{T,(\kappa)})$ as follows.

$$\begin{aligned}D_{\hat u^\rho_{i,T,(\kappa-1)}}(V^\rho_{T,i,(\kappa)}) - (D_{\hat u^\rho_{i,T,(\kappa-1)}}E_i)((\mathfrak{se})^\rho_{i,T,(\kappa-1)}, V^\rho_{T,i,(\kappa)})&\\ + \mathrm{Err}^\rho_{i,T,(\kappa-1)} \in E_i(\hat u^\rho_{i,T,(\kappa-1)}).&\end{aligned} \tag{1.82}$$

$$\mathrm{Dev}_{1,\infty}(V^\rho_{T,1,(\kappa)}) = \mathrm{Dev}_{2,\infty}(V^\rho_{T,2,(\kappa)}) = \Delta p^\rho_{T,(\kappa)}. \tag{1.83}$$

We also require

$$((V^\rho_{T,1,(\kappa)}, \Delta p^\rho_{T,(\kappa)}), (V^\rho_{T,2,(\kappa)}, \Delta p^\rho_{T,(\kappa)})) \in \mathfrak{H}(E_1, E_2). \tag{1.84}$$

Lemma 1.26 implies that such $(V^\rho_{T,1,(\kappa)}, V^\rho_{T,2,(\kappa)}, \Delta p^\rho_{T,(\kappa)})$ exists and is unique.

Remark 1.28. Note in (1.84) we use the same space $\mathfrak{H}(E_1, E_2)$ as in Definition 1.19. We may use the orthonormal complement of

$$\mathrm{Ker}(\mathrm{Dev}_{1,\infty} - \mathrm{Dev}_{2,\infty}) \cap \bigoplus_{i=1}^{2}(D_{\hat u^\rho_{i,T,(\kappa-1)}}\overline{\partial} - (D_{\hat u^\rho_{i,T,(\kappa-1)}}E_i)((\mathfrak{se})^\rho_{i,T,(\kappa-1)},\cdot))^{-1}(E_i)$$

instead. The reason why we use the same space as one in Definition 1.19 here, is that then a calculation we need to do for the exponential decay estimate of T derivative becomes a bit shorter. Since $\hat u^\rho_{i,T,(\kappa)}$ is sufficiently close to $\hat u^\rho_{i,T,(0)}$, the unique existence of $(V^\rho_{T,1,(\kappa)}, V^\rho_{T,2,(\kappa)}, \Delta p^\rho_{T,(\kappa)})$ satisfying (1.82) - (1.84) holds by (1.81).

Lemma 1.29. *If $\delta < \delta_1/10$, and $T > T(\delta, m)$ then*

$$\begin{aligned}&\|(V_{T,i,(\kappa)}, \Delta p^\rho_{T,(\kappa)})\|_{L^2_{m+1,\delta}(\Sigma_i)} \le C_{2,m}\mu^{\kappa-1}e^{-\delta T},\\ &|\Delta p^\rho_{T,(\kappa)}| \le C_{2,m}\mu^{\kappa-1}e^{-\delta T}.\end{aligned} \tag{1.85}$$

Proof. This follows from uniform boundednes of the inverse of (1.79) together with $\kappa - 1$ version of Lemma 1.22. (That is Lemma 1.31.) □

This lemma implies (1.69) and (1.70).

Step κ-2: We use $(V^\rho_{T,1,(\kappa)}, V^\rho_{T,2,(\kappa)}, \Delta p^\rho_{T,(\kappa)})$ to find an approximate solution $u^\rho_{T,(\kappa)}$ of the next level.

Definition 1.30. We define $u^\rho_{T,(\kappa)}(z)$ as follows.

(1) If $z \in K_1$ we put

$$u^\rho_{T,(\kappa)}(z) = \mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)}(z), V^\rho_{T,1,(\kappa)}(z)). \tag{1.86}$$

(2) If $z \in K_2$ we put

$$u^\rho_{T,(\kappa)}(z) = \mathrm{E}(\hat u^\rho_{2,T,(\kappa-1)}(z), V^\rho_{T,2,(\kappa)}(z)). \tag{1.87}$$

(3) If $z = (\tau,t) \in [-5T,5T]\times[0,1]$ we put

$$\begin{aligned} u^\rho_{T,(\kappa)}(\tau,t) =& \chi^{\leftarrow}_{\mathcal B}(\tau,t)(V^\rho_{T,1,(\kappa)}(\tau,t) - \Delta p^\rho_{T,(\kappa)}) \\ &+ \chi^{\rightarrow}_{\mathcal A}(\tau,t)(V^\rho_{T,2,(\kappa)}(\tau,t) - \Delta p^\rho_{T,(\kappa)}) \\ &+ u^\rho_{T,(\kappa-1)}(\tau,t) + \Delta p^\rho_{T,(\kappa)}. \end{aligned} \tag{1.88}$$

We remark that on K_1 we have $\hat u^\rho_{1,T,(\kappa-1)}(z) = u^\rho_{T,(\kappa-1)}(z)$ and on K_2 we have $\hat u^\rho_{2,T,(\kappa-1)}(z) = u^\rho_{T,(\kappa-1)}(z)$.

(1.71) is immediate from the definition and (1.69) and (1.70), since $0 < \mu < 1$.

Step κ-3:

Lemma 1.31. *For each $\epsilon_5 > 0$ we have the following. If $\delta < \delta_2$ and $T > T(\delta, m, \epsilon_5)$ then there exists $\mathfrak e^\rho_{i,T,(\kappa)} \in E_i$ such that*

$$\left\Vert \overline\partial u^\rho_{T,(\kappa)} - \sum_{a=0}^{\kappa} \mathfrak e^\rho_{1,T,(a)} - \sum_{a=0}^{\kappa} \mathfrak e^\rho_{2,T,(a)} \right\Vert_{L^2_{m,\delta}} < C_{1,m}\mu^\kappa \epsilon_5 e^{-\delta T}.$$

(Here $C_{1,m}$ is as in Lemma 1.12.) Moreover

$$\Vert \mathfrak e^\rho_{i,T,(\kappa)} \Vert_{L^2_m(K_i)} < C_{15,m}\mu^{\kappa-1}e^{-\delta T}. \tag{1.89}$$

Proof. The proof is similar to the proof of Lemma 1.22 and proceed as follows.

We have:

$$\begin{aligned} &\overline\partial(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)}, V^\rho_{T,1,(\kappa)})) \\ &\quad= \overline\partial(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)}, 0)) + \int_0^1 \frac{\partial}{\partial s}\overline\partial(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)}, sV^\rho_{T,1,(\kappa)}))ds \\ &\quad= \overline\partial(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)}, 0)) + (D_{\hat u^\rho_{1,T,(\kappa-1)}}\overline\partial)(V^\rho_{T,1,(\kappa)}) \\ &\qquad\qquad + \int_0^1 ds \int_0^s \frac{\partial^2}{\partial r^2}\overline\partial(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)}, rV^\rho_{T,1,(\kappa)}))dr. \end{aligned} \tag{1.90}$$

We remark

$$\begin{aligned} &\left\Vert \int_0^1 ds \int_0^s \frac{\partial^2}{\partial r^2}\overline\partial(\mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)}, rV^\rho_{T,1,(\kappa)}))dr \right\Vert_{L^2_m(K_1)} \\ &\le C_{4,m}\Vert V^\rho_{T,1,(\kappa)}\Vert^2_{L^2_{m+1,\delta}} \le C_{5,m}e^{-2\delta T}\mu^{2(\kappa-1)}. \end{aligned} \tag{1.91}$$

We have

$$\begin{aligned}
&\Pi^{\perp}_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(\kappa-1)},V^{\rho}_{T,1,(\kappa)}))}\\
&\qquad = \Pi^{\perp}_{E_1(\hat u^{\rho}_{1,T,(\kappa-1)})} + \int_0^1 \frac{\partial}{\partial s}\Pi^{\perp}_{E_i(\mathrm{E}(\hat u^{\rho}_{1,T,(\kappa-1)},sV^{\rho}_{T,1,(\kappa)}))}ds\\
&\qquad = \Pi^{\perp}_{E_1(\hat u^{\rho}_{1,T,(\kappa-1)})} - (D_{\hat u^{\rho}_{1,T,(\kappa-1)}}E_1)(\cdot,V^{\rho}_{T,1,(\kappa)})\\
&\qquad\qquad + \int_0^1 ds\int_0^s \frac{\partial^2}{\partial r^2}\Pi^{\perp}_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(\kappa-1)},rV^{\rho}_{T,1,(\kappa)}))}dr.
\end{aligned} \tag{1.92}$$

We can estimate the third term of the right hand side of (1.92) in the same way as (1.91).

On the other hand, (1.90) implies that

$$\left\|\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(\kappa-1)},V^{\rho}_{T,1,(\kappa)})) - \mathfrak{se}^{\rho}_{1,T,(\kappa-1)}\right\|_{L^2_m(K_1)} \le C_{6,m}e^{-\delta T}\mu^{\kappa-1}. \tag{1.93}$$

Therefore

$$\begin{aligned}
&\|\Pi^{\perp}_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(\kappa-1)},V^{\rho}_{T,1,(\kappa)}))}\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(\kappa-1)},V^{\rho}_{T,1,(\kappa)}))\\
&- \Pi^{\perp}_{E_1(\hat u^{\rho}_{1,T,(\kappa-1)},0)}\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(\kappa-1)},V^{\rho}_{T,1,(\kappa)}))\\
&+ (D_{\hat u^{\rho}_{1,T,(\kappa-1)}}E_1)(\mathfrak{se}^{\rho}_{1,T,(\kappa-1)},V^{\rho}_{T,1,(\kappa)})\|_{L^2_m(K_1)} \le C_{7,m}e^{-2\delta T}\mu^{\kappa-1}.
\end{aligned} \tag{1.94}$$

By (1.82) we have:

$$\begin{aligned}
&\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(\kappa-1)},0)) + (D_{\hat u^{\rho}_{1,T,(\kappa-1)}}\overline{\partial})(V^{\rho}_{T,1,(\kappa)})\\
&\qquad - (D_{\hat u^{\rho}_{1,T,(\kappa-1)}}E_1)(\mathfrak{se}^{\rho}_{1,T,(\kappa-1)},V^{\rho}_{T,1,(\kappa)}) \in E_1(\hat u^{\rho}_{1,T,(\kappa-1)})
\end{aligned} \tag{1.95}$$

on K_1.

Summing up we have

$$\begin{aligned}
&\|\Pi^{\perp}_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(\kappa-1)},V^{\rho}_{T,1,(\kappa)}))}(\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(\kappa-1)},V^{\rho}_{T,1,(\kappa)})))\|_{L^2_m(K_1)}\\
&\le C_{10,m}e^{-2\delta T}\mu^{\kappa-1} \le C_{1,m}e^{-\delta T}\epsilon_{5,m}\mu^{\kappa}/10
\end{aligned} \tag{1.96}$$

for $T > T_m$.

It follows from (1.93) that

$$\|\Pi_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(\kappa-1)},V^{\rho}_{T,1,(\kappa)}))}(\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(\kappa-1)},V^{\rho}_{T,1,(\kappa)}))-\mathfrak{se}^{\rho}_{1,T,(\kappa-1)}\|_{L^2_m(K_1)} \le C_{8,m}e^{-\delta T}\mu^{\kappa-1}.$$

(1.89) then follows by putting

$$\begin{aligned}
\mathfrak{e}^{\rho}_{1,T,(\kappa)} &= \Pi_{E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(\kappa-1)},V^{\rho}_{T,1,(\kappa)}))}(\overline{\partial}(\mathrm{E}(\hat u^{\rho}_{1,T,(\kappa-1)},V^{\rho}_{T,1,(\kappa)})) - \mathfrak{se}^{\rho}_{1,T,(\kappa-1)}\\
&\in E_1(\mathrm{E}(\hat u^{\rho}_{1,T,(\kappa-1)},V^{\rho}_{T,1,(\kappa)})) \cong E_1.
\end{aligned}$$

Let us estimate $\overline{\partial}u^{\rho}_{T,(\kappa)}$ on $[-T,T]\times[0,1]$. The inequality

$$\|\overline{\partial}u^{\rho}_{T,(\kappa)}\|_{L^2_{m,\delta}([-T,T]\times[0,1]\subset\Sigma_T)} < C_{1,m}\mu^{\kappa}\epsilon_{5,m}e^{-\delta T}/10$$

is also a consequene of the fact that (1.43) is the linearized equation of (1.42) and the estimate (1.85). (Note the bump functions $\chi^{\leftarrow}_{\mathcal{B}}$ and $\chi^{\rightarrow}_{\mathcal{A}}$ are $\equiv 1$ there.) On $\mathcal{A}_T$ we have

$$\overline{\partial}u^{\rho}_{T,(\kappa)} = \overline{\partial}(\chi^{\rightarrow}_{\mathcal{A}}(V^{\rho}_{T,2,(\kappa)} - \Delta p^{\rho}_{T,(\kappa)}) + V^{\rho}_{T,1,(\kappa)} + u^{\rho}_{T,(\kappa-1)}). \tag{1.97}$$

Note

$$\begin{aligned}\|\overline{\partial}(\chi_{\mathcal{A}}^{\rightarrow}(V^{\rho}_{T,2,(\kappa)} - \Delta p^{\rho}_{T,(\kappa)}))\|_{L^2_m(\mathcal{A}_T)} &\le C_{3,m}e^{-6T\delta}\|V^{\rho}_{T,2,(\kappa)} - \Delta p^{\rho}_{T,(\kappa)}\|_{L^2_{m+1,\delta}(\mathcal{A}_T\subset\Sigma_2)} \\ &\le C_{12,m}e^{-7T\delta}\mu^{\kappa-1}.\end{aligned}$$

The first inequality follows from the fact the weight function $e_{2,\delta}$ is around $e^{6T\delta}$ on $\mathcal{A}_T$. The second inequality follows from (1.85). On the other hand the weight function $e_{T,\delta}$ is around $e^{4T\delta}$ at $\mathcal{A}_T$.[5] Therefore

$$\|\overline{\partial}(\chi_{\mathcal{A}}^{\rightarrow}(V^{\rho}_{T,2,(\kappa)} - \Delta p^{\rho}_{T,(\kappa)}))\|_{L^2_{m,\delta}(\mathcal{A}_T\subset\Sigma_T)} \le C_{13,m}e^{-3T\delta}\mu^{\kappa-1}. \tag{1.98}$$

Note

$$\mathrm{Err}^{\rho}_{2,T,(\kappa-1)} = 0$$

on $\mathcal{A}_T$. Therefore in the same way as we did on K_1 we can show

$$\|\overline{\partial}(V^{\rho}_{T,1,(\kappa)} + u^{\rho}_{T,(\kappa-1)})\|_{L^2_{m,\delta}(\mathcal{A}_T\subset\Sigma_T)} \le C_{1,m}e^{-\delta T}\epsilon_{5,m}\mu^{\kappa}/20 \tag{1.99}$$

for $T > T_m$. Therefore by taking T large we have

$$\|\overline{\partial}u^{\rho}_{T,(\kappa)}\|_{L^2_{m,\delta}(\mathcal{A}_T\subset\Sigma_T)} < C_{1,m}\mu^{\kappa}\epsilon_{5,m}e^{-\delta T}/10. \tag{1.100}$$

The estimate on $\mathcal{B}_T$ and on $([-5T,-T-1]\cup[T+1,5T])\times[0,1]$ are similar. The proof of Lemma 1.31 is complete. □

Step κ-4:

Definition 1.32. We put

$$\begin{aligned}\mathrm{Err}^{\rho}_{1,T,(\kappa)} &= \chi_{\mathcal{X}}^{\leftarrow}\left(\overline{\partial}u^{\rho}_{T,(\kappa)} - \sum_{a=0}^{\kappa}\mathfrak{e}^{\rho}_{1,T,(a)}\right), \\ \mathrm{Err}^{\rho}_{2,T,(\kappa)} &= \chi_{\mathcal{X}}^{\rightarrow}\left(\overline{\partial}u^{\rho}_{T,(\kappa)} - \sum_{a=0}^{\kappa}\mathfrak{e}^{\rho}_{2,T,(a)}\right).\end{aligned}$$

We regard them as elements of the weighted Sobolev spaces $L^2_{m,\delta}(\Sigma_1;(\hat u^{\rho}_{1,T,(\kappa)})^*TX\otimes\Lambda^{01})$ and $L^2_{m,\delta}(\Sigma_2;(\hat u^{\rho}_{2,T,(\kappa)})^*TX\otimes\Lambda^{01})$ respectively. (We extend them by 0 outside compact set.)

We put $p^{\rho}_{(\kappa)} = p^{\rho}_{(\kappa-1)} + \Delta p^{\rho}_{T,(\kappa)}$.
Lemma 1.31 implies (1.72) and (1.73).

We have thus described all the induction steps. For each fixed m there exists T_m such that if $T > T_m$ then

$$\lim_{\kappa\to\infty} u^{\rho}_{T,(\kappa)}$$

coverges in $L^2_{m+1,\delta}$ sense to the solution of (1.15). The limit is automatically of C^∞ class by elliptic reguralіty. We have thus constructed the map in Theorem 1.10. We will prove its surjectivity and injectivity in Subsection 1.5 below. Before doing so we prove an exponential decay estimate of its T derivative.

[5]This drop of the weight is the main part of the idea. It was used in [FOOO, page 414]. See [FOOO, Figure 7.1.6].

1.4. **Exponential decay of T derivatives.** We first state the result of this subsection. We recall that for T sufficiently large and $\rho = (\rho_1, \rho_2) \in V_1 \times_L V_2$ we have defined $u^{\rho}_{T,(\kappa)}$. We denote its limit by

$$u^{\rho}_{T} = \lim_{\kappa \to \infty} u^{\rho}_{T,(\kappa)} : (\Sigma_T, \partial \Sigma_T) \to (X, L). \tag{1.101}$$

The main result of this subsection is an estimate of T and ρ derivative of this map. We prepare some notations to state the result.

We change the coordinate of Σ_i and Σ_T as follows. In the last subsection we put

$$\Sigma_1 = K_1 \cup ([-5T, \infty) \times [0, 1])$$

and use (τ, t) for the coordinate of $[-5T, \infty) \times [0, 1]$. This identification depends on T. So we rewrite it to

$$\Sigma_1 = K_1 \cup ([0, \infty) \times [0, 1])$$

and the coordinate for $[0, \infty) \times [0, 1]$ is (τ', t) where

$$\tau' = \tau + 5T. \tag{1.102}$$

Similarly we rewrite

$$\Sigma_2 = ((-\infty, 5T] \times [0, 1]) \cup K_2$$

to

$$\Sigma_2 = ((-\infty, 0] \times [0, 1]) \cup K_2$$

and use the coordinate (τ'', t) where

$$\tau'' = \tau - 5T. \tag{1.103}$$

We may use either (τ', t) or (τ'', t) as the coordinate of $\Sigma_T \setminus (K_1 \cup K_2)$.

Let S be a positive number. We have $K_i \subset \Sigma_T$. We put

$$\begin{aligned} K_1^{+S} &= K_1 \cup ([0, S] \times [0, 1]) \subset \Sigma_T, \\ K_2^{+S} &= ([-S, 0] \times [0, 1]) \cup K_2 \subset \Sigma_T. \end{aligned} \tag{1.104}$$

Here the inclusion $K_1 \cup ([0, S] \times [0, 1]) \subset \Sigma_T$ is by using the coordinate τ' and the inclusion $([-S, 0] \times [0, 1]) \cup K_2 \subset \Sigma_T$ is by using the coordinate τ''.

We may also regard $K_i^{+S} \subset \Sigma_i$. Note that the spaces K_i^{+S} are independent of T, as far as $10T > S$.

We restrict the map u^{ρ}_T to K_i^{+S}. We thus obtain a map

$$\text{Glures}_{i,S} : [T_m, \infty) \times V_1 \times_L V_2 \to \text{Map}_{L^2_{m+1}}((K_i^{+S}, K_i^{+S} \cap \partial \Sigma_i), (X, L))$$

by

$$\begin{cases} \text{Glures}_{1,S}(T, \rho)(x) &= u^{\rho}_T(x) \qquad x \in K_1 \\ \text{Glures}_{1,S}(T, \rho)(\tau', t) &= u^{\rho}_T(\tau', t) = u^{\rho}_T(\tau + 5T, t) \end{cases} \tag{1.105}$$

$$\begin{cases} \text{Glures}_{2,S}(T, \rho)(x) &= u^{\rho}_T(x) \qquad x \in K_2 \\ \text{Glures}_{2,S}(T, \rho)(\tau'', t) &= u^{\rho}_T(\tau'', t) = u^{\rho}_T(\tau - 5T, t) \end{cases} \tag{1.106}$$

Here $\text{Map}_{L^2_{m+1}}((K_i^{+S}, K_i^{+S} \cap \partial \Sigma_i), (X, L))$ is the space of maps of L^2_{m+1} class (m is sufficiently large, say $m > 10$.) It has a structure of Hilbert manifold in an obvious way. This Hilbert manifold is independent of T. So we can define T derivative of a family of elements of $\text{Map}_{L^2_{m+1}}((K_i^{+S}, K_i^{+S} \cap \partial \Sigma_i), (X, L))$ parametrized by T.

Remark 1.33. The domain and the target of the map $\text{Glures}_{i,S}$ depend on m. However its image actually is in the set of smooth maps. Also none of the constructions of u^ρ_T depends on m. (The proof of the convergence of (1.101) depends on m. So the number T_m depends on m.) Therefore the map $\text{Glures}_{i,S}$ is *independent* of m on the intersection of the domains. Namely the map $\text{Glures}_{i,S}$ constructed by using $L^2_{m_1}$ norm coincides with the map $\text{Glures}_{i,S}$ constructed by using $L^2_{m_2}$ norm on $[\max\{T_{m_1}, T_{m_2}\}, \infty) \times V_1 \times_L V_2$.

Theorem 1.34. *For each m and S there exist $T(m), C_{16,m,S}, \delta > 0$ such that the following holds for $T > T(m)$ and $n + \ell \le m - 10$ and $\ell > 0$.*

$$\left\| \nabla^n_\rho \frac{d^\ell}{dT^\ell} \text{Glures}_{i,S} \right\|_{L^2_{m+1-\ell}} < C_{16,m,S} e^{-\delta T}. \tag{1.107}$$

Here ∇^n_ρ is the n-th derivative in ρ direction.

Remark 1.35. Theorem 1.34 is basically equivalent to [FOOO, Lemma A1.58]. The proof below is basically the same as the one in [FOOO, page 776]. We add some more detail.

Proof. The construction of $u^\rho_{T,(\kappa)}$ was by induction on κ. We divide the inductive step of the construction of $u^\rho_{T,(\kappa+1)}$ from $u^\rho_{T,(\kappa)}$ into two.

(Part A) Start from $(V^\rho_{T,1,(\kappa)}, V^\rho_{T,2,(\kappa)}, \Delta p^\rho_{T,(\kappa)})$ and end with $\text{Err}^\rho_{1,T,(\kappa)}$ and $\text{Err}^\rho_{2,T,(\kappa)}$. This is step κ-2,κ-3,κ-4.

(Part B) Start from $\text{Err}^\rho_{1,T,(\kappa)}$ and $\text{Err}^\rho_{2,T,(\kappa)}$ and end with $(V^\rho_{T,1,(\kappa+1)}, V^\rho_{T,2,(\kappa+1)}, \Delta p^\rho_{T,(\kappa+1)})$. This is step $(\kappa+1)$-1.

We will prove the following inequality by induction on κ, under the assumption $T > T(m)$, $\ell > 0$, $n + \ell \le m - 10$.

$$\left\| \nabla^n_\rho \frac{\partial^\ell}{\partial T^\ell} (V^\rho_{T,i,(\kappa)}, \Delta p^\rho_{T,(\kappa)}) \right\|_{L^2_{m+1-\ell,\delta}(\Sigma_i)} < C_{17,m} \mu^{\kappa-1} e^{-\delta T}, \tag{1.108}$$

$$\left\| \nabla^n_\rho \frac{\partial^\ell}{\partial T^\ell} \Delta p^\rho_{T,(\kappa)} \right\| < C_{17,m} \mu^{\kappa-1} e^{-\delta T}, \tag{1.109}$$

$$\left\| \nabla^n_\rho \frac{\partial^\ell}{\partial T^\ell} u^\rho_{T,(\kappa)} \right\|_{L^2_{m+1-\ell,\delta}(K_i^{+5T+1})} < C_{18,m} e^{-\delta T}, \tag{1.110}$$

$$\left\| \nabla^n_\rho \frac{\partial^\ell}{\partial T^\ell} \text{Err}^\rho_{i,T,(\kappa)} \right\|_{L^2_{m-\ell,\delta}(\Sigma_i)} < C_{19,m} \epsilon_{6,m} \mu^\kappa e^{-\delta T}, \tag{1.111}$$

$$\left\| \nabla^n_\rho \frac{\partial^\ell}{\partial T^\ell} \mathfrak{e}^\rho_{i,T,(\kappa)} \right\|_{L^2_{m-\ell}(K_i^{\text{obst}})} < C_{19,m} \mu^{\kappa-1} e^{-\delta T}. \tag{1.112}$$

More precisely the claim we will prove is the following: For each $\epsilon_{6,m}$, we can choose $T(m)$ so that (1.108) and (1.109) imply (1.111) and (1.112) for $T > T(m)$, and we can choose $\epsilon_{6,m}$ so that (1.111) and (1.112) for κ implies (1.108) and (1.109) for $\kappa + 1$. (1.110) follows from (1.108) and (1.109).

Remark 1.36. We use L^2_{m+1} norm on K_i^{+5T+1} only in formula (1.110). Note we use coordinate (τ', t) on $K_1^{+5T+1} \setminus K_1$, and (τ'', t) on $K_2^{+5T+1} \setminus K_2$. We remark also that $\Sigma_T = K_1^{+5T+1} \cup K_2^{+5T+1}$.

Remark 1.37. Note $(V^\rho_{T,i,(\kappa)}, \Delta p^\rho_{T,(\kappa)})$ appearing in (1.108) is an element of the weighted Sobolev space $L^2_{m+1,\delta}((\Sigma_i,\partial\Sigma_i);(\hat u^\rho_{i,T,(\kappa-1)})^*TX,(\hat u^\rho_{i,T,(\kappa-1)})^*TL)$ that depends on T and ρ. To make sense of T and ρ derivatives we identify

$$\begin{aligned}&L^2_{m+1,\delta}((\Sigma_i,\partial\Sigma_i);(\hat u^\rho_{i,T,(\kappa-1)})^*TX,(\hat u^\rho_{i,T,(\kappa-1)})^*TL)\\ &\cong L^2_{m+1,\delta}((\Sigma_i,\partial\Sigma_i);u_i^*TX,u_i^*TL)\end{aligned}$$

as follows. We find V such that $\hat u^\rho_{i,T,(\kappa-1)} = \mathrm{E}(u_i,V)$. We use the parallel transport with respect to the path $r\mapsto \mathrm{E}(u_i,rV)$ and its complex linear part to define this isomorphism. The same remark applies to (1.111) and (1.112).

Remark 1.38. The left hand side of (1.108), in case $i=1$, is :

$$\begin{aligned}&\left\Vert \nabla_\rho^n \frac{\partial^\ell}{\partial T^\ell} V^\rho_{T,1,(\kappa)}\right\Vert_{L^2_{m+1-\ell}(K_1)}\\ &+\sum_{k=0}^{m+1-\ell}\int_{[0,\infty)\times[0,1]} e_{1,T}(\tau,t)\left\Vert \nabla^k_{\tau',t}\nabla^n_\rho\frac{\partial^\ell}{\partial T^\ell}\big(V^\rho_{T,i,(\kappa)} - \mathrm{Pal}(\Delta p^\rho_{T,(\kappa)})\big)\right\Vert^2 d\tau' dt.\end{aligned}$$

Note we apply Remark 1.37 to define T and ρ derivatives in the above formula.

The case $i=2$ is similar using τ'' coordinate.

(Part A) (See [FOOO, page 776 paragraph (A) and (B)].)

We assume (1.108) and (1.109).

We find that

(1)
$$\mathrm{Err}^\rho_{1,T,(\kappa)}(z) = \Pi_{E_1^\perp(\hat u^\rho_{1,T,(\kappa-1)})}\overline\partial \mathrm{E}(\hat u^\rho_{1,T,(\kappa-1)}(z), V^\rho_{T,1,(\kappa)}(z)) \tag{1.113}$$
for $z\in K_1$.

(2)
$$\begin{aligned}&\mathrm{Err}^\rho_{1,T,(\kappa)}(\tau')\\ =&(1-\chi(\tau'-5T))\overline\partial\big(\chi(\tau'-4T)(V^\rho_{T,2,(\kappa)}(\tau''+10T,t)-\Delta p^\rho_{T,(\kappa)})\\ &\qquad + V^\rho_{T,1,(\kappa)}(\tau',t)+u^\rho_{T,(\kappa-1)}(\tau',t)\big),\end{aligned}\tag{1.114}$$
for $(\tau',t)\in[0,\infty)\times[0,1]$. (Note $\tau'=\tau''+10T$.)

Here $\chi:\mathbb{R}\to[0,1]$ is a smooth function such that

$$\chi(\tau)\begin{cases} =0 & \tau<-1\\ =1 & \tau>1\\ \in[0,1] & \tau\in[-1,1].\end{cases}$$

Note in Formulas (1.108)-(1.112) the Sobolev norm in the left hand side is $L^2_{m+1-\ell,\delta}(\Sigma_i)$ etc. and is not $L^2_{m+1,\delta}(\Sigma_i)$ etc. The origin of this loss of differentiability (in the sense of Sobolev space) comes from the term $V^\rho_{T,2,(\kappa)}(\tau''+10T)$. In fact we have

$$\frac{\partial}{\partial T}V^\rho_{T_1,2,(\kappa)}(\tau''+10T) = 10\frac{\partial}{\partial\tau''}V^\rho_{T_1,2,(\kappa)}(\tau''+10T),$$

for a fixed T_1. Hence $\partial/\partial T$ is continuous as $L^2_{m+1}\to L^2_m$. We remark in (1.108) for $i=2$ we use the coordinate (τ'',t) on $(-\infty,0]\times[0,1]$ to define T derivative of $V^\rho_{T,2,(\kappa)}$.

Taking this fact into acount the proof goes as follows.

We can estimate T and ρ derivative of $\mathrm{Err}^{\rho}_{1,T,(\kappa)}$ on K_1 in the same way as the proof of Lemma 1.31.

Remark 1.39. The fact we use here is that the maps such as $(u,v) \mapsto \mathrm{E}(u,v)$, $(u,v) \to \Pi^{\perp}_{E_i(u)}(v)$ are smooth maps from $L^2_{m+1,\delta} \times L^2_{m+1,\delta} \to L^2_{m+1,\delta}$ or $L^2_{m+1,\delta} \times L^2_{m,\delta} \to L^2_{m,\delta}$ and $u \to \overline{\partial} u$ is a smooth map $L^2_{m+1,\delta} \to L^2_{m,\delta}$. (Since we assume m sufficiently large this is a well-known fact.) Moreover the map $T \mapsto u^{\rho}_{T,(\kappa-1)}$ and $T \mapsto V^{\rho}_{T,1,(\kappa)}$ are C^{ℓ} maps as a map $[T(m),\infty) \to L^2_{m+1-\ell,\delta}$ with its differential estimated by induction hypothesis (1.110) and (1.108).

We remark $\rho \mapsto u^{\rho}_{T,(\kappa-1)}$ is smooth as $V_1 \times_L V_2 \to L^2_{m+1,\delta}$.

The estimates of T and ρ derivatives of (1.114) are as follows.

We first consider the domain $\tau' \in [4T+1,\infty)$. There we have

$$\begin{aligned}\mathrm{Err}^{\rho}_{1,T,(\kappa)}(\tau',t) =&(1-\chi(\tau'-5T))\overline{\partial}(V^{\rho}_{T,2,(\kappa)}(\tau''+10T,t)\\ &+V^{\rho}_{T,1,(\kappa)}(\tau',t)+u^{\rho}_{T,(\kappa-1)}(\tau',t)-\Delta p^{\rho}_{T,(\kappa)}).\end{aligned}\tag{1.115}$$

By the same calculation as in the proof of Lemma 1.31, (1.115) is equal to

$$\begin{aligned}(1-\chi(\tau'-5T))\int_0^1 ds\int_0^s \frac{\partial^2}{\partial r^2}\overline{\partial}\big(&r(V^{\rho}_{T,2,(\kappa)}(\tau''+10T)-\Delta p^{\rho}_{T,(\kappa)})\\ &+r(V^{\rho}_{T,1,(\kappa)}(\tau',t)-\Delta p^{\rho}_{T,(\kappa)})\\ &+u^{\rho}_{T,(\kappa-1)}(\tau',t)+r\Delta p^{\rho}_{T,(\kappa)}\big)dr.\end{aligned}$$

(Note we are away from the support of E_i.)[6] Using the fact that $T \mapsto (V^{\rho}_{T,1,(\kappa)}(\tau',t)-\Delta p^{\rho}_{T,(\kappa)}) + (V^{\rho}_{T,2,(\kappa)}(\tau''+10T)-\Delta p^{\rho}_{T,(\kappa)})$ and $T \mapsto u^{\rho}_{T,(\kappa-1)}(\tau',t)$ are of C^{ℓ} class as a map to $L^2_{m+1-\ell,\delta}$, we can estimate it to obtain the required estimate (1.111) on this part. We remark $T \mapsto (V^{\rho}_{T,2,(\kappa-1)}, \Delta p^{\rho}_{T,(\kappa-1)})$ is C^{ℓ} with exponential decay estimate on T derivatives as a map $[T(m),\infty) \to L^2_{m-\ell+1,\delta}$. This follows from induction hypothesis as follows.

$$\begin{aligned}&\frac{\partial^{\ell}}{\partial T^{\ell}}\left(V^{\rho}_{T,2,(\kappa)}(\tau''+10T)\right)\Big|_{T=T_1}\\ &=\sum_{\ell_1+\ell_2=\ell}(10)^{\ell_2}\frac{\partial^{\ell_1}}{\partial T^{\ell_1}}\frac{\partial^{\ell_2}}{(\partial\tau'')^{\ell_2}}V^{\rho}_{T,2,(\kappa)}(\tau''+10T_1).\end{aligned}\tag{1.116}$$

The $L^2_{m+1-\ell,\delta}$-norm of the right hand side can be estimated by (1.108).

We next consider $\tau' \in [0,4T+1]$. There we have

$$\begin{aligned}\mathrm{Err}^{\rho}_{1,T,(\kappa)}(\tau',t) =&\overline{\partial}(\chi(\tau'-4T)(V^{\rho}_{T,2,(\kappa)}(\tau''+10T)-\Delta p^{\rho}_{T,(\kappa)})\\ &+V^{\rho}_{T,1,(\kappa)}(\tau',t)+u^{\rho}_{T,(\kappa-1)}(\tau',t)).\end{aligned}\tag{1.117}$$

Note

$$\overline{\partial}u^{\rho}_{T,(\kappa-1)}(\tau',t)) = \mathrm{Err}^{\rho}_{1,T,(\kappa-1)}(\tau',t),$$

[6]Note $\overline{\partial}$ is non-constant. So $\overline{\partial}(r(V^{\rho}_{T,2,(\kappa)}(\tau''+10T)-\Delta p^{\rho}_{T,(\kappa)})+r(V^{\rho}_{T,1,(\kappa)}(\tau',t)-\Delta p^{\rho}_{T,(\kappa)})+u^{\rho}_{T,(\kappa-1)}(\tau',t)+r\Delta p^{\rho}_{T,(\kappa)})$ is nonlinear on r.

there. Therefore we can calculate in the same way as the proof of Lemma 1.31 to find

$$\begin{aligned}&\overline{\partial}(V^{\rho}_{T,1,(\kappa)}(\tau',t)+u^{\rho}_{T,(\kappa-1)}(\tau',t))\\&=\int_0^1 ds\int_0^s \frac{\partial^2}{\partial r^2}\overline{\partial}(r(V^{\rho}_{T,1,(\kappa)}(\tau',t)-\Delta p^{\rho}_{T,(\kappa)})+u^{\rho}_{T,(\kappa-1)}(\tau',t)+r\Delta p^{\rho}_{T,(\kappa)})dr.\end{aligned}$$

We can again estimate the right hand side by using the fact that the maps $T\mapsto (V^{\rho}_{T,1,(\kappa)}(\tau',t),\Delta p^{\rho}_{T,(\kappa)})$ and $T\mapsto u^{\rho}_{T,(\kappa-1)}(\tau',t)$ are of C^{ℓ} class as a map to $L^2_{m+1-\ell,\delta}$ with estimate (1.110).

Finally we observe the ratio between weight function of $L^2_{m+1,\delta}(\Sigma_2)$ and of $L^2_{m+1,\delta}(\Sigma_T)$ is $e^{2T\delta}$ on $\tau=-T$ (that is $\tau'=4T$). We use this fact to estimate $\overline{\partial}(\chi(\tau'-4T)(V^{\rho}_{T,2,(\kappa)}(\tau''+10T)-\Delta p^{\rho}_{T,(\kappa)}))$. We thus obtain the required estimate (1.111) for $\mathrm{Err}^{\rho}_{1,T,(\kappa)}$ on $\tau'\in[0,4T+1]$.

We thus obtain an estimate for $\mathrm{Err}^{\rho}_{1,T,(\kappa)}(\tau',t)$.

The estimate of derivatives of $\mathrm{Err}^{\rho}_{2,T,(\kappa)}(\tau',t)$ is similar. Thus we have (1.111).

We remark that $\mathfrak{e}^{\rho}_{i,T,(0)}$ is independent of T as an element of E_i. Among $\mathfrak{e}^{\rho}_{i,T,(\kappa)}$'s, the term $\mathfrak{e}^{\rho}_{i,T,(0)}$ is the only one that is not of exponential decay with respect to T. Once we remark this point the rest of the proof of (1.112) is the same as the proof of Lemma 1.31.

We finally prove (1.110). On K_1 we have

$$u^{\rho}_{T,(\kappa)}=\mathrm{E}(u^{\rho}_{T,(\kappa-1)},V^{\rho}_{1,T,(\kappa)}).$$

So using $\mu<1$ (1.110) follows from (1.108) on K_1.

On $(\tau',t)\in[0,5T+1)\times[0,1]$ we have:

$$\begin{aligned}&u^{\rho}_{T,(\kappa)}(\tau',t)\\&=V^{\rho}_{T,1,(\kappa)}(\tau',t)+(1-\chi(\tau'-4T))(V^{\rho}_{T,2,(\kappa)}(\tau''+10T,t)-\Delta p^{\rho}_{T,(\kappa)})\\&\qquad\qquad+u^{\rho}_{T,(\kappa-1)}(\tau',t)\\&=\sum_{a=1}^{\kappa}V^{\rho}_{T,1,(a)}(\tau',t)+(1-\chi(\tau'-4T))\sum_{a=1}^{\kappa}(V^{\rho}_{T,2,(a)}(\tau''+10T,t)-\Delta p^{\rho}_{T,(a)})\\&\qquad\qquad+u^{\rho}_{T,(0)}(\tau',t).\end{aligned}$$

Then using a calclation similar to (1.116) we have (1.108) on $(\tau',t)\in[0,5T+1)\times[0,1]$.

Remark 1.40. In [Ab] Abouzaid used L^p_1 norm for the maps u. He then proved that the gluing map is continuous with respect to T (that is S in the notation of [Ab].) but does not prove its differentiability with respect to T. (Instead he used the technique to remove the part of the moduli space with $T>T_0$, as we mentioned at the begining of this note. This technique certainly works for the purpose of [Ab].) In fact if we use L^p_1 norm instead of L^2_m norm then the left hand side of (1.110) becomes L^p_{-1} norm which is hard to use.

Abouzaid mentioned in [Ab, Remark 5.1] that this point is related to the fact that quotients of Sobolev spaces by the diffeomorphisms in the source are not naturally equipped with the structure of smooth Banach manifold. Indeed in the situation when there is an automorpism on Σ_2, for example Σ_2 is disk with one boundary marked point $(-\infty,t)$, then the T parameter is killed by a part of the automorphism. So the shift of $V^{\rho}_{T,2,(\kappa)}$ by T that appears in the second term of (1.114) will be equivalent to the action of the automorphism group of Σ_2 in such a situation. The shift of T causes the loss of differentiability in the sense of Sobolev space in the formula (1.108) -(1.112). However at the end of the day we can still get the differentiability of C^∞ order and its exponential decay by using various Sobolev spaces with various m simultaneously. (See Remark 1.33 also.)

(Part B) (See [FOOO, page 776 the paragraph next to (B)].)

We assume (1.108)-(1.112) for κ and will prove (1.108) and (1.109) for $\kappa+1$. This part is nontrivial only because the construction here is global. (Solving linear equation.) So we first review the set up of the function space that is independent of T.

In Definition 1.18 we defined a function space $\mathfrak{H}(E_1,E_2)$, that is a subspace of (1.48). Since (1.48) is still T dependent we rewrite it a bit. We consider u^{ρ}_i : $(\Sigma_i,\partial\Sigma_i)\to(X,L)$ that is T-*independent*.

The maps $\hat u^{\rho}_{i,T,(\kappa)}$ are close to u^{ρ}_i. (Namely the C^0 distance between them is smaller than injectivity radius of X.) We take a connection of TX so that L is totally geodesic. We use the complex linear part of the parallel transport with respect to this connection, to send

$$\bigoplus_{i=1}^{2} L^2_{m,\delta}((\Sigma_i,\partial\Sigma_i);(u^{\rho}_i)^*TX,(u^{\rho}_i)^*TL).$$

to

$$\bigoplus_{i=1}^{2} L^2_{m,\delta}((\Sigma_i,\partial\Sigma_i);(\hat u^{\rho}_{i,T,(\kappa)})^*TX,(\hat u^{\rho}_{i,T,(\kappa)})^*TL).$$

Note $\mathrm{Ker}(D\mathrm{ev}_{1,\infty}-D\mathrm{ev}_{2,\infty})$ is sent to $\mathrm{Ker}(D\mathrm{ev}_{1,\infty}-D\mathrm{ev}_{2,\infty})$ by this map . Therefore we obtain an isomorphism between

$$\mathrm{Ker}(D\mathrm{ev}_{1,\infty}-D\mathrm{ev}_{2,\infty})\cap\bigoplus_{i=1}^{2} L^2_{m,\delta}((\Sigma_i,\partial\Sigma_i);(u^{\rho}_i)^*TX,(u^{\rho}_i)^*TL) \tag{1.118}$$

and

$$\mathrm{Ker}(D\mathrm{ev}_{1,\infty}-D\mathrm{ev}_{2,\infty})\cap\bigoplus_{i=1}^{2} L^2_{m,\delta}((\Sigma_i,\partial\Sigma_i);(\hat u^{\rho}_{i,T,(\kappa)})^*TX,(\hat u^{\rho}_{i,T,(\kappa)})^*TL). \tag{1.119}$$

In case $\kappa=0$ we send $\mathfrak{H}(E_1,E_2)$ by this isomorphism to obtain a subspace of (1.118) which we denote by $\mathfrak{H}(E_1,E_2)$ by an abuse of notation. We send it to the subspace of (1.119) and denote it by $\mathfrak{H}(E_1,E_2;\kappa,T)$. We thus have an isomorphism

$$I_{1,\kappa,T}:\mathfrak{H}(E_1,E_2)\to\mathfrak{H}(E_1,E_2;\kappa,T).$$

We next use the parallel transport in the same way to find an isomorphism

$$I_{2,\kappa,T}:L^2_{m,\delta}(\Sigma_i;(u^{\rho}_i)^*TX\otimes\Lambda^{01})\to L^2_{m,\delta}(\Sigma_i;(\hat u^{\rho}_{i,T,(\kappa)})^*TX\otimes\Lambda^{01}).$$

Thus the composition

$$I^{-1}_{2,\kappa,T}\circ\Big(D_{\hat u^{\rho}_{i,T,(\kappa-1)}}\overline\partial-(D_{\hat u^{\rho}_{i,T,(\kappa-1)}}E_i)((\mathfrak{se})^{\rho}_{i,T,(\kappa-1)},\cdot))\Big)\circ I_{1,\kappa,T}$$

defines an operator

$$D_{\kappa,T}:\mathfrak{H}(E_1,E_2)\to L^2_{m,\delta}(\Sigma_i;(u^{\rho}_i)^*TX\otimes\Lambda^{01}).$$

Here the domain and the target is independent of T,κ.

Remark 1.41. Note $D_{\hat u^\rho_{i,T,(\kappa-1)}}\overline\partial - (D_{\hat u^\rho_{i,T,(\kappa-1)}}E_i)((\mathfrak{se})^\rho_{i,T,(\kappa-1)},\cdot)$ is the differential operator in (1.43) and (1.44). This differential operator gives the linearization of the right hand side of (1.113).

We next eliminate T,κ dependence of E_i. We consider the finite dimensional subspace:

$$E_i(\hat u^\rho_{i,T,(\kappa)}) \subset L^2_{m,\delta}(\Sigma_i;(\hat u^\rho_{i,T,(\kappa)})^*TX\otimes\Lambda^{01}).$$

Let us consider

$$E_{i,(\kappa),T} = I^{-1}_{2,\kappa,T}(E_i(\hat u^\rho_{i,T,(\kappa)}))$$

that may depend on T. However

$$E_{i,(0)} = I^{-1}_{2,\kappa,T}(E_i(\hat u^\rho_{i,T,(0)}))$$

is independent of T since $\hat u^\rho_{i,T,(0)} = u^\rho_i$ on K_i. Let $E^\perp_{i,(0)}$ be the L^2 orthonormal complement of $E_{i,(0)}$ in $L^2_{m,\delta}(\Sigma_i;(\hat u^\rho_{i,T,(\kappa)})^*TX\otimes\Lambda^{01})$.

We have

$$E_{i,(\kappa),T}\oplus E^\perp_{i,(0)} = L^2_{m,\delta}(\Sigma_i;(u^\rho_i)^*TX\otimes\Lambda^{01}).$$

Therefore the inclusion induces an isomorphism

$$E^\perp_{i,(0)} \cong L^2_{m,\delta}(\Sigma_i;(u^\rho_i)^*TX\otimes\Lambda^{01})/E_{i,(\kappa),T}.$$

We thus obtain

$$\overline D_{\kappa,T} : \mathfrak{H}(E_1,E_2)\to E^\perp_{i,(0)}. \tag{1.120}$$

The induction hypothesis implies the following:

(1) There exists $C_{20,m},C_{21,m}>0$ such that

$$C_{20,m}\|V\|_{L^2_{m+1,\delta}} \le \|\overline D_{0,T}(V)\|_{L^2_{m,\delta}} \le C_{21,m}\|V\|_{L^2_{m+1,\delta}}. \tag{1.121}$$

(2)

$$\|\overline D_{\kappa,T}(V) - \overline D_{0,T}(V)\|_{L^2_{m,\delta}} \le C_{22,m}e^{-\delta T}\|V\|_{L^2_{m+1,\delta}}. \tag{1.122}$$

Moreover

$$\left\|\nabla^n_\rho\frac{\partial^\ell}{\partial T^\ell}\overline D_{\kappa,T}(V)\right\|_{L^2_{m-\ell,\delta}} \le C_{22,m}e^{-\delta T}\|V\|_{L^2_{m+1,\delta}}. \tag{1.123}$$

In fact (1.123) follows from

$$\left\|\nabla^n_\rho\frac{\partial^\ell}{\partial T^\ell}\hat u^\rho_{i,T,(\kappa)}\right\|_{L^2_{m-\ell}(K_i)} \le C_{23,m}e^{-\delta T}, \tag{1.124}$$

$$\left\|\nabla^n_\rho\frac{\partial^\ell}{\partial T^\ell}\hat u^\rho_{i,T,(\kappa)}\right\|_{L^2_{m-\ell}([S,S+1]\times[0,1])} \le C_{23,m}e^{-\delta T} \tag{1.125}$$

for any $S\in[0,\infty)$. Note the weighted Sobolev norm $\|\nabla^n_\rho\frac{\partial^\ell}{\partial T^\ell}\hat u^\rho_{i,T,(\kappa)}\|_{L^2_{m-\ell,\delta}(\Sigma_i)}$ can be large because

$$\frac{\partial}{\partial T}\chi^{\leftarrow}_{\mathcal B}(\tau-T,t)u^\rho_{T,(\kappa-1)}$$

is only estimated by $e^{-3\delta T}$ on the support of $\chi^{\leftarrow}_{\mathcal B}(\tau-T,t)$ but the weight $e_{1,\delta}$ is roughly $e^{7T\delta}$ on the support of $\chi^{\leftarrow}_{\mathcal B}(\tau-T,t)$. However this does not cause problem to prove (1.123). In fact the operator $\overline D_{\kappa,T}$ is a differential operator whose coefficient depends on $\hat u^\rho_{i,T,(\kappa)}$. So to estimate the operator norm of its derivatives with respect to the *weighted* Sobolev norm, we only need to estimate the local Sobolev norm without weight of $\hat u^{\rho}_{i,T,(\kappa)}$, that is provided by (1.124) and (1.125).

We remark that $\overline{D}_{0,T}$ is independent of T. So we write $\overline{D}_0$. Now we have:

$$\begin{aligned}\overline{D}_{\kappa,T}^{-1} &= \Big((1+(\overline{D}_{\kappa,T}-\overline{D}_0)\overline{D}_0^{-1})\overline{D}_0\Big)^{-1}\\ &= \overline{D}_0^{-1}\sum_{k=0}^{\infty}(-1)^k((\overline{D}_{\kappa,T}-\overline{D}_0))\overline{D}_0^{-1})^k\end{aligned} \tag{1.126}$$

Therefore

$$\left\Vert \nabla_{\rho}^{n}\frac{\partial^{\ell}}{\partial T^{\ell}}\overline{D}_{\kappa,T}^{-1}(W)\right\Vert_{L^2_{m+1-\ell,\delta}} \le C_{24,m}e^{-\delta}\Vert W\Vert_{L^2_{m,\delta}} \tag{1.127}$$

for $\ell > 0$ and $\ell + n \le m$. (Here we assume W is T independent.) Since

$$(V^{\rho}_{T,1,(\kappa+1)}, V^{\rho}_{T,2,(\kappa+1)}, \Delta p^{\rho}_{T,(\kappa+1)}) = (I_{1,\kappa,T}\circ \overline{D}_{\kappa,T}^{-1}\circ I^{-1}_{2,\kappa,T})(\mathrm{Err}^{\rho}_{1,T,(\kappa)}, \mathrm{Err}^{\rho}_{2,T,(\kappa)})$$

(1.111) and (1.127) imply (1.108) and (1.109) for $\kappa + 1$.

The proof of Theorem 1.34 is now complete. □

1.5. **Surjectivity and injectivity of the gluing map.** In this subsection we prove surjectivity and injectivity of the map Glu_T in Theorem 1.10 and complete the proof of Theorem 1.10.[7] The proof goes along the line of [D1]. (See also [FU].) The surjectivity proof is written in [FOn1, Section 14] and injectivity is proved in the same way. ([FOn1, Section 14] studies the case of pseudo-holomorphic curve without boundary. It however can be adapted easily to the bordered case as we mentioned in [FOOO, page 417 lines 21-26].) Here we explain the argument in our situation in more detail.

We begin with the following a priori estimate.

Proposition 1.42. ([FOn1, Lemma 11.2]) *There exist* $\epsilon_3, C_{25,m}, \delta_2 > 0$ *such that if* $u : (\Sigma_T, \partial\Sigma_T) \to (X, L)$ *is an element of* $\mathcal{M}^{E_1+E_2}((\Sigma_T, \vec z); \beta)_{\epsilon}$ *for* $0 < \epsilon < \epsilon_3$ *then we have*

$$\left\Vert \frac{\partial u}{\partial \tau}\right\Vert_{C^m([\tau-1,\tau+1]\times[0,1])} \le C_{25,m}e^{-\delta_2(5T-|\tau|)}. \tag{1.128}$$

The proof is the same as [FOn1, Lemma 11.2] that is proved in [FOn1, Section 14] and so is omitted.

We also have the following:

Lemma 1.43. $\mathcal{M}^{E_1+E_2}((\Sigma_T, \vec z); \beta)_{\epsilon}$ *is a smooth manifold of dimension* $\dim V_1 + \dim V_2 - \dim L$.

This is a consequence of implicit function theorem and index sum formula.

Proof of surjectivity. During this proof we take m sufficiently large and fix it. We will fix ϵ and T_0 during the proof and assume $T > T_0$. (They are chosen so that the discussion below works.) Let $u : (\Sigma_T, \partial\Sigma_T) \to (X, L)$ be an element of

[7] Here surjectivity means the second half of the statement of Theorem 1.10, that is 'The image contains $\mathcal{M}^{E_1+E_2}((\Sigma_T, \vec z); \beta)_{\epsilon_3}$.'

$\mathcal{M}^{E_1+E_2}((\Sigma_T,\vec z);\beta)_\epsilon$. The purpose here is to show that u is in the image of Glu_T. We define $u'_i : (\Sigma_i,\partial\Sigma_i) \to (X,L)$ as follows. We put $p_0^u = u(0,0) \in L$.

$$u'_1(z) = \begin{cases} \chi_{\mathcal{B}}^{\leftarrow}(\tau - T, t)u(\tau,t) + \chi_{\mathcal{B}}^{\rightarrow}(\tau - T, t)p_0^u & \text{if } z = (\tau,t) \in [-5T,5T]\times[0,1] \\ u(z) & \text{if } z \in K_1 \\ p_0^u & \text{if } z \in [5T,\infty)\times[0,1]. \end{cases}$$

$$u'_2(z) = \begin{cases} \chi_{\mathcal{A}}^{\rightarrow}(\tau + T, t)u(\tau,t) + \chi_{\mathcal{A}}^{\leftarrow}(\tau + T, t)p_0^u & \text{if } z = (\tau,t) \in [-5T,5T]\times[0,1] \\ u(z) & \text{if } z \in K_2 \\ p_0^u & \text{if } z \in (-\infty,-5T]\times[0,1]. \end{cases} \tag{1.129}$$

Proposition 1.42 implies

$$\|\Pi_{E_i(u'_i)}\overline{\partial}u'_i\|_{L^2_{m,\delta}(\Sigma_i)} \le C_{26,m}e^{-\delta T}. \tag{1.130}$$

Here we take $\delta < \delta_2/10$. On the other hand, by assumption and elliptic reguralіty we have

$$\|u'_i - u_i\|_{L^2_{m+1,\delta}(\Sigma_i)} \le C_{27,m}\epsilon. \tag{1.131}$$

Therefore by an implicit function theorem we have the following:

Lemma 1.44. *There exists $\rho_i \in V_i$ such that*

$$\|u'_i - u_i^{\rho_i}\|_{L^2_{m+1,\delta}(\Sigma_i)} \le C_{28,m}e^{-\delta T}, \tag{1.132}$$

$\rho = (\rho_1,\rho_2) \in V_1 \times_L V_2$, and

$$|\rho_i| \le C_{29,m}\epsilon. \tag{1.133}$$

(Note when $\rho_i = 0$, $u_i^{\rho_i} = u_i$.)

By (1.132) we have

$$\|u - u_T^\rho\|_{L^2_{m+1,\delta}(\Sigma_T)} \le C_{30,m}e^{-\delta T}. \tag{1.134}$$

Here $u_T^\rho = \mathrm{Glu}_T(\rho)$.

We take $V \in \Gamma((\Sigma_T,\partial\Sigma_T);(u_T^\rho)^*TX;(u_T^\rho)^*TL)$ so that

$$u(z) = \mathrm{E}(u_T^\rho(z), V(z)).$$

We define $u^s : (\Sigma_T,\partial\Sigma_T) \to (X,L)$ by

$$u^s(z) = \mathrm{E}(u_T^\rho(z), sV(z)). \tag{1.135}$$

(1.134) implies

$$\|\Pi^\perp_{(E_1+E_2)(u^s)}\overline{\partial}u^s\|_{L^2_{m,\delta}(\Sigma_T)} \le C_{31,m}e^{-\delta T} \tag{1.136}$$

and

$$\left\|\frac{\partial}{\partial s}u^s\right\|_{L^2_{m+1,\delta}(K_i^{+S})} \le C_{32,m}e^{-\delta T} \tag{1.137}$$

for each $s \in [0,1]$.

Lemma 1.45. *If T is sufficiently large then there exists $\hat u^s : (\Sigma_T,\partial\Sigma_T) \to (X,L)$ ($s \in [0,1]$) with the following properties.*

(1)

$$\overline{\partial}\hat u^s \equiv 0 \mod (E_1+E_2)(\hat u^s).$$

(2)
$$\left\|\frac{\partial}{\partial s}\hat u^s\right\|_{L^2_{m+1,\delta}(K_i^{+S})} \le 2C_{33,m}e^{-\delta T} \tag{1.138}$$
(3) $\hat u^s = u^s$ *for* $s = 0, 1$.

Proof. Run the alternating method described in Subsection 1.3 in one parameter family version. Since u^s is already a solution for $s = 0, 1$ it does not change. □

Lemma 1.46. *The map* $\mathrm{Glu}_T : V_1 \times_L V_2 \to \mathcal{M}^{E_1+E_2}((\Sigma_T, \vec z); \beta)_\epsilon$ *is an immersion if* T *is suffciently large.*

Proof. We consider the composition of Glu_T with
$$\mathcal{M}^{E_1+E_2}((\Sigma_T, \vec z); \beta)_\epsilon \to \mathrm{Map}_{L^2_{m+1}}((K_i^{+S}, K_i^{+S} \cap \partial\Sigma_i), (X, L))$$
defined by restriction. In the case $T = \infty$ this composition is obtained by restriction of maps. By unique continuation, this is certainly an immersion for $T = \infty$. Then Theorem 1.34 implies that it is an immersion for sufficiently large T. □

Now we will prove that
$$A = \{s \in [0, 1] \mid \hat u^s \in \text{image of } \mathrm{Glu}_T\}$$
is open and closed. Lemma 1.43 implies that $\mathcal{M}^{E_1+E_2}((\Sigma_T, \vec z); \beta)_\epsilon$ is a smooth manifold and has the same dimension as $V_1 \times_L V_2$. Therefore Lemma 1.46 implies that A is open. The closedness of A follows from (1.138).

Note $0 \in A$. Therefore $1 \in A$. Namely u is in the image of Glu_T as required. □

Proof of injectivity. Let $\rho^j = (\rho_1^j, \rho_2^j) \in V_1 \times_L V_2$ for $j = 0, 1$. We assume
$$\mathrm{Glu}_T(\rho^0) = \mathrm{Glu}_T(\rho^1) \tag{1.139}$$
and
$$\|\rho_i^j\| < \epsilon. \tag{1.140}$$
We will prove that $\rho^0 = \rho^1$ if T is sufficiently large and ϵ is sufficiently small. We may assume that $V_1 \times_L V_2$ is connected and simply connected. Then, we have a path $s \mapsto \rho^s = (\rho_1^s, \rho_2^s) \in V_1 \times_L V_2$ such that

(1) $\rho^s = \rho^j$ for $j = 0, 1$.
(2)
$$\left\|\frac{\partial}{\partial s}\rho^s\right\| \le \Phi_1(\epsilon)$$
where $\lim_{\epsilon\to 0}\Phi_1(\epsilon) = 0$.

We define $V(s) \in \Gamma((\Sigma_T, \partial\Sigma_T); (u_T^{\rho^0})^*TX; (u_T^{\rho^0})^*TL)$ such that
$$u_T^{\rho^s}(z) = \mathrm{E}(u_T^{\rho^0}(z), V(s)(z)).$$
(By (2) $u_T^{\rho^s}(z)$ is C^0-close to $u_T^{\rho^0}(z)$, as $\epsilon \to 0$. Therefore there exists such a unique $V(s)$ if ϵ is small.) Note $V(1) = V(0)$ since $u^{\rho^1} = u^{\rho^0}$. Therefore for $w \in D^2 = \{w \in \mathbb{C} \mid |w| \le 1\}$ there exists $V(w)$ such that

(1) $V(s) = V(w)$ if $w = e^{2\pi\sqrt{-1}s}$.

(2) We put $w = x + \sqrt{-1}y$.

$$\left\| \frac{\partial}{\partial x} V(w) \right\|_{L^2_{m+1,\delta}(\Sigma_T)} + \left\| \frac{\partial}{\partial y} V(w) \right\|_{L^2_{m+1,\delta}(\Sigma_T)} \le \Phi_2(\epsilon) \tag{1.141}$$

where $\lim_{\epsilon\to 0} \Phi_2(\epsilon) = 0$.

We put $u^w(z) = \mathrm{E}(u_T^{\rho^0}(z), V(w)(z))$.

Lemma 1.47. *If T is sufficiently large and ϵ is sufficiently small then there exists $\hat{u}^w : (\Sigma_T, \partial\Sigma_T) \to (X, L)$ $(s \in [0,1])$ with the following properties.*

(1)

$$\overline{\partial}\hat{u}^w \equiv 0 \mod (E_1 + E_2)(\hat{u}^w).$$

(2)

$$\left\| \frac{\partial}{\partial x} \hat{u}^w \right\|_{L^2_{m+1,\delta}(K_i^{+S})} + \left\| \frac{\partial}{\partial y} \hat{u}^w \right\|_{L^2_{m+1,\delta}(K_i^{+S})} \le \Phi_3(\epsilon) \tag{1.142}$$

with $\lim_{\epsilon\to 0} \Phi_3(\epsilon) = 0$.

(3) $\hat{u}^w = u^w$ *for* $w \in \partial D^2$.

Proof. Run the alternating method described in Subsection 1.3 in two parameter family version. □

Lemma 1.48. *If T is sufficiently large and ϵ is sufficiently small there exists a smooth map $F : D^2 \to V_1 \times_L V_2$ such that*

(1) $\mathrm{Glu}_T(F(w)) = \hat{u}^w$.

(2) *If $s \in [0,1]$ then we have:*

$$F(e^{2\pi\sqrt{-1}s}) = \rho^s.$$

Proof. Note $\rho \mapsto \mathrm{Glu}_T(\rho)$ is a local diffeomorphism. So we can apply the proof of homotopy lifting property as follows. Let $D_r^2 = \{z \in \mathbb{C} \mid |z - (r-1)| \le r\}$. We put

$$A = \{r \in [0,1] \mid \exists\ F : D_r^2 \to V_1 \times_L V_2 \text{ satisfying (1) above and } F(-1) = \rho^{1/2}.\}$$

Since $\mathrm{Glu}_T(\rho)$ is a local diffeomorphism A is open. We can use (1.142) to show the closednsss of A. Since $0 \in A$ it follows that $1 \in A$. The proof of Lemma 1.48 is complete. □

The proof of Theorem 1.10 is now complete. □

2. The general case

Coming later.

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