ANALYSIS OF CONTACT CAUCHY-RIEMANN MAPS III: ENERGY, BUBBLING AND FREDHOLM THEORY

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Abstract. In [OW2, OW3], the authors studied the nonlinear elliptic system

$$\overline{\partial}^\pi w = 0,\ d(w^*\lambda \circ j) = 0$$

without involving symplectization for each given contact triad (Q, λ, J), and established the a priori $W^{k,2}$ elliptic estimates and proved the asymptotic (subsequence) convergence of the map $w : \dot\Sigma \to Q$ for any solution, called a contact instanton, on $\dot\Sigma$ under the hypothesis $\|w^*\lambda\|_{C^0} < \infty$ and $d^\pi w \in L^2 \cap L^4$. The asymptotic limit of a contact instanton is a 'spiraling' instanton along a 'rotating' Reeb orbit near each puncture on a punctured Riemann surface $\dot\Sigma$. Each limiting Reeb orbit carries a 'charge' arising from the integral of $w^*\lambda \circ j$.

In this article, we further develop analysis of contact instantons, especially the $W^{1,p}$ estimate for $p > 2$ (or the C^1-estimate), which is essential for the study of compactfication of the moduli space and the relevant Fredholm theory for contact instantons. In particular, we define a Hofer-type off-shell energy $E^\lambda(j, w)$ for any pair (j, w) with a smooth map w satisfying $d(w^*\lambda \circ j) = 0$, and develop the bubbling-off analysis and prove an ϵ-regularity result. We also develop the relevant Fredholm theory and carry out index calculations (for the case of vanishing charge).

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1. Introduction and statements of main results

A contact manifold (Q, ξ) is a $2n+1$ dimensional manifold equipped with a completely non-integrable distribution of rank $2n$, called a contact structure. Complete non-integrability of ξ can be expressed by the non-vanishing property

$$\lambda \wedge (d\lambda)^n \neq 0$$

for a one-form λ which defines the distribution, i.e., $\ker \lambda = \xi$. Such a one-form λ is called a contact form associated to ξ. Each contact form λ of ξ canonically induces a splitting

$$TQ = \mathbb{R}\{X_\lambda\} \oplus \xi.$$

Here X_λ is the Reeb vector field of λ, which is uniquely determined by the equations

$$X_\lambda \rfloor \lambda \equiv 1, \quad X_\lambda \rfloor d\lambda \equiv 0.$$

We denote by $\Pi = \Pi_\lambda : TQ \to TQ$ the idempotent, i.e., an endomorphism satisfying $\Pi^2 = \Pi$ such that $\ker \Pi = \mathbb{R}\{X_\lambda\}$ and $\operatorname{Im} \Pi = \xi$. Denote by $\pi = \pi_\lambda : TQ \to \xi$ the associated projection.

In the presence of the contact form λ, one usually consider the set of J that is compatible to $d\lambda$ in the sense that the bilinear form $g_\xi = d\lambda(\cdot, J\cdot)$ defines a Hermitian vector bundle $(\xi, d\lambda|_\xi, J|_\xi)$ on Q. We call such J a CR-almost complex structure. As long as no confusion arises, we do not distinguish J and its restriction $J|_\xi$. We introduce the projection $\pi : TQ \to \xi$ with respect to the splitting $TQ = \mathbb{R}\{X_\lambda\} \oplus \xi$.

Definition 1.1. Let $J \in \operatorname{End}(TQ)$ be an endomorphism satisfying $J^2 = -\Pi$ such that $d\lambda(\cdot, J\cdot)$ is nondegenerate on ξ. We say that such J is *compatible to* λ. We define the set

$$\mathcal{J}(Q, \lambda) = \{J : \xi \to \xi \mid J^2 = -\Pi,\ J \text{ compatible to } \lambda\} \tag{1.1}$$

Following [OW1], we call any such triple (Q,λ,J) a contact triad of (Q,ξ). For each given contact triad, we equip Q with the triad metric

$$g = d\lambda(\cdot, J\cdot) + \lambda \otimes \lambda.$$

Let (Σ, j) be a Riemann surface with a finite number of marked points and let $\dot\Sigma$ be the associated punctured Riemann surface with a finite number of punctures. We call a map $w : \dot\Sigma \to Q$ a *contact Cauchy-Riemann map* if $\overline{\partial}^\pi w = 0$. Then we have the decomposition

$$dw = d^\pi w + w^*\lambda\, X_\lambda, \quad d^\pi w := \overline{\partial}^\pi w + \partial^\pi w$$

as a one-form on Σ with values in TQ. We also regard $d^\pi w$ as a ξ-valued one-form on Σ.

We introduce a nonlinear first-order differential operator

$$\overline{\partial}^\pi w = \frac{1}{2}(\pi dw + J\cdot \pi dw \cdot j),\ \partial^\pi w = \frac{1}{2}(\pi dw - J\cdot \pi dw\cdot j) \tag{1.2}$$

and consider the following variation of Cauchy-Riemann equation

$$\overline{\partial}^\pi w = 0. \tag{1.3}$$

Definition 1.2. We say a map $w : \Sigma \to Q$ is a contact Cauchy-Riemann map (with respect to J) if it satisfies (1.3).

In [OW2], Wang and the present author established the a priori $W^{k,2}$ coercive estimates for the contact Cauchy-Riemann maps by augmenting the equation $\overline{\partial}^\pi w = 0$ by the closedness condition of

$$d(w^*\lambda \circ j) = 0. \tag{1.4}$$

The standard pseudoholomorphic curve equation on the symplectization $\mathbb{R}\times Q$ equipped with the cylindrical almost complex structure $J_0 \oplus J$ with respect to the splitting

$$T(\mathbb{R}\times Q) = \mathbb{R}\cdot \frac{\partial}{\partial r} \oplus \mathbb{R}\cdot X_\lambda \oplus \xi$$

is a special case of the 'exact' contact instantons where the anti-derivative equation of $w^*\lambda\circ j$ prescribed by $a = w^*s$ with the s-coordinate of the symplectization $\mathbb{R}\times Q$ for the map $(a,w) : \dot\Sigma \to \mathbb{R}\times Q$. (See [Ho1] for the relevant calculations.)

Definition 1.3 (Contact instanton)**.** Let Σ be as above. We call a pair of (j,w) of a complex structure on Σ and a map $w : \dot\Sigma \to Q$ a *contact instanton* if it satisfies

$$\overline{\partial}^\pi w = 0, \quad d(w^*\lambda\circ j) = 0. \tag{1.5}$$

We call such (j,w) an *exact contact instanton* if the form $w^*\lambda\circ j$ is exact on $\dot\Sigma$.

Such an equation was first introduced by Hofer in [Ho2] for the case of charges vanishing at the punctures in the context of symplectization, which was further studied in [ACH], [Be] and [Ab]. We will also put this charge vanishing condition at the punctures for our study of the exponential convergence and of the Fredholm theory at least in the present paper, without involving the symplectization.

To put the research performed in the present paper in perspective, we recall the precise statement of the above mentioned a priori $W^{k,2}$ estimates established in [OW2] on the punctured Riemann surface $\dot\Sigma$ here. Denote

$$w^*\lambda = a_1^w\, d\tau + a_2^w\, dt.$$

Theorem 1.4 (Theorem 1.9 [OW2])**.** *Let $(\dot\Sigma, j)$ and w satisfying* (2.1) *on $\dot\Sigma$ as above. If $|d^\pi w| \in L^2 \cap L^4$ and $\|w^*\lambda\|_{C^0} < \infty$ on $\dot\Sigma$, then*

$$\int_{\dot\Sigma} |(\nabla)^{k+1}(dw)|^2 \leq \int_{\dot\Sigma} J'_k(d^\pi w, w^*\lambda).$$

Here J'_{k+1} a polynomial function of the norms of the covariant derivatives of $d^\pi w$, $w^*\lambda$ up to $0, \ldots, k$ with degree at most $2k+4$ whose coefficients depend on

$$\|K\|_{C^k}, \|R^\pi\|_{C^k}, \|\mathcal{L}_{X_\lambda} J\|_{C^k}, \|w^*\lambda\|_{C^0}.$$

One novel feature of this estimate is its explicit reliance on the C^0 bound of $w^*\lambda$ which concerns the X_λ component of dw. Therefore the remaining task is to complete the a priori estimates to study compactness properties of the moduli space of contact instantons is to further analyze how to control the quantities

$$\|w^*\lambda\|_{C^0}, \quad \|d^\pi w\|_{L^4}.$$

1.1. **Bubbling-off analysis and ϵ-regularity theorem.** One of the main purposes of the present article is to establish the two crucial analytical components in the construction of cementification of the moduli space of solutions of the contact instantons, one the ϵ-regularity theorem and the other the bubbling-off analysis.

To state the ϵ-regularity statement relevant to contact instantons, we recall the following standard quantity in contact geometry

Definition 1.5. Let λ be a contact form of contact manifold (Q, ξ). Denote by $\mathfrak{Reeb}(Q, \lambda)$ the set of closed Reeb orbits. We define $\operatorname{Spec}(Q, \lambda)$ to be the set

$$\operatorname{Spec}(Q, \lambda) = \left\{ \int_\gamma \lambda \mid \lambda \in \mathfrak{Reeb}(Q, \lambda) \right\}$$

and call the *action spectrum* of (Q, λ). We denote

$$T_\lambda := \inf \left\{ \int_\gamma \lambda \mid \lambda \in \mathfrak{Reeb}(Q, \lambda) \right\}.$$

We set $T_\lambda = \infty$ if there is no closed Reeb orbit. This set a priori could be empty. The Weinstein conjecture is equivalent to the statement that this set is non-empty on any compact contact manifold. A standard lemma in contact geometry says that $T_\lambda > 0$. This constant T_λ enters in a crucial way in the following ϵ-regularity type statement. In addition, we also need a Hofer-type energy, denoted by $E^\lambda(w)$ whose precise definition we refer readers to section 5.

Theorem 1.6. *Denote by D the closed disc of positive radius. Suppose that $w : D \to Q$ satisfies $\overline\partial^\pi w = 0$, $d(w^*\lambda \circ j) = 0$ with $E^\lambda(w) := K_0 < \infty$. Then for any $\epsilon > 0$ and another smaller disc $D' \subset \overline{D}' \subset D$, there exists some $K_1 = K_1(p, D', \epsilon, K_0) > 0$ such that for any contact instanton with $E^\pi(w) < T_\lambda - \epsilon$*

$$\|dw\|_{1,p;D'} \leq K_1 \tag{1.6}$$

where K_1 depends only on p, ϵ, and $D' \subset D$ and $K_0 = E^\lambda(w)$.

The proof of this theorem follows the scheme of the corresponding result in the study of pseudoholomorphic curves given by the author in [Oh1]. This proof uses the Sacks-Uhlenbeck's bubbling-off argument which essentially uses the a priori coercive $W^{k,p}$ elliptic estimates and conformal invariance of harmonic energy. In the current case of contact instanton maps, the relevant coercive estimate was established in

[OW2]. On the other hand the harmonic energy is quite irrelevant but the π-harmonic energy $E^\pi(w)$ is. However the π-harmonic energy does not have much control of the derivative dw in the Reeb direction. In the case of symplectization, Hofer [Ho1, BEHWZ] introduced the so called λ-energy for the map $u = (a, w) : \dot\Sigma \to \mathbb{R} \times Q$ for this purpose. His definition of the latter energy strongly relies on the coordinate function $a = r \circ w$ which exists only under the assumption the form $w^*\lambda \circ j$ is exact. For the non-exact case, we have to devise a different way of defining Hofer-type λ-energy. For this purpose, we introduce the notion of *contact instanton potential* whose definition relies on Zwiebach's representation of conformal structure j on the surface $\dot\Sigma$ by the minimal area metrics [Z, WZ]. See section 5 for the details. In the end, our definition of Hofer-type energy strongly depends on the complex structure j and so had better be regarded as a function for the pair (j, w) not just for w.

1.2. **Asymptotic behavior of contact instantons.** We also carry out the asymptotic study of contact instantons near the punctures. For this study of asymptotic convergence result at the punctures and the relevant index theory, it turns out to be useful to regard (1.5) as a version of gauged sigma model with abelian Hick's field. It is also important to employ the notion of asymptotic contact instantons at each puncture, which is a massless instantons on $\mathbb{R} \times S^1$ canonically associated to any finite energy contact instantons. It also gives rise to an asymptotic Hick's field, which is a holomorphic one-form that appears as the asymptotic limit of the complex-valued $(1,0)$-form
$$w^*\lambda \circ j + \sqrt{-1} w^*\lambda.$$
The following asymptotic invariants seem to be also useful to introduce in relation to the precise study of asymptotic behavior of contact instantons near punctures.

Definition 1.7 (Asymptotic Hick's charge)**.** Let (Σ, j) be a closed Riemann surface and $\dot\Sigma$ its associated punctured Riemann surface with finite energy with bounded gradient. Let p be a given puncture of $\dot\Sigma$. We define the *asymptotic Hick's charge* of the instanton $w : \dot\Sigma \to Q$ to be the complex number
$$Q(p) + \sqrt{-1} T(p)$$
defined by
$$Q(p) = -\int_{S^1} \operatorname{Re} \chi(0,t)\, dt = -\int_{\partial_{\infty;p}\overline{\Sigma}} w^*\lambda \circ j \tag{1.7}$$
$$T(p) = \int_{S^1} \operatorname{Im} \chi(0,t)\, dt = \int_{\partial_{\infty;p}\overline{\Sigma}} w^*\lambda \tag{1.8}$$
where $z = e^{-2\pi(\tau + it)}$ is the analytic coordinates of $D_r(p)$ centered at p. We call $Q(p)$ the *contact instanton charge* of w at p hand $T(p)$ the *contact instanton action* of w at p.

We define the asymptotic Hick's field (or charge) of a map $w : \mathbb{C} \to Q$ at infinity by regarding ∞ as a puncture associated $\mathbb{C} \cong \mathbb{C}P^1 \setminus \{\infty\}$.

We next prove the following removable singularity result (see Theorem 8.7).

Theorem 1.8. *Suppose $Q(p) = 0 = T(p)$. Then w is smooth across p and so the puncture p is removable.*

This theorem will be of fundamental importance in that it enables us to construct a good compactification of the moduli space of *exact contact instantons* without involving symplectization. This will be dealt in a sequel to this paper.

The theorem also allows us to make the following classification of the punctures.

Definition 1.9 (Classification of punctures)**.** Let $\dot\Sigma$ be a puncture Riemann surface with punctures $\{p_1, \cdots, p_k\}$ and let $w : \dot\Sigma \to Q$ be a contact instanton map.

(1) We call a puncture p *removable* if $T(p) = Q(p) = 0$, and *non-removable* otherwise. Among the non-removable punctures p, we call it *non-adiabatic* if $T(p) \neq 0$, *adiabatic* if $T(p) = 0$ but $Q(p) \neq 0$.
(2) We say a non-removable puncture *positive* (resp. *negative*) puncture if the function
$$\int_{\partial D_\delta(p)} w^*\lambda$$
is increasing (resp. decreasing) as $\delta \to 0$.

The appearance of adiabatic punctures is a new phenomenon when the form $w^*\lambda \circ j$ is not exact. In the exact case considered via the case of symplectization picture [Ho1, BEHWZ], the associated puncture with $T(p) = 0$ is removable and can be dropped in this classification by filling in the puncture.

Unlike the exact case, the puncture cannot be removed in general for the non-exact case, i.e., that of non-zero charge $Q(p) \neq 0$, even when $T(p) = 0$. Therefore this new asymptotic behavior has to be included in the study of moduli space of contact instantons. What happens at such a puncture is that the instanton w spirals around a leaf of Reeb foliation when the leaf is closed and chases along the leaf when it is not closed.

We would like to point out the similarity between the relationship of the forms $w^*\lambda \circ j$ and $w^*\lambda$ for the contact instanton w and the relationship between the electricity and magnetism in the electro-magnetic duality, in that in both cases the first is associated to the closed one-form while the second is not. The following highlights the similarity between the two:

$$\begin{array}{rcl} \text{electricity} & \longleftrightarrow & \text{contact instanton charge field } w^*\lambda \circ j \\ \text{electric potential} & \longleftrightarrow & \text{contact instanton potential } f \\ \text{magnetism} & \longleftrightarrow & \text{contact instanton action field } w^*\lambda \end{array} \tag{1.9}$$

1.3. **Triad connection, Fredholm theory and index calculations.** Next we establish the Fredholm theory and compute the index of the linearization map and hence the virtual dimension of the relevant moduli space of contact instantons. Establishing the Fredholm theory for the linearization map $D\Upsilon(w)$ is rather non-trivial because the operator has different orders depending on the direction of contact distribution ξ or on the Reeb direction X_λ and mixes the directions of the two. See Theorem 1.10 below. Our Fredholm theory and its index calculations strongly relies on our precise calculation of the linearization map via the contact triad connection introduced in [OW1]. We refer to section 10 for the details of the computations.

We denote by Σ either the closed Riemann surface or the punctured one. Recalling the decomposition
$$Y = Y^\pi + \lambda(Y)\, X_\lambda,$$

we have the decomposition

$$\Omega^0(w^*TQ) \cong \Omega^0(\Sigma, \mathbb{R}) \cdot X_\lambda \oplus \Omega^0(w^*\xi).$$

Here we use the splitting

$$TQ = \mathrm{span}_{\mathbb{R}}\{X_\lambda\} \oplus \xi$$

where $\mathrm{span}_{\mathbb{R}}\{X_\lambda\} := \mathcal{L}$ is a trivial line bundle and so

$$\Gamma(w^*\mathcal{L}) \cong C^\infty(\Sigma).$$

Define the map $\Upsilon(w) = (\overline{\partial}^\pi w, w^*\lambda\, X_\lambda)$. From the expression of the map $\Upsilon = (\Upsilon_1, \Upsilon_2)$, the map defines a bounded linear map

$$D\Upsilon(w) : \Omega^0(w^*TQ) \to \Omega^{(0,1)}(w^*\xi) \oplus \Omega^2(\Sigma). \tag{1.10}$$

We choose $k \geq 2$, $p > 2$. We then establish the following formula

Theorem 1.10 (Theorem 10.1)**.** *Decompose $D\Upsilon(w) = D\Upsilon_1(w) \oplus D\Upsilon_2(w)$ according to the codomain of* (1.10)*. Then we have*

$$\begin{aligned} D\Upsilon_1(w)(Y) &= \overline{\partial}^{\nabla^\pi} Y^\pi + T^{\pi,(0,1)}_{dw}(Y^\pi) + B^{(0,1)}(Y^\pi) \\ &\quad + \frac{1}{2}\lambda(Y)(\mathcal{L}_{X_\lambda} J) J(\partial^\pi w) \qquad (1.11) \\ D\Upsilon_2(w)(Y) &= -\Delta(\lambda(Y))\, dA + d((Y^\pi \rfloor d\lambda) \circ j) \qquad (1.12) \end{aligned}$$

where $T^{\pi,(0,1)}_{dw}$ and $B^{(0,1)}$ are the $(0,1)$-components of T^π_{dw} and B respectively where $B : \Omega^0(w^*TQ) \to \Omega^1(w^*\xi)$, T^π_{dw} are the zero-order differential operators given by

$$B(Y) = -\frac{1}{2} w^*\lambda\, ((\mathcal{L}_{X_\lambda} J) J Y)$$

and

$$T^\pi_{dw}(Y) = \pi T(Y, dw).$$

We denote by $\overline{\Sigma}$ the real blow-up of the punctured Riemann surface $\dot\Sigma$ associated to the set of positive and negative punctures

$$\{p_1, \cdots, p_{s^+}\}, \quad \{q_1, \cdots, q_{s^-}\}$$

and denote by $\partial_i^+\overline{\Sigma}$ and $\partial_j^-\overline{\Sigma}$ the associated boundary components. We also denote by γ_i^+ and γ_j^- the given asymptotic Reeb orbits at the punctures.

We fix a trivialization $\Phi : w^*\xi \to \overline{\Sigma} \times \mathbb{R}^{2n}$ and denote by Ψ_i^+ (resp. Ψ_j^-) the induced symplectic paths associated to the trivializations Φ_i^+ (resp. Φ_j^-) along the Reeb orbits γ_i^+ (resp. γ_j^-) at the punctures p_i (resp. q_j) respectively. Then we have the following index formula for the case of vanishing charge. We leave more accurate statements and proof to section 11, and the case of non-exact contact instantons elsewhere.

Theorem 1.11. *Consider the map Υ defined by $\Upsilon(w) = (\overline{\partial}^\pi w, d(w^*\lambda \circ j))$ on a puncture Riemann surface $\dot\Sigma$. Let w be an exact contact instanton, i.e. a solution of $\Upsilon(w) = 0$ with $Q(p_i) = 0$ for all punctures p_i.*

(1) *There exists a compact operator*

$$K : \Omega^0_{k,p}(w^*TQ) \to \Omega^{(0,1)}_{k-1,p;\delta}(w^*\xi) \oplus \Omega^2_{k-2,p;\delta}(\Sigma)$$

such that

$$\begin{aligned} \|D\Upsilon(w)Y\|_{k,p;\delta} &\leq C(\|D\Upsilon_1(w)(Y)\|_{k-1,p;\delta} + \|\pi_1 K(Y)\|_{k-1,p;\delta} \\ &\qquad + \|D\Upsilon_2(w)(Y)\|_{k-2,p;\delta} + \|\pi_2 K(Y)\|_{k-2,p}) \end{aligned}$$

and so the completed map

$$D\Upsilon(w) : \Omega^0_{k,p;\delta}(w^*TQ) \to \Omega^{(0,1)}_{k-1,p;\delta}(w^*\xi) \oplus \Omega^2_{k-2,p;\delta}(\Sigma)$$

is a Fredholm operator if $\delta \in \mathbb{R} \setminus \mathcal{D}_w$ *for some discrete subset* $\mathcal{D}_w$ *of* $\mathbb{R}$.

(2) *Furthermore, provided* $0 < \delta < \delta_0$ *for a sufficiently small* δ_0 *depending only on* w,

$$\begin{aligned} \operatorname{Index} D\Upsilon(w) &= n(2 - 2g - s^+ - s^-) + 2c_1(w^*\xi) \\ &\quad + \sum_{i=1}^{s^+} \mu_{CZ}(\Psi_i^+) - \sum_{j=1}^{s^-} \mu_{CZ}(\Psi_j^-) \\ &\quad + \sum_{i=1}^{s^+} (m(\gamma_i^+) + 1) + \sum_{j=1}^{s^-} (m(\gamma_j^-) + 1) - g \end{aligned}$$

where $\mu_{CZ}(\Psi)$ *is the Conley-Zehnder index of the symplectic path* Ψ *associated to the closed Reeb orbit* [CZ, RS, Ho1].

We would like to highlight the appearance of the second line that extracts explicit contribution depending on the mutiplicity of the closed Reeb orbits. Such an appearance in this kind of index formula seems to be new, at least such an explicit dependence on the multiplicity does not shows up in the standard index formula in symplectic field theory such as in Proposition 5.3 [Bo] (with $N = 0$.)

One important feature of our analysis of (1.5) is that we do not take symplectization of contact triad (Q, λ, J) but directly work on the contact manifold Q. Hence it enables us to get ready to construct compactification of the smooth moduli space of contact instantons (at least of exact contact instantons) with punctured Riemann surfaces as their domains and contact manifold Q as their targets, and so to define a genuinely contact topological invariants *without taking the symplectization of* Q. Indeed the question if two contact manifolds having symplectomorphic symplectization are contactomorphic or not was addressed in the book by Cieliebak and Eliashberg [CE] and recently S. Courte [Co] announced construction of two contact manifolds that have symplectomorphic symplectization which are not contactomorphic. In this regard, we hope to investigate the following question stated in [Co] in the future.

Question 1.12. Does there exist contact structures ξ and ξ' on a closed manifold M that have the same classical invariants and are not contactomorphic, but whose symplectizations are (exact) symplectomorphic?

We thank Rui Wang for the collaboration of the works [OW1, OW2, OW3] which the current research is partially based on. We also thank her for some useful comments on the present paper.

2. Three elliptic twistings of contact Cauchy-Riemann map equation

The contact Cauchy-Riemann equation itself $\overline{\partial}^\pi w = 0$ does not form an elliptic system because it is degenerate along the Reeb direction: Note that the rank of w^*TQ has $2n+1$ while that of $w^*\xi \otimes \Lambda^{0,1}(\Sigma)$ is $2n$. Therefore to develop suitable deformation theory and a priori estimates, one needs to lift the equation to an elliptic system. In hindsight, the pseudoholomorphic curve system of the pair (a, w) is one such lifting via introducing an auxiliary variable a a function on the Riemann surface, when the one-form $w^*\lambda \circ j$ is exact. Hofer [Ho1] did this by lifting the equation to the symplectization $\mathbb{R} \times Q$ and considering the pull-back function $a := s \circ w$ of the $\mathbb{R}$-coordinate function s of $\mathbb{R} \times Q$. By doing so, he *added one more variable* to the equation $\overline{\partial}^\pi w = 0$ while *adding 2 more equations* $w^*\lambda \circ j = da$ and produced an elliptic system which is exactly becomes Gromov's pseudoholomorphic curve system on the symplectization $\mathbb{R} \times Q$.

2.1. Contact instanton lifting of contact Cauchy-Riemann map.

It turns out, again by hindsight, the current contact instanton map system

$$\overline{\partial}^\pi w = 0, \quad d(w^*\lambda \circ j) = 0 \tag{2.1}$$

is such an elliptic lifting which is more natural in some respect in that it does not introduce any additional variable and keeps the original 'bulk', the contact manifold Q.

The relevant a priori (local) elliptic estimates and the global exponential decay estimates near the puncture of the punctured Riemann surface $\dot\Sigma$ have been established in [OW2]. This is the lifting whose study is the main theme of the present paper and is also closely related to the following lifting of *gauged sigma model with abelian Hick's field*. We would like to emphasize that this lifting includes the study of pseudoholomorphic curves in symplectization as the special case of exact $w^*\lambda \circ j$.

2.2. Gauged sigma model lifting of contact Cauchy-Riemann map.

There is another lifting of w this time involving a section of complex line bundle

$$\mathcal{L}_\lambda \to Q \tag{2.2}$$

whose fiber at $q \in Q$ is given by

$$\mathcal{L}_{\lambda,q} = \mathbb{R}_{\lambda,q} \otimes \mathbb{C}$$

where $\mathbb{R}_\lambda \to Q$ is the trivial real line bundle whose fiber at q is given by

$$\mathbb{R}_{\lambda,q} = \mathbb{R}\{X_\lambda(q)\}.$$

Note that $\mathcal{L}_q$ has a canonical identification with the bundle

$$T(\mathbb{R}_+ \times Q)|_{\{r=1\}} = \mathbb{R} \cdot \frac{\partial}{\partial s} \oplus \mathbb{R} \cdot X_\lambda \oplus \xi$$

in the symplectization $\mathbb{R}_+ \times Q$.

Now let $w : \Sigma \to Q$ be a smooth map where Σ is either closed or a punctured Riemann surface, and χ be a section of the pull-back bundle $w^*\mathcal{L}_\lambda$.

Definition 2.1. We call a triple (w, j, χ) consisting of a complex structure j on Σ, $w : \Sigma \to Q$ and a $\mathbb{C}$-valued one-form χ a *gauged contact instanton* if they satisfy

$$\begin{cases} \overline{\partial}^\pi w = 0 \\ \overline{\partial}\chi = 0, \quad \operatorname{Im}\chi = w^*\lambda. \end{cases} \tag{2.3}$$

This system is a coupled system of the contact Cauchy-Riemann map equation and the well-known Riemann-Hilbert problem of the type which solves the real part in terms of the imaginary part of holomorphic functions in complex variable theory. A detailed study of this elliptic system will be carried out elsewhere.

2.3. **Pseudoholomorphic lifting of contact Cauchy-Riemann map.** The above two liftings do not have any restriction on the cohomology class $[w^*\lambda\circ j] \in H^1(\dot\Sigma;\mathbb{R})$. On the other hand, there is the more commonly known elliptic twisting *under the restriction that* $w^*\lambda\circ j$ *is exact*, and *with the specification of the anti-derivative of* $w^*\lambda\circ j$ as an auxiliary variable $a : \Sigma \to \mathbb{R}$ by requiring

$$w^*\lambda\circ j = da$$

whose expounding is now in order. We call a contact instanton *exact* if $[w^*\lambda\circ j] = 0$.

Remark 2.2. We would like to point out that the exact case itself forms a closed realm in the study of contact instantons and does not need to involve symplectization in its study. If we restrict to the exact contact instantons, any adiabatic puncture with $T = 0$ will be removable as in the case of pseudoholomorphic curves by Theorem 1.8. This enables us to perform the standard Gromov-Floer theory type compactification of the moduli space of exact contact instantons and to define a Floer homology type invariants. However the geometry of contact instantons is not exactly the same as that of pseudoholomorphic curves in symplectization and so we do not expect the algebraic structures of the contact homology type invariants coincide. We will come back elsewhere to the study of compactification of the moduli space of exact contact instantons and of construction of the relevant contact homology type invariants.

We consider the canonical symplectization $E \to Q$ (see section 3).

Note that in the presence of contact form λ, any smooth map $w : \dot\Sigma \to Q$ can be naturally lifted to a map $\widetilde w : \dot\Sigma \to W$ so that

$$\widetilde w(z) = a(z)\lambda(w(z)) \in W_{w(z)} \subset T^*_{w(z)}Q \tag{2.4}$$

for some function $a : \dot\Sigma \to \mathbb{R}_+$ or equivalently to a map

$$(a, w) : \dot\Sigma \to \mathbb{R}\times Q$$

via the trivialization $\exp\circ\Phi_\lambda : \mathbb{R}\times Q \to W$.

Now we equip (Q,ξ) with a triad (Q,λ,J) and the cylindrical almost complex structure $\widetilde J = J_0 \oplus J$. Then the derivative $d\widetilde w = da\frac{\partial}{\partial s} \oplus dw$ can be further decomposed to

$$d\widetilde w(z) = da\frac{\partial}{\partial s} \oplus w^*\lambda X_\lambda \oplus d^\pi w. \tag{2.5}$$

as a TW-valued 1-form with respect to the splitting

$$\mathrm{Hom}(T_z\dot\Sigma, T_{\widetilde w(z)}W) = \mathrm{Hom}(T_z\dot\Sigma, VT_{\widetilde w(z)}W) \oplus \mathrm{Hom}(T_z\dot\Sigma, HT_{\widetilde w(z)}W).$$

By definition, we have

$$d\pi\widetilde{dw} = dw.$$

It was derived by Hofer [Ho1] that $\widetilde w$ is $\widetilde J$-holomorphic if and only if (a, w) satisfies

$$\begin{cases} \overline\partial^\pi w = 0 \\ w^*\lambda\circ j = da. \end{cases} \tag{2.6}$$

3. Canonical symplectization and Hofer's λ-energy; revisit

In this subsection, we first recall the canonical symplectization of contact manifold (Q,ξ) explained in Appendix 4 [Ar], which does not involve the choice of contact form. We denote this canonical symplectization by (W,ω_W) which is defined to be

$$\{\alpha \in T^*Q \mid \alpha \neq 0, \ker\alpha = \xi\} \subset T^*Q \setminus \{0\}. \tag{3.1}$$

When Q is oriented and a positive contact form λ is given, we can canonically lift a map $w : \dot\Sigma \to Q$ to a map $\widehat w : \dot\Sigma \to W$. We then examine the relationship between w being a contact instanton and $\widehat w$ being a pseudoholomorphic curves on W with respect to scale-invariant almost complex structure on W. We give a geometric description of Hofer's remarkable energy introduced in [Ho1] in terms of this canonical symplectization. This energy is the key ingredient needed in the bubbling-off analysis and so in the construction of the compactification of the moduli spaces of pseudoholomorphic curves needed to develop the symplectic field theory [EGH], [BEHWZ]. In section 5, we will then introduce its variant for the study of contact instanton maps whose charge is not necessarily vanishing, i.e. $w^*\lambda\circ j$ does not have to be exact.

Consider the $(2n+2)$-dimensional submanifold W of T^*Q defined in (3.1). When we fix an orientation Q, we can consider

$$W = \{\alpha \in T^*Q \setminus \{0\} \mid \ker\alpha = \xi,\ \alpha(\vec n) > 0\} \tag{3.2}$$

where $\vec n$ is a vector such that $\mathbb{R}\{\vec n\} \oplus \xi$ becomes a positively oriented basis. Note that W is a principal $\mathbb{R}_+$-bundle over Q that is trivial.

We denote by $i_W : W \hookrightarrow T^*Q$ and by Θ the Liouville one-form on T^*Q. The basic proposition is that W carries the canonical symplectic form

$$\omega_W = -i_W^* d\Theta.$$

One important point of this canonical symplectization is the fact that it depends only on the orientation of Q but does not depend on the choice of contact form λ. The symplectic form ω_W provides a natural symplectic (Ehresman) connection provided by the splitting

$$TW = VTW \oplus \widetilde{TQ} \tag{3.3}$$

where VTW is the vertical tangent bundle and

$$\widetilde{TQ}|_\alpha = \{\eta \in T_\alpha W \mid \omega_W(\eta,\cdot) \equiv 0\}. \tag{3.4}$$

Now we choose a contact form λ so that $\lambda\wedge(d\lambda)^n$ is positive with respect to the given orientation. Since λ provides a section of of $W \to Q$, it induces a trivialization of W as the principal $\mathbb{R}_+$-bundle

$$\Phi_\lambda : \mathbb{R}_+ \times Q \to W; \quad (r,q) \mapsto r\,\lambda(q)$$

which in turn leads to the natural isomorphism

$$\mathbb{R} \oplus TQ \cong TW = \mathbb{R}\cdot\lambda \oplus \widetilde{TQ} \tag{3.5}$$

defined by $(c,Z) \mapsto c\lambda \oplus \widetilde Z$. Combining this with (3.3), we obtain the splitting

$$TW = \widetilde{\mathbb{R}\cdot\lambda} \oplus \widetilde{\mathbb{R}\cdot X_\lambda} \oplus \widetilde\xi. \tag{3.6}$$

We note that there is a canonical paring on $\widetilde{\mathbb{R}\cdot\lambda} \oplus \widetilde{\mathbb{R}\cdot X_\lambda}$ given by

$$\langle \lambda, X_\lambda\rangle = 1$$

and so it carries the canonical symplectic form thereon. We summarize the above discussion into

Proposition 3.1. *Suppose Q is given an orientation and a positive contact form λ. Then it provides a natural $\mathbb{R}_+$-equivariant symplectomorphism*

$$\Phi_\lambda : \mathbb{R}_+ \times Q \to W$$

whose derivative induces a canonical $\mathbb{R}_+$-equivariant symplectic vector bundle isomorphism

$$d\Phi : (\mathbb{R}^2 \oplus TQ, \omega_{0,2} \oplus d\lambda) \to (TW, \omega_W)$$

$$d\Phi_\lambda(a, b, Z) = a\,\widetilde{\lambda} + b\,\widetilde{X}_\lambda + \widetilde{Z}. \tag{3.7}$$

The usual symplectization of (Q, λ) used in the literature is nothing but $\mathbb{R}_- \times Q$ with the pull-back symplectic form $(\Phi_\lambda)^*\omega_W$ thereto, which can be explicitly written as

$$(\Phi_\lambda)^*\omega_W = (\Phi_\lambda \circ i_W)_*\Theta = d(r\,\pi^*\lambda)$$

where $r = r_\lambda \in \mathbb{R}^+$ is the radial coordinate such that the embedding $Q \hookrightarrow W$ corresponds to the hypersurface $r = 1$ and $\pi : \mathbb{R}_+ \times Q \to Q$ the projection. If we now pull-back this form to $\mathbb{R} \times Q$ by the diffeomorphism $\exp : \mathbb{R} \times Q \to \mathbb{R}_+ \times Q$ defined by $\exp(s, q) = (e^s, q)$, then the corresponding symplectic form becomes

$$e^s(\pi^* d\lambda + ds \wedge \pi^*\lambda), \quad \pi : \mathbb{R} \times Q \to Q.$$

Next we involve an endomorphism $J : \xi \to \xi$ with $J^2 = -id$ such that (ξ, J, g_λ) with $g_\lambda = d\lambda(\cdot, J\cdot)$ becomes a Hermitian vector bundle. For the purpose of doing analysis on $\mathbb{R} \times Q$, we need to provide a *cylindrical metric* thereon which we choose

$$g_\lambda + dr^2 = d\lambda(\cdot, J\cdot) + \lambda \otimes \lambda + dr^2$$

and cylindrical almost complex structure

$$\tilde{J} = J_0 \oplus J$$

on $T(\mathbb{R} \times Q) \cong \mathbb{R}\{\frac{\partial}{\partial r}\} \oplus \mathbb{R}\{X_\lambda\} \oplus \xi$. On the other hand, the pull-back symplectic form becomes

$$e^s(\pi^* d\lambda + ds \wedge \pi^*\lambda)$$

which is *not cylindrical*. The above fact that the pull-back symplectic form is not cylindrical makes the topological control of the *full* harmonic energy of a $\widetilde{J}$-holomorphic map $u : \Sigma \to \mathbb{R} \times Q$ by the symplectic area of this symplectic form not possible in general, *unless one has the control of the coordinate* $a = s \circ w$.

Instead one tries to control the *local* (in target) harmonic energy by considering the map

$$\widehat{\psi} : \mathbb{R} \times Q \to W; \quad \widehat{\psi}(s, x) = \psi(s)\,(\pi^*\lambda)(x)$$

associated to each monotonically increasing function ψ such that

$$\psi(s) = \begin{cases} 1 & \text{for } s \geq R_1 \\ \frac{1}{2} & \text{for } s \leq R_0 \end{cases} \tag{3.8}$$

for any pair $R_0 < R_1$ of real numbers. We measure the symplectic area of the composition $\widehat{\psi} \circ w : \dot{\Sigma} \to W$ for all possible variations of such ψ. Hofer's original definition of this type of energy then can be expressed as the integral

$$\begin{aligned} E_{\mathcal{C}}(u) &:= \sup_{\psi} \int_{\dot\Sigma} (\widehat{\psi} \circ u)^* \omega_W & (3.9)\\ &= \sup_{\psi} \int_{\dot\Sigma} (\widehat{\psi} \circ u)^* d(r\, \pi^*\lambda) \\ &= \sup_{\psi} \int_{\dot\Sigma} d(\psi(s)\, \pi^*\lambda) & (3.10)\\ &= \sup_{\psi} \left(\int_{\dot\Sigma} \psi(a) dw^*\lambda + \psi'(a)\, da \wedge w^*\lambda \right). & (3.11) \end{aligned}$$

Note that (3.10) is precisely the same as Hofer's original definition of his energy given in [Ho1]. Later in [BEHWZ], the authors split this energy into two parts, one purely depending on w

$$E^\pi(w) = \int_{\dot\Sigma} dw^*\lambda$$

and the other

$$E^\lambda(u) = \sup_{\psi} \int_{\dot\Sigma} \psi'(a)\, da \wedge w^*\lambda.$$

In retrospect, it was an amazing insight of Hofer [Ho1] that this way of considering nicely controls the bubbling-off analysis when there is no apparent way of controlling the asymptotic behaviour of the bubble map $\mathbb{C} \to \mathbb{R} \times Q$ when the bubble map is not confined in a compact domain of $\mathbb{R} \times Q$.

4. Jenkins-Strebel quadratic differential and minimal area metrics

For any given marked Riemann surface $(\Sigma, \{r_1, \cdots, r_k\})$, we denote by $\dot\Sigma$ the associated punctured Riemann surface. We assume either genus $\Sigma \geq 1$ or genus $\Sigma = 0$ with $k \geq 2$.

Following Zwiebach [Z], we give a description of the notion of minimal area metric associated to the given punctured Riemann surface $\dot\Sigma$ and its relationship with the Jenkins-Strebel quadratic differentials. We also refer to section 2 of Bergmann's preprint [Be] for some discussion that is in the similar spirit as that of this section.

Definition 4.1. A metric $h = \rho\, |dz|$ is called *admissible* for a set of constants A_j if

$$\int_{\gamma; \gamma\sim\gamma_j} \rho\, |dz| \geq A_j$$

for any curve γ homotopic to γ_j in $\dot\Sigma$.

In this metric, one has the semi-infinite tubes of circumference $\ell \geq A_j$ at each puncture r_j. Near the puncture r_j, one must have

$$\rho^2(z) \sim (A_j/2\pi|z|)^2.$$

Definition 4.2 (Reduced area [Z])**.** The reduced area, denoted by $\mathrm{Area}^{red}(\Sigma, h)$ is given by

$$\mathrm{Area}^{red}(\Sigma, h) = \lim_{\delta\to 0} \left(\int\int_{\Sigma(\delta)} dA + \frac{1}{2\pi} \ln\delta \sum_{j=1}^k A_j^2 \right) \tag{4.1}$$

where $\Sigma(\delta)$ denotes the surface obtained by excising the discs $|z_j| \leq \delta$ from Σ.

Definition 4.3 (Minimal area metric)**.** A metric h on $\dot\Sigma$ is called a *minimal area metric* if the reduced area is minimal among all possible metrics arising from quadratic differentials.

From now on, we restrict ourselves to the case of $g = 0$. We will need the following basic existence and the uniqueness result proved in [Z]

Theorem 4.4 (Zwiebach [Z])**.** *When $g = 0$ and $k \geq 3$, there exists a unique minimal area metric associated to each $(\Sigma, j) \in \mathcal{M}_{0,k}$, which continuously extends to the compactification $\overline{\mathcal{M}}_{0,k}$.*

In other words, the minimal area metric provides a natural slice to the well-known isomorphism between the set of complex structures and the set of conformal isomorphism classes of associated metrics, which respects the sewing rule of the degeneration of conformal structures. A similar representation of the conformal structure on the boundary punctured discs, the open string analogue of the above theorem, was used in Fukaya and the author's work [FO] in their study of adiabatic degeneration of pseudo-holomorphic polygons with Lagrangian boundaries on the cotangent bundle.

It is also shown that each minimal area metric arises from Jenkins-Strebel quadratic differential [J], [St] whose singularities are at most a pole. Some brief account on Jenkin-Strebel quadratic differential should be in order. A quadratic differential φ on a Riemann surface Σ is a set of function elements $\phi_i(z_i)$, meromorphic in the local coordinates $z_i = x_i + iy_i$ with transformation property

$$\phi_i(z_i)(dz_i)^2 = \phi_j(z_j)(dz_j)^2, \tag{4.2}$$

under a change of local coordinates. A quadratic differential defines a metric $|\phi_i(z_i)||dz_i|^2$.

A horizontal trajectory of a quadratic differential is a curve along which $\phi(z)(dz)^2$ is real and positive.

Definition 4.5. A Jenkins-Strebel quadratic differential is a quadratic differential for which the nonclosed trajectories cover a set of measure zero on the surface.

A JS quadratic differential decomposes a surface into characteristic ring domains, the maximal ring domains swept by the closed trajectories. These ring domains can be annuli or punctured discs.

On a punctured discs $D(1)\setminus\{0\}$ with coordinates w, the JS quadratic differential is given by the form

$$\phi_{JS}(z)\, dz^2, \quad \phi_{JS}(z) = -\frac{a^2}{(2\pi)^2}\frac{1}{z^2} \tag{4.3}$$

where $2\pi a$ is the length of the horizontal trajectory of the associated minimal area metric. The metric is flat and isometric to the semi-infinite tube $(-\infty, 0] \times S^1$ with coordinates (τ, t) with $u = \tau + i\,t$ and proportional to the standard metric $d\tau^2 + dt^2$. This is nothing but the canonical isothermal coordinate of the metric and satisfies

$$du^2 = -\frac{a^2}{(2\pi)^2}\frac{1}{z^2}dz^2. \tag{4.4}$$

Under the minimal area metric for the case of $g = 0$, $\dot\Sigma$ is a finite union of k semi-infinite cylinders and a finite set of cylinders with finite height of circumference

2π. So each cylinder is isometric to the standard cylinders, either $[0,\infty)\times S^1$ or $[0,\ell]\times S^1$ with metric

$$h = \left(\frac{a}{2\pi}\right)^2 (d\tau^2 + dt^2)$$

where (τ, t) is the standard coordinates on the cylinder. On each cylinder it carries the vector field $\frac{\partial}{\partial\tau}$, $\frac{\partial}{\partial t}$ which are invariant under the transformation

$$(\tau, t) \mapsto (\tau + \tau_0, t + t_0)$$

and so depends only on the metric. Denote by $S \subset \dot\Sigma$ the union of sewing seams of the set of cylinders given above. Then $\dot\Sigma$ carries a vector field $V = V(j)$ that restrict to the coordinate vector field $\frac{\partial}{\partial\tau}$ on each cylinder. As a result $V(j)$ is discontinuous along the S but its flow lines form a foliation those leaves are continuous even across the seams. The vector field V is called the vertical vector field and the associated foliation is called the vertical foliation of the quadratic differential associated to the minimal area metric [St]. Similarly the vector field $\frac{\partial}{\partial t}$ glues to define a global vector field $H(j)$ called the horizontal vector field, which is continuous except at a finite number of points.

We would like to mention that when we give a distinguished marked point r_0 as the 'output' and put the rest as the 'input' marked points as in the definition of A_∞-structures as in [FO, FOOO], the flow of the vector field $V(j)$ becomes an oriented foliation whose leaves consist of the flow lines of $V(j)$. Then the flow become continuous even across the seams.

We will also need to consider the case $g = 0$ and $k = 2$. (See [WZ] for the relevant discussion.) In this case, $\dot\Sigma$ with the minimal area metric is isometric to the standard cylinder $\mathbb{R}\times S^1$ with the metric $d\tau^2 + dt^2$. While the metric is uniquely determined, its associated flat coordinates are defined uniquely modulo the translations and rotations

$$(\tau, t) \mapsto (\tau + \tau_0, t + t_0), \quad \tau_0 \in \mathbb{R},\, t_0 \in S^1.$$

5. Off-shell energy of contact instantons

Fix a Kähler metric h on (Σ, j). The norm $|dw|$ of the map

$$dw : (T\Sigma, h) \to (TQ, g)$$

with respect to the metric g is defined by

$$|dw|_g^2 := \sum_{i=1}^2 |dw(e_i)|_g{}^2,$$

where $\{e_1, e_2\}$ is an orthonormal frame of $T\Sigma$ with respect to h.

The following are the consequences from the definition of contact Cauchy-Riemann map and the compatibility of J to $d\lambda$ on ξ, whose proofs we omit but refer to [OW1].

Proposition 5.1. *Denote $g_J = \omega(\cdot, J\cdot)$ and the associated norm by $|\cdot| = |\cdot|_J$. Fix a Hermitian metric h of (Σ, j), and consider a smooth map $u : \Sigma \to M$. Then we have*

(1) $|d^\pi w|^2 = |\partial^\pi w|^2 + |\overline\partial^\pi w|^2$,
(2) $2w^*d\lambda = (-|\overline\partial^\pi w|^2 + |\partial^\pi w|^2)\, dA$ *where dA is the area form of the metric h on Σ.*
(3) $w^*\lambda \wedge w^*\lambda \circ j = |w^*\lambda|^2\, dA$

(4) $|\nabla w^*\lambda|^2 = |dw^*\lambda|^2 + |\delta w^*\lambda|^2$.

We then introduce the ξ-component of the harmonic energy, which we call the π-harmonic energy. This energy equals the contact area $\int w^* d\lambda$ 'on shell' i.e., for any contact Cauchy-Riemann map, which satisfies $\overline{\partial}^\pi w = 0$

Definition 5.2. For a smooth map $\dot\Sigma \to Q$, we define the π-energy of w by

$$E^\pi(j,w) = \frac{1}{2}\int_{\dot\Sigma} |d^\pi w|^2. \tag{5.1}$$

As discovered by Hofer in [Ho1] in the context of symplectization, the π-harmonic energy itself is not enough for the crucial bubbling-off analysis needed for the equation (2.1). This is only because the bubbling-off analysis requires the study of asymptotic behavior of contact instantons on the complex place $\mathbb{C}$. A crucial difference between the current case of contact instantons from Gromov's theory of pseudoholomorphic curves on symplectic manifolds is that there is no removal singularity result of the type of harmonic maps (or pseudoholomorphic maps). Because of this, one needs to examine the X_λ-part of energy that controls the asymptotic behavior of contact instantons near the puncture. For this purpose, the Hofer-type energy introduced in [Ho1] is crucial. In this section, we generalize this energy to the general context of non-exact case without involving the symplectization.

Following the modification made in [BEHWZ] of Hofer's original definition [Ho1] (and denoting $\varphi = \psi'$ for the function ψ given in section 3), we introduce the following class of test functions

Definition 5.3. We define

$$\mathcal{C} = \{\varphi : \mathbb{R} \to \mathbb{R}_{\geq 0} \mid \operatorname{supp}\varphi \text{ is compact}, \int_{\mathbb{R}} \varphi = 1\}. \tag{5.2}$$

Let $w : \dot\Sigma \to Q$ be a contact instanton with the asymptotic charge $Q(p)$ at the puncture. Recall this number depends only on the homology class $[\gamma]$ of the loop $\gamma = w|_{D_\delta(p)}(\tau,\cdot) \subset \dot\Sigma \setminus \{p\}$ by the closedness equation of $w^*\lambda \circ j$, which does not depend on τ either.

Then on the given cylindrical neighborhood $D_\delta(p) \setminus \{p\}$, we can write

$$w^*\lambda \circ j + Q(p)\, dt = df$$

for some function $f : [0,\infty) \times S^1 \to \mathbb{R}$. Here dt is the one-form that is made of the one-form dt defined before on each cylinder. The form is globally continuous except at the finite number points at which the vector field $\frac{\partial}{\partial t}$ is not continuous. We call f the *contact instanton potential.*

We remark that when w is given, the function f on $D_\delta(p) \setminus \{p\}$ is uniquely determined modulo the shift by a constant.

Definition 5.4 ($E_{\mathcal{C}}$-energy)**.** Let w satisfy $d(w^*\lambda \circ j) = 0$. Then we define

$$E_{\mathcal{C}}(j,w) = \sup_{\varphi\in\mathcal{C}} \int_\Sigma d(\psi(f)) \wedge df \circ j = \sup_{\varphi\in\mathcal{C}} \int_\Sigma d(\psi(f)) \wedge (-w^*\lambda + Q(p)\, d\tau).$$

We note that

$$d(\psi(f)) \wedge df \circ j = \psi'(f) df \wedge df \circ j = \varphi(f) df \wedge df \circ j \geq 0$$

and hence we can rewrite $E_{\mathcal{C}}(j,w)$ into

$$E_{\mathcal{C}}(j,w) = \sup_{\varphi\in\mathcal{C}} \int_\Sigma \varphi(f) df \wedge df \circ j.$$

Proposition 5.5. *For a given smooth map w satisfying $d(w^*\lambda \circ j) = 0$, we have $E_{\mathcal{C};f}(w) = E_{\mathcal{C},g}(w)$ whenever $df = w^*\lambda \circ j + Q(p)\, dt = dg$ on $D^2_\delta(p) \setminus \{p\}$ (and so $g(z) = f(z) + c$ for some constant c on each connected component of Q).*

Proof. Certainly df or $df \circ j$ are independent of the addition by constant c. On the other hand, we have

$$\varphi(g) = \varphi(f + c)$$

and the function $a \mapsto \varphi(a + c)$ still lie in $\mathcal{C}$. Therefore after taking the supremum over $\mathcal{C}$, we have derived

$$E_{\mathcal{C},f}(j, w) = E_{\mathcal{C},g}(j, w).$$

This finishes the proof. □

This proposition enables us to introduce the following

Definition 5.6 (λ-energy at a puncture p)**.** We denote the common value of $E_{\mathcal{C},f}(j, w)$ by $E^\lambda_p(w)$, and call the λ*-energy at* p.

The following then would be the preliminary definition of the total energy.

Definition 5.7 (Total energy)**.** Let $w : \dot\Sigma \to Q$ be any smooth map. We define the total energy of w by

$$E(j, w) = E^\pi(j, w) + \sum_{l=1}^{k} E^\lambda_{p_l}(j, w). \tag{5.3}$$

We denote

$$E^\lambda(j, w) = \sum_{l=1}^{k} E^\lambda_{p_l}(j, w).$$

Remark 5.8.
1. To take further analogy with physics, one may regard the π-harmonic energy as the 'kinetic energy' of the contact instanton and the λ-energy as the 'potential energy' thereof respectively.
2. The above definition is unsatisfying and incomplete as an off-shell energy of the pair (j, w) *when we vary complex structure* j on the punctured surface $\dot\Sigma$. For this purpose, we need to involve the complex structure in the definition of E^λ also like $E^\pi(j, w)$ does. This is where the Zwiebach's notion of minimal area metric [Z], [WZ] enters which extends the cylindrical structure to the full Riemann surface not just to the punctured neighborhoods.

In the rest of the section, we assume Σ has genus 0. The reason for this restriction is only because for the higher genus case, the minimal area metric representation of conformal structure is over-counting [Z]. Other than this, the discussion below is equally applied to any conformal structure represented by a minimal area metric.

First, we assume $k \geq 2$, i.e., the number of marked points at least 2. In this case, the conformal structure carries the minimal area metric representation [Z]. Under the minimal area metric for the case of $g = 0$, $\dot\Sigma$ is a finite union of k semi-infinite cylinders and a finite set of cylinders with finite height of circumference 2π. So each cylinder is isometric to the standard cylinders, either $[0, \infty) \times S^1$ or $[0, \ell] \times S^1$ with metric

$$h = \left(\frac{a}{2\pi}\right)^2 (d\tau^2 + dt^2)$$

where (τ, t) is the standard coordinates on the cylinder. On each cylinder it carries the vector field $\frac{\partial}{\partial \tau}$ which is invariant under the transformation

$$(\tau, t) \mapsto (\tau + \tau_0, t + t_0)$$

and so depends only on the metric. Denote by $S \subset \dot{\Sigma}$ the union of sewing seams of the set of cylinders given above. We label the marked points as $\{r_0, \cdots, r_k\}$ for $k \geq 1$ so that r_0 is incoming and the rest are outgoing. Then $\dot{\Sigma}$ carries a vector field $V = V(j)$ that is rotationally invariant and restricts to the coordinate vector field $\frac{\partial}{\partial \tau}$ on each cylinder. (Here 'V' stands for 'vertical' since the meridian circles are often called 'horizontal foliation'.) (See section 4 and [J, St, Z].)

We can also associate a tree T consisting of the cores of the above cylinders that is naturally oriented consistently with the unique incoming assignment of the puncture r_0. We denote by $\ell(e)$ the length of the edge e of the tree. There is also the unique incoming exterior edge incident to r_0 and the unique interior vertex of the exterior edge. We denote by v^{dist} the unique distinguished interior vertex.

Denote by $Q(r_i) = Q(e_i^{ext})$ the charge at the puncture r_i, and assign these numbers to the exterior edges incident to the punctures respectively. We then associate charge $Q(e)$ to each interior edge e so that the following balancing condition holds

$$\sum_{e \in E(v)} Q(e) = 0 \tag{5.4}$$

for all interior vertex $v \in V^{int}(T)$ where $E(v)$ is the set of edges incident to the vertex v. This uniquely determines the charge function $Q : E(T) \to \mathbb{R}$. Furthermore this balancing condition makes the following lemma hold.

Lemma 5.9. *Consider the current $\sum_{e \in E(T)} Q(e)\, dt_e$, i.e., the distributional one-form on $\dot{\Sigma}$. Then it is closed as a current, provided* (5.4) *holds at every interior vertex $v \in V(T)$.*

We remark that the coordinate t_e defined up to the rotation of S^1 can be uniquely determined by assigning a tangent direction at each puncture. But the one-form $Q(e)\, dt_e$ is well-defined independently of the rotations. In particular the current

$$\sum_{e \in E(T)} Q(e)\, dt_e$$

is smooth away from a finite number of Lipschitz singularities located in the sewing seams.

Next we associate the charges $Q(w; e)$ of contact instanton w by the integrals

$$Q(w; e) = -\int_{S_e^1} w^* \lambda \circ j$$

where S_e^1 is a meridian circle of the cylinder associated to the edge $e \in E(T)$. Then we consider the one-form

$$w^* \lambda \circ j + \sum_{e \in E(T)} Q(e)\, dt_e$$

as a current, where $(\tau_e, t_e) \in [0, \ell(e)] \times S^1$ the natural cylindrical coordinates on the cylinder associated to the edge e. By construction this current is exact and so we can solve the distributional equation

$$w^*\lambda \circ j + \sum_{e \in E(T)} Q(w;e)\, dt_e = df$$

a priori for some distribution f.

Proposition 5.10. *The distribution f is a continuous function on $\dot\Sigma$ which is smooth away from the singularities mentioned above.*

Proof. By the property of the minimal area metric which is rotationally symmetric on each cylinder, the function f depends only on the coordinate τ_e and can be uniquely determined by the integral formula

$$f(z) = \int_{v^{dist}}^{z} \left(w^*\lambda \circ j + \sum_{e \in E(T)} Q(w;e)\, dt_e \right)$$

and setting the normalization condition

$$f(v^{dist}) = 0. \tag{5.5}$$

This integral is path-independent by the exactness of the current and so is well-defined. All the properties stated then immediately follows from the expression of f. ☐

This function f seems to deserve a name.

Definition 5.11 (Contact instanton potential)**.** We call the above normalized function f the *contact instanton potential* of the contact instanton charge form $w^*\lambda \circ j$.

If $\dot\Sigma$ carries only one puncture, $\dot\Sigma \cong \mathbb{C}$ and so cannot carry the above minimal area representation but in this case the closed form $w^*\lambda \circ j$ is automatically exact. Therefore there exists a function $f : \dot\Sigma \to \mathbb{R}$ such that $w^*\lambda \circ j = df$ in which case we may regard the pair (f, w) as a pseudoholomorphic map to the symplectization as in [Ho1].

Now we define the final form of the off-shell energy. Let $w : \dot\Sigma \to Q$ be any smooth map. We define the total energy of w by

$$E(j, w) = E^\pi(j, w) + E^\lambda(j, w) \tag{5.6}$$

We define

$$E^\lambda(j, w) = \sup_{\varphi \in \mathcal{C}} \int_{\Sigma} d(\varphi(f)) \wedge df \circ j. \tag{5.7}$$

This energy will be used in our construction of the compactification of moduli space of contact instantons of genus 0 in a sequel. In the rest of the paper, we suppress j from the arguments of the energy $E(j, w)$ and just write $E(w)$.

6. Contact instantons on the plane

As in Hofer's bubbling-off analysis in pseudo-holomorphic curves on symplectization [Ho1], it turns out that study of contact instantons on the plane plays a crucial role in the bubbling-off analysis of contact instantons too.

We recall the following useful lemma from [HV] whose proof we refer thereto.

Lemma 6.1. *Let (X,d) be a complete metric space, $f : X \to \mathbb{R}$ be a nonnegative continuous function, $x \in X$ and $\delta > 0$. Then there exists $y \in X$ and a positive number $\epsilon \leq \delta$ such that*

$$d(x,y) < 2\delta, \max_{B_y(\epsilon)} f \leq 2f(y), \epsilon f(y) \geq \delta f(x).$$

For this purpose, we start with a proposition which is an analog to Theorem 31 [Ho1]. Our proof is a slight modification and some simplification of Hofer's proof of Theorem 31 [Ho1] in our generalized context.

Proposition 6.2. *Let $w : \mathbb{C} \to Q$ be a solution of* (2.1)*. Regard ∞ as a puncture of $\mathbb{C} = \mathbb{C}P^1 \setminus \{\infty\}$. Suppose $|dw|_{C^0} < \infty$ and*

$$E^\pi(w) = 0, \quad E^\lambda_\infty(w) < \infty. \tag{6.1}$$

Then w is a constant map.

Proof. From the equality $|d^\pi w|^2 \, dA = d(w^*\lambda)$ and the hypothesis $E^\pi(w) = 0$ imply $|d^\pi w|^2 = 0 = d(w^*\lambda)$ in addition to $d(w^*\lambda \circ j) = 0$. Therefore we derive that $d^\pi w = 0$. This implies

$$dw = w^*\lambda \, X_\lambda(w)$$

with $w^*\lambda$ a bounded harmonic one-form. The boundedness of $w^*\lambda$ follows from the hypothesis $|dw|_{C^0} < \infty$. Since $\mathbb{C}$ is connected, the image of w must be contained in a single leaf of Reeb foliation. We parameterize the leaf by $\gamma : \mathbb{R} \to Q$, $\gamma = \gamma(t)$.

Then there is a smooth function $b = b(z)$ such that

$$w(z) = \gamma(b(z)).$$

Since $w^*\lambda$ is exact on $\mathbb{C}$, $w^*\lambda = db$ for some function b. Since we also have $d(w^*\lambda \circ j) = 0$,

$$d(db \circ j) = 0$$

i.e., $b : \mathbb{C} \to \mathbb{R}$ is a harmonic function and hence b is the imaginary part of a holomorphic function f, i.e., $f(z) = a(z) + ib(z)$. Since b has bounded gradient. the gradient of f is also bounded on $\mathbb{C}$. Therefore $f(z) = \alpha z + \beta$ for some constants α, $\beta \in \mathbb{C}$.

Once this is achieved, the rest of the argument is exactly the same as Hofer's proof of Lemma 28 [Ho1] via the usage of the λ-energy bound $E^\lambda_\infty(w) < \infty$ and so omitted. □

Using the above proposition, we prove the following fundamental result.

Theorem 6.3. *Let $w : \mathbb{C} \to Q$ be a solution of* (2.1)*. Suppose*

$$E(w) = E^\pi(w) + E^\lambda_\infty(w) < \infty. \tag{6.2}$$

Then $|dw|_{C^0} < \infty$.

Proof. Suppose to the contrary that $|dw|_{C^0} = \infty$ and let z_α be a blowing-up sequence. We denote $R_\alpha = |dw(z_\alpha)| \to \infty$. Then by applying Lemma 6.1, we can choose another such sequence z'_α and $\epsilon_\alpha \to 0$ such that

$$|dw(z'_\alpha)| \to \infty, \quad \max_{z \in D_{\epsilon_\alpha}(z'_\alpha)} |dw(z)| \leq 2R_\alpha, \quad \epsilon_\alpha R_\alpha \to 0. \tag{6.3}$$

We consider the re-scaling maps $\widetilde{w}_\alpha : D^2_{\epsilon_\alpha R_\alpha}(0) \to Q$ defined by

$$w_\alpha(z) = w\left(z'_\alpha + \frac{z}{R_\alpha}\right).$$

Then we have

$$|dw_\alpha|_{C^0;\epsilon_\alpha R_\alpha} \leq 2, \quad |dw_\alpha(0)| = 1.$$

Applying Ascoli-Arzela theorem, there exists a continuous map $w_\infty : \mathbb{C} \to Q$ such that $w_\alpha \to w_\infty$ uniformly on compact subsets. Then by the a priori $W^{k,2}$-estimates, Theorem 1.4, the convergence is in compact C^∞ topology and w_∞ is smooth. Furthermore w_∞ satisfies $\overline{\partial}^\pi w_\infty = 0 = d(w_\infty^* \lambda \circ j) = 0$, $E^\lambda(w_\infty) \leq E(w) < \infty$ and

$$|dw_\infty|_{C^0;\mathbb{C}} \leq 2, \quad |dw_\infty(0)| = 1.$$

On the other hand, by the finite π-energy hypothesis and density identity $|d^\pi w|^2\, dA = d(w^*\lambda)$, we derive

$$\begin{aligned} 0 &= \lim_{\alpha\to\infty} \int_{D_{\epsilon_\alpha}(z'_\alpha)} d(w^*\lambda) = \lim_{\alpha\to\infty} \int_{D_{\epsilon_\alpha R_\alpha}(z'_\alpha)} d(w_\alpha^*\lambda) \\ &= \lim_{\alpha\to\infty} \int_{D_{\epsilon_\alpha R_\alpha}(z'_\alpha)} |d^\pi \widetilde{w}_\alpha|^2 = \int_{\mathbb{C}} |d^\pi w_\infty|^2. \end{aligned}$$

Therefore we derive

$$E^\pi(w_\infty) = 0.$$

Then Proposition 6.2 implies w_∞ is a constant map which contradicts to $|dw_\infty(0)| = 1$. This finishes the proof. $\square$

An immediate corollary of this theorem and Proposition 6.2 is the following

Corollary 6.4. *For any non-constant contact instanton $w : \mathbb{C} \to Q$ with the energy bound $E(w) < \infty$, we obtain*

$$E^\pi(w) = \int z^*\lambda > 0$$

for $z = \lim_{R\to\infty} w(Re^{2\pi i t})$. In particular $E^\pi(w) \geq T_\lambda > 0$.

Now we have the following refinement of the asymptotic convergence result from [Ho1] and [OW1]. It is a refinement of Theorem 6.3 of [OW1] in that the derivative bound $|dw|_{C^0} < \infty$ imposed therein is replaced by the more natural energy bound $E(w) < \infty$.

Theorem 6.5 (Compare with Theorem 31 [Ho1], Theorem 6.3 [OW2])**.** *Let $\dot\Sigma$ be a punctured Riemann surface equipped with a Kähler metric that is cylindrical around punctures. Let $w : \dot\Sigma \to Q$ be a solution of* (2.1)*. Let p be a given puncture. Suppose*

$$E(w) < \infty. \tag{6.4}$$

Then for any given sequence $R_i \to \infty$, there exists a subsequence, again denoted by R_i, and a map $w_\infty : \mathbb{R} \times S^1 \to Q$ such that

1. *for any given $K > 0$, w_i defined by $w_i(\tau,t) = w(\tau + \tau_i, t)$ converges to w_∞ uniformly on $[-K,K] \times S^1$,*
2. *the image of w_∞ is contained in a single leaf of Reeb foliation. Therefore if we fix a parametrization of this leaf by $\gamma = \gamma(t)$ for $t \in \mathbb{R}$, then*

$$w_\infty(\tau,t) = \gamma(Q(p)\tau + T(p)t).$$

Furthermore one of the following alternatives holds: Consider

$$T(p) = \int w(0,\cdot)^*\lambda + \frac{1}{2}\int_{[0,\infty)\times S^1} |d^\pi w|^2 = \lim_{i\to\infty}\int w(\tau_i,\cdot)^*\lambda \tag{6.5}$$

$$Q(p) = -\int_{S^1} (w(0,\cdot))^*\lambda\circ j \tag{6.6}$$

(1) *When $T \neq 0$, there exists a Reeb orbit γ of period T such that*

$$w_\infty(\tau,t) = \gamma(Q(p)\tau + T(p)t)$$

as $i\to\infty$ where $z_{R_i}(t) = w(\tau_i,t)$ at each puncture p, and its period is given by T. In this case, $w(\tau_i,\cdot)\to\gamma(T(\cdot))$ as $i\to\infty$.

(2) *When $T = 0$, $w_\infty(\tau,t) = \gamma(Q(p)\,\tau)$. In this case, $w(\tau_i,\cdot)$ converges to a point in the leaf.*

Combining Theorem 6.5 and Theorem 6.3, we immediately derive

Corollary 6.6. *Let w be a non-constant contact instanton on $\mathbb{C}$ with*

$$E(w) < \infty. \tag{6.7}$$

Then there exists a sequence $R_j\to\infty$ and a Reeb orbit γ such that $z_{R_j}\to\gamma(T(\cdot))$ with $T\neq 0$ and

$$T = E^\pi(w), \quad Q = \int_z w^*\lambda\circ j = 0.$$

Proof. If $T = 0$, the above theorem shows that there exists a sequence $\tau_i\to\infty$ such that $w(\tau_i,\cdot)$ converges to a constant in C^∞ topology and so

$$\int_{\{\tau=\tau_i\}} w^*\lambda \to 0$$

as $i\to\infty$. By Stokes' formula, we derive

$$\int_{D_{e^{\tau_i}}(0)} w^* d\lambda = \int_{\tau=\tau_i} w^*\lambda\to 0.$$

On the other hand, we have

$$E^\pi(w) = \lim_{i\to\infty}\int_{D_{e^{\tau_i}}(0)} |d^\pi w|^2 = \lim_{i\to\infty}\int_{D_{e^{\tau_i}}} w^* d\lambda = 0.$$

This contradicts to Corollary 6.4, which finishes the proof. $\square$

The following is the analog to Proposition 30 [Ho1].

Corollary 6.7. *Let w be a contact instanton on $\mathbb{R}\times S^1$ with $E(w) < \infty$. Then $|dw|_{C^0} < \infty$.*

Proof. As in Hofer's proof of Proposition 30 [Ho1], we apply the same kind of bubbling-off argument as that of Theorem 6.3 and derive the same conclusion. $\square$

7. Bubbling-off analysis and ϵ-regularity theorem

We recall from [OW2] that the local a priori $W^{k,2}$-regularity estimates are established with respect to the bounds of $\|dw\|_{L^4}$ and $\|dw\|_{L^2}$. Therefore in addition to the local a priori $W^{k,2}$-regularity estimates, one should establish another crucial ingredient, the ϵ-regularity result, for the study of moduli problem as usual in any of conformally invariant geometric non-linear PDE's. This will in turn establish the $W^{1,p}$-bound with $p > 2$ (say $p = 4$) appears in many problems in geometry and physics under the suitable smallness hypothesis on the relevant energy. (See [SU].)

In the current setting of contact instanton map, it is not obvious what would be the precise form of relevant ϵ-regularity statement is. We formulate this ϵ-regularity theorem in the setting of contact instantons. It turns out that the relevant energy is the π-harmonic energy.

Definition 7.1. Let λ be a contact form of contact manifold (Q,ξ). Denote by $\mathfrak{Reeb}(Q,\lambda)$ the set of closed Reeb orbits. We define $\operatorname{Spec}(Q,\lambda)$ to be the set

$$\operatorname{Spec}(Q,\lambda) = \left\{ \int_\gamma \lambda \mid \lambda \in \mathfrak{Reeb}(Q,\lambda) \right\}$$

and call the *action spectrum* of (Q,λ). We denote

$$T_\lambda := \inf \left\{ \int_\gamma \lambda \mid \lambda \in \mathfrak{Reeb}(Q,\lambda) \right\}.$$

We set $T_\lambda = \infty$ if there is no closed Reeb orbit.

The following is a standard lemma in contact geometry

Lemma 7.2. *Let (Q,ξ) be a closed contact manifold. Then $\operatorname{Spec}(Q,\lambda)$ is either empty or a countable nowhere dense subset of $\mathbb{R}_+$ and $T_\lambda > 0$. Moreover the subset*

$$\operatorname{Spec}^K(Q,\lambda) = \operatorname{Spec}(Q,\lambda) \cap (0,K]$$

is finite for each $K > 0$.

Remark 7.3. A priori we cannot rule out the possibility $\operatorname{Spec}(Q,\lambda) = \emptyset$. Nonemptyness of this set is precisely the content of Weinstein's conjecture: Any contact form λ of a contact manifold (Q,ξ) carries a closed Reeb orbit. The conjecture has been proved by Taubes [T] in 3 dimensional case after other scattered results obtained earlier.

The constant T_λ will enter in a crucial way in the following ϵ-regularity statement. The proof of this theorem will closely follow the argument used in [Oh3, section 8.4] and [Oh2] by adapting it to the proof of the current ϵ-regularity theorem with the replacement of the standard harmonic energy by the π-harmonic energy. However there is one marked difference between the current ϵ-regularity statement and that of pseudoholomorphic curves because of the second order part $d(w^*\lambda \circ j) = 0$ of contact instanton map: The local $W^{k,2}$ a priori estimate given in Theorem 1.4 plays a crucial role in establishing that the limit map of a subsequence obtained via application of Ascoli-Arzela theorem still satisfies the equation $\overline{\partial}^\pi w = 0$, $d(w^*\lambda \circ j) = 0$.

Theorem 7.4. *Denote by $D^2(1)$ the closed unit disc. Let $w : D^2(1) \to Q$ satisfy*

$$\overline{\partial}^\pi w = 0,\ d(w^*\lambda \circ j) = 0,\ E^\lambda(w) < K_0.$$

Then for any given $0 < \epsilon < T_\lambda$ and w satisfying $E^\pi(w) < T_\lambda - \epsilon$, and for a smaller disc $D' \subset \overline{D}' \subset D$, there exists some $K_1 = K_1(D', \epsilon, K_0) > 0$

$$\|dw\|_{C^0;D'} \leq K_1 \tag{7.1}$$

where K_1 depends only on (Q, λ, J), ϵ, $D' \subset D$.

Proof. Suppose to the contrary that there exists a disc $D' \subset D$ with $\overline{D'} \subset \overset{\circ}{D}$ and a sequence $\{w_\alpha\}$ such that

$$\overline{\partial}^\pi w_\alpha = 0, \quad d(w_\alpha \circ j) = 0$$

and satisfy

$$E^\pi_{\lambda,J;D}(w_\alpha) < T_\lambda - \epsilon, , E^\lambda(w_\alpha) < K_0, \quad \|dw_\alpha\|_{C^0,D'} \to \infty \tag{7.2}$$

as $\alpha \to \infty$. Let $x_\alpha \in D'$ such that $|dw_\alpha(x_\alpha)| \to \infty$. By choosing a subsequence, we may assume that $x_\alpha \to x_\infty \in \overline{D}' \subset \overset{\circ}{D}$. We take a coordinate chart centered at x_∞ on $D_{x_\infty}(\delta) \subset \overset{\circ}{D}$ and identify $D_{x_\infty}(\delta)$ with the disc $D^2(\delta) \subset \mathbb{C}$ and x_∞ with $0 \in \mathbb{C}$. This can be done by choosing $\delta > 0$ sufficiently small since we assume $\overline{D}' \subset \overset{\circ}{D}$. Then $x_\alpha \to 0$. We choose $\delta_\alpha \to 0$ so that $\delta_\alpha |dw_\alpha(x_\alpha)| \to \infty$.

We adjust the sequence x_α to y_α by applying Hofer's lemma, Lemma 6.1, so that $y_\alpha \to 0$ and

$$\max_{x \in B_{y_\alpha}(\epsilon_\alpha)} |dw_\alpha| \leq 2|dw_\alpha(y_\alpha)|, \quad \delta_\alpha |dw_\alpha(y_\alpha)| \to \infty. \tag{7.3}$$

We denote $R_\alpha = |dw_\alpha(y_\alpha)|$ and consider the re-scaled map

$$v_\alpha(z) = w_\alpha\left(y_\alpha + \frac{z}{R_\alpha}\right).$$

Then the domain of w_α at least includes $z \in \mathbb{C}$ such that

$$y_\alpha + \frac{z}{R_\alpha} \in D^2(\delta),$$

i.e., those z's satisfying

$$\left| y_\alpha + \frac{z}{R_\alpha} \right| \leq \delta.$$

In particular, if $|z| \leq R_\alpha(\delta - |y_\alpha|)$, $v_\alpha(z)$ is defined. Since $y_\alpha \to 0$ and $\delta_\alpha \to 0$ as $\alpha \to \infty$, $R_\alpha(\delta - |y_\alpha|) > R_\alpha \epsilon_\alpha$ eventually, v_α is defined on $D^2(\epsilon_\alpha R_\alpha)$ for all sufficiently large α's. Since $\delta_\alpha R_\alpha \to \infty$ by (7.3), for any given $R > 0$, $D^2(\delta_\alpha R_\alpha)$ of $v_\alpha(z)$ eventually contains $B_{R+1}(0)$.

Furthermore, we may assume,

$$B_{R+1}(0) \subset \left\{ z \in \mathbb{C} \mid \eta_\alpha z + y_\alpha \in \overline{D}' \right\}$$

Therefore, the maps

$$v_\alpha : B_{R+1}(0) \subset \mathbb{C} \to M$$

satisfy the following properties:

(i) $E^\pi(v_\alpha) < T_\lambda - \epsilon$, $\overline{\partial}^\pi v_\alpha = 0$, $E^\lambda(v_\alpha) \leq K_0$, (from the scale invariance)
(ii) $|dv_\alpha(0)| = 1$ by definition of v_α and R_α,
(iii) $\|dv_\alpha\|_{C^0,B_1(x)} \leq 2$ for all $x \in B_R(0) \subset D^2(\epsilon_\alpha R_\alpha)$,
(iv) $\overline{\partial}^\pi v_\alpha = 0$ and $d(v_\alpha^* \lambda \circ j) = 0$.

For each fixed R, we take the limit of $v_\alpha|_{B_R}$, which we denote by w_R. Applying (iii) and then the local $W^{k,2}$ estimates, Theorem 1.4, we obtain

$$\|dv_\alpha\|_{k,2;B_{\frac{9}{10}}(x)} \leq C$$

for some $C = C(R)$. By the Sobolev embedding theorem, we have a subsequence that converges in C^2 in each $B_{\frac{8}{10}}(x), x \in \overline{D}'$. Then we derive that the convergence is in C^2-topology on $B_{\frac{8}{10}}(x)$ for all $x \in \overline{D}'$ and in turn on $B_R(0)$.

Therefore the limit $w_R : B_R(0) \to M$ of $v_\alpha|_{B_R(0)}$ satisfies

(1) $E^\pi(w_R) \leq T_\lambda - \epsilon$, $\overline{\partial}^\pi w_R = 0$, $d(w_R^*\lambda \circ j) = 0$ and $E^\lambda(v_\alpha) \leq K_0$,
(2) $E^\pi(w_R) \leq \limsup_\alpha E^\pi_{(\lambda,J;B_R(0))}(v_\alpha) \leq T_\lambda - \epsilon$,
(3) Since $v_\alpha \to w_R$ converges in C^2, we have

$$\|dw_R\|^2_{p,B_1(0)} = \lim_{\alpha\to\infty} \|dv_\alpha\|^2_{p,B_1(0)} \geq \frac{1}{2}.$$

By letting $R \to \infty$ and taking a diagonal subsequence argument, we have derived nonconstant contact instanton map $w_\infty : \mathbb{C} \to Q$. Therefore by definition of T_λ, we must have $E^\pi(w_\infty) \geq T_\lambda$.

On the other hand, the bound $E^\pi(w_R) \leq T_\lambda - \epsilon$ for all R and again by Fatou's lemma implies

$$E^\pi(w_\infty) \leq T_\lambda - \epsilon$$

which gives rise to a contradiction. This finishes the proof of (7.1). □

8. Asymptotic behaviors of finite energy contact instantons

In this section, we study the asymptotic behavior of contact instanton $w : \dot\Sigma \to Q$ with finite energy $E(w) < \infty$ near the punctures. We start with classifying the solutions of (2.1) of zero energy on the cylinder $\mathbb{R} \times S^1$.

We start with the following lemma

Lemma 8.1. *Suppose $E(w) = E^\pi(w) + E^\lambda(w) < \infty$. Then*

$$|dw|_{C^0} < \infty.$$

Proof. By the finiteness $E^\pi(w) < \infty$, we can choose sufficiently small $\delta > 0$ such that

$$E^\pi(w|_{\Sigma\setminus\Sigma(\delta)}) < \frac{1}{2}T_\lambda.$$

Denote

$$\Sigma(\delta) = \dot\Sigma \setminus \cup_{\ell=1}^k D_{r_\ell}(\delta).$$

Then we apply the ϵ-regularity theorem, Theorem 7.4, to w on $\cup_{\ell=1}^k D_{r_\ell}(\delta) = \dot\Sigma \setminus \Sigma(\delta)$ to derive

$$|dw|_{\cup_{\ell=1}^k D_{r_\ell}} < \infty.$$

Obviously $|dw|_{\Sigma(\delta)}|_{C^0} < \infty$ and hence the proof. □

8.1. **Massless contact instantons.** The following is a key lemma in which the closed condition $w^*\lambda \circ j$ plays a crucial role.

Lemma 8.2. *Let $\dot\Sigma$ be any punctured Riemann surface. Suppose $w : \dot\Sigma \to Q$ is a massless contact instanton on $\dot\Sigma$. Then $w^*\lambda$ is a harmonic 1-form and the image of w lies in a single leaf of the Reeb foliation.*

Proof. From the equation, we have $\overline{\partial}^\pi w = 0$. We also have $\partial^\pi w = 0$ from the massless condition and so $d^\pi w = \pi dw = 0$. This implies the values of dw are parallel to X_λ at all points of $\dot\Sigma$. By the connectedness of $\dot\Sigma$, this implies that the image of w must be contained in a leaf.

Next we obtain $d(w^*\lambda) = 0$ from $E^\pi_{(\lambda,J)}(w) = 0$ and the identity $|d^\pi w|^2 = |\partial^\pi w|^2\, dA = d(w^*\lambda)$ since $\overline{\partial}^\pi w = 0$. We also have

$$\delta(w^*\lambda)\, dA = -d(w^*\lambda \circ j) = 0$$

where the first follows since the metric h on $\dot\Sigma$ is Kähler with respect to j and the second equality follows from the equation. This finishes the proof. $\square$

The following result connects the basic hypotheses for the a priori $W^{k,2}$-estimates to the study of structure of singularities of contact instanton.

Proposition 8.3. *Let w be a contact instanton on $\dot\Sigma$ with punctures $p \in \{p_1, \cdots, p_k\}$. Let $p \in \{p_1, \cdots, p_k\}$ and let z be an analytic coordinate at p. Suppose*

$$E(w) = E^\pi(w) + E^\lambda(w) < \infty.$$

Then for any given sequence $\delta_j \to 0$ there exists a subsequence, still denoted by δ_j, and a conformal diffeomorphism $\varphi_j : [-\frac{1}{\delta_j}, \infty) \times S^1 \to D_{\delta_j}(p) \setminus \{p\}$ such that the one form $\varphi_j^*\chi$ converges to a bounded holomorphic one-form χ_∞ on $(-\infty, \infty) \times S^1$.

Proof. By Lemma 8.1, $|dw|_{C^0} < \infty$. Let $C = |dw|_{C^0}$. Then $|w^*\lambda|_{C^0} \leq C$.

By the finiteness $E^\pi(w) < \infty$, Fatou's lemma implies

$$\lim_{r\to 0} \int_{D_r(p)\setminus\{0\}} |d^\pi w|^2 = 0.$$

We fix a sequence $r_j \to 0$ and fix a conformal diffeomorphism

$$\varphi_j : \left[-\frac{1}{\delta_j}, \infty\right) \times S^1 \to D_{r_j}(p) \setminus \{0\}, \quad \varphi_r(\tau, t) = \delta_0 e^{-\frac{1}{\delta_j}} e^{-2\pi(\tau + it)} = z$$

for each $j > 0$. In particular, the map $(\varphi_j^* w, \varphi_j^*\chi)$ are contact-instantons on $[0, \infty) \times S^1$ which satisfy

$$E^\pi(\varphi_j^* w) \to 0.$$

By $W^{k,2}$ a priori estimates, Theorem 1.4, and the ϵ-regularity theorem, Theorem 7.4, we obtain the gradient bound $|d(\varphi_j^* w)|_{[-1/\delta_j,\infty)\times S^1} \leq C$ and in particular $|(\varphi_j^* w)^*\lambda|_{C^0} \leq C$ for all j.

Applying the diagonal subsequence argument, we can select a sequence $\delta_j \to 0$ such that $\varphi_{\delta_j}^* w$ converges to $w_\infty : (-\infty, \infty) \times S^1 \to Q$ and $\varphi_{\delta_j}^*\chi \to \chi_\infty$ in compact C^∞ topology so that the pair (w_∞, χ_∞) is a contact instanton satisfying

$$E^\pi(w_\infty) = 0, \quad |\chi_\infty|_{C^0} \leq \frac{3C}{2}. \tag{8.1}$$

Since $|d^\pi w_\infty|^2\, dA = d(w_\infty^*\lambda)$, this implies

$$d(w_\infty^*\lambda) = 0.$$

Together with $d(w_\infty^*\lambda \circ j) = 0$, this implies that χ_∞ is a non-zero holomorphic one-form that is bounded on $\mathbb{R} \times S^1$. This finishes the proof. □

We would like to emphasize that at the moment, the limiting holomorphic one-form χ_∞ may depend on the choice of subsequence.

The following theorem slightly strengthens the convergence results from [Ho1], [OW2].

Theorem 8.4. *Let Σ be a closed Riemann surface of genus 0 with a finite number of marked points $\{p_1, \cdots, p_k\}$ for $k \geq 3$, and let $\dot\Sigma = \Sigma \setminus \{p_1, \cdots, p_k\}$ be the associated punctured Riemann surface equipped with a metric as before. Suppose that w is a contact instanton map $w : (\dot\Sigma, j) \to (Q, J)$ with finite total energy $E(w) = E^\pi(w) + E^\lambda(w)$ and fix a puncture $p \in \{p_1, \cdots, p_k\}$.*

Then for any given sequence $I = \{\tau_k\}$ with $\tau_k \to \infty$, there exists a subsequence $I' \subset I$ and a closed parameterized Reeb orbit $\gamma = \gamma_{I'}$ of period T and some $(\tau_0, t_0) \in \mathbb{R} \times S^1$ such that such that

$$\lim_{i\to\infty} w(\tau + \tau_{k_i}, t) = \gamma(Q(p)\, \tau + T(p)\, t)$$

in compact C^∞ topology.

If λ is nondegenerate and $T \neq 0$, then the convergence $w(\tau, \cdot) \to \gamma(T\cdot)$ is uniform.

Proof. The finiteness of $E(w)$ and the ϵ-regularity implies the C^1 bound $|dw|_{C^0} < \infty$ on $[R, \infty) \times S^1$ for a sufficiently large $R > 0$. Once this bound is established, the same proof as that of Theorem 6.3 of [OW2] proves that there exists a closed Reeb orbit (T, γ) and a subsequence $k_i \to \infty$ such that

$$w(\tau_{k_i} + \tau, \cdot) \to \gamma(Q(p)(\tau_{k_i} + \tau), T(p)t)$$

uniformly on $[-K, K] \times S^1$ in C^∞ topology for any given $K \geq 0$. Once we have established this subsequence convergence result, the same proof as that of Theorem 6.5 [OW2] applies to conclude the theorem. We refer to [OW2] for the complete detail of the proof and the proof of uniform convergence for the nondegenerate case. □

We would like to call the readers' attention to the case where $T(p) = 0$. In this case the asymptotic limit w_∞ is t-independent, i.e., $w_\infty(\tau, t) \equiv \gamma(Q(p)\tau)$. In particular, the image of the instanton is 1 dimensional.

8.2. **Classification of punctures.** Assume that λ is nondegenerate. We would like to further analyze the asymptotic behavior of the instanton w.

Associated to the splitting

$$TQ = \mathrm{span}\{X_\lambda\} \oplus \xi,$$

Q carries the canonical (trivial) complex line bundle $\mathcal{L} \to Q$ with connection form $\sqrt{-1}\lambda$. When we are given a map $w : \dot\Sigma \to Q$, it induces the pull-back bundle $w^*\mathcal{L}$ with the pull-back connection $\sqrt{-1}w^*\lambda$. The associated (abelian) Yang-Mills equation is nothing but

$$\delta w^*\lambda = 0$$

with respect to the Kähler metric associated to the complex structure j on the surface Σ is precisely equivalent to $d(w^*\lambda \circ j) = 0$.

Now we introduce the complex valued one-form

$$\chi = w^*\lambda \circ j + \sqrt{-1}w^*\lambda. \tag{8.2}$$

It appears to be worthwhile to give a name to the complex valued $(1,0)$-form in the general context.

Definition 8.5. Let (Σ, j) be a closed Riemann surface with finite number of marked points $\{p_1, \cdots, r_k\}$. Denote by $\dot\Sigma$ the associated punctured Riemann surface with cylindrical metric near the punctures, and let $\overline{\Sigma}$ the real blow-up of Σ along the punctures. Let w be a contact instanton map. Let $p \in \{p_1, \cdots, r_k\}$. We call the integrals

$$Q(p) := -\int_{\partial_{\infty;r}\Sigma} w^*\lambda \circ j \tag{8.3}$$

$$T(p) := \int_{\partial_{\infty;r}\Sigma} w^*\lambda \tag{8.4}$$

the *contact instanton charge* and *contact instanton action* at p respectively. Here $\partial_{\infty;r}\Sigma$ is the boundary component corresponding to p of the real blow-up $\overline{\Sigma}$ of $\dot\Sigma$. Then we call the form $\chi = w^*\lambda \circ j + \sqrt{-1}w^*\lambda$ the *contact Hick's field* of w and

$$Q(p) + \sqrt{-1}T(p)$$

the *charge* of the Hick's field of the instanton w at the puncture p.

Note that by the closedness $d(w^*\lambda \circ j) = 0$, the charge $Q(p)$ is the same as the initial integral

$$\int_{\{\tau=0\}} w^*\lambda \circ j$$

which does not depend on the choice of subsequence but is determined by the initial condition at $\tau = 0$ and homology class of the loop $w|_{\tau=0} \in H_1(\dot\Sigma) = H_1(\Sigma \setminus \{p_1, \cdots, p_k\}$.

Proposition 8.6. *For any finite energy contact instanton w, we have*

$$\sum_{l=1}^{N} Q(p_\ell) = 0. \tag{8.5}$$

We call this equation the balancing condition *of the contact Hick's charge.*

Proof. This is an immediate consequence of Stokes' formula applied to the closed 1-form $w^*\lambda \circ j$ on the real blow-up $\overline{\Sigma}$ of $\dot\Sigma$. □

Now we consider the asymptotic Hick's field χ_∞ associated to the asymptotic instanton w_∞ obtained in the proof of Proposition 8.3, and call χ_∞ the asymptotic Hick's field of w at the puncture p. Because w_∞ is massless and has bounded derivatives on $\mathbb{R} \times S^1$, χ_∞ becomes a bounded holomorphic one-form. Therefore we derive

$$\chi_\infty = c\,(d\tau + i\,dt) \tag{8.6}$$

for some complex number $c \in \mathbb{C}$. We denote $c = b + ia$ for $a, b \in \mathbb{R}$. Equivalently, we obtain

$$w^*\lambda = a\,d\tau + b\,dt.$$

Here a, b are nothing but the period integrals

$$a = -\int_{S^1} (w(\tau,\cdot))^*\lambda \circ j,\, b = \int_{S^1} (w(\tau,\cdot))^*\lambda$$

which do not depend on τ for the massless instantons, thanks to the closedness of $w^*\lambda$, $w^*\lambda \circ j$. We denote them by $a = Q(p)$, $b = T(p)$ and call them as the *Hick's charge* at p.

We now examine the various cases arising depending on the constant c. Let $\chi_\infty = c\,(d\tau + i\,dt)$ as above.

Theorem 8.7. *Suppose $c = 0$. Then w is smooth across p and so the puncture p is removable.*

Proof. When $c = 0$, we obtain $dw_\infty = d^\pi w_\infty + \lambda^* w_\infty\, X_\lambda = 0$ and so w_∞ must be a constant map $q \in Q$. By the convergence $w_j \to w_\infty$ in compact C^∞ topology, it follows that $w_j(0,\cdot) \to q$ or equivalently

$$d(w|_{r=\delta_j}, q) \to 0$$

and $w_j^*\lambda \to 0$ converges uniformly. Using the compactness of Q and applying Ascoli-Arzela theorem, we can choose a sequence $z_i \to p$ in $D_\delta(p) \setminus \{p\}$ such that $w(z_i) \to p$ and $w^*\lambda|_{r=\delta_j} \to 0$ uniformly. Then this continuity of $w^*\lambda$ at p in turn implies dw is continuous at p by the expression

$$dw = d^\pi w + w^*\lambda\, X_\lambda(w)$$

In particular $|dw|_{D_\delta(r)}$ is bounded and so lies in $L^2 \cap L^4$ on $D_\delta(r)$. Then the local $W^{k,2}$ a priori estimate implies that w is indeed smooth across p. This finishes the proof. $\square$

If $c \neq 0$, we obtain

$$\int_{S^1} \chi_\infty|_\tau \equiv c$$

for all τ. In particular, we derive

$$\lim_{j\to\infty} \int_{S^1} (\chi|_{r=\delta_j})^*\lambda = c$$

and so

$$\begin{aligned} \lim_{k\to\infty} \int_{S^1} (w|_{r=\delta_k})^*\lambda \circ j &= \operatorname{Re} c \\ \lim_{k\to\infty} \int_{S^1} (w|_{r=\delta_k})^*\lambda &= \operatorname{Im} c. \end{aligned}$$

In fact by the closedness of $w^*\lambda \circ j$ and convergence of $w|_{r=\delta_j} \to p$, the integral $(w|_{r=\delta_k})^*\lambda \circ j$ does not depend on k's eventually.

We divide our consideration of the remaining cases into two different cases, one with $b = \operatorname{Im} c = 0$ and the other with $b = \operatorname{Im} c \neq 0$.

Proposition 8.8. *Suppose $b \neq 0$. Then there exists a closed Reeb orbit γ of period $T = \frac{b}{2\pi}$ such that there exists a sequence $\tau_k \to \infty$ for which $w(\tau_k,\cdot) \to \gamma(T(\cdot))$ in C^∞ topology.*

Proof. When $b \neq 0$, we obtain

$$dw_\infty = (a\, d\tau + b\, dt)\, X_\lambda.$$

Again by the connectedness of $[0, \infty) \times S^1$, it follows that the image of w_∞ must be contained in a single leaf of the Reeb foliation and so

$$w_\infty(\tau, t) = \gamma(a\tau + b\, t)$$

for a parameterized Reeb orbit γ such that $\dot\gamma = X_\lambda(\gamma)$. Such a parameterization is unique modulo the time-shift. Since the map w is one-periodic for any τ, we derive

$$\gamma(b) = \gamma(0).$$

This implies first that γ is a periodic Reeb orbit of period g. $\square$

If we denote by $T > 0$ its minimal period, then we obtain

$$2\pi\, b = m\, T$$

for some integer m. Since we assume $b \neq 0$, it follows that $m\, T \neq 0$.

Proposition 8.9. *Suppose $b = 0$, $a \neq 0$. Then w_∞ does not depend on the t-variable and the map $\tau \to w_\infty(\tau)$ becomes a Reeb trajectory which is not necessarily closed.*

Proof. In this case, $w_\infty^*\lambda = a\, d\tau$. Therefore w_∞ does not depend on t and satisfies

$$\frac{\partial w_\infty}{\partial \tau} = a\, X_\lambda(w(\tau, t))$$

and so $w(\tau, t) \equiv z(a\tau)$ for a path satisfying $\dot z = X_\lambda(z)$. This finishes the proof. $\square$

Remark 8.10.
(1) We would like to remark that all the above three scenarios can actually occur and have to be examined in the asymptotic study of contact instantons. For the exact case, we have $a = 0$.
(2) Each massless contact instanton on $\mathbb{R} \times S^1$ induces a linear foliation thereon. When the charge is zero, the foliation becomes the standard foliation but when the instanton carries a non-trivial charge the 'horizontal' foliation is skewed. This could be interpreted as the change of conformal structure (or 'gravity' by physical terms) of the cylinder that is powered by non-trivial charge carried by the instanton. This phenomenon seems to be worthwhile to further study which is a subject of future study.
(3) Presence of the above non-trivial 'spiraling' massless instantons on the cylinder which does not exist in the exact case, makes the asymptotic study of contact instantons for the non-exact case more complicated but also makes more interesting.

Now we are ready to define the notion of positive and negative punctures of contact instanton map w. Assume λ is nondegenerate.

Let p be one of the punctures of $\dot\Sigma$. In the disc $D_\delta(p) \subset \mathbb{C}$ with the standard orientation, we consider the function

$$\int_{\partial D_\delta(p)} w^*\lambda$$

as a function of $\delta > 0$. This function is either decreasing or increasing by the Stokes' formula, the positivity $w^*d\lambda \geq 0$ and the finiteness of π-energy

$$\frac{1}{2}\int_{\dot\Sigma} |d^\pi w|^2 = \int_{\dot\Sigma} w^* d\lambda < \infty.$$

Definition 8.11 (Classification of punctures)**.** Let $\dot\Sigma$ be a puncture Riemann surface with punctures $\{p_1, \cdots, p_k\}$ and let $w : \dot\Sigma \to Q$ be a contact instanton map.

(1) We call a puncture p *removable* if $T(p) = Q(p) = 0$, and *non-removable* otherwise. Among the non-removable punctures p, we call it *non-adiabatic* if $T(p) \neq 0$, *adiabatic* if $T(p) = 0$ but $Q(p) \neq 0$.
(2) We say a non-removable puncture *positive* (resp. *negative*) puncture if the function

$$\int_{\partial D_\delta(p)} w^*\lambda$$

is increasing (resp. decreasing) as $\delta \to 0$.

The appearance of adiabatic punctures is a new phenomenon when the form $w^*\lambda \circ j$ is not exact. In the latter case considered via the case of symplectization picture [Ho1], the associated puncture is removable and can be dropped in this classification by removing the puncture. However in the non-exact case, such a puncture is not necessarily removable and so has to be considered separately.

9. Properness of contact instanton potential function and λ-energy

In this section, we examine the relationship between the π-energy, the λ-energy and the contact instanton potential function f.

We first note that the function $f : \dot\Sigma \to \mathbb{R}$ is proper if and only if

$$f(v_j) = \pm\infty \tag{9.1}$$

for all exterior vertex $v_j \in V(T)$. One immediate corollary of Lemma 8.1 is the following C^1-bound of the contact potential function f.

Corollary 9.1. *Suppose that $E(w) < \infty$ and let f be the function defined in section 5. Then $|df|_{C^0} < \infty$.*

Proof. From Lemma 8.1 and the defining equation of f

$$w^*\lambda \circ j + \sum_{e \in E(T)} Q(w; e)\, dt_e = df,$$

we obtain $|df|_{C^0} < |dw|_{C^0} + \max_{e\in E(T)} |Q(w; e)| < \infty$. □

The following proposition is the analog to Lemma 5.15 [BEHWZ] whose proof is also similar.

Proposition 9.2. *Suppose that $E^\pi(w) < \infty$ and the function $f : \dot\Sigma \to \mathbb{R}$ is proper. Then $E(w) < \infty$.*

Proof. Since f is assumed to be proper, $f(r_\ell) = \pm\infty$ for each puncture r_ℓ of $\dot\Sigma$ depending on whether the puncture is positive or negative.

The rest of the argument is very similar to that of the proof of Lemma 5.15 [BEHWZ] with replacement of a and the equation $dw^*\lambda \circ j = da$ therein by f and the equation

$$dw^*\lambda \circ j + \sum_{e \in E(T)} Q(w;e)\, dt_e = df$$

respectively in our current context. (We would also like point out that [BEHWZ] used the letter 'f' for the map w which should not confuse the readers with our notation f for the function which corresponds to a in their notation.)

Since our setting does not use the setting of symplectization, we provide the full details of the proof in Appendix. □

By the same argument as the derivation of Lemma 5.16 [BEHWZ], we obtain

Lemma 9.3. *Suppose $E^\pi(w) < \infty$ and f is proper. Denote by $\gamma_1^+, \cdots, \gamma_k^+$ (resp. $\gamma_1^-, \cdots, \gamma_\ell^-$) the periodic orbits of X_λ asymptotic to the positive (resp. negative punctures) of $\dot\Sigma$. Then*

$$\begin{aligned} E^\pi(w) &= \sum_{j=1}^k \int \overline{\gamma}_j^*\lambda - \sum_{i=1}^\ell \int \underline{\gamma}_i^*\lambda \\ E^\lambda(w) &= \sum_{j=1}^k \int \overline{\gamma}_j^*\lambda \\ E(w) &= 2\sum_{j=1}^k \int \overline{\gamma}_j^*\lambda - \sum_{i=1}^\ell \int \underline{\gamma}_i^*\lambda. \end{aligned}$$

10. Calculation of the linearization map with contact triad connection

Let Σ be a closed Riemann surface and $\dot\Sigma$ be its associated punctured Riemann surface. We allow the set of whose punctures to be empty, i.e., $\dot\Sigma = \Sigma$. We would like to regard the assignment

$$w \mapsto \left(\overline{\partial}^\pi w, d(w^*\lambda \circ j)\right)$$

for a map $w : \dot\Sigma \to Q$ as a section of the (infinite dimensional) vector bundle over the space of maps of w. In this section, we lay out the precise relevant off-shell framework of functional analysis.

Let $(\dot\Sigma, j)$ be a punctured Riemann surface, the set of whose punctures may be empty, i.e., $\dot\Sigma = \Sigma$ is either a closed or a punctured Riemann surface. We will fix j and its associated Kähler metric h.

We consider the map

$$\Upsilon(w) = \left(\overline{\partial}^\pi w, d(w^*\lambda \circ j)\right)$$

which defines a section of the vector bundle

$$\mathcal{H} \to \mathcal{F} = C^\infty(\Sigma, Q)$$

whose fiber at $w \in C^\infty(\Sigma, Q)$ is given by

$$\mathcal{H}_w := \Omega^{(0,1)}(w^*\xi) \oplus \Omega^2(\Sigma).$$

We decompose $\Upsilon = (\Upsilon_1, \Upsilon_2)$ where

$$\Upsilon_1 : \Omega^0(w^*TQ) \to \Omega^{(0,1)}(w^*\xi); \quad \Upsilon_1(w) = \overline{\partial}^\pi(w) \tag{10.1}$$

and

$$\Upsilon_2 : \Omega^0(w^*TQ) \to \Omega^2(\dot\Sigma); \quad \Upsilon_2(w) = d(w^*\lambda \circ j). \tag{10.2}$$

We first compute the linearization map which defines a linear map

$$D\Upsilon(w) : \Omega^0(w^*TQ) \to \Omega^{(0,1)}(w^*\xi) \oplus \Omega^2(\Sigma)$$

where we have

$$T_w\mathcal{F} = \Omega^0(w^*TQ).$$

We note

$$\begin{aligned} \operatorname{rank} \Lambda^0(w^*TQ) &= 2n+1 \\ \operatorname{rank} \Lambda^{(0,1)}(w^*\xi) \oplus \Lambda^2(\Sigma) &= 2n+1. \end{aligned}$$

For the optimal expression of the linearization map and its relevant calculations, we use the contact triad connection ∇ of (Q, λ, J) and the contact Hermitian connection ∇^π for (ξ, J) introduced in [OW2].

Theorem 10.1. *In terms of the decomposition $d\pi = d^\pi w + w^*\lambda\, X_\lambda$ and $Y = Y^\pi + \lambda(Y)X_\lambda$, we have*

$$\begin{aligned} D\Upsilon_1(w)(Y) &= \overline{\partial}^{\nabla^\pi} Y^\pi + B^{(0,1)}(Y^\pi) + T^{\pi,(0,1)}_{dw}(Y^\pi) &\quad (10.3)\\ &\quad + \frac{1}{2}\lambda(Y)(\mathcal{L}_{X_\lambda}J)J(\partial^\pi w) &\quad (10.4)\\ D\Upsilon_2(w)(Y) &= -\Delta(\lambda(Y))\, dA + d((Y^\pi \rfloor d\lambda) \circ j) &\quad (10.5) \end{aligned}$$

where $B^{(0,1)}$ and $T^{\pi,(0,1)}_{dw}$ are the $(0,1)$-components of B and $T^{\pi,(0,1)}_{dw}$, where B, T^π_{dw} : $\Omega^0(w^*TQ) \to \Omega^1(w^*\xi)$ are zero-order differential operators given by

$$B(Y) = -\frac{1}{2} w^*\lambda\, ((\mathcal{L}_{X_\lambda}J)JY)$$

and

$$T^\pi_{dw}(Y) = \pi T(Y, dw)$$

respectively.

Proof. Let Y be a vector field over w and w_s be a family of maps $w_s : \Sigma \to Q$ with $w_0 = w$ and $Y = \frac{d}{ds}\Big|_{s=0} w^s$, and $a = \frac{d\gamma}{dt}\Big|_{t=0}$ for a curve γ with $\gamma(0) = z$. We decompose

$$Y = Y^\pi + \lambda(Y)\, X_\lambda$$

into the sum of ξ-component and X_λ-component. Now we calculate

$$D_w(d^\pi)(Y) := \nabla^\pi_s(\pi dw_s)\Big|_{s=0} = \pi\nabla_s(\pi dw_s)\Big|_{s=0} \tag{10.6}$$

We will evaluate

$$\begin{aligned} \nabla^\pi_s(\pi dw_s) &= \pi\nabla_s(\Pi dw_s) \\ &= \pi(\nabla_s\Pi)(dw_s) + \pi\nabla_s(dw_s). \end{aligned}$$

To evaluate this, we recall the following basic identity

Lemma 10.2 (Equations (5.2) & (5.3) [OW1]). *Let ∇ be the contact triad connection. Then*

$$\Pi(\nabla\Pi)Y = 0 \tag{10.7}$$

for all $Y \in \xi$, and

$$(\nabla\Pi)X_\lambda = -\Pi\nabla X_\lambda = -\Pi\left(\frac{1}{2}(\mathcal{L}_{X_\lambda}J)J\right). \tag{10.8}$$

Using this lemma, we compute

$$\begin{aligned} \pi(\nabla_s\Pi)(dw_s) &= \pi(\nabla_s\Pi)(d^\pi w_s + w_s^*\lambda\, X_\lambda) \\ &= \pi(\nabla_s\Pi)(w_s^*\lambda\, X_\lambda) = w_s^*\lambda\,\pi(\nabla_s\Pi)(X_\lambda) \\ &= -w_s^*\lambda\,\pi\left(\frac{1}{2}(\mathcal{L}_{X_\lambda}J)JY\right). \end{aligned} \tag{10.9}$$

Next, the standard computation of $\nabla_s(dw_s)|_{s=0}$ gives rise to

$$\begin{aligned} \pi\nabla_s(dw_s)|_{s=0}(a) &= \pi\nabla_s\left(dw_s\left(\frac{d\gamma}{dt}\right)\right)\Big|_{(s,t)=(0,0)} \\ &= \pi\nabla_s\frac{d}{dt}(w_s\circ\gamma)\Big|_{(s,t)=(0,0)} \\ &= \pi(\nabla_a Y + T(Y, dw(a)) \\ &= \pi(\nabla_a Y) + \pi(T(Y, dw(a)). \end{aligned} \tag{10.10}$$

On the other hand, we compute

$$\begin{aligned} \pi(\nabla_a Y) &= \pi(\nabla_a Y^\pi + \nabla_a(\lambda(Y)\, X_\lambda) \\ &= \nabla_a^\pi Y^\pi + \lambda(Y)\nabla_a X_\lambda \\ &= \nabla_a^\pi Y^\pi + \lambda(Y)\nabla_{d^\pi w(a)} X_\lambda \\ &= \nabla_a^\pi Y^\pi + \frac{1}{2}\lambda(Y)(\mathcal{L}_{X_\lambda}J)Jd^\pi w(a) \end{aligned}$$

where we used the formula $\nabla X_\lambda = \frac{1}{2}(\mathcal{L}_{X_\lambda}J)J$ for the second equality. This proves

$$\pi(\nabla Y) = \nabla^\pi Y^\pi + \frac{1}{2}\lambda(Y)(\mathcal{L}_{X_\lambda}J)Jd^\pi w.$$

Substituting this into (10.10), we derive

$$\pi\nabla_s(dw_s)|_{s=0} = \nabla^\pi Y^\pi + \frac{1}{2}\lambda(Y)(\mathcal{L}_{X_\lambda}J)Jd^\pi w.$$

Combining this with (10.9), we obtain

$$\nabla_s^\pi(\pi dw_s)|_{s=0} = \nabla^\pi Y^\pi + T^\pi(Y, dw) + \frac{1}{2}\lambda(Y)(\mathcal{L}_{X_\lambda}J)J\pi dw - w^*\lambda\left(\frac{1}{2}(\mathcal{L}_{X_\lambda}J)JY\right).$$

Therefore we have derived

$$\begin{aligned} D_w(d^\pi)(Y) &= \nabla_s^\pi(\pi dw_s)|_{s=0} \\ &= \nabla^\pi Y^\pi + T^\pi(Y, dw) + \frac{1}{2}\lambda(Y)\pi(\mathcal{L}_{X_\lambda}J)Jdw - \frac{1}{2}w^*\lambda\left((\mathcal{L}_{X_\lambda}J)JY\right). \end{aligned}$$

We note that

$$\begin{aligned}\frac{1}{2}\left(\lambda(Y)(\mathcal{L}_{X_\lambda}J)J\pi dw\right)^{(0,1)} &= \frac{1}{2}\lambda(Y)\left(\left(\frac{\mathcal{L}_{X_\lambda}J)J\pi dw + J(\mathcal{L}_{X_\lambda}J)J\pi dw\circ j}{2}\right)\right.\\ &= \frac{1}{2}\lambda(Y)\mathcal{L}_{X_\lambda}JJ\left(\frac{\pi dw - J\pi dw\circ j}{2}\right)\\ &= \frac{1}{2}\lambda(Y)(\mathcal{L}_{X_\lambda}J)J\partial^\pi w\end{aligned}$$

where $\partial^\pi w = (\pi dw)^{(1,0)}$. By taking the $(0,1)$-projection, we have proved (10.4).

Next we compute $D\Upsilon_2(w)$ and prove (10.5). We compute $\frac{d}{ds}|_{s=0} d(w_s^*\lambda\circ j)$

$$\frac{d}{ds}\Big|_{s=0} d(w_s^*\lambda\circ j) = d\left(\frac{d}{ds}\Big|_{s=0} w_s^*\lambda\circ j\right). \tag{10.11}$$

By Cartan's formula applied to the *vector field* Y *over the map* w, we obtain

$$\frac{d}{ds}\Big|_{s=0} w_s^*\lambda = Y\rfloor d\lambda + d(Y\rfloor\lambda)$$

where $\rfloor$ is the interior product over the map w. Substituting this into (10.11), we derive

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0} d(w_s^*\lambda\circ j) &= d(d(\lambda(Y))\circ j) + d((Y\rfloor d\lambda)\circ j)\\ &= -\Delta(\lambda(Y))\,dA + d((Y\rfloor d\lambda)\circ j).\end{aligned}$$

This proves

$$D\Upsilon_2(w)(Y) = -\Delta(\lambda(Y))\,dA + d((Y\rfloor d\lambda)\circ j) = -\Delta(\lambda(Y))\,dA + d((Y^\pi\rfloor d\lambda)\circ j) \tag{10.12}$$

which finishes the proof of Theorem 10.1. ☐

Now we evaluate the $D\Upsilon_1(w)$ more explicitly. We have

$$\overline{\partial}^{\nabla^\pi} Y = \frac{1}{2}\left(\nabla^\pi Y + J\nabla^\pi_{j(\cdot)}Y\right)$$

and $B^{(0,1)}(Y)$ becomes

$$-\frac{1}{4}\left(w^*\lambda\,\pi((\mathcal{L}_{X_\lambda}J)JY) + w^*\lambda\circ j\,\pi(\mathcal{L}_{X_\lambda}J)Y\right).$$

11. Fredholm theory and index calculations

We divide our discussion into the closed case and the punctured case.

11.1. The closed case.

We start with the following classification result. This is stated by Abbas as a part of [Ab, Proposition 1.4]. A somewhat different proof is also given in [OW2]. (See Proposition 3.3 [OW2].)

Proposition 11.1. *Assume $w : \Sigma \to M$ is a smooth contact instanton from a closed Riemann surface. Then*

1. *If $g(\Sigma) = 0$, w can only be a constant map;*
2. *If $g(\Sigma) \geq 1$, w is either a constant or has its locus of its image is a* closed *Reeb orbit.*

In particular, any such instanton is massless and satisfies $[w] = 0$ in $H_2(Q;\mathbb{Z})$.

From the expression of the map $\Upsilon = (\Upsilon_1, \Upsilon_2)$, the map defines a bounded linear map

$$D\Upsilon(w) : \Omega^0_{k,p}(w^*TQ) \to \Omega^{(0,1)}_{k-1,p}(w^*\xi) \oplus \Omega^2_{k-2,p}(\Sigma). \tag{11.1}$$

We choose $k \geq 2$, $p > 2$. Recalling the decomposition

$$Y = Y^\pi + \lambda(Y)\, X_\lambda,$$

we have the decomposition

$$\Omega^0_{k,p}(w^*TQ) \cong \Omega^0_{k,p}(w^*\xi) \oplus \Omega^0_{k,p}(\dot\Sigma, \mathbb{R}) \cdot X_\lambda.$$

Here we use the splitting

$$TQ = \operatorname{span}_{\mathbb{R}}\{X_\lambda\} \oplus \xi$$

where $\operatorname{span}_{\mathbb{R}}\{X_\lambda\} := \mathcal{L}$ is a trivial line bundle and so

$$\Gamma(w^*\mathcal{L}) \cong C^\infty(\Sigma).$$

By definition as the linearization operator $D\Upsilon_2(w)$ acts trivially for the section Y tangent to the Reeb direction.

It follows that the map $D\Upsilon(w)$ is a partial differential operator whose symbol map is given by $\sigma(D\Upsilon) = \sigma(D\Upsilon_1) \oplus \sigma(D\Upsilon_2)$ where

$$\begin{aligned} \sigma(D\Upsilon_1(w))(\eta) &= J\Pi^*\eta \\ \sigma(D\Upsilon_2(w))(\eta) &= \langle \lambda, \eta\rangle^2 = (\eta(X_\lambda))^2 \end{aligned} \tag{11.2}$$

where η is a cotangent vector in $T^*Q \setminus \{0\}$ and has decomposition

$$\eta = \eta^\pi + \eta(X_\lambda(\pi(\eta))\, \lambda(\pi(\eta)).$$

Therefore $D\Upsilon(w)$ can be written into the matrix form

$$\begin{pmatrix} \overline\partial^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)} & \frac{1}{2}\lambda(\cdot)(\mathcal{L}_{X_\lambda}J)J\partial^\pi w \\ d\left((\cdot)\rfloor d\lambda\right)\circ j & -\Delta(\lambda(\cdot))\, dA \end{pmatrix} \tag{11.3}$$

where

$$\begin{aligned} \overline\partial^{\nabla^\pi} + B^{(0,1)} &: \Omega^0_{k,p}(w^*\xi) \to \Omega^{(0,1)}_{k-1,p}(w^*\xi) \\ -*\Delta &: \Omega^0_{k,p}(\Sigma) \to \Omega^2_{k-2,p}(\Sigma) \\ d\left((\cdot)\rfloor d\lambda\right)\circ j &: \Omega^0_{k,p}(w^*\xi) \to \Omega^2_{k-1,p}(\Sigma) \hookrightarrow \Omega^2_{k-2,p}(\Sigma). \end{aligned}$$

In particular we note that the restriction $D\Upsilon_1(w)|_{\Omega^0(w^*\xi)}$ has the same symbol as that of

$$\overline\partial^{\nabla^\pi} : \Omega^0(w^*\xi) \to \Omega^{(0,1)}(w^*\xi)$$

which is the first order elliptic operator of Cauchy-Riemann type, and $D\Upsilon_2(w)$ has the symbol of the Hodge Laplacian acting on zero forms

$$*\Delta : \Omega^0(\Sigma) \to \Omega^2(\Sigma).$$

We now establish Fredholm property and the index formula of the operator $D\Upsilon(w)$ by dividing the study into the closed and the punctured cases.

For the closed case, we derive

Proposition 11.2. *Consider the completion of $D\Upsilon(w)$, which we still denote by $D\Upsilon(w)$, as a bounded linear map from $\Omega^0_{k,p}(w^*TQ)$ to $\Omega^{(0,1)}(w^*\xi)\oplus\Omega^2(\Sigma)$ for $k \geq 2$ and $p \geq 2$. Then the operator $D\Upsilon(w)$ is homotopic to the operator*

$$\begin{pmatrix} \overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)} & 0 \\ 0 & -\Delta(\lambda(\cdot))\, dA \end{pmatrix} \tag{11.4}$$

via the homotopy

$$s \in [0,1] \mapsto \begin{pmatrix} \overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)} & \frac{s}{2}\lambda(\cdot)(\mathcal{L}_{X_\lambda}J)J(\pi dw)^{(1,0)} \\ s\, d\left((\cdot)\rfloor d\lambda\right)\circ j) & -\Delta(\lambda(\cdot))\, dA \end{pmatrix} =: L_s \tag{11.5}$$

which is a continuous family of Fredholm operators. And the principal symbol

$$\sigma(z,\eta) : w^*TQ|_z \to w^*\xi|_z \oplus \Lambda^2(T_z\Sigma), \quad 0 \neq \eta \in T^*_z\Sigma$$

of (11.4) *is given by the matrix*

$$\begin{pmatrix} \frac{\eta + i\eta\circ j}{2} Id & 0 \\ 0 & |\eta|^2 \end{pmatrix}$$

after applying the isomorphism $* : \Omega^2(\Sigma) \to \Omega^0(\Sigma)$ and so is elliptic.

Proof. It is enough to establish the inequality

$$\begin{aligned} \|Y\|_{k,p} \quad \leq \quad & C(\|\pi_1(L_s(Y))\|_{k-1,p} + \|\pi_1(K_s(Y))\|_{k-1,p}) \\ & +\|\pi_2(L_s(Y))\|_{k-2,p} + \|\pi_2(K_s(Y))\|_{k-2,p}) \end{aligned} \tag{11.6}$$

for a family of compact operators $K_s : \Omega^0_{k,p}(w^*TQ) \to \Omega^{(0,1)}_{k-1,p}(w^*\xi) \oplus \Omega^2_{k-2,p}(\Sigma)$ and a constant C independent of $s \in [0,1]$ for all $Y \in \Omega_{k,p}(w^*TQ)$.

We decompose $Y = Y^\pi + \lambda(Y)\, X_\lambda$. We have already computed above

$$\begin{aligned} \pi_1(L_s(Y)) &= (\overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)})(Y^\pi) + \frac{s}{2}\lambda(Y)(\mathcal{L}_X J)J(\pi dw)^{(1,0)} \\ \pi_2(L_s(Y)) &= s\, d(Y\rfloor d\lambda)\circ j - \Delta(\lambda(Y))\, dA. \end{aligned}$$

By the ellipticity of $\overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)} : \Omega^0(w^*\xi) \to \Omega^{(0,1)}(w^*\xi)$ and of $\Delta : \Omega^0(\Sigma) \to \Omega^0(\Sigma)$, we have

$$\|Y^\pi\|_{k,p} \leq C(\|(\overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)})(Y^\pi)\|_{k-1,p} + \|Y^\pi\|_{k-1,p}) \tag{11.7}$$

and

$$\|\lambda(Y)\|_{k,p} \leq C(\|\Delta(\lambda(Y))\|_{k-2,p} + \|\lambda(Y)\|_{k-2,p}). \tag{11.8}$$

Then we get

$$\begin{aligned} & \|\lambda(Y)(\mathcal{L}_X J)J(\pi dw)^{(1,0)}\|_{k-1,p} \\ \leq \;& C_k(\|(\mathcal{L}_X J)J(\pi dw)^{(1,0)}\|_{k-2,\infty}\|\lambda(Y)\|_{k-1,p} + \|(\mathcal{L}_X J)J(\pi dw)^{(1,0)}\|_{k-1,\infty}\|\lambda(Y)\|_{k-2,p}) \\ \leq \;& C_k\|(\mathcal{L}_X J)J(\pi dw)^{(1,0)}\|_{k-2,\infty}\,(C(\|\Delta(\lambda(Y))\|_{k-2,p} + \|\lambda(Y)\|_{k-2,p}) \end{aligned}$$

(Here the last line can be improved by $k-3$ for $k \geq 3$ but $k-2$ will be enough for our purpose which we have to use anyway for $k=2$), and

$$\|d(Y\rfloor d\lambda)\circ j\|_{k-2,p} \leq C_k(\|Y^\pi\|_{k-1,p}\|d\lambda\|_{k-2,\infty} + \|Y^\pi\|_{k-1,p}\|d\lambda\|_{k-1,\infty})$$

for some constant C_k depending only on k (and dw) but independent of Y. Combining all the above, using the bounds for $\|(\mathcal{L}_X J)J(\pi dw)^{(1,0)}\|_{k-2,\infty}$ and $\|d\lambda\|_{k-1,\infty}$ and substituting

$$(\overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)})(Y^\pi) = \pi_1(L_s(Y)) - \frac{s}{2}\lambda(Y)(\mathcal{L}_X J)J(\pi dw)^{(1,0)}$$

and

$$-\Delta(\lambda(Y))\, dA = \pi_2(L_s(Y)) - s\, d(Y \rfloor d\lambda) \circ j$$

into (11.7) and (11.8) and then rearranging terms, we derive

$$\|Y\|_{k,p} \leq C(\|\pi_1(L_s(Y))\|_{k-1,p} + \|Y^\pi\|_{k-1,p} + \|\pi_2(L_s(Y))\|_{k-2,p} + \|Y\|_{k-2,p}) \quad (11.9)$$

for a constant C independent of $s \in [0,1]$ for all $Y \in \Omega_{k,p}(w^*TQ)$. By the compactness of the Sobolev embedding $W^{l,p}$ into $W^{l-1,p}$ for $l = k,\, k-1$ (on compact Σ), we have finished the proof of (11.6) by taking the operator $K_s = K_{1,s} + K_{2,s}$: Here $K_{1,s}$ is the composition of the bounded map

$$\Omega^0_{k,p}(w^*TQ) \to \Omega^{(0,1)}_{k,p}(w^*\xi) \oplus \Omega^2_{k-1,p}(\Sigma)$$

defined by

$$Y \mapsto \begin{pmatrix} \frac{s}{2}\lambda(Y)(\mathcal{L}_X J)J(\pi dw)^{(1,0)} \\ s\, d\left((\cdot)\rfloor d\lambda\right) \circ j) \end{pmatrix}$$

and the inclusion map

$$\Omega^{(0,1)}_{k,p}(w^*\xi) \oplus \Omega^2_{k-1,p}(\Sigma) \to \Omega^{(0,1)}_{k-1,p}(w^*\xi) \oplus \Omega^2_{k-2,p}(\Sigma)$$

which is compact. In particular, $K_{1,s}$ is a compact operator.

And we define $K_{2,s}$ is just the inclusion map

$$\Omega^0_{k,p}(w^*TQ) \cong \Omega^0_{k,p}(w^*\xi) \oplus \Omega^0_{k,p}(\Sigma) \hookrightarrow \Omega^0_{k-1,p}(w^*\xi) \oplus \Omega^0_{k-2,p}(\Sigma)$$

which is also compact. Obviously

$$\|Y^\pi\|_{k-1,p} + \|\lambda(Y)\|_{k-2,p} \leq \|\pi_1(K_{2,s}(Y))\|_{k-1,p} + \|\pi_2(K_{2,s}(Y)\|_{k-2,p}.$$

Therefore combining all the above, we have established (11.6) which finishes the proof. ◻

From this, we immediately derive the following index formula for $D\Upsilon(w)$ from the homotopy invariance of the index

Theorem 11.3. *Let Σ be any closed Riemann surface of genus g, and let $w : \Sigma \to Q$ be a solution to* (2.1) *with finite energy. Then the operator* (11.1) *is a Fredholm operator whose index is given by*

$$\operatorname{Index} D\Upsilon(w) = 2n(1-g). \quad (11.10)$$

Proof. We already know that the operators $\overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)}$ and $-\Delta$ are Fredholm. Furthermore we can homotope the operator (11.3) to the direct sum operator

$$(\overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)} + \frac{1}{2}\lambda(\cdot)(\mathcal{L}_{X_\lambda} J)J\partial^\pi w \oplus (- * \Delta(\lambda(\cdot)))$$

by considering the continuous deformation of Fredholm operators

$$s \mapsto \begin{pmatrix} \overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)} & \frac{1}{2}\lambda(\cdot)(\mathcal{L}_{X_\lambda} J)J\partial^\pi w \\ s\, d\left((\cdot)\rfloor d\lambda\right) \circ j) & - * \Delta(\lambda(\cdot)) \end{pmatrix}$$

from $s = 1$ to $s = 0$. From this, the Fredholm property immediately follows. Then the index is given by

$$\operatorname{Index} \overline{\partial}^{\nabla^\pi} + \operatorname{Index}(-\Delta) = 2c_1(w^*\xi) + 2n(1-g) + 0 = 2c_1(w^*\xi) + 2n(1-g)$$

in general. But since $[w] = 0$ in $H_2(Q;\mathbb{Z})$ by Proposition 11.1, this is reduced to (11.10). This finishes the proof. ☐

We would like to call attention of readers that the index $\operatorname{Index} \overline{\partial}^{\nabla^\pi} = 2n$ when $g = 0$ is 1 smaller than the dimension of Q.

11.2. **The punctured case.** For the punctured case, we need to make some preparation. For the exposition of this section, we adapt the exposition given by Bourgeois and Mohnke in [BM] to the current context of contact Cauchy-Riemann maps. Because the structure of the linearization of (2.1) is significantly different, establishing the Fredholm property of the linearization map and its index calculation is also different. In particular, a priori the ellipticity itself of the linearization map is not obvious.

From now on in the rest of the paper, we will restrict ourselves to the case of vanishing charge, i.e., we put the following hypothesis.

Hypothesis 11.4 (Charge vanishing)**.** We assume the asymptotic charges of w at all ends vanish, i.e.,

$$-a = \lim_{\tau\to\infty} \int_{\partial_\ell \Sigma(\rho)} w(\tau,\cdot)^*\lambda \circ j = 0 \tag{11.11}$$

for all $\ell = 1, \cdots, k$ where $\rho = e^{-2\pi\tau}$.

Let $(\dot\Sigma, j)$ be a punctured Riemann surface and let

$$p_1, \cdots, p_{s^+}, q_1, \cdots, q_{s^-}$$

be the positive and negative punctures. Fix an elongation function $\rho : \mathbb{R} \to [0,1]$ so that

$$\begin{aligned} \rho(\tau) &= \begin{cases} 1 & \tau \geq 1 \\ 0 & \tau \leq 0 \end{cases} \\ 0 &\leq \rho'(\tau) \leq 2. \end{aligned}$$

Let γ_i^+ for $i = 1, \cdots, s^+$ and γ_j^- for $j = 1, \cdots, s^-$ be two given collections of Reeb orbits. For each p_i (resp. q_j), we associate the isothermal coordinates $(\tau, t) \in [0,\infty) \times S^1$ (resp. $(\tau, t) \in (-\infty, 0] \times S^1$) on the punctured disc $D_{e^{-2\pi R_0}}(p_i) \setminus \{p_i\}$ (resp. on $D_{e^{-2\pi R_0}}(q_i) \setminus \{q_i\}$) for some sufficiently large $R_0 > 0$. Then we consider sections of w^*TQ by

$$\overline{Y}_i = \rho(\tau - R_0) X_\lambda(\gamma_k^+(t)), \quad \underline{Y}_j = \rho(\tau + R_0) X_\lambda(\gamma_k^+(t)) \tag{11.12}$$

and denote by $\Gamma_{s^+,s^-} \subset \Gamma(w^*TQ)$ the subspace defined by

$$\Gamma_{s^+,s^-} = \bigoplus_{i=1}^{s^+} \mathbb{R}\{\overline{Y}_i\} \oplus \bigoplus_{j=1}^{s^-} \mathbb{R}\{\underline{Y}_j\}.$$

Let $k \geq 2$ and $p > 2$. We denote by

$$\mathcal{W}^{k,p}_\delta(\dot\Sigma, Q; J; \gamma^+, \gamma^-), \quad k \geq 2$$

the Banach manifold such that

$$\lim_{\tau\to\infty} w((\tau,t)_i) = \gamma_i^+(T_i(t+t_i)), \quad \lim_{\tau\to-\infty} w((\tau,t)_j) = \gamma_j^-(T_j(t-t_j)) \tag{11.13}$$

for some t_i, $t_j \in S^1$, where

$$T_i = \int_{S^1} (\gamma_i^+)^*\lambda,\ T_j = \int_{S^1} (\gamma_j^-)^*\lambda.$$

Here t_i, t_j depends on the given analytic coordinate and the parameterization of the Reeb orbits.

The local model of the tangent space of $\mathcal{W}^{k,p}_\delta(\dot\Sigma, Q; J; \gamma^+, \gamma^-)$ at $w \in C^\infty_\delta(\dot\Sigma, Q) \subset W^{k,p}_\delta(\dot\Sigma, Q)$ is given by

$$\Gamma_{s^+,s^-} \oplus W^{k,p}_\delta(w^*TQ) \tag{11.14}$$

where $W^{k,p}_\delta(w^*TQ)$ is the Banach space

$$\begin{aligned} &\{Y = (Y^\pi, \lambda(Y)\, X_\lambda) \mid e^{\frac{\delta}{p}|\tau|} Y^\pi \in W^{k,p}(\dot\Sigma, w^*\xi),\ \lambda(Y) \in W^{k,p}(\dot\Sigma, \mathbb{R})\} \\ \cong\quad & W^{k,p}(\dot\Sigma, \mathbb{R}) \cdot X_\lambda(w) \oplus W^{k,p}(\dot\Sigma, w^*\xi). \end{aligned}$$

Here we measure the various norms in terms of the triad metric of the triad (Q, λ, J). To describe the choice of $\delta > 0$, we need to recall the covariant linearization of the map $D\Upsilon_{\lambda,T} : W^{1,2}(z^*\xi) \to L^2(z^*\xi)$ of the map

$$\Upsilon_{\lambda,T} : z \mapsto \dot z - T\, X_\lambda(z)$$

for a given T-periodic Reeb orbit (T, z). The operator has the expression

$$D\Upsilon_{\lambda,T} = \frac{D^\pi}{dt} - \frac{T}{2}(\mathcal{L}_{X_\lambda} J)J =: A_{(T,z)} \tag{11.15}$$

where $\frac{D^\pi}{dt}$ is the covariant derivative with respect to the pull-back connection $z^*\nabla^\pi$ along the Reeb orbit z and $(\mathcal{L}_{X_\lambda} J)J$ is (pointwise) symmetric operator with respect to the triad metric. (See Lemma 3.4 [OW1].) We choose $\delta > 0$ so that $0 < \delta/p < 1$ is smaller than the spectral gap

$$\operatorname{gap}(\gamma^+, \gamma^-) := \min_{i,j}\{d_{\mathrm{H}}(\operatorname{spec} A_{(T_i,z_i)}, 0),\ d_{\mathrm{H}}(\operatorname{spec} A_{(T_j,z_j)}, 0)\}. \tag{11.16}$$

Now for each given $w \in \mathcal{W}^{k,p}_\delta := \mathcal{W}^{k,p}_\delta(\dot\Sigma, Q; J; \gamma^+, \gamma^-)$, we consider the Banach space

$$\Omega^{(0,1)}_{k-1,p;\delta}(w^*\xi)$$

the $W^{k-1,p}_\delta$-completion of $\Omega^{(0,1)}(w^*\xi)$ and form the bundle

$$\mathcal{H}^{(0,1)}_{k-1,p;\delta}(\xi) = \bigcup_{w \in \mathcal{W}^{k,p}_\delta} \Omega^{(0,1)}_{k-1,p;\delta}(w^*\xi)$$

over $\mathcal{W}^{k,p}_\delta$. Then we can regard the assignment

$$\Upsilon_1 : w \mapsto \overline\partial^\pi w$$

as a smooth section of the bundle $\mathcal{H}^{(0,1)}_{k-1,p;\delta}(\xi) \to \mathcal{W}^{k,p}_\delta$. Furthermore the assignment

$$\Upsilon_2 : w \mapsto d(w^*\lambda \circ j)$$

defines a smooth section of the trivial bundle

$$\Omega^2_{k-2,p}(\Sigma) \times \mathcal{W}^{k,p}_\delta \to \mathcal{W}^{k,p}_\delta.$$

We have already computed the linearization of each of these maps in the previous section.

With these preparations, the following is a corollary of exponential estimates established in Part II [OW2] for the case $Q(p_i) = 0$. We hope that the relevant off-shell analytical framework for the case $Q(p_i) \neq 0$ can be treated elsewhere.

Proposition 11.5 (Theorem 1.12 [OW2]). *Assume λ is nondegenerate and $Q(p_i) = 0$. Let $w : \dot\Sigma \to Q$ be a contact instanton and let $w^*\lambda = a_1\, d\tau + a_2\, dt$. Suppose*

$$\lim_{\tau\to\infty} a_{1,i} = -Q(p_i), \qquad \lim_{\tau\to\infty} a_{2,i} = T(p_i)$$
$$\lim_{\tau\to-\infty} a_{1,j} = -Q(q_j), \qquad \lim_{\tau\to-\infty} a_{2,j} = T(p_j) \tag{11.17}$$

at each puncture p_i and q_j. Then $w \in \mathcal{W}^{k,p}_\delta(\dot\Sigma, Q; J; \gamma^+, \gamma^-)$.

Now we are ready to define the moduli space of contact instantons with prescribed asymptotic condition as the zero set

$$\mathcal{M}(\dot\Sigma, Q; J; \gamma^+, \gamma^-) = \mathcal{W}^{k,p}_\delta(\dot\Sigma, Q; J; \gamma^+, \gamma^-) \cap \Upsilon^{-1}(0) \tag{11.18}$$

whose definition does not depend on the choice of k, p or δ as long as $k \geq 2$, $p > 2$ and $\delta > 0$ is sufficiently small. One can also vary λ and J and define the universal moduli space whose detailed discussion is postponed.

In the rest of this section, we establish the Fredholm property of the linearization map

$$D\Upsilon_{(\lambda,T)}(w) : \Omega^0_{k,p;\delta}(w^*TQ; J; \gamma^+, \gamma^-) \to \Omega^{(0,1)}_{k-1,p;\delta}(w^*\xi) \oplus \Omega^2_{k-2,p}(\Sigma)$$

and compute its index. Here we also denote

$$\Omega^0_{k-2,p;\delta}(w^*TQ; J; \gamma^+, \gamma^-) = W^{k-2,p}_\delta(w^*TQ; J; \gamma^+, \gamma^-)$$

for the semantic reason.

For this purpose, we remark that as long as the set of punctures is non-empty, the symplectic vector bundle $w^*\xi \to \dot\Sigma$ is trivial. We denote by $\Phi : E \to \overline{\Sigma} \times \mathbb{R}^{2n}$ and by

$$\Phi_i^+ := \Phi|_{\partial_i^+\overline{\Sigma}}, \quad \Phi_j^- = \Phi|_{\partial_j^-\overline{\Sigma}}$$

its restrictions on the corresponding boundary components of $\partial\overline{\Sigma}$. Using the cylindrical structure near the punctures, we can extend the bundle to the bundle $E \to \overline{\Sigma}$ where $\overline{\Sigma}$ is the real blow-up of the punctured Riemann surface $\dot\Sigma$.

We then consider the following set

$$\mathcal{S} := \{A : [0,1] \to Sp(2n, \mathbb{R}) \mid 1 \notin \operatorname{spec}(A(1)),\, A(0) = id,\, \dot A(0)A(0)^{-1} = \dot A(1)A(1)^{-1}\}$$

of regular paths in $Sp(2n, \mathbb{R})$ and denote by $\mu_{CZ}(A)$ the Conley-Zehnder index of the paths following [RS]. Recall that for each closed Reeb orbit γ with a fixed trivialization of ξ, the covariant linearization $A_{(T,z)}$ of the Reeb flow along γ determines an element $A_\gamma \in \mathcal{S}$. We denote by Ψ_i^+ and Ψ_j^- the corresponding paths induced from the trivializations Φ_i^+ and Φ_j^- respectively.

We have the decomposition

$$\Omega^0_{k,p;\delta}(w^*TQ; J; \gamma^+, \gamma^-) = \Omega^0_{k,p;\delta}(w^*\xi) \oplus \Omega^0_{k,p;\delta}(\Sigma)$$

and again the operator

$$D\Upsilon_{(\lambda,T)}(w) : \Omega^0_{k,p;\delta}(w^*TQ; J; \gamma^+, \gamma^-) \to \Omega^{(0,1)}_{k-1,p;\delta}(w^*\xi) \oplus \Omega^2_{k-2,p;\delta}(\Sigma)$$

can be written into the matrix

$$\begin{pmatrix} \overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)} & \frac{1}{2}\lambda(\cdot)(\mathcal{L}_{X_\lambda} J) J \partial^\pi w \\ d\left((\cdot) \rfloor d\lambda\right) \circ j & - * \Delta(\lambda(\cdot)) \end{pmatrix} \tag{11.19}$$

where

$$\begin{aligned} \overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)} &: \Omega^0_{k,p;\delta}(w^*\xi; J; \gamma^+, \gamma^-) \to \Omega^{(0,1)}_{k-1,p;\delta}(w^*\xi) \\ - * \Delta &: \Omega^0_{k,p;\delta}(\Sigma) \to \Omega^2_{k-2,p;\delta}(\Sigma) \\ d\left((\cdot) \rfloor d\lambda\right) \circ j &: \Omega^0_{k,p;\delta}(w^*\xi; J; \gamma^+, \gamma^-) \to \Omega^2_{k-1,p;\delta}(\Sigma) \hookrightarrow \Omega^2_{k-2,p;\delta}(\Sigma). \end{aligned}$$

The following proposition can be derived from the arguments used by Lockhart and McOwen [LM]. However before applying their general theory, one needs to pay some preliminary measure to handle the fact that the order the operators $D\Upsilon(w)$ are different depending on the direction of ξ or on that of X_λ.

Proposition 11.6. *Suppose $\delta > 0$ satisfies the inequality*

$$0 < \delta < \min\left\{\frac{\operatorname{gap}(\gamma^+, \gamma^-)}{p}, \frac{2\pi}{p}\right\}$$

where $\operatorname{gap}(\gamma^+, \gamma^-)$ *is the spectral gap, given in* (11.16)*, of the asymptotic operators $A_{(T_j, z_j)}$ or $A_{(T_i, z_i)}$ associated to the corresponding punctures. Then the operator* (11.19) *is Fredholm.*

Proof. We first note that the operators $\overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)}$ and $-\Delta$ are Fredholm: The relevant a priori coercive $W^{k,2}$-estimates for any integer $k \geq 1$ for the derivative dw on the punctured Riemann surface $\dot\Sigma$ with cylindrical metric near the punctures are established in [OW2] for the operator $\overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)}$ and the one for $-\Delta$ is standard. From this, the standard interpolation inequality establishes the $W^{k,p}$-estimates for $D\Upsilon(w)$ for all $k \geq 2$ and $p \geq 2$.

Secondly, it follows that the operator (11.19) can be homotoped to the direct sum operator

$$(\overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)}) \oplus (-\Delta)$$

by considering the continuous deformation of operators

$$s \mapsto \begin{pmatrix} \overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)} & \frac{s}{2}\lambda(\cdot)(\mathcal{L}_{X_\lambda} J) J \partial^\pi w \\ s\, d\left((\cdot) \rfloor d\lambda\right) \circ j & - * \Delta(\lambda(\cdot)) \end{pmatrix}$$

from $s = 1$ to $s = 0$. Once these two are established, the proof of the proposition is parallel to that of Proposition 11.2 and so omitted. □

Then by the continuous invariance of the Fredholm index, we obtain

$$\operatorname{Index} D\Upsilon_{(\lambda,T)}(w) = \operatorname{Index}(\overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)}) + \operatorname{Index}(-\Delta). \tag{11.20}$$

Therefore it remains to compute the latter two indices. For this, we obtain

Theorem 11.7. *We fix a trivialization $\Phi : E \to \overline{\Sigma}$ and denote by Ψ_i^+ (resp. Ψ_j^-) the induced symplectic paths associated to the trivializations Φ_i^+ (resp. Φ_j^-) along* *the Reeb orbits* γ_i^+ *(resp.* γ_j^-*) at the punctures* p_i *(resp.* q_j*) respectively. Then we have*

$$\begin{aligned}\operatorname{Index}(\overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)}) &= n(2-2g-s^+-s^-) + 2c_1(w^*\xi) + (s^+ + s^-)\\ &\quad + \sum_{i=1}^{s^+} \mu_{CZ}(\Psi_i^+) - \sum_{j=1}^{s^-} \mu_{CZ}(\Psi_j^-) \qquad (11.21)\\ \operatorname{Index}(-\Delta) &= \sum_{i=1}^{s^+} m(\gamma_i^+) + \sum_{j=1}^{s^-} m(\gamma_j^-) - g. \qquad (11.22)\end{aligned}$$

In particular,

$$\begin{aligned}\operatorname{Index} D\Upsilon_{(\lambda,T)}(w) &= n(2-2g-s^+-s^-) + 2c_1(w^*\xi)\\ &\quad + \sum_{i=1}^{s^+} \mu_{CZ}(\Psi_i^+) - \sum_{j=1}^{s^-} \mu_{CZ}(\Psi_j^-)\\ &\quad + \sum_{i=1}^{s^+} (m(\gamma_i^+)+1) + \sum_{j=1}^{s^-} (m(\gamma_j^-)+1) - g. \qquad (11.23)\end{aligned}$$

Proof. The formula (11.21) can be immediately derived from the general formula given in the top of p. 52 of Bourgeois's thesis [Bo]: The summand $(s^+ + s^-)$ comes from the factor Γ_{s^+,s^-} in the decomposition (11.14) which has dimension $s^+ + s^-$.

So it remains to compute the index (11.22). We recall that any harmonic function on $\dot\Sigma$ can be written as the imaginary part of a holomorphic function on $\dot\Sigma$ with the same orders of zeros and poles respectively. (The converse also holds.) Therefore to compute the (real) index of $-\Delta$, we consider the Dolbeault complex

$$0 \to \Omega^0(\Sigma; D) \to \Omega^1(\Sigma; D) \to 0$$

where $D = D^+ + D^-$ is the divisor associated to the set of punctures

$$D^+ = \sum_{i=1}^{s^+} m(\gamma_i^+) p_i, \quad D^- = \sum_{j=1}^{s^-} m(\gamma_j^-) q_j$$

where $m(\gamma_i^+)$ (resp. $m(\gamma_j^-)$) is the multiplicity of the Reeb orbit γ_i^+ (resp. γ_j^-). The standard Riemann-Roch formula then gives rise to the formula for the Euler characteritic

$$\begin{aligned}\chi(D) &= \dim_{\mathbb C} H^0(D) - \dim_{\mathbb C} H^1(D) = \deg(D) - g\\ &= \sum_{i=1}^{s^+} m(\gamma_i^+) + \sum_{j=1}^{s^-} m(\gamma_j^-) - g.\end{aligned}$$

This finishes the proof.

□

12. Generic transversality under the perturbation of J

We start with recalling the linearization of the equation $\dot x = X_\lambda(x)$ along a closed Reeb orbit. Let z be a closed Reeb orbit of period $T > 0$. In other words, $z : \mathbb{R} \to Q$ is a periodic solution of $\dot z = X_\lambda(z)$ with period T, thus satisfying $z(T) = z(0)$.

Denote the Reeb flow $\phi^t = \phi^t_{X_\lambda}$ of the Reeb vector field X_λ, we can write $z(t) = \phi^t_{X_\lambda}(z(0))$. In particular $p := z(0)$ is a fixed point of the diffeomorphism ϕ^T. Further, since $L_{X_\lambda}\lambda = 0$, the contact diffeomorphism ϕ^T induces the isomorphism

$$\Psi_z := d\phi^T(p)|_{\xi_p} : \xi_p \to \xi_p$$

which is the tangent map of the Poincaré return map ϕ^T restricted to ξ_p.

Definition 12.1. We say a Reeb orbit with period T is *nondegenerate* if $\Psi_z : \xi_p \to \xi_p$ with $p = z(0)$ has no eigenvalue 1.

Denote $\mathrm{Cont}(Q,\xi)$ the set of contact 1 forms with respect to the contact structure ξ and $\mathcal{L}(Q) = C^\infty(S^1, Q)$ the space of loops $z : S^1 = \mathbb{R}/\mathbb{Z} \to Q$. Let $\mathcal{L}^{1,2}(Q)$ be the $W^{1,2}$-completion of $\mathcal{L}(Q)$. We would like to consider some Banach vector bundle $\mathcal{L}$ over the Banach manifold $(0,\infty) \times \mathcal{L}^{1,2}(Q) \times \mathrm{Cont}(Q,\xi)$ whose fiber at (T, z, λ) is given by $L^2(z^*TQ)$. We consider the assignment

$$\Upsilon : (T, z, \lambda) \mapsto \dot z - T\, X_\lambda(z)$$

which is section of $\mathcal{L}$.

Denote D the covariant derivative. Then we have the following expression of the full linearization.

Lemma 12.2.
$$d(T,z,\lambda)\Upsilon(a,Y,B) = \frac{DY}{dt} - TDX_\lambda(z)(Y) - aX_\lambda - T\delta_\lambda X_\lambda(B),$$
where $a \in \mathbb{R}$, $Y \in T_z\mathcal{L}^{1,2}(Q) = W^{1,2}(z^*TQ)$ and $B \in T_\lambda \mathrm{Cont}(Q,\xi)$ and the last term $\delta_\lambda X_\lambda$ is some linear operator.

By using this full linearization, one can study the generic existence of the contact one-forms which make all Reeb orbits nondegenerate. We refer to Appendix of [ABW] for its complete proof. We now assume that λ is such a generic contact form.

Now we involve the set $\mathcal{J}(Q,\lambda)$ given in (1.1). We study the linearization of the map Υ^{univ} which is the map Υ augmented by the argument $J \in \mathcal{J}(Q,\lambda)$. More precisely, we define

$$\Upsilon^{univ}(j,w,J) = \left(\overline{\partial}^\pi_J w, d(w^*\lambda \circ j)\right)$$

$\overline{\partial}$ at each $(j,w,J) \in \overline{\partial}^{-1}(0)$. In the discussion below, we will fix the complex structure j on Σ, and so suppress j from the argument of Υ^{univ}.

We denote the zero set $(\Upsilon^{univ})^{-1}(0)$ by

$$\mathcal{M}(Q,\lambda;\overline{\gamma},\underline{\gamma}) = \left\{(w,J) \in \mathcal{W}^{k,p}_\delta(\dot\Sigma, Q;\overline{\gamma},\underline{\gamma}) \times \mathcal{J}^\ell(Q,\lambda) \,\middle|\, \Upsilon^{univ}(w,J) = 0\right\}$$

which we call the universal moduli space. Denote by

$$\pi_2 : \mathcal{W}^{k,p}(\dot\Sigma, Q;\overline{\gamma},\underline{\gamma}) \times \mathcal{J}^\ell(Q,\lambda) \to \mathcal{J}^\ell(Q,\lambda)$$

the projection. Then we have

$$\mathcal{M}(J;\overline{\gamma},\underline{\gamma}) = \mathcal{M}(Q,\lambda,J;\overline{\gamma},\underline{\gamma}) = \pi_2^{-1}(J) \cap \mathcal{M}(Q,\lambda;\overline{\gamma},\underline{\gamma}). \tag{12.1}$$

One essential ingredient for the generic transversality under the perturbation of $J \in \mathcal{J}(Q,\lambda)$ is the usage of the following unique continuation result. We take a short cut in its proof relating the (local) contact instanton to a (local) pseudoholomorphic curves in a (local) symplectization exploiting the well-known unique continuation result for the pseudoholomorphic maps. Here again the closedness condition $d(w^*\lambda\circ j)$ for the contact instanton map w enters in an essential way.

Proposition 12.3 (Unique continuation lemma)**.** *Any non-constant contact Cauchy-Riemann map does not have an accumulation point in the zero set of dw.*

Proof. Suppose to the contrary that there exists a point $z_0 \in \Sigma$ and a sequence $z \to z_0$ such that $dw(z) = 0$ for all i. Since $w^*\lambda \circ j$ is closed on Σ, it can be written as $w^*\lambda \circ j = da$ on a neighborhood of z_0 for some locally defined function a. Then the pair (a, w) defines a pseudo-holomorphic map to $\mathbb{R} \times Q$. From the equation $w^*\lambda \circ j = da$, we also have $da(z) = 0$ too. This implies z are critical points of the pseudoholomorphic map (a, w) with z_0 as an accumulation point of z which are critical points of (a, w). Then the unique continuation lemma applied to (a, w) implies $(a, w) \equiv const$ and so w must be constant, a contradiction to the hypothesis. This finishes the proof. □

The following theorem summarizes the main transversality scheme needed for the study of the moduli problem of contact instanton map, whose proof is not very different from that of pseudo-holomorphic curves, once the above unique continuation result is established, and so omitted.

Theorem 12.4. *Let $0 < \ell < k - \frac{2}{p}$. Consider the moduli space $\mathcal{M}(Q, \lambda; \overline{\gamma}, \underline{\gamma})$. Then*

1. *$\mathcal{M}(Q, \lambda; \overline{\gamma}, \underline{\gamma})$ is an infinite dimensional C^ℓ Banach manifold.*
2. *The projection $\Pi_\alpha = \pi_2|_{\mathcal{M}(Q,\lambda,J;\overline{\gamma},\underline{\gamma})} : \mathcal{M}(Q, \lambda, J; \overline{\gamma}, \underline{\gamma})) \to \mathcal{J}^\ell(Q, \lambda)$ is a Fredholm map and its index is the same as that of $D\Upsilon(w)$ for a (and so any) $w \in \mathcal{M}(Q, \lambda, J; \overline{\gamma}, \underline{\gamma})$.*

One should compare this with the corresponding statement for Floer's perturbed Cauchy-Riemann equations on symplectic manifolds.

13. Appendix: Proof of energy bound for the case of proper potential

In this appendix, we give the proof of Proposition 9.2.

Since f is assumed to be proper, $f(r) = \pm\infty$ for each puncture r_ℓ of $\dot\Sigma$ depending on whether the puncture is positive or negative.

The proof is entirely similar to the proof of Lemma 5.15 [BEHWZ] verbatim with replacement of a and the equation $dw^*\lambda \circ j = da$ therein by f and the equation

$$dw^*\lambda \circ j + \sum_{e\in E(T)} Q(w; e)\, dt_e = df$$

respectively in our current context. (*We would also like point out that* [BEHWZ] *used the letter 'f' for the map w while our notation f is for the contact instanton potential function which corresponds to a in their notation.* This should not confuse the readers, hopefully.)

In a neighborhood $D_\delta(p) \subset \mathbb{C}$ of a given puncture p with analytic coordinate z centered at p and $C_\delta(p) = \partial D_\delta(p)$, with oriented positively for a positive puncture, and negatively for a negative puncture. Consider the function

$$\delta \mapsto \int_{C_\delta(p)} w^*\lambda.$$

It is increasing and bounded above (resp. decreasing and bounded below), if the puncture is positive (resp. negative), since $d\lambda \geq 0$ on any contact Cauchy-Riemann map w and $\int_{D_\delta(p)} dw^*\lambda \leq E^\pi(w) < \infty$. Therefore the integral

$$\int_{C_\delta(p)} w^*\lambda$$

has a finite limit as $\delta \to 0$ for all punctures. Now let $\varphi \in \mathcal{C}$ and let $\varphi_n \in \mathcal{C}$ such that $\|\varphi - \varphi_n\|_{C^0} \to 0$ and $\varphi_n \circ f = 0$ on $D_{\frac{1}{n}}(p)$ for all punctures p. Such function exists by the assumption on properness of potential function f. Moreover we can choose φ_n so that

$$\int_{\dot\Sigma} (\varphi_n \circ f)\, df \wedge w^*\lambda = \int_{\dot\Sigma} w^* d(\psi_n w^*\lambda) - \int_{\dot\Sigma} (\psi_n \circ f) w^* d\lambda,$$

where $\psi_n(s) = \int_{-\infty}^s \varphi_n(\sigma)\, d\sigma$. Notice that $\psi_n \circ f = 1$ in $D_{\frac{1}{n}}(p)$ when p is a positive puncture and $\psi_n \circ f = 0$ therein when p is negative. By Stokes' theorem,

$$\int_{\dot\Sigma} w^* d(\psi_n \lambda) = \lim_{\delta \to 0} \sum_{\ell^+} \int_{\partial_{\ell^+}\Sigma(\delta)} w^*\lambda$$

where the sum is taken over all positive punctures p_{ℓ^+}. Therefore

$$\begin{aligned} \int_{\dot\Sigma} (\varphi_n \circ f)\, df \wedge w^*\lambda &= \lim_{\delta \to 0} \sum_{\ell^+} \int_{\partial_{\ell^+}\Sigma(\delta)} w^*\lambda - \int_{\dot\Sigma} (\psi_n \circ f)\, w^* d\lambda \\ &\leq \lim_{\delta \to 0} \sum_{\ell^+} \int_{\partial_{\ell^+} D_\delta(p)} w^*\lambda < C' < \infty. \end{aligned}$$

Moreover

$$\int_{\dot\Sigma} (\varphi_n \circ f)\, df \wedge w^*\lambda \to \int_{\dot\Sigma} (\varphi \circ f)\, df \wedge w^*\lambda$$

as $n \to \infty$, which implies

$$\int_{\dot\Sigma} (\varphi \circ f)\, df \leq C',$$

and so $E(w) \leq E^\pi(w) + C' < \infty$. This finishes the proof.

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