ANSWERS TO THE QUESTIONS FROM KATRIN WEHRHEIM ON KURANISHI STRUCTURE.

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Dear all the members of google group 'kuranishi'.

I am very happy that at last I heard some serious questions on Kuranishi structure, after 16 years. So far I heard only a rumor and nobody asked me anything about Kuranishi strucrture itself directly, (except when I asked Dominic by writing directly an E-mail to ask and discuss several points piecefully, some of which he already mentioned in his E-mail.)

I am very happy to discuss to clarify mathematical points. As we write explicitly in our book [FOOO1] there are certainly some errors in our paper [FOn] written in 1996. But I am pretty sure that all the errors are correctable without changing the main idea of the paper, so are not fatal. I hope this discussion group clarifies the technical points around the definition of virtual fundamental chain and cycle so that it becomes easier for everybody to use it.

I think virtual technique should be the basic tool and should be used by many of the symplectic geometers. It should contribute for the symplectic geometry to make more progress. It was rather unfortunate that not many people have used it for 16 years. We have to improve this situation. That is my only purpose to be in this group.

While writing this note I discussed with Oh-Ohta-Ono. They gave me various important comments.

I use the notation of [FOOO1] appendix. If you do not have [FOOO1], its appendix can be downloaded from my home page. (It may be an old version but should be OK to see the notation.)

Question 1

(a) Yes. (b) We need to assume f to be strongly continuous and Kuranishi structure has tangent bundle[1] and orientation.

Question 2

(a)-(c)

Since there is an automorpism of Kuranishi neighborhood the correct notion of germ is not for Kuranishi neighborhood but for Kuranishi structure itself.

We define Kuranishi struture in the way introduced in our book [FOOO1]. It does not use germ. (We began to be aware of its danger at some point.) This is enough for most of the purposes. (Including the proof of all the results in [FOn].) In particular the cocycle condition

$$\overline{\phi}_{pq} \circ \overline{\phi}_{qr} = \overline{\phi}_{pr}$$

[1]We need to take the version of [FOOO1] not of [FOn] for the definition of the existence of tangent bundle

is the exact equality and *not* modulo automorphism of Kuranishi neighborhood. This is important to avoid 2-category. ($\overline{\phi}_{pq} : U_{pq} \to U_p$ is an embedding of the orbifold $U_{pq} = V_q/\Gamma_q \to U_p = V_p/\Gamma_p$, that is induced by the $h_{pq} : \Gamma_q \to \Gamma_p$ equivariant map $\phi_{pq} : V_{pq} \to V_p$.)

Actually I want to avoid using 2 category unless it is absolutely necessary because it makes things complicated and harder to use.

On the other hand, I am very happy for somebody like Dominic takes his own way to use 2 category or any higher category to built related but different theory of virtual fundamental chain.

There is one point that is not so good in the way taken in [FOOO1]. The point is that then the definition for the two Kuranishi structures to be the same (that is isomorphic as I will define below) is too restrictive. For example under that definition the fiber produt of Kuranishi structures (as defined in [FOOO1, Section A1.2]) is not associative. (Also the definition of fiber product will involve choices [2].) We can see however that ambiguity does not exist if we restrict to a neighborhood of moduli space (original space itself). In case we want a notion to formulate it, a germ of Kuranishi structure provides one. Its definition is as follows.

First two Kuranishi structures as in [FOOO1] are said to be *isomorphic* to each other if there exists a diffeomorphisms between (effective) orbifolds V_p/Γ_p, V'_p/Γ'_p for all p that is covered by an isomorphism of bundles E_p and E'_p so that all maps (coordinate change etc) commutes and V_{qp}/Γ_p etc. are sent to V'_{qp}/Γ'_p.

We need to be careful about one point. Maps between (effective) orbifolds are said to be equal to one other if its underlying map between sets are the same. This is a correct definition since we assume that orbifolds are effective. (Two different equivariant maps from V_{pq} to V_p may be the same as an orbifold map. That is the case when two maps are transformed by the Γ_q action.)

We next define open sub Kuranishi structure.

It is defined as follows. We take open neighborhood of p in V_p. We call it V'_p. We assume that it is Γ_p invariant. We put $V'_{qp} = V_{qp} \cap V'_p \cap \phi_{qp}^{-1}(V'_q)$ and all the other data is obtained by restricting the original one.

We say two Kuranishi structures are equivalent if their open substructures are isomorphic in the above sense.

This is an equivalence relation since we can take intersection of two open substructures and isomorphisms are composable.

The equivalence class is by definition a germ of Kuranishi structure.

This equivalence is somehow related to what Dr.Dingyu Yang wrote in his note. However it is different from it and is much simpler thing. In fact we include open embedding only. Yang includes the process to increase the dimension of the Kuranishi neighborhood and obstruction bundle by the same number. This second process is more dangerous as Dominic already mentioned. Namely there is a condition for quotient category construction to work. (See [KS, Chapter 7] for example.)

The equivalence I explained above is not so strong and only a slightly better than the too much strict one in our book. If the equivalence relation that Yang mentioned

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[2] After I looked Dominic's mail on March 18th, I think to add the following may be useful to clarify the issue of fiber product of Kuranishi structure. (This point I told to Dominic during our discussion over E-mail before.) The fiber product I mean is not in the sense of category theory. It is defined in the way we wrote in [FOOO1] . So the well-defined-ness and associativity is *not* a consequnce of general result of category theory.

works it would be a much better equivalence relation and it could be expected to give a canonical Kuranishi structure of the moduli space up to equivalence. The one I exlained does *not* give canonical Kuranishi structure.

(d) We can use the definition written in our book [FOOO1]. To clarify the point it may be better to explicitely state the following condition:

Condition 0.1 (Joyce[Jyo] formula (32))**.**

$$\overline{\phi}_{pq}(V_{pq}/\Gamma_q) \cap \overline{\phi}_{pr}(V_{pr}/\Gamma_r) = \overline{\phi}_{pr}(\overline{\phi}_{qr}^{-1}(V_{pq}/\Gamma_q) \cap V_{pr}/\Gamma_r).$$

This is due to Joyce [Jyo] (32)[3]. See [Ya] page 2. Yang said 'A good coordinate system will inherit this condition.' Maybe it is. But let me put it as a part of definition explicitely.

As for the Kuranihi structure we start with, we may assume the following

$$\psi_p(U_{qp} \cap \overline{s}_p^{-1}(0)) = \psi_p(\overline{s}_p^{-1}(0)) \cap \psi_q(\overline{s}_q^{-1}(0)). \tag{0.1}$$

in addition to [FOOO1] Definition A1.3, for the Kuranishi structure we start with.

Remark 0.2. Here $\overline{s}_p$ is a section of orbibundle $(V_p \times E_p)/\Gamma_p \to U_p$ on $U_p = V_p/\Gamma_p$ induced by the Kuranishi map $V_p \to E_p$.

$U_{qp} = V_{qp}/\Gamma_p$. Here $V_{qp} \subset V_p$ is an open set where the coordinate transformation $\phi_{qp} : V_{qp} \to V_q$ is defined. V_{qp} is Γ_p invariant.

Note (0.1) is somewhat similar to Condition 0.1. However (0.1) contains a condition on the moduli space only. In fact if we have a Kuranishi struture which may not satisfy (0.1) we replace U_p by

$$U_p \setminus \bigcup_q \left(\psi_p^{-1}(\psi_q(\overline{s}_q^{-1}(0))) \setminus U_{qp}\right).$$

Here the sum over q is taken appropriately according to the situation we apply it. Since it is cumbersome to check that this process works we simply add (0.1) as a part of the definition.

Let me explain why I mention Condtion 0.1. Kaoru told me that Katrin concerns with the construction of multisection (on each chart maybe). After thinking for a while, looking question 4 and Yang's note, I guess the following might be her concern. (Please let me know if her concern is on different point.)

Let $r < q < p$ as as bove. Then there is a following trouble. The construction of multisection is by induction on $<$ on charts. So suppose we have one for q, r and try to construct one for p. Let us denote the image of U_q, U_r in U_p by U_q, U_r for simplicity.

We need to extend the multisections on U_q and on U_r to one on U_p. If Condition 0.1 above holds then since $U_q \supseteq U_r$ we only need to extend one on U_q and forget U_r. However if Condition 0.1 does not hold then the extended multisection on U_q and on U_r may be inconsistent.

Given Condition 0.1, the construction of multisection by induction on $<$ works as follows. First note the following property Property 0.5 follows from two conditions Conditions 0.3 and 0.4 we mention below.

Let me first remark the following point. Let P be the index set of the Kuranishi chart of our good coordinate system. P is a partially ordered set. Let $p, q \in P$. In

[3] I thank Dominic Joyce very much who pointed out this important condition.

some case when U_p and U_q (the charts of good coordinate system) intersect (namly $\overline{s}_p^{-1}(0)$ and $\overline{s}_q^{-1}(0)$ intersect in X after we send them by ψ_p, ψ_q), it may happen that dimesions of U_p and of U_q are the same and Γ_p is isomorphic to Γ_q by the map appearing in the definition of coordinate change (from p to q say). (Note in our formulation $\overline{\phi}_{qp}$ is induced by $\phi_{qp} : V_p \to V_q$ and $h_{qp} : \Gamma_p \to \Gamma_q$ and h_{qp} is an isomorphism.) Then the map $\overline{\phi}_{qp}$ in the coordiante change are diffeomorphism to open sets. So we can invert it. In this case we redefine the partial order so that $p \le q$ and $q \le p$ are both satisfied. It is s slight abuse of notation to call it partial order, since $p \le q$ and $q \le p$ implies $p = q$ does not hold. But it seems harmless here, if the following Condition 0.3 is satisfied.

Condition 0.3. Let $p, q \in P$. If $p \le q$ and $q \le p$ then $\overline{\phi}_{qp}(U_{qp}) = U_{pq}$ and $\overline{\phi}_{pq} = \overline{\phi}_{pq}^{-1}$

It seems that in a way we wrote our paper [FOn] the following condition is not so clearly stated. So I will state it here as one of the assumptions for the good coordinate system to satisfy.

Condition 0.4. Let $p, q \in P$. If $\mathcal{U}_p \cap \mathcal{U}_q \ne \emptyset$ then either $p \le q$ or $q \le p$ holds. (Here $\mathcal{U}_p = \psi_p(\overline{s}_p^{-1}(0)$.)

Now the linearity property is stated as follows.

Property 0.5 (Linearity of partial order)**.** Let $p_i \in P$. Suppose

$$\bigcap_{i=1}^{N} \psi_{p_i}(\overline{s}_{p_i}^{-1}(0)) \ne \emptyset. \tag{0.2}$$

Then the set $\{p_i \mid i = 1, \dots, N\}$ are linearly ordered. (Namely for each p_i, p_j at least one of $p_i \le p_j$, $p_j \le p_i$ holds.)

It is easy to see that Property 0.5 follows from Conditions 0.3 and 0.4.

Let us take an inductive step to construct multisection on U_r. Suppose we already constructed one for all U_p with $p < r$. We denote the image of U_{rp} in U_r by U_p for simplicity. Let NU_p be a tubular neighborhood of U_p in U_r. We will extend the multisections (defined on the union of images of U_p's) to the union of NU_p's by downward induction on p.

I explain the reason why Conditions 0.1, 0.3, 0.4 and Property 0.5 are enough to construct multisection by induction. Let p be maximal among q's with $q < r$. We extend multisection defined on U_p to NU_p as follows. We have $E_r = E_p \oplus E_p^{\perp}$ on NU_p. (Here we extend this decomposition to the tubular neighborhood.) Fot the $E_p^{\perp}$ component, we use the component of the original Kuranishi map itself. (This is required by the compatibility of multisection.) On E_p component we extend the given multisection on U_p. This is transversal if NU_p is sufficiently small.

Note Condition 0.5 implies that NU_p are disjoint among maximal p's. More precisely if NU_p intersect with $NU_{p'}$ and both p and p' are maximal, then both $p \le p'$ and $p' \le p$ hold. Namely the coordinate change are diffeomorphism on the overlapped part. So using Conditions 0.1, 0.3, we can perform the above construction on the union of such NU_p's.

Now we assume we have already extended to NU_q for all q with $p < q < r$. We will extend the multisection to NU_p. We remark that if there are two such p's say p_1 and p_2 then one of the following holds.

(1) $NU_{p_1} \cap NU_{p_2} = \emptyset$.
(2) The coordinate change between p_1 and p_2 is a diffeomorphism on open subsets.

We collect all such p's among which coordinate changes are diffeomorphisms. Then we can repeat the first step in the relative setting to extend it on the union of such NU_p's.

We have thus constructed multisection on the union of all NU_p's.

Then we can use the relative version of existence theorem of multisection (orbifold case) to extend it to U_r.

We remark that we need to shrink U_p several times in the above construction. But we need to do it only finitely many times.

(e)(f) This is proved in page 957 - 958 of [FOn]. I will explain it in more detail below and at the end of this note.

First a few words about a general strategy. It seems that Yang and maybe some of other people's idea to construct good coordinate system is first to glue U_p's and obtain some space which we call M (following Yang in page 2 of his note) and then we work on it. In other words to construct coordinate change etc. on V_{qr} they look what happens in U_p for $p > q$. This is related to Condition 0.1.

On the other hand, the method I will explain below do not use the space M (the space obtained by glueing U_p's) at all. (Such a space never appeared in our paper [FOn] or [FOOO1], except when we construct the zero section of the perturbed multisection. That is a step which starts after the construction of good coordinate system is completed.) On the contrary, to construct coordinate change from U_p to $U_{p'}$ we use finer cover $\{U'_q\}$ of $U_p \cap U_{p'}$. To see its properties, we take even finer cover $\{U'_r\}$. In other words we do not care what will happen for U_o with $o > p, p'$. We can do it as far as we stay in a neighborhood of the moduli space. Let me explain this method more explicitely.

Any point $p \in X$ has well defined Γ_p and $\dim U_p$. We put $d_p = \dim U_p$ and $m_p = \#\Gamma_p$. We put

$$X(d,m) = \{p \mid d_p = d,\ m_p = m\}.$$

Note

$$CX(d_0,m_0) = \bigcup_{d \ge d_0, m \ge m_0} X(d,m)$$

is a closed subset of X. We will construct Kuranishi neighborhood on $CX(d,m)$ by downward induction of (d,m).[4] (We say $(d,m) \le (d',m')$ if both $d \le d'$ and $m \le m'$ holds.)

We start with the case $X(d,m)$ for which (d,m) is maximal. Such $X(d,m)$ is compact and disjoint from one another. We cover it by $\mathcal{U}_{p_i}$. Note we set

$$\psi_{p_i}(\overline{s}_{p_i}^{-1}(0)) = \mathcal{U}_{p_i}$$

By a standard arguement in general topology we can choose it so that the following holds.

Condition 0.6. If $\mathcal{U}_{p_i} \cap \mathcal{U}_{p_j} \neq \emptyset$, then $\mathcal{U}_{p_i} \cap \mathcal{U}_{p_j} \cap X(d,m) \neq \emptyset$.

[4] In this note we use induction on (d,m). In [FOn] induction on d is used. The reason for this difference is as follows. In [FOn] the charts U_p are orbifolds in general and $\overline{\phi}_{qp}$ is an embedding of orbifold. In [FOOO1] (whose formulation we follow here) the chart U_p is a global quotient V_p/Γ_p and $\overline{\phi}_{qp}$ is obtained from an equivariant map $V_q \to V_p$.

For simplicity let me assume that the covering is by only two members $\mathcal{U}_{p_1}$ and $\mathcal{U}_{p_2}$. (The general case will be discussed at the last part of this note.) Our task is to construct a coordinate change among them after shrinking it appropriately. (After shrinking it should still cover $X(d,m)$.)

Let $q \in \mathcal{U}_{p_i} \cap \mathcal{U}_{p_j} \cap X(d,m)$. There exists U_q^1 a Kuranishi neighborhood (and is an open subset of the Kuranishi neighborhood we starded with) so that there exist coordinate change from U_q^1 to both U_{p_1} and U_{p_2}. Since the cardinalities of the isotropy groups and the dimensions of the Kuranishi neighborhoods of U_q^1 and U_{p_i} are both the same, the coordinate change has an inverse from the image. We put $U_{p_1}^q$, $U_{p_2}^q$ the image of coordinate change, and

$$\overline{\phi}^{-1}_{q;p_2p_1} = \overline{\phi}_{p_2q}\overline{\phi}^{-1}_{qp_1} : U_{p_1}^q \to U_{p_2}^q.$$

(Note in [FOOO1] the coordinate change is given by an equivariant map $V_{qp} \to V_q$ where V_{qp} is an open subset of V_p. Here and hereafter U_{qp} etc. is a quotient space U_{qp}/Γ_p.) Choose finitely many q_j, $j = 1,\dots,J$ so that $\mathcal{U}_{p_1} \cap \mathcal{U}_{p_2} \cap X(d,m)$ is covered by $\mathcal{U}_{q_j}^1$'s, where

$$\mathcal{U}_{q_j}^1 = \psi_{q_j}(\overline{s}_{q_j}^{-1}(0) \cap U_{q_j}^1).$$

(We shrink U_{p_i} slightly here so that we can use compactness to obtain finiteness of the cover.) We may also assume $\mathcal{U}_{q_j}$ satisfies Conidition 0.6. (See **Figure 1** in the separate sheet.)

Put

$$U_{p_2p_1}^1 = \bigcup_{j\in J} U_{p_1}^{q_j}.$$

If

$$\overline{\phi}^{-1}_{q_j;p_2p_1} = \overline{\phi}^{-1}_{q_{j'};p_2p_1} \qquad \text{on } U_{p_1}^{q_j} \cap U_{p_1}^{q_{j'}} \tag{0.3}$$

holds for each j and j' then $U_{p_2p_1}^1 \subset U_{p_1}$ together with $\overline{\phi}^{-1}_{q_j;p_2p_1}$ glued gives required coordinate change. However (0.3) may not hold.

But we can show that (0.3) holds if we restrict the maps in (0.3) to a smaller neighborhood of $\mathcal{U}_{p_i} \cap \mathcal{U}_{p_j} \cap X(d,m)$, as follows. We put

$$\mathcal{U}_{q_j}^1 = \psi_{q_j}(\overline{s}_{q_j}^{-1}(0) \cap U_{q_j}^1).$$

Let $r \in \mathcal{U}_{p_1} \cap \mathcal{U}_{p_2} \cap X(d,m)$. We take U_r^2 so that we have a coordinate change $U_r^2 \to U_{q_j}^1$ for any j with $r \in \mathcal{U}_{q_j}^1$. Then the cocycle condition implies that the composition

$$U_r^2 \xrightarrow{\overline{\phi}_{q_jr}} U_{q_j}^1 \xrightarrow{\overline{\phi}_{p_iq_j}} U_{p_i}$$

is independent of j with $r \in U_{q_j}^1$. (See **Figures 2 and 3**.)

Therefore we have (0.3) on the image $\overline{\phi}_{q_jr}(U_r^2)$ for various j.

We cover $\mathcal{U}_{p_1} \cap \mathcal{U}_{p_2} \cap X(d,m)$ by finitely many $\mathcal{U}_r^2$'s for $r \in R$. Here

$$\mathcal{U}_r^2 = \psi_r(\overline{s}_r^{-1}(0) \cap U_r^2).$$

(We shrink U_{p_i} slightly for compactness argument.) Then on the union of $\overline{\phi}_{p_1r}(U_r^2)$ over various $r \in R$, the equality (0.3) holds. We put

$$U_{p_2p_1} = \bigcup_{r\in R} \overline{\phi}_{p_1r}(U_r^2),$$

and define $\overline{\phi}_{p_2p_1}$ on it by (0.3). We thus have constructed coordinate change.

Note, since we shrink $U_{p_2p_1}$ two sets $\mathcal{U}_{p_2}$ and $\mathcal{U}_{p_1}$ may intersect away from $\psi_{p_1}(U_{p_2p_1} \cap \overline{s}_{p_1}^{-1}(0))$. But this occurs only away from $X(d,m)$. We remove $\overline{s}_{p_1}^{-1}(0) \setminus U_{p_2p_1}$ from U_{p_1} and $\overline{s}_{p_2}^{-1}(0) \setminus \overline{\phi}_{p_2p_1}(U_{p_2p_1})$ from U_{p_2}. Then (0.1) is satsified and they still cover $X(d,m)$.

We thus have constructed good coordinate system on $X(d,m)$.

The inductive step of the constrution is similar to this first step. Suppose we have defined a good coordinate system on a neighborhood $\mathcal{U}^{>(d,m)}$ of

$$X^{>(d,m)} = \bigcup_{(d',m')>(d,m)} X(d',m').$$

We will construct a good coordinate system on $CX(d,m)$. Choose $\mathcal{K}^{>(d,m)}$ that is a neighborhood of $X^{>(d,m)}$ and its closure is compact and contained in $\mathcal{U}^{>(d,m)}$. (See **Figures 4 and 5**.) We construct Kuranishi neighborhood of the complement $X(d,m) \setminus \mathcal{U}^{>(d,m)}$ that is contained in $X(d,m) \setminus \mathcal{K}^{>(d,m)}$. This construction is the same as the first step above.

We need to define a coordinate change from U_p to U_q, (after shrinking U_p appropriately). Here U_q is obtained in earlier step of induction ($q \in X(d',m')$, $(d',m') > (d,m)$ and $p \in X(d,m)$). In this step we need to construct the coordinate change only in one direction. (From p to q.) So a construction together with the proof of cocycle condition is the same as the first step. (Again we need to take V's for Kuranishi chart of $X(d,m)$ smaller.)

Thus we are done. □

I will write more about this point, espacially how we obtain Condition 0.1, in the end of this note. Maybe this is the point several people want to hear.

Let me add a remark about Dr.Dingyu Yang's note especially its page 2. As the above construction shows we *never* use the space obtained by glueing various U_p's in the construction (the space written as M in the last paragraph of page 2 of his note.) We use the topology of X (the space on which we define Kuranishi structure) only [5]. In the last part of this note, we will actully *construct* a Hausdorff metrizable space to which our Kuranishi neighborhoods are embedded without assuming its existence a priori.

Question 3

No further condition on s_p^ϵ is necessary if it is close enough to original Kuranishi map and if we shrink U_p's during the construction.

Since I heard indirectly that somebody has a question about Hausdorffness let me explain a bit more about it.

The Hausdorffness of $\bigcup_p \left((\overline{s}_p^\epsilon)^{-1}(0)/\sim\right)$ is broken if the following holds. ($\sim$ is defined by using coordinate change $\overline{\phi}_{qp}$.) Hereafter we denote by $\partial_A B$ the set $\overline{B} \setminus B$ where $\overline{B}$ is the closure on B in A. (Here $B \subset A$ and A, B are topological spaces.)

Phenomenon 1 There exists a sequence $x_i \in (\overline{s}_p^\epsilon)^{-1}(0) \cap U_{qp}$ such that

$$\lim_{i\to\infty} x_i = x \in \partial_{U_p} U_{qp}.$$

$$\lim_{i\to\infty} \overline{\phi}_{qp}(x_i) = y \in \partial_{U_q}(\overline{\phi}_{qp}(U_{qp}))$$

[5] I thank Dr.Dingyu Yang very much. He takes much time to try to understand our text and found several important points which are very useful to clarify some points of the theory of virtual fundamental chain.

□

See **Figure 6**.

In fact then any neighborhood of y and x intersect and y is not in the image of $\overline{\phi}_{qp}$ and so is not equivalent (with respect to $\sim$) to x.[6] I claim that we can shrink each of U_q a bit (V_{qp} also at the same time) so that all the properties are still satisfied and the above does not occur as far as $(s^{\epsilon}_p)^{-1}(0)$ is sufficiently close to $s_p^{-1}(0)$.

We can show it as follows. Let U'_p be a Γ_p invariant open subset of U'_p such that its closure $\overline{U}'_p$ in U_p is compact. We put

$$U'_{qp} = \overline{\phi}_{qp}^{-1}(U'_q) \cap U'_p \cap U_{qp}. \tag{0.4}$$

This choice works obviously in the case when there are only two charts to glue. Otherwise more complicated phenomena may occur. In the construction of good coordinate system satisfying Condition 0.1, which I will explain at the end of this note, we only need to consider the situation where we glue two spaces at each step of the construction. So (0.4) is certainly the correct choice in the case we use.

I claim that Phenomenon 1 does not occur if $(\overline{s}^{\epsilon}_p)^{-1}(0)$ is sufficiently close to $\overline{s}_p^{-1}(0)$. In fact we observe that the closure of $U'_{qp} \cap \overline{s}_p^{-1}(0)$ is compact and contained in U_{qp}. (This is because the space X to which $\overline{s}_p^{-1}(0)$ is embedded is compact and Hausdorff.) So if ϵ is small enough then a neighborhood W of $U'_{qp} \cap (\overline{s}^{\epsilon}_p)^{-1}(0)$ does not intersect with ∂U_{pq}. Moreover for $x_i \in W$ the sequence $\overline{\phi}_{qp}(x_i)$ does not converge to a point in $\partial_{U_q}(\overline{\phi}_{qp}(U_{qp}))$.

Therefore a sequence x_i in $U'_{qp} \cap (\overline{s}^{\epsilon}_p)^{-1}(0)$ has a convergent subsequence unless one of the following holds.

(1) x_i has a subsequence converging to $x \in \partial U'_p$.
(2) $\overline{\phi}_{qp}(x_i)$ has a subsequence convergin to $y \in \partial U'_q$

Therefore it is impossible that

$$x \in \partial_{U'_p} U'_{qp}, \quad y \in \partial_{U'_q}(\overline{\phi}_{qp}(U'_{qp}))$$

both hold.

Let me remark that the trouble of non-Haudorff-ness occurs typically in the following way. Let

$$U_p = \{(x,y) \in \mathbb{R}^2 \mid |y| > -x, \text{ or } x > 0\}$$

$$U_q = \{(x,y) \in \mathbb{R}^2 \mid |y| > x - 1, \text{ or } x < 1\}$$

$s(x,y) = y$, $\Gamma_p = \Gamma_q = \{1\}$. Moreover we take

$$U_{qp} = \{(x,y) \mid 1 > x > 0\}.$$

The coordinate change is the identity map. (See **Figure 7**.)

Non-Hausdorff-ness occurs on lines $x = 0, 1$ minus $y = 0$.

Note also if we perturb s to $s_\epsilon(x,y) = y - \epsilon$, then its zero set after glued becomes as in **Figure 8** and is not a cycle. Undesirable noncompactness also occurs in a similar way.

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[6]The equivalence relation is defined as follows. If $x \in U_p$ and $y \in U_q$ then $x \sim y$ if $x \in U_{qp}$ and $y = \overline{\phi}_{qp}(x)$. Note in the case when $p \le q$ and $q \le p$, we required that $\overline{\phi}_{qp}(U_{qp}) = U_{pq}$ and $\overline{\phi}_{pq} = \overline{\phi}_{pq}^{-1}$. (Condition 0.3.)

The solution above is to replace U_p and U_q by

$$U'_p = \{(x,y) \in \mathbb{R}^2 \mid |y| > -(x-\epsilon), \text{ or } x-\epsilon > 0\}$$

$$U'_q = \{(x,y) \in \mathbb{R}^2 \mid |y| > (x+\epsilon)-1, \text{ or } x+\epsilon < 1\}$$

$$U'_{qp} = U'_p \cap U'_q \cap U_{qp}.$$

The trouble now has gone, since there will be no longer non-Hausdorff-ness in the neighborhood of $y=0$. (See **Figure 9**.)

Question 4

(a) This seems to be related to page 979 of [FOn]. There is a bit more discussion in [FOOO1, page 424]. This question is a bit vague and it is hard for me to see which part you want to know in more detail. Let me explain one point according to my guess that this may be your concern.

We cover the moduli space by finitely many sufficiently small closed sets W_i each of which are centered at p_i that is represented by $((\Sigma_i,\vec z_i),u_i)$. ($\Sigma_i$ is a (bordered) Riemann surface and $\vec z_i = (z_{i,1},\dots,z_{i,m_i})$ are marked points. (Interior or bounary marked points.) $u_i : \Sigma_i \to M$ is a pseudo-holomorphic map.) We fix a subspace E_i of $\Gamma(\Sigma_i; u_i^*TM \otimes \Lambda^{0,1})$ as in (12.7) in page 979 of [FOn]. For $p = ((\Sigma_p,\vec z_p),u_p)$ we collect E_i for all i with $p \in W_i$ and the sum of them is E_p. The Kuranishi neighborhood of p is a set of solutions of

$$\overline{\partial} u \equiv 0 \mod E_p. \tag{0.5}$$

I will disuss the way how we identify E_i to a subset of $\Gamma(\Sigma; u^*TM \otimes \Lambda^{0,1})$ in case $((\Sigma,\vec z),u)$ is close to $((\Sigma_p,\vec z_p),u_p)$. (Please let me know if your concern is on different point.)

When we fix E_i we also fix finitely many additional marked points $\vec z_{i+} = (z_{ij})$ where $z_{ij} \in \Sigma_i$, $j = 1,\dots,k_i$ at the same time and take transversals $\mathfrak{D}_{ij}$ to $u_i(\Sigma_i)$ at $u_i(z_{ij})$ as in appendix [FOn]. We take it sufficiently many so that after adding those marked points $(\Sigma_i, \vec z_i \cup \vec z_{i+})$ becomes stable.

We consider $((\Sigma,\vec z),u)$. For each i we add marked points $\vec z'_i = (z'_{ij})$, $z'_{ij} \in \Sigma$, $j = 1,\dots,k_i$ to $(\Sigma,\vec z)$ so that $u(z'_{ij})$ is on the slice $\mathfrak{D}_{ij}$. We add them to obtain $(\Sigma, \vec z \cup \vec z'_i)$ that becomes stable, for each i. We require that it is close to $(\Sigma_i, \vec z_i \cup \vec z_{i+})$ in Deligne-Mumford moduli space (or its bordered version). Then we obtain a diffeomorphism (outside the neck region) between Σ and Σ_i which sends $\vec z_i \cup \vec z_{i+}$ to $\vec z \cup \vec z'_i$, preserving the enumeration. (See [FOn] the discussion of the identification right before Definition 10.2.) Using this diffeomorphism and (complex linear part of) the parallel transport on M (the sympletic manifold) with respect to the Levi-Civita connection along the minimal geodesic joining $u(w)$ with $u_i(w_i)$ (where $w_i \in \Sigma_i$ is identified with $w \in \Sigma$ by the above mentioned diffeomorphism), we send E_i to a subspace of $\Gamma(\Sigma; u^*TM \otimes \Lambda^{0,1})$. We do it for each of i. (In other words the stabilization we use *depends* on i.) Thus each of E_i is identified with a subspace of $\Gamma(\Sigma; u^*TM \otimes \Lambda^{0,1})$. We take its sum and that is E_p at $(\Sigma,\vec z)$. We may perturb E_i a bit so that $\dim E_p = \sum_i \dim E_i$.

We thus make sense of (0.5).

An important point here is that the subspace $E_i \subset \Gamma(\Sigma; u^*TM \otimes \Lambda^{0,1})$ at $(\Sigma,\vec z)$ is *independent* of p as far as $(\Sigma,\vec z)$ is close to p. The data we use for stabilization is chosen on p_i (not on p) once and for all. This is essential for cocycle condition to hold[7].

(b)(c) Once (a) is understood the coordinate chang ϕ_{qp} is just a map which send an element $((\Sigma, \vec{z}), u)$ to the same element. So the cocycle condition is fairly obvious.

Question 5

Let me mention that there are two proofs of isomorphism between Floer homology of periodic Hamiltonian system and ordinary homology of M, now. One is in [FOn] and uses identification with Morse complex in the case when Hamiltonian is small and time independent. The other uses Bott-Morse theorey and de Rham theory and is in our paper [FOOO2]. (Several other proofs are written in 1996 by Ruan [Ru], Liu-Tian [LT] also.)

(a)(b) I do not think it is possible in completely abstract setting. At least I do not know how to do it. In a geometric setting such as one appearing in page 1036 [FOn], Kuranishi structure is obtained by specifying the choice of the obstruction space E_p for each p. We can take E_p in an S^1 equivariant way so the Kuranishi structure on the quotient X/S^1 is obtained. And it is a quotient of an S^1 equivariant Kuranishi structure on X. S^1 equivariant multisection can be constructed in an abstract setting so if the quotient has virtual dimension -1 the zero set is empty.

(c) We can take a direct sum of the obstruction bundles, the support of which is disjoint from the points where two maps are glued. In the situation where two solutions of perturbed Cauchy-Riemann equation that are not of Morse trajectory (that is the situation of (1)) are glued, this obstruction bundle is $S^1 \times S^1$ equivariant. The symmetry is compatible with the diagonal S^1 action nearby.

(d) It is Theorem 20.5 [FOn].

Construction of good coordinate system (continued)

Let me go back to the construction of the good coordinate system and add more explanation espacially on the way how to show that resulting good coordinate system satisfies Condition 0.1.

As a short cut[8] we can take the following way.

We first consider the construction of Kuranishi neighborhood of $X(d,m)$ with maximal (d,m). We cover $X(d,m)$ by $\mathcal{U}_{p_i}$. We already explained how to glue two of them, say U_{p_1} and U_{p_2}. Condition 0.1 is satisfied since we have only two charts. We then shrink them a bit as in the answer to Question 3. Then it becomes Hausdorff in a neighborhood of X. In other words, the union of U_{p_1} and U_{p_2} after glued becomes an effective orbifold in the usual sense. (We can throw away everything away from X as follows. Put a metric of a neighborhood of $\bigcup_{i=1}^2 \overline{s}_{p_i}^{-1}(0)$ in the glued union of U_{p_1} and U_{p_2}. Throw away everything outside the ϵ neighborhood of $\bigcup_{i=1}^2 \overline{s}_{p_i}^{-1}(0)$ in this space. Then it becomes Hausdorff.) We denote it by $U_{\{p_1,p_2\}}$. Now we take U_3. In the same way as before we can glue U_{p_3} with $U_{\{p_1,p_2\}}$.

[7]Since Equation (0.5) makes sense in a way independent of p it seems possible to simply take the union of its solution space to obtain some Hausdorff metrizable space. That can play a role of the metric space in which all the Kuranishi neighborhoods is contained. However I insiste that we should *not* built the general theory of Kuranishi structure under the assumption of the existence of such space, since it spoils the flexibility of the definition of the general story we have.

[8]There is a way to glue coordinate carefully so that it satisfies Condition 0.1. But the way below looks shorter and closer to the way many people in this googole group get used to.

Remark 0.7. To repeat the process described in the anwer 3 (e)(f) for the purpose of this step, we remark the following. Let $q \in \mathcal{U}_{p_3} \cap \mathcal{U}_{\{p_1,p_2\}}$. To define a coordinate change between U_{p_3} and $U_{\{p_1,p_2\}}$, we choose a sufficiently small Kuranishi neighborhood of q. More precisely we require the following. Suppose $q \in \mathcal{U}_{p_3} \cap \mathcal{U}_{p_1} \cap \mathcal{U}_{p_2}$. (Other cases are simpler.) We used a cover of $\mathcal{U}_{p_1} \cap \mathcal{U}_{p_2}$ to glue U_{p_1} and U_{p_2}. Let $\{U'_{q_j}\}$ be those covers (that is a Kuranishi neighborhood). We take U''_q a Kuranishi neighborhood of q so that there exists a coordinate change $U''_q \to U'_{q_j}$ for any j with $q \in U'_{q_j}$. (Note the set U'_{q_j} is the Kuranishi neighborhood that we wrote $U^1_{q_j}$ before. It was shrinked several times afterwards during the construction of earlier steps.) We also assume that there exists a coordinate change $U''_q \to U_{p_3}$. By this choice, the Kuranishi chart U''_q and the coordinate change from it determine a coordinate change from the image of U''_q in U_{p_3} to a Kuranishi neighborhood $U_{\{p_1,p_2\}}$ of $\mathcal{U}_{\{p_1,p_2\}}$. To find a domain where various maps $U''_q \to U_{\{p_1,p_2\}}$ are compatible, we take U'_r as before and use it.

In each of further steps of the construction, we choose finer and finer neighborhoods as construction goes.

We go back to our construction. We shrink U_{p_3} and $U_{\{p_1,p_2\}}$ again to make the union Hausdorff. Thus we obtain an effective orbifold $U_{\{p_1,p_2,p_3\}}$ together with an obstruction bundle and a section so that the zero set of the section goes to $\bigcup_{i=1}^3 \mathcal{U}_{p_i}$ by a homeomorphism. Repeating it finitely many times, we obtain a Hausdorff and effective orbifold together with an obstruction bundle and a section so that the zero set of the section goes to $\bigcup_{i=1}^N \mathcal{U}_{p_i}$ that contains $X(d,m)$. We can use this orbifold as one of the charts. Or (if we want a chart that is a global quotient V/Γ, for examaple) we can work in this orbifold safely to obtain a good coordinate system on a neighborhood of $X(d,m)$ satisfying Condition 0.1.

The inductive step is as follows. We consider the step to find a Kuranishi neighborhood of $X(d,m)$. We assume that, for $(d',m') > (d,m)$ we constructed $U(d',m')$ a Kuranishi neighborhood of $X(d',m')$ that is an orbifold. They are glued together by induction hypothesis. Also they satisfy Condition 0.1. We then can use the previous argument that we can glue those orbifolds $U(d',m')$ to obtain a Hausdorff space, that we call $\mathcal{M}(d,m)$. (See **Figure 10**.)

Note when we perform this glueing we can add $U(d',m')$ one by one inductively (downward induction on $<$) so that we need to glue only two spaces in each step. So the process to shrink a bit to obtain Hausdorff space, that we explained in the answer to 3, works safely here.

The space $\mathcal{M}(d,m)$ is a union of various orbifolds with various dimensions. So $\mathcal{M}(d,m)$ itself is not an orbifold. This is a typical situation of Kuranishi structure.

Let $\mathcal{U}^{>(d,m)}$ be the union of the images to X of the zero set of Kuranishi maps in Kuranishi neighborhoods of $X(d',m')$ with $(d',m') > (d,m)$. We choose a relatively compact neighborhood $\mathcal{K}^{>(d,m)}$ of $X^{>(d,m)}$ in $\mathcal{U}^{>(d,m)}$. (See **Figures 4, 5**.) We take an open neighborhood $\mathcal{K}_+^{>(d,m)}$ of $\overline{\mathcal{K}^{>(d,m)}}$ so that its closure is compact in $\mathcal{U}^{>(d,m)}$. We also take $\mathcal{U}_-^{>(d,m)} \subset \mathcal{U}^{>(d,m)}$ such that the closure of $\mathcal{U}_-^{>(d,m)}$ is in $\mathcal{U}^{>(d,m)}$ and the closure of $\mathcal{K}_+^{>(d,m)}$ is in $\mathcal{U}_-^{>(d,m)}$. (See **Figure 11**.)

We take points $p \in X(d,m) \cap \left(\overline{\mathcal{U}_-^{>(d,m)}} \setminus \mathcal{K}_+^{>(d,m)}\right)$ and ist Kuranishi neighborhoods U'_p so that $\mathcal{U}'_p \subset \mathcal{U}^{>(d,m)} \setminus \overline{\mathcal{K}^{>(d,m)}}$. (Here $\mathcal{U}'_p = \psi_p(\overline{s}_p^{-1}(0) \cap U'_p)$.) We take finitely many of them so that $\{\mathcal{U}'_p\}$ covers $X(d,m) \cap \left(\overline{\mathcal{U}_-^{>(d,m)}} \setminus \mathcal{K}_+^{>(d,m)}\right)$. We glue them to obtain an orbifold $L(d,m)$. (This process is exactly the same as before.)

Using the fact that we already have a Hausdorff metrizable space $\mathcal{M}(d,m)$ it is easy to glue the embeding of each U'_p to $\mathcal{M}(d,m)$ to obtain an embedding $L(d,m) \to \mathcal{M}(d,m)$. (See **Figure 12**). We need to shrink $L(d,m)$ to construct this embedding without changing the intersection of its image in X and $X(d,m)$.

We remove a small neighborhood of $\partial\mathcal{K}_+^{>(d,m)}$ from $\mathcal{M}(d,m)$ and obtain another Hausdorff metrizable space $\mathcal{M}(d,m)^-$. (See **Figure 13**.)

We then glue $\mathcal{M}(d,m)^-$ and $L(d,m)$ in $\mathcal{M}(d,m)$. The glued union of them is a Hausdorff metrizable space that we call $\mathcal{M}(d,m)^+$. (See **Figure 14**.)

We observe that the intersection of $\mathcal{M}(d,m)^+$ with a neighborhood of $\partial\mathcal{U}^{>(d,m)}$ is an orbifold of dimension d. This is because we removed the part of $\mathcal{M}(d,m)$ which is close to $\partial\mathcal{U}^{>(d,m)}$.

Now we start glueing U'_p's for $p \in X(d,m) \setminus \mathcal{U}^{>(d,m)}$ to $\mathcal{M}(d,m)^+$. During this glueing, we glue U'_q with $L(d,m)$ both of which are orbifolds of dimension d. (This is because $\mathcal{M}(d,m)^-$ is away from $\partial\mathcal{U}^{>(d,m)}$.) Therefore we obtain a Hausdorff metrizable space after this glueing. (See **Figure 15**.) It is an orbifold outside a neighbhorfoold of $X^{>(d,m)}$. We can use this space to extend a good coordinate system to $X(d,m)$ that has required properties.

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