INTRODUCTION TO SYMPLECTIC ALGEBRAIC TOPOLOGY

YONG-GEUN OH

Contents

1991 *Mathematics Subject Classification.* 53D37, 53D40, 53D45, 14B07, 14M25, 37J05.

Key words and phrases. A_∞ structure, Lagrangian submanifolds, Floer cohomology, genus 0 bordered Riemann surface, open Gromov-Witten invariant, Hochschild cohomology.

Supported byu the IBS project IBS-R003-D1.

1. Based loop space and Stasheff polytope

Let X be a topological space with a base point x_0. We consider a based loop space ΩX is given by

$$\Omega X = \{\gamma : [0,1] \longrightarrow X \,|\, \gamma(0) = \gamma(1) = x_0\}.$$

For any two based loops α, β in ΩX, a loop product $*$ can be defined by concatenating two loops. Explicitly, a loop product $* : \Omega X \times \Omega X \longrightarrow \Omega X$ is defined by

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

In general, the associativity of loop product does not hold, i.e., $(\alpha*\beta)*\gamma \neq \alpha*(\beta*\gamma)$ in ΩX, where

$$((\alpha * \beta) * \gamma)(t) = \begin{cases} \alpha(4t) & \text{if } t \in \left[0, \frac{1}{4}\right] \\ \beta(4\left(t - \frac{1}{4}\right)) & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ \gamma(2t-1) & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

$$(\alpha * (\beta * \gamma))(t) = \begin{cases} \alpha(2t) & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(4\left(t - \frac{1}{2}\right)) & \text{if } t \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ \gamma(4\left(t - \frac{3}{4}\right)) & \text{if } t \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

However, $(\alpha*\beta)*\gamma$ and $\alpha*(\beta*\gamma)$ are homotopic in ΩX. Indeed, a linear homotopy can be consctructed by staring Figure 1. Here the mid region is given by the inequality

$$\frac{1}{4} + \frac{s}{4} \le t \le \frac{1}{2} + \frac{s}{4}.$$

Using $s \in [0,1]$ as a homotopy parameter, we denote such a homotopy at level s

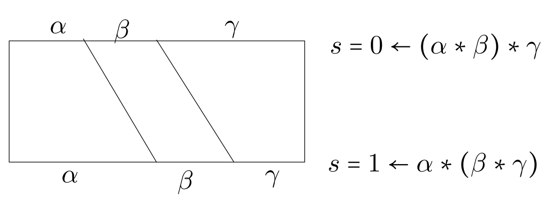

Figure 1. Homotopy

by $M(\alpha,\beta,\gamma)_s$. We then see that ΩX carries a natural continuous map

$$M_3 : \Omega X \times \Omega X \times \Omega X \times [0,1] \longrightarrow \Omega X$$

given by

$$M_3(\alpha,\beta,\gamma;s) := M(\alpha,\beta,\gamma)_s.$$

Now, we consider the case when four loops are given. Note that the way of concatenating them is exactly equivalent to the way of inserting suitable parentheses on four letters a, b, c and d. First, adorn each vertex with a letter of two parentheses (Figure 2). We then associate a 2-dimensional polytope K_4 whose edges are homotopy between ways of putting two parenthese on four letters. Generally, n loops can be associated to $(n-2)$ dimensional polytope denoted by K_n $(n \ge 2)$. Such a polytope is called a **Stasheff polytope** or **associahedron**. Later on, we will discuss an inductive construction of K_n.

Example 1.1. $K_2 = \{*\}$, $K_3 = [0,1]$ and K_4 is a pentagon.

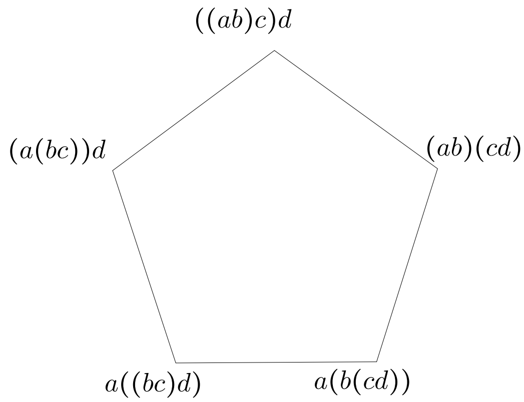

FIGURE 2. K_4

Going back to based loop spaces, let us look at motivating question: What condition on a topological space Y makes the following statement true?

Y is homotopy equivalent to ΩX for some topological space X.

The answer is given by Stasheff.

Theorem 1.2. (Stasheff) A topological space Y is homotopy equivalent to ΩX for some topological space X if and only if there exists a family of maps

$$M_n : Y^n \times K_n \longrightarrow Y \quad \text{for } n \geq 2$$

satisfying some relations (A_∞-relation) where $Y^n = Y \times Y \times \cdots \times Y$ (n times) and K_n is the associahedron.

We are going to look at how to construct a $(n-2)$ dimensional associahedron K_n. Thinking K_n labelled by k letters $x_1, x_2, \cdots, x_n$ with suitable parentheses, we assign a word $x_1 x_2 \cdots x_n$ without parenthese to the $(n-2)$ dimensional cell K_n. Next, a word consisting of k letters $x_1, x_2, \cdots, x_k$ with one parenthesis is assigned to facets as follows:

$$x_1 x_2 \cdots (x_k x_{k+1} \cdots x_{k+s-1}) \cdots x_n$$

where $2 \leq s \leq n-1$ and $1 \leq k \leq n-s-1$. Inductively, a word with parentheses can be assigned to lower dimensional faces by inserting parenthesis with the rule that the next insertion of parenthesis is within a pair of parentheses or outside a pair of parentheses.

Example 1.3. Following the above procedure, a word with suitable parentheses can be assigned to each cell in K_4. (See Figure 7)

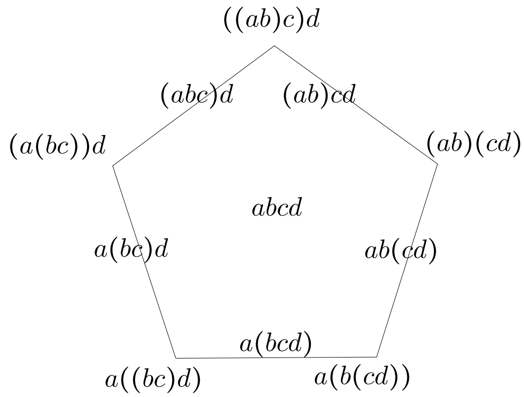

FIGURE 3. K_4

2. Rooted ribbon tree

Now, we associate to a word with suitable parentheses a *stable rooted tree* whose definition is now in order.

Definition 2.1. A tree T is called a **rooted tree** if one vertex of T has been designated the root, in which case the edges have a natural orientation, towards the root. A rooted tree T is called **stable** if T does not contain any vectices of valence 2.

Definition 2.2. Let T be a stable rooted tree. A vertex with valence 1 is called an **exterior vertex** and a vertex with valence > 2 is called an **interior vertex**. The set of exterior vertices is denoted by V_{ext} and the set of interior vertices is denoted by V_{int}.

Similarly, an edge containing exterior vertex is called an **exterior edge** and an edge not containing exterior vertex is called an **interior edge**. Also, the set of exterior edges is denoted by E_{ext} and the set of interior edges is denoted by E_{int}.

Finally, an exterior vertex with incoming orientation is called the **root** and an exterior vertex with outgoing orientation is called a **leaf**.

Stable rooted trees are assigned to vertices of K_4 as follows.

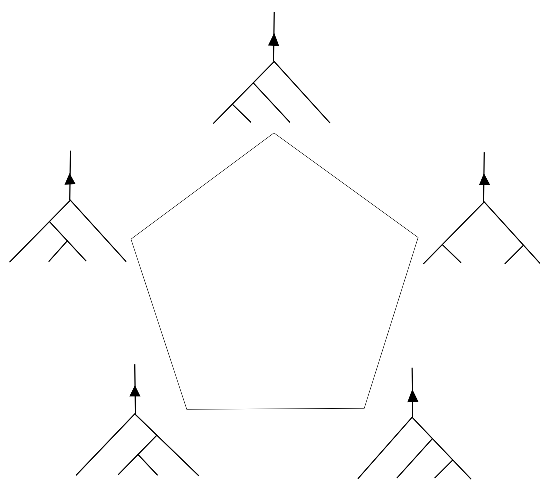

FIGURE 4. K_4

Definition 2.3. A **ribbon tree** is a pair (T, i) consisting of

(i) T is a tree.
(ii) $i : T \longrightarrow D^2$ is an embedding such that

$$i^{-1}(\partial D^2) = V_{ext}(T).$$

A ribbon tree is called stable if T is stable. We denote by $[T, i]$ the isotopy class of (T, i) and call it the combinatorial type thereof.

Definition 2.4. A **rooted ribbon tree** is a pair $((T, i), v_0)$ consisting of

(i) (T, i) is a ribbon tree.
(ii) $v_0 \in V_{ext}(T)$

with the orientation on T is given by the rule that

(1) the ordering of exterior vertices starting from v_0 counterclockwise in D^2,
(2) v_0 is the unique incoming exterior vertex and all others are outgoing
(3) there exists a unique outgoing edges at all interior vertices.

Exercise 2.5. Prove that there exists a unique orientation on a rooted tree satisfying the rule mentioned above.

We denote by G_{n+1} the set of $([T,i],v_0)$ where n is the number of letters. Note that G_{n+1} is the set of different combinatorial types of rooted ribbon tree.

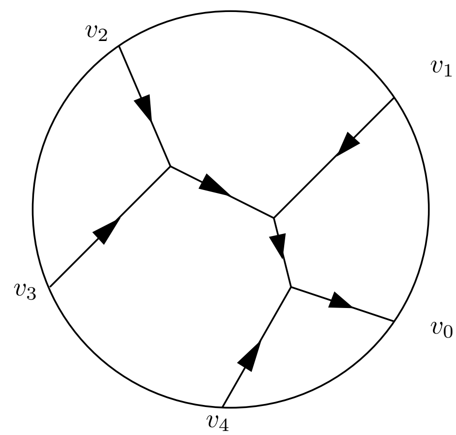

FIGURE 5. Rooted Ribbon Tree

Example 2.6. $\#(G_3) = 1$ (See Figure 5)

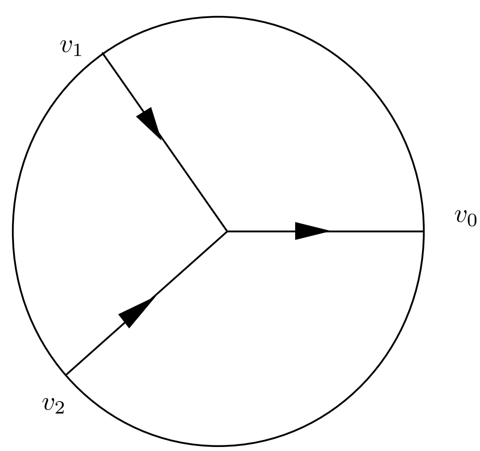

FIGURE 6. G_3

Example 2.7. $\#(G_4) = 3$ (See Figure 6)

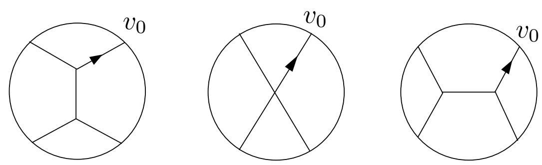

FIGURE 7. G_4

Proposition 2.8. For fixed $k \geq 2$,

$$\#(G_{k+1}) < \infty.$$

In fact, we have

$$\#(V(T)) \leq 2k$$
$$\#(E(T)) \leq 2k - 1$$

and so

$$\#(G_{k+1}) < (2k-1)^{2k}.$$

Proof. We denote by $val(v)$ the number of edges incident to $v \in V(T)$. Then, we have

$$\#(E(T)) = \frac{1}{2} \sum_{v \in V(T)} val(v)$$

Since T is a tree, we know that $b_0(T) = 1, b_1(T) = 0$. The Euler's formula yields

$$\begin{aligned} 1 &= b_0(T) - b_1(T) \\ &= \#(V(T)) - \#(E(T)) \\ &= \frac{1}{2} \sum_{v \in V(T)} (2 - val(v)). \end{aligned}$$

Therefore,

$$\sum_{v \in V(T)} (2 - val(v)) = 2$$

We rewrite

$$\begin{aligned} 2 &= \sum_{v \in V_{ext}(T)} (2 - val(v)) + \sum_{v \in V_{int}(T)} (2 - val(v)) \\ &= \#(V_{ext}(T)) + \sum_{v \in V_{int}(T)} (2 - val(v)) \\ &= k + 1 + \sum_{v \in V_{int}(T)} (2 - val(v)). \end{aligned}$$

By stability, $val(v) \geq 3$ for any $v \in V_{int}(T)$ and hence

$$\#(V_{int}(T)) \leq \sum_{v \in V_{int}(T)} (val(v) - 2) = k - 1$$

Thus, we obtain

$$\begin{aligned} \#(V_{int}(T)) &\leq k - 1 \\ \#(V(T)) &\leq k + 1 + k - 1 = 2k \\ \#(E(T)) &\leq 2k - 1. \end{aligned}$$

□

We introduce some notations. For a combinatorial type $\mathfrak{t} := ([T,i],v_0)$ in G_{k+1}, we set

$$C^0(\mathfrak{t}) := V(T),\ C^0_{int}(\mathfrak{t}) := V_{int}(T),\ C^0_{ext}(\mathfrak{t}) := V_{ext}(T),$$
$$C^1(\mathfrak{t}) := E(T),\ C^1_{int}(\mathfrak{t}) := E_{int}(T),\ C^1_{ext}(\mathfrak{t}) := E_{ext}(T).$$

3. Stasheff polytopes and A_n-space

Now we state the axiomatic description of $K_n (n \geq 2)$. Roughly speaking, K_n is a convex polytope with one vertex for each way of inserting parentheses in a word of n letters in the following way.

Definition 3.1. Denote by $I = (i, i+1, \cdots, j)$ an interval of natural numbers, We call two such intervals I, J *compatible* if they satisfy either (1) $J \subset I$, (2) $I \subset J$ or (3) $I \cup J$ is not an interval.

With this definition, there is one-one correspondence with G_{n+1} and the way of bracketing $b(I_1, \cdots I_p)$ of k letters for the set of compatible intervals $\{I_1, \cdots, I_p\}$. We note that if I_j and I_k are compatible, the bracketing $b(I_j)$ and $b(I_k)$ are either nested or disjoint.

We have the following 5 axioms for the construction of K_n,

(1) The set of vertices of K_k has one to one correspondence with the set of binary trees in G_{k+1}.
(2) There exists one to one correspondence between G_{k+1} and the set of faces of K_k.
   Then we denote $F(\mathfrak{t})$ by the face corresponding to $\mathfrak{t}$.
(3) Each $F(\mathfrak{t})$ is an open cell of codimension $\#(E_{int}(\mathfrak{t}))$.
(4)
$$\overline{F(\mathfrak{t})} = \bigcup_{\mathfrak{t}' \leq \mathfrak{t}} F(\mathfrak{t})$$
$$\partial \overline{F(\mathfrak{t})} = \bigcup_{\mathfrak{t}' < \mathfrak{t}} F(\mathfrak{t})$$
(5) K_k is the cone over ∂K_k.

We take the realization of K_n by Gr_{n+1} and consider the CW-structure induced from this realization.

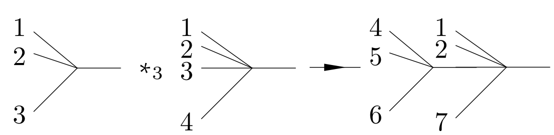

Figure 8. Grafting

Regarding K_k as the collection of stable rooted trees, we can construct the full map $o_i : K_k \times K_l \longrightarrow K_{k+l-1}$ by iterating tree grafting procedures. Inductively, we have

$$K_n = \text{ cone on } \partial K_n$$
$$\partial K_n = \bigcup_{k,s} (K_{n-s-1} \times K_s)_k = \bigcup_{k,s} *_k (K_{n-s-1} \times K_s).$$

Proposition 3.2. Each facet of K_n is of the form $o_i(K_{n-s+1} \times K_s)$.

Now we go back to the maps

$$M_n : K_n \times Y^n \longrightarrow Y \quad \text{for } n \geq 2$$

and define $m = M_2$. For $n = 2$, m is a map from $Y \times Y$ to Y since K_2 is a one point set. We require (Y, m) to be a H-space which will be defined soon. (1) and (2) are translated from the composition

$$Y \xrightarrow{i_L} Y \times Y \xrightarrow{m} Y \;\; i_L(x) = (x_0, x)$$
$$Y \xrightarrow{i_R} Y \times Y \xrightarrow{m} Y \;\; i_R(x) = (x, x_0)$$

Definition 3.3. A space with a multiplication m and a base point satisfying

$$m \circ i_L \sim id \sim m \circ i_R$$

is called a **H-space**.

Definition 3.4. We say a multiplication map M_2 on X

$$M_2 : X \times X \longrightarrow X$$

satisfies $\mathbf{A_k}$**-relation** if it promotes to a family of maps

$$M_k : K_k \times X^k \longrightarrow X$$

for $2 \leq k \leq n$ such that

(i) M_2 is a H-space multiplication
(ii) M_k's are compatible in the following sense:

$$M_k(x_1, \cdots, x_k; \alpha *_i \beta) = M_{k-s+1}(x_1, \cdots, x_{i-1}, M_s(x_i, \cdots, x_{i+s-1}; \alpha), x_{i+s}, \cdots, x_k; \beta)$$

where $\alpha \in G_{s+1}$ and $\beta \in G_{k-s+2}$.

Here, K_k is a k-th Stasheff polytope and $\alpha *_i \beta$ is the operation corresponding to the grafting in Stasheff polytope.

For, $\alpha \in G_{s+1}$ and $\beta \in G_{k-s+2}$, the compatibility condition of M_k can be represented by Figure 11.

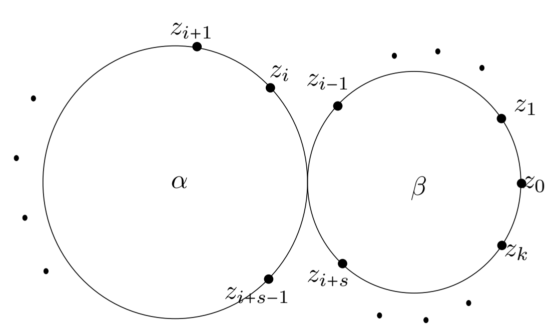

FIGURE 9. a

Here is an example when the Figure 11 comes out.

Example 3.5. Let M be a manifold and L be a submanifold. Consider

$$w : (D^2, \partial D^2) \longrightarrow (M, L)$$

with marked points $\{z_0, z_1, \cdots, z_k\}$ in ∂D^2. In this situation, the disc picture can be decorated by the homotopy class of w in $\pi_2(M, L)$.

4. Two realizations of Stasheff polytopes K_n

Now, we consider the 2nd realization of K_n (resp. W_n) via a compactification of the configuration spaces of $(0,1)$ (resp. S^1).

We define the configuration space $\mathrm{Conf}_{k+1}(S^1)$

$$\mathrm{Conf}_{k+1}(S^1) = \{(z_0,\cdots,z_k) | z_i \neq z_j \text{ for } i \neq j\} \subset (S^1)^{k+1} - \Delta,$$

where $\Delta = \{(z_0,\cdots,z_k)|z_i = z_j \text{ for all } i,j\}$. Note that $PSL(2,\mathbb{R})$ acts on $\mathrm{Conf}_{k+1}(S^1)$ by Mobius transformation. Consider the orbit space of this action

$$\mathcal{M}_{k+1}(D^2) = \mathrm{Conf}_{k+1}(S^1)/\sim$$

where $\sim$ is the orbit equivalence of the action of $Aut(D^2) \cong PSL(2,\mathbb{R})$ given by

$$g\cdot(z_0,\cdots,z_k) = (g(z_0),\cdots g(z_k)).$$

Now we introduce a compactification of $\mathcal{M}_{k+1}(D^2)$ for $k \geq 2$ by considering the notion of stable curves of genus 0 bordered Riemann surface.

4.1. **Moduli space of (bordered) stable curves.** Suppose that we have a set of compact (bordered) surfaces Σ_v equipped with a complex structure j_v indexed by a finite set V. We recall that in 2 dimensional surface a complex structure j_v can be identified with an almost complex structure which is defined to be an endomorphism of $T\Sigma_v$ such that $j_v^2 = -\mathrm{id}$.

We start from a disjoint union

$$\coprod_{v\in V} \Sigma_v.$$

To give information about gluing, we consider a set of unordered pairs $\{x_e, y_e\} = \{y_e, x_e\}$ indexed by a finite set E satisfying

(i) $x_e \in \coprod_{v\in V}\Sigma_v$, $y_e \in \coprod_{v\in V}\Sigma_v$ and $x_e \neq y_e$ for all $e \in E$.

(ii) $\{x_e, y_e\} \cap \{x_{e'}, y_{e'}\} = \varnothing$, whenever $e \neq e'$.

Next, we define an equivalence relation on $\coprod_{v\in V}\Sigma_v$ by

$$x \sim y \Leftrightarrow \{x,y\} = \{x_e, y_e\} \text{ for some } e \in E$$

We then obtain a glued surface

$$\Sigma = \coprod_{v\in V}\Sigma_v/\sim.$$

A glued point $[x_e] = [y_e]$ is called a **double point** (or a **node**) and the set of all such points is denoted by $\mathrm{Sing}(\Sigma)$.

Now, we equip the glued surface Σ with a complex structure as a (singular) complex variety. For this, we consider a homeomorphism from a neighborhood of each double point $[x_e] = [y_e]$ onto a local model given by $xy = 0$ in $\mathbb{C}^2$ near the origin such that $[x_e]$ corresponds to the origin in $\mathbb{C}^2$. We can define a complex structure on a neighborhood of double point by pulling back the standard complex structure of the graph $\{x,y) \in \mathbb{C}^2 \,|\, xy = 0\}$. We denote by j the glued complex structure.

Definition 4.1. A **nodal Riemann surface** (with boundary) is a pair (Σ, j) defined as above. We call each component (Σ_v, j_v) an **irreducible component** of (Σ, j).

From construction, it immediately follows that Σ carries a normalization $\pi_v : \Sigma_v \longrightarrow \Sigma$. Every nodal Riemann surface can be represented by its dual graph whose vertices are decorated by a geometric genus of each irreducible component. Namely, each irreducible component of a nodal Riemann surface (Σ, j) is corresponding to a vertex of the dual graph decorated by its genus, and each double point is matched with an edge of the dual graph (See Figure 15).

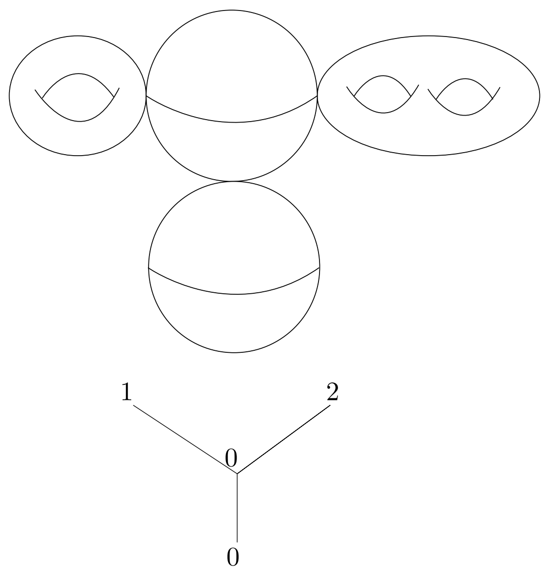

FIGURE 10. Nodal Riemann surface

We now equip a nodal Riemann surface with marked points. A nodal Riemann surface equipped with marked points are called a pre-stable curve.

Definition 4.2. A **pre-stable curve** is a pair $(\Sigma, \mathbf{z}, \mathbf{z}^+)$ consisting of

(i) $\Sigma = (\Sigma, j)$ is a nodal Riemann surface (with bondary).
(ii) $\mathbf{z} = \{z_1, \cdots, z_m\} \subset \partial\Sigma$ (boundary marked points)
(iii) $\mathbf{z}^+ = \{z_1^+, \cdots, z_l^+\} \subset \text{ Int } \Sigma$ (interior marked points)

such that all marked points are disticnt and $z_i, z_j^+ \in \Sigma \setminus \text{Sing}(\Sigma)$. A point z in a nodal Riemann surface is called **special** if z is either a nodal point or a marked point. We denote by g_v the genus of the irreducible component Σ_v and

$$k_v := \text{mark}(\Sigma_v) + \text{sing}(\Sigma_v)$$

where $\text{mark}(\Sigma_v)$ is the number of marked points from $(\Sigma, \vec{z})$ and $\text{sing}(\Sigma_v)$ is the number of nodal points.

Definition 4.3. Suppose that we have two pre-stable maps $(\Sigma, \mathbf{z}, \mathbf{z}^+)$ and $(\Sigma', \mathbf{z}', \mathbf{z}^{+\prime})$. A continuous map $\phi : \Sigma \longrightarrow \Sigma'$ is called an **isomorphism** if it satisfies

(i) ϕ is a homeomorphism.
(ii) $\phi \circ \pi_v$ lifts to a biholomorphism onto some irreducible component Σ'_w where π_v is a normalization.
(iii) $\phi(z_i) = z_i'$ and $\phi(z_j^+) = z_j^{+\prime}$ for each i and j.

A self-isomorphism $\phi : (\Sigma, \mathbf{z}, \mathbf{z}^+) \longrightarrow (\Sigma, \mathbf{z}, \mathbf{z}^+)$ is called an **automorphism**. The set of automorphism is denoted by $\mathrm{Aut}(\Sigma, \mathbf{z}, \mathbf{z}^+)$.

Definition 4.4. A pre-stable curve is called **stable** if $\#\mathrm{Aut}(\Sigma, \mathbf{z}, \mathbf{z}^+)$ is finite, and is called **unstable** otherwise.

The following is a useful numerical criterion of the stability $(\Sigma, \vec{z})$.

Proposition 4.5. A prestable curve $(\Sigma, \vec{z})$ is stable if and only if $k_v + 2g_v \geq 3$ for all irreducible component Σ_v.

We will be especially interested in a pre-stable curve of genus 0 case with a single boundary component later. To see stability condition, observe the following examples.

Example 4.6. We consider a unit sphere S^2 with some marked points.
(1) $\mathrm{Aut}(S^2) \simeq \mathrm{PSL}(2, \mathbb{C})$.
(2) $\dim_{\mathbb{C}} \mathrm{Aut}(S^2, \{z_1\}) = 2$.
(3) $\dim_{\mathbb{C}} \mathrm{Aut}(S^2, \{z_1, z_2\}) = 1$.
(4) $\mathrm{Aut}(S^2, \{z_1, z_2, z_3\}) = \{\mathrm{id}\}$.

Example 4.7. We consider a unit disk D^2 with some marked points.
(1) $\mathrm{Aut}(D^2) \simeq \mathrm{PSL}(2, \mathbb{R})$.
(2) $\dim_{\mathbb{R}} \mathrm{Aut}(D^2, \{z_1\}) = 2$.
(3) $\dim_{\mathbb{R}} \mathrm{Aut}(D^2, \{z_1, z_2\}) = 1$.
(4) $\mathrm{Aut}(D^2, \{z_1, z_2, z_3\}) = \{\mathrm{id}\}$.
(5) $\dim_{\mathbb{R}} \mathrm{Aut}(D^2, \{z_1^+\}) = 1$
(6) $\mathrm{Aut}(D^2, \{z_1\}, \{z_1^+\}) = \{\mathrm{id}\}$
Here z_i's are boundary marked points and z_i^+'s are interior marked points.

Example 4.8. Let $(D^2 \vee D^2, \mathbf{z})$ be given as follows where $\mathbf{z} = \{z_1, z_2, z_3, z_4\}$. We

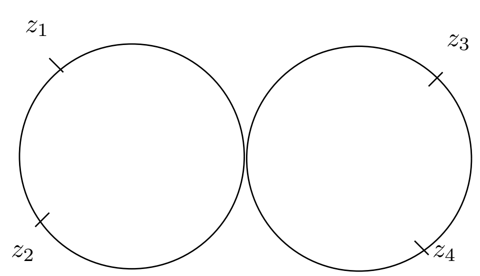

FIGURE 11. $(D^2 \vee D^2, \mathbf{z})$

have $\mathrm{Aut}(D^2 \vee D^2, \{z_1, z_2, z_3, z_4\}) = \{\mathrm{id}\}$.

4.2. **Configuration space of S^1 and $\mathcal{M}^b_{k+1}$.** Suppose that we are given a disk Σ with $(k+1)$ boundary marked points (See Figure 17). It can be considered as a nodal Riemann surface of genus 0 with a single boundary component adorned with marked points. Throughout this lecture, it will be denoted by $(\Sigma, \mathbf{z})$ where $\mathbf{z}$ is the set of boundary marked points.

Let $\mathrm{Aut}(\Sigma, \mathbf{z})$ be the set of automorphisms acting on $(\Sigma, \mathbf{z})$, which gives rise to an equivalence relation on the set of disk with $(k+1)$ boundary marked points

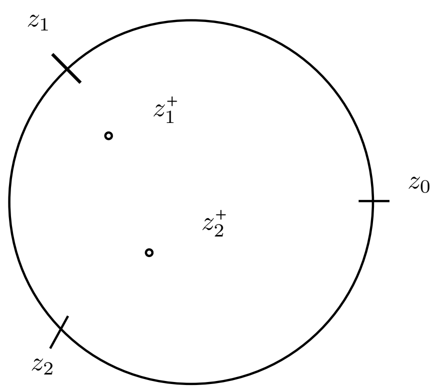

FIGURE 12. $(\Sigma, \mathbf{z}, \mathbf{z}^+)$

and m interior marked points. We then define the set of all isomorphism classes, which is denoted by $\mathcal{M}^b_{k+1}$. Here, b stands for the boundary, and $k+1 = \#\mathbf{z}$. Note that $\mathcal{M}^b_{k+1}$ (with one distinguished vertex) can be identified with the set of all conformal structures $\mathrm{Conf}_k(\mathbb{H}^+)$ on an upper half plane $\mathbb{H}$. We state basic topological properties on the moduli space $\mathcal{M}^b_{k+1}$.

Lemma 4.9. (Topological properties of $\mathcal{M}^b_{k+1}$)

(i) The moduli space $\mathcal{M}^b_{k+1}$ of isomorphism classes has exactly $k!$ components.
(ii) The dimension of $\mathcal{M}^b_{k+1}$ is given by $(k+1) - 3 = k - 2$.

We call the one with $z_0, \cdots, z_k$ cyclically ordered counterclockwise the **main component** of $\mathcal{M}^b_{k+1}$. We denote by $\mathcal{M}^b_{k+1}$ this particular component. Based on the observation of last time (See Example 13.6), we can expect a numerical criterion for stability of $(\Sigma, \mathbf{z})$.

Proposition 4.10. Let $(\Sigma, \mathbf{z})$ be a disk together with $(k+1)$ boundary marked points. Then, $(\Sigma, \mathbf{z})$ is stable if and only if $(k+1) \geq 3$.

For a compactification of $\mathcal{M}^b_{k+1}$, we need to include degeneration of disks and so need to consider Σ which is acheived by glueing disks at boundary points.

Let $(\Sigma, \mathbf{z})$ be a prestable curve obtained by glueing disks at boundary point(s) consisting following data

(i) Σ is a glued disk as above,
(ii) $\mathbf{z}$ is the set of boundary marked points,

Here is a numerical criterion of stability for more general situaion.

Proposition 4.11. Let $(\Sigma, \mathbf{z})$ be a prestable curve as above. Then, $(\Sigma, \mathbf{z})$ is stable if $(\Sigma_v, \mathbf{z}_v)$ is stable for each $v \in V$. More specifically, Σ_v is stable if $k_v = \mathrm{mark}(v) + \mathrm{sing}(v) \geq 3$.

For example, $\overline{\mathcal{M}}_3(D^2)$ is a one point set, and $\overline{\mathcal{M}}_4(D^2) \cong [0,1]$. Each disc is stable in that it carries at least 3 special points which consists of either marked points or double points. We denote this union of discs by Σ. The counterclockwise orientation of each disc induces the orientation of Σ. Also, the ordering of z_i coincides with this orientation.

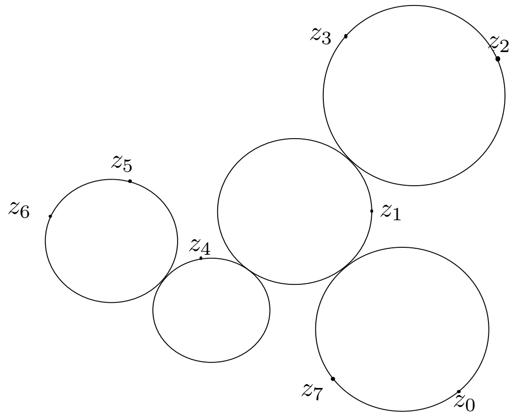

FIGURE 13. Bordered Niemann surface with marked points

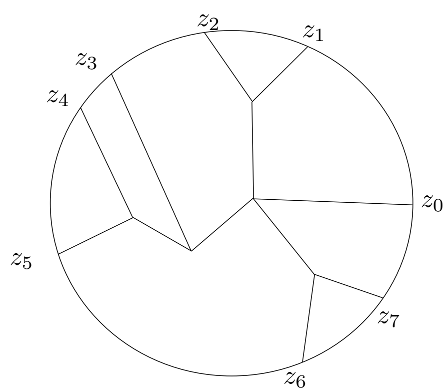

FIGURE 14. Dual graph

Next, we consider the dual graph associated to $\overline{\mathcal{M}}_{k+1}(D^2)$. The dual graph has the following correspondence. (See Figure 9 and Figure 10.)

a disc component $\longleftrightarrow$ a vertex
a double point $\longleftrightarrow$ an interior edge
a marked point $\longleftrightarrow$ an exterior edge(flag)

In this way, we associate each stable curve in $\overline{\mathcal{M}}_{k+1}(D^2)$ a ribbon graph. As before we denote by $\mathfrak{t}$ the topological type of ribbon graph. When the stable curve is rooted, we get a rooted ribbon graph.

For each element $\mathfrak{t} \in G_{k+1}$, we denoted

$$\mathcal{M}(\mathfrak{t}) = \left\{\vec{z} = (z_0, \cdots, z_k) | z_i \in \partial\Sigma,\, T_{(\Sigma,\vec{z})} = \mathfrak{t}\right\} / \sim,$$

where T is the rooted ribbon tree associated to the bordered Riemann surface with marked points.

First, assume that $\mathfrak{t}$ is a corolla. The corresponding Σ has only one disc. We say $\vec{z} = (z_0, \cdots, z_k) \sim \vec{z'} = (z'_0, \cdots, z'_k)$ if and only if there exists $\phi \in PSL(2, \mathbb{R})$ such that $z'_i = \phi(z_i)$. Since $PSL(2, \mathbb{R})$ acts freely on $\mathrm{Conf}_{k+1}(S^1)$, $\mathcal{M}(\mathfrak{t})$ has smooth manifold structure and

$$\dim M(\mathfrak{t}) = k+1-3 = k-2.$$

For general $\mathfrak{t} \in G_{k+1}$, $\vec{z} = (z_0, \cdots, z_k) \sim \vec{z'} = (z'_0, \cdots, z'_k)$ if and only if the following holds: Let D_l be the lth component in Σ and m be the number of special points on D_l. Then there exist $\phi_i \in PSL(2,\mathbb{R})(i = 1, 2, \cdots, n$, n is the number of disc components) such that ϕ_i maps $(D_l, \vec{z}_l)$ to $(D'_l, \vec{z'_l})$, where $\vec{z'_l}$ consists of special points on ∂D_l. Now we have the decomposition

$$\overline{\mathcal{M}}^b_{k+1} = \bigcup_{\mathfrak{t} \in G_{k+1}} \mathcal{M}(\mathfrak{t}).$$

Now we provide the structure of CW-complex by requiring that

$$\overline{\mathcal{M}(\mathfrak{t})} = \bigcup_{\mathfrak{t}' \geq \mathfrak{t}} \mathcal{M}(\mathfrak{t}).$$

For example, whenever $\mathfrak{t}$ is minimal, $\mathcal{M}(\mathfrak{t})$ is a closed point.

Theorem 4.12. The $\overline{\mathcal{M}}^b_{k+1}$ has a cell decomposition exactly the same combinatorics of cell structure as that of Stasheff polytope, more specifically K_k (or an associatehedron).

Proof. We just note that each $F(\mathfrak{t}) := \mathcal{M}(\mathfrak{t})$ is an open cell of codimension $\#(E_{int}(\mathfrak{t}))$ that satisfies all the 5 axioms given in section 3. In the next section, we will give further details of the description of this cell structgure in terms of the metric ribbon trees, which carries a cell decomposition dual to the one present in $\overline{\mathcal{M}}^b_{k+1}$. □

4.3. **Metric ribbon trees.** Recall that G_{k+1} is the set of rooted ribbon trees. We define a partial order on G_{k+1}. That is, for $\mathfrak{t}, \mathfrak{t}' \in G_{k+1}$, $\mathfrak{t} < \mathfrak{t}'$ if and only if $\mathfrak{t}'$ is obtained by collapsing a sequence of interior edges of $\mathfrak{t}$.

Definition 4.13. A binary tree is a tree with $val(v) = 3$ for all interior vertices v.

The following lemma follows immediately from definition.

Lemma 4.14.

(i) This defines a partial order on G_{k+1}.
(ii) A minimal element is a binary tree.
(iii) A maximal element is a corolla which is unique in G_{k+1}.

We now associate the length $\ell : C^1_{int}(\mathfrak{t}) \longrightarrow (0,\infty)\}$ with the set of interior edges.

Definition 4.15. To each $\mathfrak{t} := ([T,i], v_0)$, we associate an open cell

$$Gr(\mathfrak{t}) = \{l : C^1_{int}(\mathfrak{t}) \longrightarrow (0,\infty)\} \simeq (0,\infty)^{\#(C^1_{int}(\mathfrak{t}))} \tag{4.1}$$

For each $\mathfrak{t} \in G_{k+1}$, we assign $Gr(\mathfrak{t})$ thereto. We form the union

$$Gr_{k+1} = \coprod_{\mathfrak{t} \in G_{k+1}} Gr(\mathfrak{t}).$$

We call each element in Gr_{k+1} a metric ribbon tree.

Note that $Gr(\mathfrak{t}) \cong (0,1)^{\#(C^1_{int}(\mathfrak{t}))}$. For example, $\dim Gr(\mathfrak{t}) = 0$ if $\mathfrak{t}$ is a corolla. A corolla corresponds to the interior of the cell which has dimension $k-2$ for $K_k \cong \overline{\mathcal{M}}^b_{k+1}$.

To give a CW structure of Gr_{k+1}, we define a grafting operation.

$$\star_i : K_k \times K_l \longrightarrow K_{k+l-1}$$

As a building block for stable rooted trees in K_k, we define a **corolla**.

Definition 4.16. A stable rooted tree is called a **corolla** if $\#(V_{int}(T)) = 1$.

We define a grafting operation $\star_i$ of two stable rooted trees T_1 in K_k and T_2 in K_l. Here, a subscript i indicates a position of the second tree where the first tree is glued. First, we attach the root of T_1 to a i-th leaf of T_2. After attaching, we get rid of the vertex with valence 2 to make a stable rooted tree. The constructed stable rooted tree is the output of grafting operation $\star_i$. Combining them all over i, we have the following

Theorem 4.17. There exists a compactification $\overline{Gr}_{k+1}$ of Gr_{k+1} such that $\overline{Gr}_{k+1}$ carries a CW structure with the face maps

$$\star : Gr_{k+1} \times Gr_{l_1+1} \times Gr_{l_2+1} \times \cdots \times Gr_{l_k+1} \longrightarrow \partial \overline{Gr}_{k+\sum_{i=1}^k l+1}$$

forms a $(k-2)$-cell. Furthermore Gr_{k+1} carries a smooth structure with respect to which it is diffeomorphic to $\mathbb{R}^{k-2}$.

In the remaining section, we will give the construction of $\overline{Gr}_{n+1}$ with $n \geq 2$. prove the theorem.

To describe the cell structure, we need to discuss the gluing map $\star_{(i)}$ precisely.

Let $\mathfrak{t} = [(T, i, p)] \in Gr_{k+1}$ where p is the root vertex of T. Let $k_v = \mathrm{val}(v)$ be the valence of $v \in C^0_{\mathrm{int}}(T)$. We define a map

$$\Phi_{\mathfrak{t}} : Gr(\mathfrak{t}) \times \prod_{v \in C^0_{\mathrm{int}}} Gr_{k_v+1} \to Gr_{k+1} :$$

Let

$$(\ell, (\ell_v)) \in Gr(\mathfrak{t}) \times \prod_{v \in C^0_{\mathrm{int}}} Gr_{k_v+1}$$

where $\ell_v \in Gr(\mathfrak{c}_v + 1) \subset Gr_{k_v+1}$ and $\mathfrak{c}_v = (C_v, i_v, p_v))$. We replace the vertex $v \in T$ by the tree C_v. Namely we identify $k_v + 1$ edges containing v and $k_v + 1$ exterior edges of $\mathfrak{c}_v$ with the output edge of v glued to the output flag of $\mathfrak{c}_v$ respecting the ordering of the edges. We denote by $\widehat{\mathfrak{t}}$ the resulting ribbon graph with the induced ribbon structure and the order.

We note that the set of interior edges of $\widehat{t}$ consist of the union of interior edges of $\widehat{t}$ and those of $\mathfrak{c}_v$, $v \in C^1_{\mathrm{int}}(T)$. Then we define the value $\Phi_{\mathfrak{t}}(\ell, (\ell_v)) =; \ell' \in Gr(\widehat{t})$ by the formula

$$\ell'(e) = \begin{cases} \ell(e) & e \in C^1_{\mathrm{int}}(T) \\ \ell_v(e) & e \in C^1_{\mathrm{int}}(C_v). \end{cases} \tag{4.2}$$

This defines the glued metric ribbon trees in Gr_k.

Now consider iteration of the above gluing. Let $\mathfrak{c}_v \in G_{k_v+1}$ and $\mathfrak{c}_{u,v} \in G_{k_{u,v}+1}$ where $u \in C^0_{\mathrm{int}}(\mathfrak{c}_v)$ and $k_{u,v} + 1 = \mathrm{val}(u)$. We then have the following commutative diagram

$$\begin{array}{ccc} Gr(\mathfrak{t}) \times \Pi Gr(\mathfrak{c}_v) \times \Pi Gr(\mathfrak{c}_{u,v}) & \xrightarrow{1 \times \Pi \Phi_{\mathfrak{t}_v}} & Gr(\mathfrak{t}) \times \Pi Gr_{k_v} \\ \downarrow{\scriptstyle \Phi_{\mathfrak{t}} \times id} & & \downarrow \\ Gr(\widehat{\mathfrak{t}}) \times \Pi Gr(\mathfrak{c}_{u,v}) & \xrightarrow{\Phi_{\widehat{\mathfrak{t}}}} & Gr_{k+1} \end{array} \tag{4.3}$$

We will prove Theorem 4.17 by induction over k with $k \geq 2$.

When $k = 2$, G_3 contains a unique element which has no interior edge. Therefore $Gr3$ is a point. Next assume $k > 2$ and suppose the theorem holds. Let $\mathfrak{t}_{k+1}$ be the corolla and then $Gr(\mathfrak{t}_{k+1})$ is a point. We first prove that $Gr_{k+1} \setminus Gr(\mathfrak{t}_{k+1})$ is a topological manifold.

Let $\mathfrak{t} \neq \mathfrak{t}_{k+1}$ be in G_{k+1}. Then $k_v < k$ for all $v \in C^0_{int}(\mathfrak{t})$. By the induction hypothesis, Gr_{k_v+1} is homeomorphic to $\mathbb{R}^{k_v-2}$. Recalling that $Gr(\mathfrak{t})$ is a cell, we derive that $Gr(\mathfrak{t}) \times \Pi Gr(k_v)$ is homeomorphic to $\mathbb{R}^{k-2}$. Therefore $\Phi_\mathfrak{t}$ defines a topological open embedding whose image provides a neighborhood of $Gr(\mathfrak{t})$ in Gr_{k+1}. Regard them together with $Gr(\mathfrak{t})$ as coordinate charts of Gr_{k+1} which provides a structure of topological manifold with $Gr_{k+1} \setminus Gr(\mathfrak{t}_{k+1})$.

We next provide a smooth structure with $Gr_{k+1} - Gr(\mathfrak{t}_{k+1})$ again by induction. The C^∞ compatibilty of the above charts follows from the commutativity of the above diagram. Finally we will construct a diffeomorphism $Gr_{k+1} \cong \mathbb{R}^{k-2}$ later.

4.4. **Duality between the cell structures of $\overline{\mathcal{M}}^b_{k+1}$ and $\overline{Gr}_{k+1}$.** We start with the definition of the dual cell decomposition.

Definition 4.18. Let X be a smooth manifold and X_a, $a \in I$ being a some indexing set, be smooth submanifolds such that their closures $\overline{X}_a$ become smooth submanifolds with corners. We call them *smooth cell decomposition* of X if they satisfy the following:

(1) X_a are disjoint from one another and $\coprod_{a\in I} X_a = X$
(2) X_a is diffeomorphic to $\mathbb{R}^{|a|}$, and $\overline{X_a}$ is diffeomorphic to $D^{|a|}$ after smoothing our their corners.
(3) The boundary $\partial D^{|a|}$ is a union of some of the X_b' with $|b| < |a|$.

Given such a decomposition, its dual decomposition is defined as follows. The definition will be inductively given over the dimension of X. Suppose that the dual decomposition is defined for manifolds of dimension $< n$. Let $a \in I$ and $p \in X_a$. We consider the normal space N_pX_a and its unit sphere SN_pX_a. For each X_b with $\overline{X_b} \supset X_a$ we consider the intersection $SN_pX_a \cap T_p\overline{X_b}$ which forms a cell. These cells define a cell decomposition of $SN_pX_a = S^{n-|a|-1}$. Certainly $n - |a| - 1 < n$ and so by the induction hypothesis, we obtain a dual cell decomposition thereof. We add one more cell of dimension $n - |a|$ to the dual cell decomposition given above. Then we get a cell decomposition of $D^{n-|a|}$ which we define to be $\overline{Y_a}$.

Now we explain how these cells are glued to one another. For each given $a, b \in I$ with $\overline{X_b} \supset X_a$, we construct an embedding $\overline{Y_a} \subset \overline{Y_b}$. (We recall $\dim Y_a = n - |a|$.) We consider the case $|a| = |b| + 1$. Let $q \in X_b$ and choose a unit vector $v \in SN_qX_b$ that is tangent to $\overline{X_a}$. For sufficiently small $\epsilon > 0$, we may assume $\exp(\epsilon v) \in X_a$ with respect to a suitably chosen Riemannian metric. Since $|a| = |b| + 1$, such a vector is unique (modulo the sign). This way the decomposition of X induces one on SN_qX_b. Then the latter induces a decomposition on $SN_v(SN_qX_b)$.

Lemma 4.19. The above defined decomposition of $SN_v(SN_qX_b)$ is can be naturally identified with that of SN_pX_a.

Therefore by construction, the cell of the dula decomposition of SN_qX_b corresponding to $v \in SN_qX_b$ is isomorphic to $\overline{Y_a}$. This identification defines the embedding $\overline{Y_a} \subset \overline{Y_b}$. By gluing these cells $\overline{Y_a}$, we have obtain a cell complex Y which carries a smooth structure for which $\{Y_a\}$ gives a smooth cell decomposition on $Y = \coprod_{a\in I} Y_a$.

Proposition 4.20. X is diffeomorphic to Y.

Proof. It is enough note that the barycentric subdivision of X also becomes one of Y. $\square$

Now we compactify Gr_{k+1} using the $\mathbb{R}_+$-action on $Gr_{k+1} - Gr(\mathfrak{t}_{k+1})$ and denote by $\overline{G}r_{k+1}$ the resulting compactification. We denote by $\overline{Gr(\mathfrak{t})}$ the closure of $Gr(\mathfrak{t})$ in $\overline{G}r_{k+1}$.

Theorem 4.21. Each $\overline{Gr(\mathfrak{t})}$ is a cell and $(\overline{Gr_{k+1}}, \{Gr(\mathfrak{t})\})$ is the dual cell decomposition of $(\overline{\mathcal{M}}^b_{k+1}, \{\overline{\mathcal{M}}(\mathfrak{t})\})$.

5. The based loop space is an A_∞-space

Let X be a connected topological space with x_0 a base point. We consider the based loop space $Y = \Omega X$ and identify the base point x_0 with the constant loop based at x_0.

Proposition 5.1. *Y is a H-space.*

Proof. We consider the constant loop x_0 as a base point of $Y = \Omega X$, and let $m : \Omega X \times \Omega X \longrightarrow \Omega X$ be the concatenation defined in Lecture 1. This operation will be multiplication in Y. Note that $m(\gamma, x_0) \sim \gamma$, and $m(x_0, \gamma) \sim \gamma$, and this implies that $m \circ i_L \sim id \sim m \circ i_R$ in Y. □

Theorem 5.2. *(Y, m) is an A_∞-space.*

Proof. We will use another related space $Z = \Theta X$.

$$\Theta X = \{(r, \alpha) \,|\, r \geq 0,\ \alpha : [0, r] \longrightarrow X \text{ with } \alpha(0) = x_0 = \alpha(r)\}$$

Z has an associative multiplication which is just concatenation without doing any reparameterization of the domain. That is, the multiplication $\mu : Z \times Z \longrightarrow Z$ is defined by

$$\mu((r, \alpha), (s, \beta))(t) = \begin{cases} \alpha(t), & 0 \leq t \leq r \\ \beta(t - r), & r \leq t \leq r + s \end{cases}$$

We have natural maps $f : Y \longrightarrow Z$ and $g : Z \longrightarrow Y$ defined by $g(r, \alpha)(u) = \alpha(ru)$ and $f(\gamma) = (1, \gamma)$.

Lemma 5.3. *There is a homotopic between $f \circ g$ and id_Z.*

Proof. We define $H : [0, 1] \times Z \longrightarrow Z$ by

$$H(t, (r, \alpha))(u) = ((1 - t)r + t, \alpha((1 - t) + tr)u).$$

$$H(0, (r, \alpha)(u) = (r, \alpha(u)) = (r, \alpha)(u)$$
$$H(1, (r, \alpha)(u) = (1, \alpha(ru)) = (f \circ g)(r, \alpha)(u)$$

Hence, H defines a homotopic between them. □

Now, we apply the following theorem and this finishes the proof.

Theorem 5.4. *Let (Y, m) be a H-space and suppose there exists a space Z with associated multiplication with identity and $f : Y \longrightarrow Z$ and $g : Z \longrightarrow Y$ as in the lemma above. Then (Y, m) is an A_∞-space.*

□

Theorem 5.5. *Let Y be a topological space and Z is an associate H-space with associative multiplication $\mu : Z \times Z \longrightarrow Z$. Suppose that there is a pair of maps*

$$f : Y \longrightarrow Z, \quad g : Z \longrightarrow Y$$

such that $f \circ g$ and id_Z are homotopic. Now, we define a multiplication $m : Y \times Y \longrightarrow Y$ by

$$m(y_1, y_2) = g(\mu(f(y_1), f(y_2))).$$

In other word, we define a multiplication m on Y to make the following am commute. Then, $(Y.m)$ becomes an A_∞-space.

Proof. The multiplication of k-elements is denoted by μ_k. Since we assume μ is associative, this is well-defined irrespective the order of multiplication. Using μ_k, we are going to define $M_k(\cdot,\lambda)$ for $\lambda \in K_k$ inductively and show that the constructed μ_k's satisfy A_∞-relation.

(1) $(M_2 : Y \times Y \longrightarrow Y)$ It will be given by $M_2 := m$.

(2) $(M_3 : Y^3 \times K_3 \longrightarrow Y)$ First, we construct M_3 at the boundary of K_3 and extend it to interior of K_3. Recall that

$$\partial K_3 = (K_2 *_1 K_2) \cup (K_2 *_2 K_2)$$

We define $M_3(x,y,z;\partial_{K_3})$ as follows:

$$M_3(x,y,z;\{0\}) = M_2(M_2(x,y),z) \text{ in } K_2 *_1 K_2,$$
$$M_3(x,y,z;\{1\}) = M_2(x,M_2(y,z)) \text{ in } K_2 *_2 K_2.$$

Let h be a homotopy between $f\circ g$ to id_Z,

$$h : [0,1]\times Z \longrightarrow Z \text{ with } h(0) = f\circ g, h(1) = id_Z.$$

Now, we observe that $M_2(M_2(x,y),z)$ and $M_2(x,M_2(y,z))$ are homotopic. Moreover, using a homotopy h, an explicit homotopy between them can be constructed as follows.

$$\begin{aligned} M_2(M_2(\cdot,\cdot),\cdot) &= g\circ\mu_2(f(g\circ\mu_2(f\times f)\times f)) \\ &= g\circ\mu_2((f\circ g)\circ\mu_2(f\times f)\times f) \\ &\overset{h}{\simeq} g\circ\mu_2(\mu_2(f\times f)\times f) \\ &= g\circ\mu_3(f\times f\times f) \end{aligned}$$

Here, the last equality follows from the associativity of μ. By the same way, we obtain

$$M_2(\cdot,M_2(\cdot,\cdot)) \overset{h}{\simeq} g\circ\mu_3(f\times f\times f).$$

Composing two homotopies, we get an explicit homotopy as we desired.

For $t\in[0,1] = K_3$, $M_3(x,y,z;t)$ can be assigned as

$$M_3(x,y,z;t) = \begin{cases} g\circ\mu_2(h(2t,\mu_2(f(x),f(y))),f(z)) & \text{for } 0\le t\le \frac{1}{2} \\ g\circ\mu_2(f(x),h(2-2t,\mu_2(f(y),f(z)))) & \text{for } \frac{1}{2}\le t\le 1 \end{cases}$$

and this gives rise to a map $M_3 : Y^3\times K_3 \longrightarrow Y$. By construction, M_2 and M_3 satisfy A_3-relation.

(3) $(M_4 : Y^4\times K_4 \longrightarrow Y)$ By the previous construction, $M_4(\cdot,\cdot,\cdot,\cdot,\lambda)$ can be constructed for any $\lambda\in\partial K_4$ using a homotopy h. Let v be a barycenter of K_4. We define

$$M_4(\cdot,\cdot,\cdot,\cdot,v) := g\circ\mu_4(f\times f\times f\times f)$$

Note that at the midpoint m labelled with $(abc)d$ (See Figure 12) M_4 is defined as

$$M_4(\cdot,\cdot,\cdot,\cdot,m) = g\circ\mu_2(f(g\circ\mu_3(f\times f\times f))\times f).$$

We connect the midpoint m and the barycenter v. Using the following homotopy, we can define M_4 at every point in the connected line as we did before.

$$\begin{aligned} g\circ\mu_2(f(g\circ\mu_3(f\times f\times f))\times f) &\overset{h}{\simeq} g\circ\mu_2(\mu_3(f\times f\times f)\times f) \\ &= g(\mu_4(f\times f\times f\times f). \end{aligned}$$

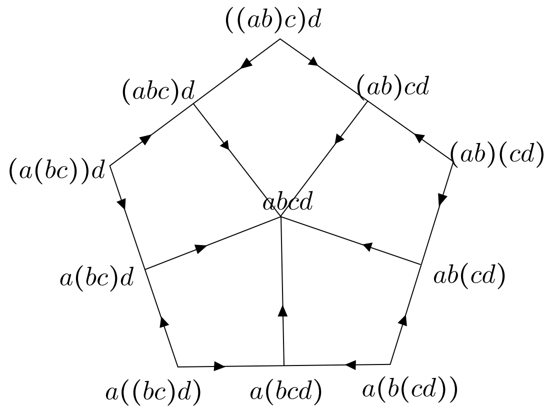

FIGURE 15. a

Similary, we can construct M_4 at each point of lines between a midpoint and the barycenter. Finally, M_4 can be defined all points in K_4 be considering a homotopy between homotopies. We leave construction of M_k $(k > 4)$ as an exercise.

□

Exercise 5.6. Finish the proof of Theorem 5.3.

Corollary 5.7. ΩX is an A_∞-space.

Proof. Apply the theorem to $Y = \Omega X$ and $Z = \Theta X$. □

Moral: taking the chain complex of A_∞-space gives rise to an A_∞-algebra.

6. Definition of A_∞ algebra

Let R be a commutative ring with unit and A be a graded R-module

$$A = \bigoplus_{i=0}^{\infty} A^i$$

with $\mathbb{K}$-linear map

$$m_k : A^{\otimes k} \longrightarrow A$$

of degree $(2-k)$. Recall that

$$\deg(x_1 \otimes \cdots \otimes x_k) := \deg(x_1) + \cdots + \deg(x_k).$$

Let $A[k]$ denote the shifted grading module given by

$$(A[k])^i := A^{k+i}$$

for $k \in \mathbb{Z}$.

Lemma 6.1. Suppose that we have a graded R-module with $\mathbb{K}$-linear map $m_k : A^{\otimes k} \longrightarrow A$ of degree $(2-k)$ as above. We define the shifted map $m_k' : A[1]^{\otimes k} \longrightarrow A[1]$ as follows

$$m_k' := s \circ m_k((s^{-1})^{\otimes k})$$

where $s : A[1] \longrightarrow A$. Then, m_k' has degree 1 for all k.

Proof. We introduce notations

$$|x_i| := \text{ the original degree}, \quad |x_i|' := |x_i| - 1.$$

We want to prove

$$|x_1 \otimes \cdots \otimes x_k|' = |x_1|' + \cdots + |x_k|' + 1.$$

Since m_k has degree $(2-k)$, we have

$$|x_1 \otimes \cdots \otimes x_k|' = |x_1 \otimes \cdots \otimes x_k| - 1 = \sum_1^k |x_k| + (2-k)$$

$$= \sum_1^k (|x_k| - 1) + 2 = \sum_1^k |x_k|' + 1.$$

□

Definition 6.2. Let A be a graded ring and let $m = \{m_k\}$ be the collection of $\mathbb{K}$-linear maps for integer $k \geq 0$. We call (A, m) **A_∞-algebra** if m_k's satisfy

$$\sum_{k+s=n+1} \sum_{s=1}^{n-s+1} (-1)^{\varepsilon} m_k(x_1, \cdots, x_{i-1}, m_s(x_1, \cdots, x_{i+s-1}), x_{i+s}, \cdots, x_n) = 0.$$

where $\varepsilon = |x_1|' + \cdots + |x_{i-1}|'$. Moreover, (A, m) is called a **strict A_∞-algebra** if $m_0 = 0$ and (A, m) is called a **weak A_∞-algebra** or **curved A_∞-algebra** if $m_0 \neq 0$.

Definition 6.3. (**Unit**)
Let (A, m) be an A_∞-algebra. An element of $\mathbf{e} \in A^0 = A[1]^{-1}$ is called a unit of (A, m) if $\mathbf{e}$ satisfies

(1) $m_{k+1}(x_1, \cdots, \mathbf{e}, \cdots, x_k) = 0$ for $k \geq 2$ or $k = 0$.
(2) $m_2(\mathbf{e}, x) = (-1)^{\deg x} m_2(x, \mathbf{e}) = x$

In this case, $(A, m, \mathbf{e})$ is called a unital A_∞-algebra.

Example 6.4. Let (A, m) be a strict A_∞-algebra. Then, we have

(1) $m_1 \circ m_1 = 0$ so that m_1 is a differential.
(2) $m_1(m_2(x,y)) = m_2(m_1(x),y) \pm m_2(x,m_1(y))$ so that m_1 is a derivation with respect to m_2.

Remark 6.5. In the curved A_∞-algebra, m_1 does not give a differential.

Example 6.6. Consider an A_∞-space Y. We apply cohomology functor, with field coefficients, to $M_n : K_n \times Y^n \to Y$ and obtain

$$(M_n)^* : H^*(Y) \to H^*(K_n \times Y^n)).$$

Composing this map with the isomorphism

$$H^*(K_n \times Y^n) \cong H^{*-(n-2)}(Y^n) \cong (H^*(Y))^{\otimes n}[2-n]$$

where the last isomorphism is from the Künneth formula. By taking the adjoint of the map and shifting the degree of the complex $H_*(Y)$ by 1, the map is equivalent to

$$(H_*(Y)[1])^{\otimes n} \to H_*(Y)[1]$$

of degree 1 *for all* n.

We now consider the case $k = 2$, which induces a product

$$m_2 : H_*(Y) \times H_*(Y) \cong H_*(Y \times Y) \to H_*(Y)$$

Here we use a field $\mathbb{K}$ as coefficient ring. Note that m_2 is a multiplications. When Y carries a base point $x_0 \in Y$, $x_0 \hookrightarrow Y$ induces a unit

$$\epsilon : \mathbb{R} \cong H_*(x_0) \longrightarrow H_*(Y)$$

The following algebraic fact arises from the A_∞-relation for the A_∞-space.

Theorem 6.7. For any A_∞-space, its homology complex $C = H_*(Y)$ carries an A_∞-algebra structure.

We have defined A_∞-algebra. Note that A_∞-algebra is a kind of generalization of the following object.

Definition 6.8. A **differential graded algebra(DGA)** A is a graded vector space with

(1) $d : A \longrightarrow A$ such that $d^2 = 0$
(2) There is an associative product $A \otimes A \longrightarrow A$ that satisfies Leibnitz rule

$$d(ab) = (da)b + (-1)^{|a|}(db),$$

where $|a|$ is the degree of a.

If we put

$$m_1(a) = (-1)^{|a|} da$$
$$m_2(a,b) = (-1)^{|a|(|b|+1)} ab$$
$$m_k = 0 \text{ for all } k \geq 3$$

then $(A, \{m_k\}_{k=1}^\infty)$ becomes an A_∞-algebra.

Now, suppose that an $(A, \{m_k\}_{k=1}^\infty)$ is given and

$$m_k : A[1]^{\otimes k} \longrightarrow A[1]$$

is degree 1.

Definition 6.9. The **bar complex** BA is defined by

$$BA = \bigoplus_{k=0}^{\infty} B_k A,$$

where $B_k A = A[1]^{\otimes k}$.

Here, we introduce the dual notion of an algebra.

Definition 6.10. (1) A **coalgebra** over a field $\mathbb{K}$ is a vector space C with $\mathbb{K}$-linear maps $\Delta : C \longrightarrow C \otimes C$ such that $(\mathrm{id}_C \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_C) \circ \Delta$ Equivalently, the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \downarrow\Delta & & \downarrow\mathrm{id}_C\otimes\Delta \\ C \otimes C & \xrightarrow{\Delta\otimes\mathrm{id}_C} & C \otimes C \otimes C \end{array}$$

Such Δ is called a coproduct. The condition is called **coassociativity** (dual of associativity),

(2) A *counit* is a R-linear map $\epsilon : C \longrightarrow R$ such that $(\mathrm{id}_C \otimes \epsilon) \circ \Delta = \mathrm{id}_C = (\epsilon \otimes \mathrm{id}_C) \circ \Delta$. Equivalently the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \downarrow\Delta & & \downarrow\mathrm{id}_C\otimes\epsilon \\ C \otimes C & \xrightarrow{\epsilon\otimes\mathrm{id}_C} & \mathbb{K} \otimes C = C = C \otimes \mathbb{K} \end{array}$$

The map ϵ in the second condition is called *counit*. Counit is the dual notion of the obvious map $\mathbb{K} \longrightarrow C$ associated to the unit of the algebra.

Then BA is an example of a coalgebra.

Lemma 6.11. The bar complex BA is a coalgebra with respect to the coproduct $\Delta : BA \longrightarrow BA \otimes BA$ defined by

$$\Delta(x_1 \otimes \cdots \otimes x_n) = \sum_{i=0}^{n} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_n) \in BA \otimes BA$$

for indecomposable element $x_1 \otimes \cdots \otimes x_n$ extended R-linearly to BA.

Proof. Take the count $\epsilon : BA \longrightarrow \mathbb{K}$ to be the obvious projection. Then it can be checked easily that Δ and ϵ satisfy the conditions in the definition of a subalgebra. □

Sweedler notation: We write the coproduct Δ by

$$\Delta(\mathbf{x}) = \sum_c \mathbf{x}^{(c;1)} \otimes \mathbf{x}^{(c;2)} \tag{6.1}$$

for a general element $\mathbf{x} \in BA$. More generally, we denote

$$\Delta^{k-1}(\mathbf{x}) = \sum_c \mathbf{x}_c^{(k;1)} \otimes \mathbf{x}_c^{(k;2)} \otimes \cdots \otimes \mathbf{x}_c^{(k;k)}. \tag{6.2}$$

7. Massey product and Borromian ring

Why A_∞-algebra is important? We give one example.

Example 7.1. Let B be a Borromean ring and L be a trivial link. (See Figure 13 and 14)

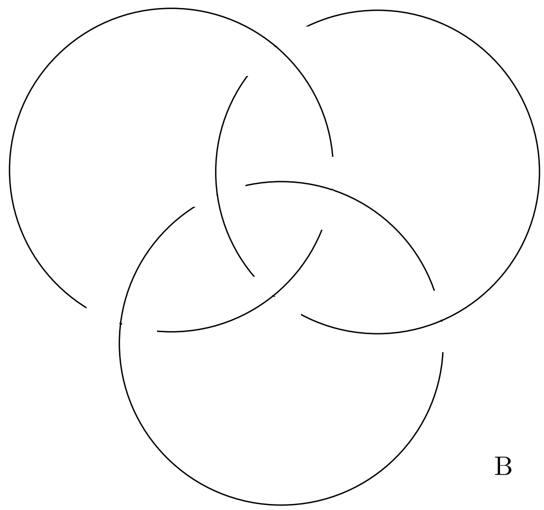

Figure 16. Borromean Ring

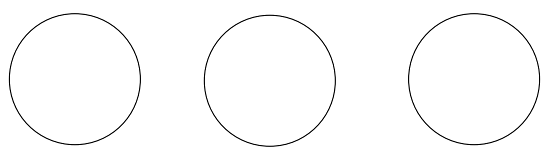

Figure 17. Trivial Link

We are going to see that $S^3 - B$ and $S^3 - L$ have same cohomology ring structure. Thus, cohomology ring cannot distinguish B and L. However, A_∞-algebra structures are different.

First, we calculate cohomology groups of $S^3 - B$ and $S^3 - L$. Recall the Alexander duality: For any good compact pair (A, B) in an oriented manifold X, we have an isomorphism

$$H_q(X - B, X - A) \longrightarrow H^{n-q}(A, B).$$

By applying the Alexander duality to $X = S^3, A = S^3$ and $B = S^3 - B$, we have

$$H^i(S^3, S^3 - B) \simeq H_{3-i}(B)$$

Then, we obtain

$$H^i(S^3, S^3 - B) = \begin{cases} \mathbb{Z}^3 & \text{for } i = 3 \\ \mathbb{Z}^3 & \text{for } i = 2 \\ 0 & \text{for } i = 1 \\ 0 & \text{for } i = 0 \end{cases}$$

Consider a long exact sequence of a pair $(S^3, S^3 - B)$:

$$\begin{aligned} \cdots \longrightarrow H^1(S^3, S^3 - B) &\longrightarrow H^1(S^3; \mathbb{Z}) \longrightarrow H^1(S^3 - B; \mathbb{Z}) \\ \longrightarrow H^2(S^3, S^3 - B) &\longrightarrow H^2(S^3; \mathbb{Z}) \longrightarrow H^2(S^3 - B; \mathbb{Z}) \\ \longrightarrow H^3(S^3, S^3 - B) &\longrightarrow H^3(S^3; \mathbb{Z}) \longrightarrow H^3(S^3 - B; \mathbb{Z}) \longrightarrow \cdots \end{aligned}$$

Inverstigating boundary map, we derive

$$H^i(S^3 - B; \mathbb{Z}) = \begin{cases} 0 & \text{for } i = 3 \\ \mathbb{Z}^2 & \text{for } i = 2 \\ \mathbb{Z}^3 & \text{for } i = 1 \\ \mathbb{Z} & \text{for } i = 0 \end{cases}$$

In the same way, we can calculate $H^i(S^3 - L; \mathbb{Z})$.

Next time, we will look at ring structure and A_∞-algebra structure of $(S^3 - B)$ and $(S^3 - L)$

We saw that codicology groups of $S^3 - B$ and $S^3 - L$ are isomorphic, where B is the Borromean rings and L is the space of unlinked three rings. Now we consider the ring structure of their codicology rings. In this case, it is easier to look at intersection pairing of some homology groups which are isomorphic to $H^1(S^3 - B)$ and $H^1(S^3 - L)$. (Here we assume that the coefficient ring is $\mathbb{Z}$.)

First, we consider the codicology ring of $S^3 - B$. We denote each circle in B by B_1, B_2, and B_3. Then take regular neighborhoods U_1, U_2, and U_3 containing B_1, B_2, and B_3, respectively. Let $U = U_1 \cup U_2 \cup U_3$, and $M = S^3 - U$. Then M is a manifold with boundary. Note that

$$H^r(M) \cong H_{3-r}(M, \partial M)$$

by Alexander duality, and the cup product paring

$$H^r(M) \otimes H^s(M) \xrightarrow{\cup} H^{r+s}(M)$$

is equivalent to the intersection pairing

$$H_{3-r}(M, \partial M) \otimes H_{3-s}(M, \partial M) \xrightarrow{\cap} H_{3-r-s}(M, \partial M)$$

That is, the following diagram commutes.

$$\begin{array}{ccc} H^r(M) \otimes H^s(M) & \xrightarrow{\cup} & H^{r+s}(M) \\ \downarrow\cong & & \downarrow\cong \\ H_{3-r}(M, \partial M) \otimes H_{3-s}(M, \partial M) & \xrightarrow{\cap} & H_{3-r-s}(M, \partial M) \end{array}$$

But note that

$$H^i(M) \cong H^i(S^3 - B)$$

by deformation retraction, and

$$H_i(M, \partial M) \cong H_i(S^3, \bar{U}) \cong H_i(S^3, B)$$

by excision and deformation retraction.
Therefore, the cup product of $H^1(S^3 - B)$ is equivalent to the intersection pairing in $H_2(S^3, B)$. Note that generators of $H_2(S^3, B)$ are D_1, D_2, and D_3, where D_i is a disc whose boundary is B_i with proper orientation.($i = 1, 2, 3$.) However, any two circles in B are unlinked, and so we can take any two discs to be disjoint. Hence,

$$[D_i] \cap [D_j] = 0 \text{ in } H_1(S^3, B) \text{ for all } i, j$$

Hence, the corresponding cup product in $H^1(S^3 - B)$ is trivial. Moreover $H^3(S^3 - B) \cong 0$, and so

$$H^1(S^3 - B) \otimes H^2(S^3 - B) \xrightarrow{\cup} H^3(S^3 - B).$$

is a trivial product. This shows that the ring structure of $H^*(S^3 - B)$ is trivial.

Also, we can show that the ring structure of $H^*(S^3 - L)$ is trivial by the same way.(All argument will be exactly same except replacing B by L.) So we conclude that the ring structure of codicology cannot distinguish B and L.

Next, we consider chain level cup product as follows. We regard the circles B_i in B as chains and realize these circles in $\mathbb{R}^3$. Here, we consider S^3 as one point cementification of $\mathbb{R}^3$. Then B_i's can be defined by the following equations.

$$B_1 = \{(x, y, z) |\ x = 0,\ y^2 + \frac{z^2}{4} = 1\}$$
$$B_2 = \{(x, y, z) |\ y = 0,\ z^2 + \frac{x^2}{4} = 1\}$$
$$B_3 = \{(x, y, z) |\ z = 0,\ x^2 + \frac{y^2}{4} = 1\}$$

Then D_i's can be defined by the following equations.

$$D_1 = \{(x, y, z) |\ x = 0,\ y^2 + \frac{z^2}{4} \le 1\}$$
$$D_2 = \{(x, y, z) |\ y = 0,\ z^2 + \frac{x^2}{4} \le 1\}$$
$$D_3 = \{(x, y, z) |\ z = 0,\ x^2 + \frac{y^2}{4} \le 1\}$$

Then we can express the intersection pairs of D_i's.

$$D_1 \cap D_2 = \{(x, y, z) |\ x = y = 0,\ |z| \le 1\}$$
$$D_2 \cap D_3 = \{(x, y, z) |\ y = z = 0,\ |x| \le 1\}$$
$$D_3 \cap D_1 = \{(x, y, z) |\ x = z = 0,\ |y| \le 1\}$$

Then $D_1 \cap D_2 = \partial \Sigma_1$, and $D_2 \cap D_3 = \partial \Sigma_2$, where

$$\Sigma_1 = \{(x, y, z) |\ z^2 + \frac{x^2}{4} \le 1,\ x \ge 0,\ y = 0\} \subset D_2$$
$$\Sigma_2 = \{(x, y, z) |\ x^2 + \frac{y^2}{4} \le 1,\ y \ge 0,\ z = 0\} \subset D_3$$

Here, the equations $D_1 \cap D_2 = \partial \Sigma_1$, and $D_2 \cap D_3 = \partial \Sigma_2$ hold in relative sense. That is, the equations make sense in $C_2(S^3, B) = C_2(S^3)/C_2(B)$. Here, we recall

Definition 7.2. Suppose that α, β and γ are homogeneous elements in some codicology ring C^*, and satisfy

$$\alpha \cup \beta = df \text{ and } \beta \cup \gamma = dg$$

Then **Massey product** is defined by choosing cocycles α, β, γ

$$< \alpha, \beta, \gamma >= f \cup \gamma - (-1)^{|\alpha|} \alpha \cup g,$$

where $|\alpha|$ is the degree of α

Lemma 7.3. If $\delta\alpha = 0 = \delta\gamma = 0$, then $\langle \alpha, \beta, \gamma \rangle = 0$.

By equalizing we compute the triple Masse product in homology $H_2(S^3, B)$

$$< B_1, B_2, B_3 >= D_1 \cap \Sigma_2 \pm \Sigma_1 \cap D_3$$

Note that

$$D_1 \cap \Sigma_2 = \{(x, y, z)|\ x = z = 0,\ 0 \leq y \leq 1\}$$
$$\Sigma_2 \cap D_3 = \{(x, y, z)|\ y = z = 0,\ 0 \leq x \leq 1\}$$

Hence. the triple Masse product is a path connecting two distinct components of B, and the endpoints of this path lie in B_1 and B_3. Also, $< B_1, B_2, B_3 >$ represents a nonzero element in $H_1(S^3, B)$ since if we take the boundary homomorphism

$$\partial : H_1(S^3, B) \longrightarrow H_0(B),$$

then

$$\partial(< B_1, B_2, B_3 >) = [(1, 0, 0)] \pm [(0, 1, 0)] \neq 0 \text{ in } H_0(B).$$

This triple product represents the dual of a certain triple product in $H^*(S^3 - B)$, and the fact that it is nonzero implies that the three circles in B cannot be pulled apart unlike the space of three circles unlinked L. Therefore, the Masse product distinguishes B and L.

8. Coalgebra and coderivations

Suppose that we are given an A_∞-algebra A together with a sequence of R-module homomorphisms $\{m_k\}_{k=1}^{\infty}$ of (shifted) degree 1 where

$$m_k : A[1]^{\otimes k} \longrightarrow A[1]$$

(Here, m_0 is assumed to be zero). A bar complex BA of A is defined by

$$BA := \bigoplus_{k=0}^{\infty} B_k A$$

where

$$B_0 A := R \qquad (A \text{ is a graded R-module.})$$
$$B_k A := A[1]^{\otimes k} = \bigoplus_{m_1,\cdots,m_k} A[1]^{m_1} \otimes \cdots \otimes A[1]^{m_k}.$$

Note that $B_k A$ carries a natural degree inherited from $A[1]$. Namely, the (shifted) degree of homogenous element in $A[1]^{m_1} \otimes \cdots \otimes A[1]^{m_k}$ is $m_1 + \cdots + m_k$.

Definition 8.1. A **Hochschild cochain module** $CH(A,A)$ of A is the set of all sequences $\{\varphi_k\}_{k=0}^{\infty}$ of graded R-module homomorphisms with homomorphism

$$\varphi_k : B_k A \longrightarrow A[1].$$

That is,

$$CH(A,A) := \prod_{k=0}^{\infty} \mathrm{Hom}(B_k A, A[1])$$

Denote by $CH^a(A,A)$ the set of degree a elements. Then we have

$$CH(A,A) = \prod_{a\in\mathbb{Z}} CH^a(A,A).$$

Let A be a (graded) R-module with a **coproduct** Δ, i.e., $\Delta : A \longrightarrow A \otimes A$ is a R-module homomorphism of degree 0. A coproduct $\Delta : A \longrightarrow A \otimes A$ is called **coassociative** if it makes the following diagram commute: which is exactly the dual notion of associativity. Then, an R-module A along with coassociative coproduct Δ is called a **(graded) coalgebra**.

Lemma 8.2. A bar complex BA forms a graded coalgebra with respect to $\Delta : BA \longrightarrow BA \bigotimes BA$ which is obtained by extending the following formula linearly:

$$\begin{aligned}\Delta(x_1 \otimes \cdots \otimes x_n) &:= \sum_{i=0}^{n} (x_1 \otimes \cdots \otimes x_i) \bigotimes (x_{i+1} \otimes \cdots \otimes x_n) \\ &= 1 \bigotimes (x_1 \otimes \cdots \otimes x_n) + x_1 \bigotimes (x_2 \otimes \cdots \otimes x_n) + \cdots + (x_1 \otimes \cdots \otimes x_n) \bigotimes 1.\end{aligned}$$

Here, two tensor products are involved: a small tensor product $\otimes$ denotes a tensor product in A_∞ algebra A and a big tensor product $\bigotimes$ denotes a tensor product in a bar complex BA.

Suppose that we have two graded R-module homomorphisms $F, G : BA \longrightarrow BA$. We define a **graded tensor product** of F and G as follows:

$$(F \widehat{\bigotimes} G)(\mathbf{x} \bigotimes \mathbf{y}) := (-1)^{(\deg G)\cdot|\mathbf{x}|'} (F(\mathbf{x}) \bigotimes G(\mathbf{y})) \tag{8.1}$$

and **graded Lie bracket**

$$[F, G] := F \widehat{\bigotimes} G - (-1)^{|F||G|} G \widehat{\bigotimes} F. \tag{8.2}$$

Definition 8.3. A **coderivation** $D : BA \longrightarrow BA$ is a graded R-module homomorphism satisfying

(i) $(D\widehat{\otimes}\mathrm{id} + \mathrm{id}\widehat{\otimes}D) \circ \Delta = \Delta \circ D$. In other words, the following diagram is commutative:

(ii) The B_0A-component of $D(\mathbf{x})$ is zero for any $\mathbf{x} \in BA$. That is,

$$D(BA) \subset \bigoplus_{k=1}^{\infty} B_k A.$$

The set of coderivations from BA to BA is denoted by $\mathrm{CoDer}(BA, BA)$.

Similarly as $CH(A, A)$, we have

$$\mathrm{CoDer}(BA, BA) = \prod_{a \in \mathbb{Z}} \mathrm{CoDer}_a(BA, BA).$$

Now, we are going to show

Theorem 8.4. The Hochschild cochain module $CH(A, A)$ of A is isomorphic to the set $\mathrm{CoDer}(BA, BA)$ of coderivations of the bar complex BA.

Proof. In order to construct an isomorphism between them, we need to define some notations. We denote the projection from BA to B_kA by

$$\pi_k : BA \longrightarrow B_k A$$

and more generally the projection from BA to B_IA by

$$\pi_I : BA \longrightarrow B_I A := \bigoplus_{k \in I} B_k A$$

where I is a subset of $\{0\} \cup \mathbb{N}$. Also, we have an obvious inclusion ι_k from B_kA to BA where

$$\iota_k : B_k A \longrightarrow BA.$$

First, we construct $\Phi : CH(A, A) \longrightarrow \mathrm{CoDer}(BA, BA)$. For any $\varphi = (\varphi_0, \varphi_1, \cdots) \in CH(A, A)$, the associated coderivation $\widehat{\varphi}$ is defined as follows:

$$\widehat{\varphi} := \sum_{k=0}^{\infty} \widehat{\varphi}_k,$$

each of which is obtained by extension of the following formula:

$$\widehat{\varphi}_k(x_1 \otimes \cdots \otimes x_n) := \sum_{l=1}^{n-k+1} (-1)^{|x_1|' + \cdots + |x_{l-1}|'} x_1 \otimes \cdots \otimes x_{l-1} \otimes \varphi_k(x_l, \cdots, x_{l+k-1}) \otimes x_{l+k} \otimes \cdots \otimes x_n.$$

Using the Sweedler's notation, we can write

$$\widehat{\varphi}(\mathbf{x}) = \sum (-1)^{|\mathbf{x}_c^{(3:1)}|'} (\mathbf{x}_c^{(3:1)} \otimes \varphi(\mathbf{x}^{(3:2)}) \otimes \mathbf{x}_c^{(3:3)})$$

when

$$\Delta^2(\mathbf{x}) = \sum_c \mathbf{x}_c^{(3:1)} \otimes \mathbf{x}_c^{(3:2)} \otimes \mathbf{x}_c^{(3:3)}.$$

The first component $\varphi_0 : R \longrightarrow A[1]$ is called a **coaugmentation**, which is completely determined by $\varphi_0(1) \in A[1]$. Due to the definition of the associated coderivation, $\widehat{\varphi}_0$ is given by

$$\widehat{\varphi}_0(x_1 \otimes \cdots \otimes x_n) = \sum_{l=1}^{n+1} (-1)^{|x_1|' + \cdots + |x_{l-1}|'} x_1 \otimes \cdots x_l \otimes \varphi_0(1) \otimes x_l \otimes \cdots \otimes x_n.$$

It can be easily observed that $\widehat{\varphi}$ is a coderivation from BA to BA. In particular, $\widehat{\varphi}$ has no $B_0A = R$ component in its image. We then set $\Phi : CH(A,A) \longrightarrow \mathrm{CoDer}(BA,BA)$ by

$$\Phi(\varphi) := \widehat{\varphi}.$$

Conversely, we build up the inverse map Ψ of Φ as follows: For $D \in \mathrm{CoDer}(BA,BA)$, we define

$$\Psi : \mathrm{CoDer}(BA,BA) \to CH(A,A)$$

by

$$\Psi(D) = (\varphi_0, \cdots, \varphi_k, \cdots)$$

where $\varphi_k = \pi_1 \circ D \circ \iota_k$. From construction of Φ and Ψ, it immediately follows that $\Psi \circ \Phi = \mathrm{id}_{CH(A,A)}$ yielding that Ψ is surjective.

Finally, we will show that Ψ is injective and so Ψ is an isomorphism. We denote the projection from BA to B_IA by

$$\pi_I : BA \longrightarrow B_IA$$

where I is a subset of $\{0\} \cup \mathbb{N}$.

Denote $\underline{n} = \{1, \cdots, n\}$ and

$$B_{\underline{n}}A = \bigoplus_{k=1}^{n} B_kA.$$

We will use the following identity.

Lemma 8.5. Let $n \geq 1$. Then

$$\Delta \circ \iota_{\underline{n}} \pi_{\underline{n}} = \sum_{1 \leq \ell+k \leq n} (\pi_l \bigotimes \pi_k) \circ \Delta$$

Proof. Consider $\mathbf{x} \in BA$ and write $\pi_{\underline{n}}(\mathbf{x})$.

We first consider the case $\mathbf{x} = x_1 \otimes \cdots \otimes x_n \in B_nA$. $\iota_{\underline{n}}\pi_{\underline{n}}(\mathbf{x}) = \mathbf{x}$.

By definition of Δ, we have

$$\Delta(x_1 \otimes \cdots \otimes x_n) = \sum_{\ell=0}^{n} (x_1 \otimes \cdots \otimes x_\ell) \bigotimes (x_{\ell+1} \otimes \cdots \otimes x_n).$$

In Sheedler's notation, $\Delta(\mathbf{x}) = \sum_c \mathbf{x}_c^{(2;1)} \otimes \mathbf{x}_c^{(2;2)}$. Therefore

$$\begin{aligned} \Delta \circ \iota_n(\mathbf{x}) &= \Delta(\mathbf{x}) = \sum_{\ell=0}^{n} (x_1 \otimes \cdots \otimes x_\ell) \bigotimes (x_{\ell+1} \otimes \cdots \otimes x_n) \\ &= \sum_{\ell=0}^{n} \pi_\ell \otimes \pi_{n-\ell}(\Delta(\mathbf{x})) \end{aligned}$$

where the last equality follows because

$$\pi_\ell \otimes \pi_{n-\ell}(\mathbf{x}_c^{(2;1)} \otimes \mathbf{x}_c^{(2;2)}) = \pi_\ell(\mathbf{x}_c^{(2;1)}) \otimes \pi_{n-\ell}(\mathbf{x}_c^{(2;2)}) = 0$$

unless $|\mathbf{x}_c^{(2;1)}| = \ell$, $|\mathbf{x}_c^{(2;2)}| = n - \ell$. Similar calculation gives rise to the identity for general elements $\mathbf{x} \in B_{\underline{n}}A$.

If $\mathbf{x} \in B_{\mathbb{N} \setminus \underline{n}}$, then the left hand side is obviously vanishes. On the other other hand in the coproduct expansion $\Delta(\mathbf{x}) = \sum_c \mathbf{x}_c^{(2;1)} \otimes \mathbf{x}_c^{(2;2)}$, the length of each summand cannot match $\pi_l \bigotimes \pi_k$ to produce nontrivial outcome and so the right hand side also vanishes.

This finishes the proof. ∎

It remains to check Ψ is injective. Consider the coordinate expression

$$\Psi(D) = (\psi_0, \cdots, \psi_\ell, \cdots)$$

i.e., $\varphi_k := \pi_1 \circ D \circ \iota_k$.

Suppose that $\Psi(D) = 0$. To prove $D = 0$, it suffices to prove that for all $n \geq 1$

$$\pi_{\underline{n}} \circ D = 0.$$

(We note $\pi_{\underline{n}} \circ D = \pi_{0,\cdots,n} \circ D$ because the image of D lies in $B_{\mathbb{N}}A$ (We recall $\pi_0 D = 0$).

We will use induction on n. When $n = 1$, it follows from $\Psi(D) = 0$ since $\pi_1 \circ D|_{B_k A} = \psi_k$.

Now suppose $\pi_{\underline{k}} \circ D = 0$ for $1 \leq k \leq n$. We want to prove $\pi_{\underline{n+1}} \circ D = 0$.

Clearly $\Delta : BA \to BA \otimes BA$ is injective. Since both Δ and $\iota_{\underline{n+1}}$ are injective, it is enough to check

$$(\Delta \circ \iota_{\underline{n+1}}) \circ \pi_{\underline{n+1}} D = 0$$

instead of directly showing $\pi_{\underline{n+1}} D = 0$. But we obtain

$$\Delta \circ \iota_{\underline{n+1}} \circ \pi_{\underline{n+1}} D = \sum_{1 \leq \ell + k \leq n+1} (\pi_\ell \bigotimes \pi_k) \circ \Delta \circ D$$

from Lemma 8.5. Substituting $\Delta \circ D = (D \widehat{\otimes} id + id \otimes D) \circ \Delta$ hereinto, we derive

$$\begin{aligned} (\pi_\ell \bigotimes \pi_k) \circ \Delta \circ D &= (\pi_\ell \bigotimes \pi_k) \circ (D \widehat{\bigotimes} id + id \widehat{\bigotimes} D) \circ \Delta \\ &= ((\pi_\ell \circ D) \widehat{\bigotimes} \pi_k + \pi_\ell \widehat{\bigotimes} (\pi_k \circ D)) \circ \Delta. \end{aligned}$$

Then by the induction hypothesis $\pi_{\underline{n}} \circ D = 0$,

$$(\pi_\ell \circ D) \widehat{\bigotimes} \pi_k + \pi_\ell \widehat{\bigotimes} (\pi_k \circ D) = 0$$

for all $1 \leq k + \ell \leq n + 1$, except for $(\ell, k) = (0, n+1)$ or $(n+1, 0)$. Furthermore $\pi_0 \circ D = 0$ by Definition 8.3. Therefore we obtain

$$\sum_{1 \leq \ell + k \leq n+1} (\pi_\ell \bigotimes \pi_k) \circ \Delta \circ D = ((\pi_0 \bigotimes \pi_{n+1} + \pi_{n+1} \bigotimes \pi_0)((D \widehat{\bigotimes} id + id \widehat{\bigotimes} D) \circ \Delta.$$

This leads us to

$$\Delta \circ \iota_{\underline{n+1}} \circ \pi_{\underline{n+1}} D = (\pi_0 \bigotimes \pi_{n+1} + \pi_{n+1} \bigotimes \pi_0)(D \widehat{\bigotimes} id + id \widehat{\bigotimes} D) \circ \Delta.$$

We now evaluate this against $\mathbf{x} = x_1 \otimes \cdots, x_k$. We write

$$\Delta(\mathbf{x}) = 1 \otimes \mathbf{x} + \sum_{\ell=1}^{k-1} (x_1 \otimes \cdots \otimes x_\ell) \bigotimes (x_{\ell+1} \otimes \cdots \otimes x_k) + \mathbf{x} \bigotimes 1.$$

Therefore we compute

$$\begin{aligned} & (\pi_0 \bigotimes \pi_{n+1} + \pi_{n+1} \bigotimes \pi_0)(D \widehat{\bigotimes} id + id \widehat{\bigotimes} D) \circ \Delta(\mathbf{x}) \\ = \ & (\pi_0 \bigotimes \pi_{n+1} + \pi_{n+1} \bigotimes \pi_0)(D \widehat{\bigotimes} id(\mathbf{x} \bigotimes 1)) \\ & \quad + (\pi_0 \bigotimes \pi_{n+1} + \pi_{n+1} \bigotimes \pi_0)(id \widehat{\bigotimes} D(1 \bigotimes \mathbf{x})) \\ = \ & (\pi_0 \bigotimes \pi_{n+1} + \pi_{n+1} \bigotimes \pi_0)((D(\mathbf{x}) \bigotimes 1) + (\pi_0 \bigotimes \pi_{n+1} + \pi_{n+1} \bigotimes \pi_0)(1 \bigotimes D(\mathbf{x})) \\ = \ & (\pi_{n+1} D)(\mathbf{x}) \bigotimes 1 + 1 \bigotimes (\pi_{n+1} D)(\mathbf{x}). \end{aligned}$$

Here we use the degree consideration $\deg id = 0$ and hence the graded tensor product does not pick up any sign in the equality next to the last one. The last term vanishes unless the length of $\mathbf{x}$ is $n+1$. For those $\mathbf{x}$ with length $n+1$, it becomes $\psi_{n+1}(\mathbf{x}) \otimes 1 + 1 \otimes \psi_{n+1}(\mathbf{x})$. But this vanishes by the hypothesis $\Psi(D) = 0$, which in particular implies $\psi_{n+1}(\mathbf{x}) = 0$.

Combining the above discussion, we have proved $(\Delta \circ \iota_{\underline{n+1}}) \circ \pi_{\underline{n+1}} D = 0$ provided $\Psi(D) = 0$. Therefore Ψ is injective and we finally establish Theorem 8.4. □

Applying Theorem 8.4 to $\phi = m = (m_0, m_1, \cdots)$, we have

Proposition 8.6. (A, m) is an A_∞-algebra if and only if $\widehat{m}^2 = 0$, where $\widehat{m} : BA \longrightarrow BA$.

Proof. We compute $(\widehat{m} \circ \widehat{m})$. Recall that $\widehat{m} = \sum_{k=1}^{\infty} \widehat{m}_k$, where

$$\begin{aligned}
&\widehat{m}_k(x_1 \otimes \cdots \otimes x_n) \\
&= \sum_{l=1}^{n-k+1} (-1)^{|x_1|' + \cdots + |x_{l-1}|'} x_1 \otimes \cdots \otimes x_{l-1} \otimes m_k(x_l, \cdots, x_{l+k-1}) \otimes x_{l+k} \otimes \cdots \otimes x_n
\end{aligned}$$

Note that $\widehat{m}_k : B_n A \longrightarrow B_{n-k+1} A$. By definition,

$$\begin{aligned}
(\widehat{m} \circ \widehat{m})(x_1, \cdots, x_n) &= (\widehat{m} \circ \widehat{m}(x_1 \otimes \cdots \otimes x_n)) \\
&= \widehat{m}(\widehat{m}_1(x_1, \cdots, x_n) + \cdots + \widehat{m}_n(x_1, \cdots, x_n))
\end{aligned}$$

Hence, the $B_1 A = A[1]$ component of $(\widehat{m} \circ \widehat{m})(x_1, \cdots, x_n)$ is

$$\sum_{i+j=n+1} \sum_{l=1}^{n-j+1} (-1)^{|x_1|' + \cdots + |x_{l-1}|'} m_i(x_1 \otimes \cdots \otimes x_{l-1} \otimes m_j(x_l, \cdots, x_{l+k-1}) \otimes x_{l+k} \otimes \cdots \otimes x_n)$$

Now, we consider a graded commutator $\widehat{m} \circ \widehat{m} - (-1)^{\deg \widehat{m}} \widehat{m} \circ \widehat{m} = 2\widehat{m} \circ \widehat{m}$.(Note that $\deg \widehat{m}$=1.) We know that a graded commutator of coderivations is again a coderivation.(The proof is same as the proof of Proposition 12.1.) By the proof of Theorem 8.4, $\widehat{m} \circ \widehat{m} = 0$ if and only if $B_1 A = A[1]$ component of $(\widehat{m} \circ \widehat{m})$ is 0. Therefore, $\widehat{m} \circ \widehat{m} = 0$ if and only if A_∞ relation holds. □

It turns out that $CH(A, A)$ carries additional algebraic structures which will be important in applications.

Proposition 8.7. For any $F, G \in \mathrm{CoDer}(BA, BA)$, $[F, G] \in \mathrm{CoDer}(BA, BA)$. The graded Lie bracket induces a (graded) Lie algebra structure on $\mathrm{CoDer}(BA, BA) \cong CH(A, A)$.

Proposition 8.8. Consider an A_∞-algebra $(A, \mathfrak{m} = \{\mathfrak{m}_k\}_{k=0}^{\infty}$. Set $d_A := \widehat{m} : BA \to BA$.

(1) For any $D \in \mathrm{CoDer}(BA, BA)$, $d^A \circ D - (-1)^{|D|} D \circ d^A$ is a coderivation of degree $|D| + 1$.
(2) We denote by $\delta : \mathrm{CoDer}(BA, BA) \longrightarrow \mathrm{CoDer}(BA, BA)$ this assignment. That is,

$$\delta(D) = d^A \circ D - (-1)^{|D|} D \circ d^A$$

Then $\delta \circ \delta = 0$.

Proof. We will give the proof of a more general version of this theorem in Section 10 and so omit the proof here. □

Therefore the triple $(CH(A, A), \delta, [\cdot, \cdot])$ defines a differential graded Lie algebra (DGLA).

9. A_∞ HOMOMORPHISMS

Definition 9.1. (**A_∞ homomorphism**) Let (A, m^A) and (C, m^C) be A_∞-algebras, and $f_k : B_k A \longrightarrow C[1]$ of degree 0 satisfy

$$\sum_{i_1+\cdots+i_n=k} m_n^C(f_{i_1}(a_1, \cdots, a_{i_1}), \cdots, f_{i_n}(a_{k-i_n+1}, \cdots, a_k))$$
$$= \sum_{p+\ell=k+1} \sum_{p=1}^{l} (-1)^{|a_1|'+\cdots+|a_{p-1}|'} f_\ell(a_1, \cdots, a_{p-1}, m_\ell^A(a_p, \cdots, a_{p+\ell-1}), a_{p+\ell}, \cdots, a_k)$$

Then we call a collection $f = \{f_k\}_{k=1}^\infty$ an A_∞ homomorphism. Let (A, m^A) and (C, m^C) be unital. We say f is unital if $f_1(e_A) = e_B$ and $f_k(\cdots, \mathbf{e}, \cdots) = 0$ for $k \geq 2$.

Definition 9.2. Let $CH(A, C)$ be the set of homomorphisms from $B_k A$ to $C[1]$ of degree $0.(k = 1, 2, \cdots)$ Each $f = \{f_k\}_{k=1}^\infty \in CH(A, C)$ can be extended to a unique subalgebra map $BA \longrightarrow BC$ by

$$\hat{f}(a_1 \otimes \cdots \otimes a_k) \qquad = \sum_{i_1+\cdots+i_n=k} f_{i_1}(a_1, \cdots, a_{i_1}) \otimes \cdots \otimes f_{i_n}(a_{k-i_n+1} \otimes \cdots \otimes a_k)$$

Lemma 9.3. $f = \{f_k\}_{k=1}^\infty$ is an A_∞ homomorphism if and only if $\hat{f} : BA \longrightarrow BC$ is a chain map, i.e, $\hat{f} \circ m^A = m^C \circ \hat{f}$.

Definition 9.4. (**Composition**) Let $f \in CH(A, B)$ and $g \in CH(B, C)$. Then we define $g \circ f$ by

$$(g \circ f)_k(a_1, \cdots a_k)$$
$$= \sum_m \sum_{k_1+\cdots+k_m=k} g_m(f_{k_1}(a_1, \cdots, a_{k_1}), \cdots, f_{k_m}(a_{k-k_m+1}, \cdots, a_k))$$

Proposition 9.5.

(1) $g \circ f$ is an A_∞ homomorphism if f and g are.
(2) Composition is associative.
(3) We define $id_k : B_k A \longrightarrow A[1]$ by id_1 is the identity on $A[1]$ and $id_k = 0$ for $k \geq 2$. Then $\widehat{id} : BA \longrightarrow BA$ is the identity map.
(4) An A_∞ homomorphism $f : A \longrightarrow C$ induces a graded algebra homomorphism
$$f_* : H(A, m_1^A) \longrightarrow H(C, m_1^C),$$
where (A, m^A) and (C, m^C) are strict A_∞-algebras

Proof. (1) We need to show that $\widehat{g \circ f}$ is a chain map. We can check $\widehat{g \circ f} = \hat{g} \circ \hat{f}$.
Then $m_C \circ (\widehat{g \circ f}) = m_C \circ \hat{g} \circ \hat{f} = \hat{g} \circ m_B \circ \hat{f} = \hat{g} \circ \hat{f} \circ m_A = (\widehat{g \circ f}) \circ m_A$ as desired.
The proofs of (2), (3), and (4) are left as exercise. □

Now, recall the definition of a differential graded algebra. Also, recall that we can regard a differential graded algebra as an A_∞-algebra. This point of view has some advantages.

Definition 9.6. Let (A, d^A) and (B, d^B) be differential graded algebras, and $f : A \longrightarrow B$ be a chain map respecting the product. Then f is called a **quasi-isomorphism** if $f_* : H(A, d^A) \longrightarrow H(B, d^B)$ is an isomorphism

Definition 9.7. A differential graded algebra (A, d^A) is called **quasi-isomorphic** to a differential graded algebra (B, d^B) if there exist a differential graded algebra (C, d^C) and $f : C \longrightarrow A$ and $g : C \longrightarrow B$ such that f and g are quasi-isomorphisms.

Definition 9.8. Let (A, m^A) and (B, m^B) be strict A_∞-algebras. We call an A_∞ homomorphism $\hat{f} : (A, m^A) \longrightarrow (B, m^B)$ a **quasi-isomorphism** if $(f_1)_* : H(A, m_1^A) \longleftrightarrow H(B, m_1^B)$ is an isomorphism. $\hat{f}$ is called an **isomorphism** if f_1 is an isomorphism and there exists an A_∞ homomorphism $\hat{g} : (B, m^B) \longrightarrow (A, m^A)$ such that $f_1 \circ g_1$ and $g_1 \circ f_1$ are identity maps on B and A, respectively.

Theorem 9.9. Any A_∞ quasi-isomorphism is an isomorphism.

This theorem implies that if we regard a differential graded algebra as an A_∞ algebra, a quasi-isomorphism between differential graded algebras is an isomorphism between A_∞ algebras.

Now, recall the definition of a differential graded algebra. Also, recall that we can regard a differential graded algebra as an A_∞-algebra. This point of view has some advantages.

Definition 9.10. Let (A, d^A) and (B, d^B) be differential graded algebras, and $f : A \longrightarrow B$ be a chain map respecting the product. Then f is called a **quasi-isomorphism** if $f_* : H(A, d^A) \longrightarrow H(B, d^B)$ is an isomorphism

Definition 9.11. A differential graded algebra (A, d^A) is called **quasi-isomorphic** to a differential graded algebra (B, d^B) if there exist a differential graded algebra (C, d^C) and $f : C \longrightarrow A$ and $g : C \longrightarrow B$ such that f and g are quasi-isomorphisms.

Definition 9.12. Let (A, m^A) and (B, m^B) be strict A_∞-algebras. We call an A_∞ homomorphism $\hat{f} : (A, m^A) \longrightarrow (B, m^B)$ a **quasi-isomorphism** if $(f_1)_* : H(A, m_1^A) \longleftrightarrow H(B, m_1^B)$ is an isomorphism. $\hat{f}$ is called an **isomorphism** if f_1 is an isomorphism and there exists an A_∞ homomorphism $\hat{g} : (B, m^B) \longrightarrow (A, m^A)$ such that $f_1 \circ g_1$ and $g_1 \circ f_1$ are identity maps on B and A, respectively.

Theorem 9.13. Any A_∞ quasi-isomorphism is an isomorphism.

This theorem implies that if we regard a differential graded algebra as an A_∞ algebra, a quasi-isomorphism between differential graded algebras is an isomorphism between A_∞ algebras.

10. Hochschild cohomology of A_∞-homomorphisms

In this section, we consider the notion of *(graded) coderivation over $\hat{f}$* for a given A_∞-homomorphism from $(A, \mathfrak{m}^A)$ to $(C, \mathfrak{m}^C)$.

Definition 10.1. $D : BA \longrightarrow BC$ is called a graded coderivation over $\hat{f}$ with values in $BC = \bigoplus_{n=1}^{\infty} B_n C$ if

$$(\hat{f}\widehat{\otimes}D + D\widehat{\otimes}\hat{f}) \circ \Delta = \Delta \circ D$$

We denote by $\mathrm{CoDer}(BA, BC; \hat{f})$ the set of graded coderivation over $\hat{f}$ with values in BC.

Note that the A_∞ operation $m : BA \longrightarrow BA$ is a graded codeclination over $\widehat{id}$ with degree 1. Also, we do not put any degree restriction on $\mathrm{CoDer}(BA, BC; \hat{f})$. That is,

$$\mathrm{CoDer}(BA, BC; \hat{f}) = \bigoplus_{k=-\infty}^{\infty} \mathrm{CoDer}_k(BA, BC; \hat{f}),$$

where $\mathrm{CoDer}_k(BA, BC; \hat{f})$ is the set of conservations of degree k over $\hat{f}$.

For given $\phi = (\phi_1, \phi_2, \cdots) \in \prod_{k=1}^{\infty} \mathrm{Hom}(B_k A, C[1])$, we define

$$\begin{aligned} &\widehat{\phi}_i(x_1, \cdots, x_k) \\ = &\sum_{l=1}^{k-i+1} (-1)^{|x_1|'+\cdots+|x_{l-1}|'} \hat{f}(x_1, \cdots, x_{l-1}) \otimes \phi_i(x_l, \cdots, x_{l+i-1}) \otimes \hat{f}(x_{l+i}, \cdots, x_k) \end{aligned} \tag{10.1}$$

for $i \le k$ and 0 for $i > k$. We then set $\widehat{\phi} = \sum_{i=1}^{\infty} \widehat{\phi}_i$.

Lemma 10.2. Consider $\prod_{k=1}^{\infty} \mathrm{Hom}(B_k A, C[1])$, where $\mathrm{Hom}(B_k A, C[1])$ is the set of graded homomorphisms from $B_k A$ to $C[1]$.

(1) Then the definition (10.1) defines a coderivation over $\hat{f}$ if f is an A_∞-homomorphism.
(2) $\hat{f}$ induces a natural homomorphism

$$\prod_{k=1}^{\infty} \mathrm{Hom}(B_k A, C[1]) \to \mathrm{CoDer}(BA, BC; \hat{f})$$

given by (10.1).

Proof. A straightforward calculation proves statement (1). Then the correspondence $\phi \mapsto \widehat{\phi}$ gives the desired isomorphism between $\prod_{k=1}^{\infty} \mathrm{Hom}(B_k A, C[1])$ and $\mathrm{CoDer}(BA, BC, \hat{f})$, and this homomorphism is grade preserving. The proof of isomorphism property is similar to that of Theorem 8.4, and we omit. □

Remark 10.3. We would like to point out that the space $\prod_{k=1}^{\infty} \mathrm{Hom}(B_k A, C[1])$ does not depend on $\widehat{f}$ while $\mathrm{CoDer}(BA, BC; \hat{f})$ does.

Let (A, d^A) and (C, d^C) be A_∞ algebras with $d^A = \widehat{m}^A$, $d^C = \widehat{m}^C$. We consider $\mathrm{CoDer}(BA, BC, \hat{f})$ for a fixed A_∞ homomorphism $\hat{f} : BA \longrightarrow BC$.

Proposition 10.4.

(1) For any $D \in \mathrm{CoDer}(BA, BC, \hat{f})$, $d^C \circ D - (-1)^{|D|} D \circ d^A$ is a coderivation of degree $|D| + 1$ over $\hat{f}$.

(2) We denote by $\delta^f : \mathrm{CoDer}(BA, BC, \hat{f}) \longrightarrow \mathrm{CoDer}(BA, BC, \hat{f})$ this assignment. That is,

$$\delta^f(D) = d^C \circ D - (-1)^{|D|} D \circ d^A$$

Then $\delta^f \circ \delta^f = 0$.

Proof.

(1) The degree of $d^C \circ D - (-1)^{|D|} D \circ d^A$ is $|D| + 1$ since d^A and d^C are of degree 1. Next we compute

$$\begin{aligned}
&\Delta \circ (d^C \circ D - (-1)^{|D|} D \circ d^A)\\
&= (\Delta \circ d^C) \circ D - (-1)^{|D|} (\Delta \circ D) \circ d^A\\
&= (id \widehat{\otimes} d^C + d^C \widehat{\otimes} id) \circ \Delta \circ D - (-1)^{|D|} (\hat{f} \widehat{\otimes} D + D \widehat{\otimes} \hat{f}) \circ \Delta \circ d^A\\
&= (id \widehat{\otimes} d^C + d^C \widehat{\otimes} id) \circ (\hat{f} \widehat{\otimes} D + D \widehat{\otimes} \hat{f}) \circ \Delta\\
&\quad - (-1)^{|D|} (\hat{f} \widehat{\otimes} D + D \widehat{\otimes} \hat{f}) \circ (id \widehat{\otimes} d^A + d^A \widehat{\otimes} id) \circ \Delta\\
&= (\hat{f} \widehat{\otimes} (d^C \circ D) + (-1)^{|D|} D \widehat{\otimes} (d^C \circ \hat{f}) + (d^C \circ \hat{f}) \widehat{\otimes} D + (d^C \circ D) \widehat{\otimes} \hat{f}) \circ \Delta\\
&\quad - (-1)^{|D|} (\hat{f} \widehat{\otimes} (D \circ d^A) + (-1)^{|D|} (\hat{f} \circ d^A) \widehat{\otimes} D + D \widehat{\otimes} (\hat{f} \circ d^A) + (D \circ d^A) \widehat{\otimes} \hat{f}) \circ \Delta\\
&= \hat{f} \widehat{\otimes} (d^C \circ D - (-1)^{|D|} D \circ d^A) + (d^C \circ D - (-1)^{|D|} D \circ d^A) \widehat{\otimes} \hat{f}
\end{aligned}$$

as desired. We note that

$$(A \widehat{\otimes} B) \circ (C \widehat{\otimes} D) = (-1)^{|B||C|} (A \circ C) \widehat{\otimes} (B \circ D)$$

(2) We compute

$$\begin{aligned}
&\delta^f(\delta^f(D))\\
&= \delta^f(d^C \circ D - (-1)^{|D|} D \circ d^A)\\
&= d^C \circ (d^C \circ D) - (-1)|D| + 1 d^C \circ D \circ d^A - (-1)^{|D|} d^C \circ D \circ d^A - (-1)^{|D|} D \circ d^A \circ d^A\\
&= 0
\end{aligned}$$

□

By the previous proposition, we can define

Definition 10.5. (**Hochschild cohomology of** f)
We define Hochschild cohomology of an A_∞-homomorphism f by

$$HH(A, C; \hat{f}) = \ker\delta^f / \mathrm{im}\delta^f$$

Also, we note that $D \in \ker\delta^f$ if and only if D is a graded chain map of degree $|D|$.

By considering $(C, \mathfrak{m}^C) = (A, \mathfrak{m}^A)$ and $f = id$, we give the definition of Hochschild cohomology of $(A, \mathfrak{m})$ as follows.

Definition 10.6. Hochschild cohomology of $(A, \mathfrak{m})$ We define the Hochschild cohomology of an A_∞-algebra $(A, \mathfrak{m})$ by

$$HH(A, A) := HH(A, A; \hat{id}).$$

11. A_∞ MODULES

Now we consider a module over an A_∞-algebra. Recall that the usual right A-module M has the structure map $\eta : M \times A \longrightarrow M$ satisfying

$$\eta(\eta(v,a),b) = \eta(v,ab)$$

for $v \in M$, and $a, b \in A$. Here is the definition of a module over an A_∞-algebra. Note that $\deg' = \deg - 1$

Definition 11.1.

(1) Let (A, m) be a strict A_∞-algebra and M be a graded R-module, where R is a commutative ring. Suppose that we have the structure maps $\eta_k : M \otimes A^{\otimes k} \longrightarrow M$ of degree $1-k$. We say $(M, \{\eta_k\}_{k=0}^{\infty})$ is a right A_∞-module over A if

$$\begin{aligned}&\sum_{i=0}^{k} \eta_{k-i}(\eta_i(v, a_1, \cdots, a_i), a_{i+1}, \cdots, a_k)\\ &+\sum_{j=1}^{k}\sum_{i=1}^{k-j+1} (-1)^{\star} \eta_{k-j+1}(v, a_1, \cdots, a_{i-1}, m_j(a_i, \cdots, a_{i+j-1}), a_{i+j}, \cdots, a_k) = 0,\end{aligned}$$

where $v \in M$, $a_i \in A$ and $\star = \deg' v + \deg' a_1 + \cdots + \deg' a_{i-1}$.

(2) Let (A_1, m^1) and (A_2, m^2) be A_∞-algebras and M be a graded R-module, where R is a commutative ring. We say M is an (A_1, A_2)-bimodule if the structure maps $\eta_{k_1,k_2} : A_1^{\otimes k_1} \otimes M \otimes A_2^{\otimes k_2} \longrightarrow M$ of degree $1 - k_1 - k_2$ satisfy

$$\begin{aligned}&\sum_{\substack{0\le i\le k_1\\ 0\le j\le k_2}} (-1)^{\star_1} \eta_{k_1-i,k_2-j}(a_1, \cdots, a_{k_1-i}, \eta_{i,j}(a_{k_1-i}, \cdots, a_{k_1}, v, b_1, \cdots, b_j), b_{j+1}, \cdots, b_{k_2})\\ &+\sum_{\substack{1\le i\le k_1-j+1\\ 0\le j\le k_1}} (-1)^{\star_2} \eta_{k_1-j+1,k_2}(a_1, \cdots, a_{i-1}, m_j(a_i, \cdots, a_{i+j-1}), a_{i+j}, \cdots, a_{k_1}, v, b_1, \cdots, b_{k_2})\\ &+\sum_{\substack{1\le i\le k_2-j+1\\ 0\le j\le k_2}} (-1)^{\star_3} \eta_{k_1,k_2-j+1}(a_1, \cdots, a_{k_1}, v, b_1, \cdots, b_{i-1}, m_j(b_i, \cdots, b_{i+j-1}), b_{i+j}, \cdots, b_{k_2})\\ &= 0,\end{aligned}$$

where $v \in M, a_i \in A_1$, and $b_i \in A_2$, and

$$\begin{aligned}\star_1 &= \deg' a_1 + \cdots + \deg' a_{k_1-i}\\ \star_2 &= \deg' a_1 + \cdots + \deg' a_{i-1}\\ \star_3 &= \deg' a_1 + \cdots + \deg' a_{k_1} + \deg' v + \deg' b_1 + \cdots + \deg' b_{i-1},\end{aligned}$$

Remark 11.2. We note that if we consider shifted degree, then the degree of structure maps is 1. That is, if we regard $\eta_k : M[1] \otimes A[1]^{\otimes k} \longrightarrow M[1]$, then $\deg' \eta_k = 1$. Moreover, if we shift only the degree of A, that is, we regard $\eta_k : M \otimes A[1]^{\otimes k} \longrightarrow M$, the degree of η_k in this case is also 1. This is the usual convention for a right A_∞-module. In other words, we usually do not shift the degree of M, but we shift the degree of A. The same holds for the bimodule case.

Example 11.3.

(1) Let (A, d^A) be a differential graded algebra(DGA) and (M, d^M) a differential graded(DG) module over A. We recall from the definition of a right DG module $d^M \circ d^M = 0$ and

$$d^M(v \cdot a) = d^M(v) \cdot a + (-1)^{\deg v} v \cdot d^A(a),$$

where $v \in M$, and $a \in A$. Also we know from Definition 8.1 that A is a strict A_∞-algebra. Now we define the structure maps $\eta_k : M \otimes A^{\otimes k} \longrightarrow M$ by

$$\eta_0(v) = (-1)^{\deg v} d^M$$
$$\eta_1(v, a) = (-1)^{\deg v(\deg a+1)} v \cdot a$$
$$\eta_k = 0 \text{ for } k \geq 2.$$

Then $(M, \{\eta_k\}_{k=0}^\infty)$ is a right A_∞-module over A.

(2) Let (A, m_k) be a strict A_∞-algebra. Then A is a right A_∞-module over A with the structure map $\eta_k = m_{k+1}$. Note that if A is not strict, then A is not necessarily a right A_∞-module over itself with the obvious structure maps.

(3) Let (A, m_k) be a A_∞-algebra, which is not necessarily strict. Then A is a (A, A)-bimodule over A with the structure map $\eta_{k_1,k_2} = m_{k_1+k_2+1}$.

Example 11.4. Let $(M, \{\eta_k\})$ be a right A_∞-module over a strict A_∞-algebra $(A, \{m_k\})$. Then $\eta_0 : M \longrightarrow M$ satisfies $\eta_0 \circ \eta_0 = 0$, and thus we can define $H^*(M, \eta_0)$. Also, m_1 defines $H^*(A, m_1)$ since A is strict. We recall the relation

$$\eta_0(\eta_1(v, a)) + \eta_1(\eta_(v), a) \pm \eta_1(v, m_1(a)) = 0$$

for $v \in M$, and $a \in A$. Therefore, if $m_1(a) = 0$ and $\eta_0(v) = 0$, then $\eta_1(v, a)$ is η_0-cocycle. It is easy to check $\eta_1 : M \otimes A \longrightarrow M$ induces a homomorphism $H^*(M, \eta_0) \otimes H^*(A, m_1) \longrightarrow H^*(M, \eta_0)$, and thus graded $H^*(A, m_1)$-module structure on $H^*(M, \eta_0)$.

Definition 11.5. Let A be a strict unital A_∞-algebra with unit $\mathbf{e}$. We call a right A_∞-module M over A **unital** if for $v \in M$

(1) $\eta_1(v, \mathbf{e}) = (-1)^{\deg v} v$
(2) $\eta_k(\cdots, \mathbf{e}, \cdots) = 0$ for $k \geq 2$

Recall that we constructed an A_∞-algebra related to a spin Lagrangian submanifold in a symplectic manifold M. Geometric realization of A_∞-bimodule is related to a relatively spin pair of Lagrangian submanifolds in M. Consider a relatively spin pair of Lagrangian submanifolds L_1 and L_2 in M intersecting transversally, and free-$\lambda_{0,nov}$ module generated by intersection points $C(L_1, L_2)$. This module has $(C(L_1), C(L_2))$-bimodule structure. In this case, the structure maps η_{k_1,k_2} essentially counts the "number" of moduli space of holomorphic strips passing through chains in $C(L_1)$ and $C(L_2)$. This number is counted with sign determined by orientation of the moduli space.

12. Hochschild cohomology and A_∞ Whitehead theorem

Recall that $(A,\{m_k\})$ is an A_∞-algebra if and only if the sum of associated coderivations $\widehat{d}=\sum_{k=1}^{\infty}\widehat{m}_k$ satisfies $\widehat{d}\circ\widehat{d}=0$. We want to find similar result for a right A_∞-module M over a strict A_∞-algebra A. We need some definitions. We assume that A_∞-algebra $(A,\{m_k\})$ is strict in this lecture.

Definition 12.1. Let R be a commutative ring with 1, and A be a coalgebra over R with comuliplication Δ and counit ϵ. A right R-module M is called a **right comodule over** A if there exists a linear map $\rho: M\longrightarrow M\otimes A$ such that

(1) $(id\otimes\Delta)\circ\rho=(\rho\otimes id)\circ\rho$
(2) $(id\otimes\epsilon)\circ\rho=id$

Example 12.2. Recall that the bar complex BA is a coalgebra with the comultiplication Δ and the counit ϵ for an A_∞-algebra $(A,\{m_k\})$ over a base ring R. Let M be a usual right R-module. We put $B_AM:=M\otimes BA$ and define $\rho: B_AM\longrightarrow B_AM\otimes BA$ by $\rho(v\otimes\mathbf{x})=v\otimes\Delta(\mathbf{x})$. Then B_AM is a right comodule over BA.

Definition 12.3. Let M and N be right comodules over a coalgebra A with linear maps $\rho_M: M\longrightarrow M\otimes A$ and $\rho_N: N\longrightarrow N\otimes A$ determining comodule structure, respectively. Then a R-linear map $\phi: M\longrightarrow N$ is called a **comodule homomorphism** if $(\phi\otimes id)\circ\rho_M=\rho_N\circ\phi$.

Example 12.4. Let M be a usual right R-module, and consider the linear maps $\eta_k: M\otimes B_kA\longrightarrow M$. Then we extend η_k to $\widehat{\eta}_k: B_AM\longrightarrow B_AM$ defined by

$$\begin{aligned}&\widehat{\eta}_k(v,x_1,\cdots,x_n)\\=&\eta_k(v,x_1,\cdots,x_k)\otimes x_{k+1}\otimes\cdots\otimes x_n\\+&\sum_{i=1}^{n-k+1}(-1)^{\star}v\otimes x_1\otimes\cdots\otimes x_{i-1}\otimes m_k(x_i,\cdots,x_{i+k-1})\otimes x_{i+k}\otimes\cdots\otimes x_n,\end{aligned}$$

where $v\in M$, $x_i\in A$, and $\star=\deg' v+\deg' x_1+\cdots+\deg' x_{i-1}$. Then $\widehat{\eta}_k$ is a comodule homomorphism. Now, we define $\widehat{\eta}=\sum_{k=0}^{\infty}\widehat{\eta}_k$. Then we have the following proposition.

Proposition 12.5. M is a right A_∞-module over A if and only if $\widehat{\eta}\circ\widehat{\eta}=0$.

Now, we consider a map between two right A_∞-modules.

Definition 12.6. Let M and N be right A_∞-module over an A_∞-algebra A. We denote by $CH_A(M,N)$ the set of sequence of maps $\rho_k: M\otimes B_kA\longrightarrow N$. We call $\{\rho_k\}_{k=0}^{\infty}\in CH_A(M,N)$ of degree′ 0 (degree 0 with respect to the shifted degree) a **prehomomorphism**.

Let $\{\rho_k\}_{k=0}^{\infty}\in CH_A(M,N)$ be a prehomomorphism. As before, we extend ρ_k to a comodule homomorphism $\widehat{\rho_k}: M\otimes BA\longrightarrow N\otimes BA$ by

$$\begin{aligned}&\widehat{\rho_k}(v,x_1,\cdots,x_n)\\=&\rho_k(v,x_1,\cdots,x_k)\otimes x_{k+1}\otimes\cdots\otimes x_n,\end{aligned}$$

where $v\in M$ and $x_i\in A$. Then we define $\widehat{\rho}=\sum_{k=0}^{\infty}\rho_k$.

Definition 12.7. We call $\{\rho_k\}_{k=0}^{\infty}$ an A_∞-module homomorphism if $\widehat{\eta}^N\circ\widehat{\rho}=\widehat{\rho}\circ\widehat{\eta}^M$, where $\widehat{\eta}^M$ and $\widehat{\eta}^N$ are the differential associated to the A_∞-module structure maps of M and N, respectively.

Remark 12.8. We define an A_∞-module homomorphism only for degree$'$ 0 elements in $CH_A(M,N)$. However, even if degree$'$ of $\{\rho_k\}_{k=0}^\infty \in CH_A(M,N)$ in not 0, we can extend ρ_k to a comodule homomorphism $\widehat{\rho_k} : M \otimes BA \longrightarrow N \otimes BA$ exactly same way as above and define $\widehat{\rho} = \sum_{k=0}^\infty \rho_k$. This gives an isomorphism between $CH_A(M,N)$ and $CoMod_A(M,N)$, where $CoMod_A(M,N)$ is the set of comodule homomorphisms from M to N. The proof of this fact is similar to the one of Theorem 10.2.

Now, we turn our attention to the interpretation of A_∞-module homomorphism in terms of **Hochschild** differential on $CH_A(M,N)$.

Definition 12.9. Let M and N be right A_∞-module over an A_∞-algebra (A,m) with structure maps η^M and η^N, respectively. We define $\delta : CH_A(M,N) \longrightarrow CH_A(M,N)$, the **Hochschild** differential on $CH_A(M,N)$ by

$$\begin{aligned}
&(\delta\rho)_k(v,a_1,\cdots,a_k)\\
&= \sum_{i=0}^k \eta^N_{k-i}(\rho_i(v,a_1,\cdots,a_i),a_{i+1},\cdots,a_k)\\
&+ \sum_{i=0}^k (-1)^{|\rho|+1}\rho_{k-i}(\eta^M_i(v,a_1,\cdots,a_i),a_{i+1},\cdots,a_k)\\
&+ \sum_{j=1}^k \sum_{i=1}^{k-j+1} (-1)^{|\rho|+1+\star}\rho_{k-j+1}(v,a_1,\cdots,a_{i-1},m_j(a_i,\cdots,a_{i+j-1},a_{i+j},\cdots,a_k),
\end{aligned}$$

where $v \in M$, $a_i \in A$ and $\star = \deg' v + \deg' a_1 + \cdots + \deg' a_{i-1}$.

The following lemma explains why we call δ a Hochschild "differential".

Lemma 12.10. The map δ in Definition 24.9 satisfies $\delta \circ \delta = 0$.

Moreover, the Hochschild differential gives another definition of an A_∞-module homomorphism.

Proposition 12.11. Let the degree$'$ of $\{\rho_k\}_{k=0}^\infty \in CH_A(M,N)$ is 0. Then $\{\rho_k\}_{k=0}^\infty$ is an A_∞-module homomorphism if and only if $\delta\rho = 0$.

Now, let $\{\rho_k\}_{k=0}^\infty \in CH_A(M,N)$ be an A_∞-module homomorphism, that is, $\delta\rho = 0$. We compute for $v \in M$,

$$0 = (\delta\rho)_0(v) = \eta^N_0(\rho_0(v)) - \rho_0(\eta^M_0(v)),$$

and so $\eta^N_0(\rho_0(v)) = \rho_0(\eta^M_0(v))$. Recall that $\eta^M_0 \circ \eta^M_0 = 0$ and $\eta^N_0 \circ \eta^N_0 = 0$. Therefore ρ_0 is a chain map between η^M_0-complex and η^N_0-complex. In other words, ρ_0 induces $(\rho_0)_* : H^*(M,\eta^M_0) \longrightarrow H^*(N,\eta^N_0)$. This enables us to define the following.

Definition 12.12. An A_∞-module homomorphism $\{\rho_k\}_{k=0}^\infty \in CH_A(M,N)$ is called an A_∞**-quasi-isomorphism** if $(\rho_0)_* : H^*(M,\eta^M_0) \longrightarrow H^*(N,\eta^N_0)$ is an isomorphism.

Next, we define composition of A_∞-module homomorphisms.

Definition 12.13. Let $\{\tau_k\}_{k=0}^\infty \in CH_A(M,N)$ and $\{\rho_k\}_{k=0}^\infty \in CH_A(N,P)$. Then we define $\{(\rho\circ\tau)_k\}_{k=0}^\infty \in CH_A(M,P)$ by

$$\begin{aligned}
&(\rho\circ\tau)_k(v,a_1,\cdots,a_k)\\
&= \sum_{i=0}^k \rho_{k-i}(\tau_i(v,a_1,\cdots,a_i),a_{i+1},\cdots,a_k)
\end{aligned}$$

Proposition 12.14.
(1) The composition is associative.
(2) $\delta(\rho \circ \tau) = \delta\rho \circ \tau + (-1)^{|\rho|}\rho \circ \delta\tau$

Corollary 12.15. $CH_A(M, N)$ is a differential graded algebra with composition as product.

More generally, the set of A_∞-modules over an A_∞-algebra A forms a differential graded category. The objects in this category are A_∞-modules, and a morphism between two objects M and N is given by an element $CH_A(M, N)$. The product in $CH_A(M, N)$ is composition. Also, the proposition implies that composition of A_∞-module homomorphisms is again an A_∞-module homomorphism.

Definition 12.16. Let $\{\rho_k\}_{k=0}^\infty$ and $\{\tau_k\}_{k=0}^\infty$ in $CH_A(M, N)$ be A_∞-module homomorphisms. We say ρ is **homotopic** to τ if there exists $T \in CH_A(M, N)$ such that $\rho - \tau = \delta(T)$.

Proposition 12.17.
(1) Homotopy is an equivalence relation.
(2) If ρ is homotopic to τ, then $\rho \circ \psi$ is homotopic to $\tau \circ \psi$ and $\psi \circ \rho$ is homotopic to $\psi \circ \tau$, where $\{\rho_k\}, \{\tau_k\}$, and $\{\psi_k\}$ are in $CH_A(M, N)$.

The following is the A_∞-analog to the Whitehead theorem whose proof we postpone until later in section 22, 23.

Theorem 12.18. If $\{\rho_k\} \in CH_A(M, N)$ is an A_∞-quasi-isomorphism, then there exists an A_∞ module homomorphism $\{\psi_k\} \in CH_A(N, M)$ such that $\rho \circ \psi$ is homotopic to id and $\psi \circ \rho$ is homotopic to id.

We briefly discuss the strategy of the proof of the theorem.
(1) By the assumption, $(\rho_0)_* : H^*(M, \eta_0^M) \longrightarrow H^*(N, \eta_0^N)$ is an isomorphism. Using this, we construct a chain map $\psi_0 : N \longrightarrow M$ such that $\rho_0 \circ \psi_0$ is homotopic to id and $\psi_0 \circ \rho_0$ is homotopic to id. Here, we need to use that the base ring is a field and each graded piece is finite dimensional.
(2) We construct ψ_k inductively so that $\rho \circ \psi$ is homotopic to id up to $k+1$, assuming the existence of A_k homotopy.
(3) To get ψ_{k+1}, we write the corresponding A_{k+1} relation $\rho \circ \psi - id = \delta(T)$ and solve the equation. In general, the equation is written in the way $\delta(\cdot) = (\cdot)$.

13. Definition of symplectic manifolds

For motivation, we will look at basic objects of study in symplectic geometry. Let L be an oriented manifold and $T^\star L$ be its cotangent bundle, which is called a quantization of L.

Definition 13.1. A **symplectic manifold** is a manifold M with a nondegenerate and closed 2-form ω.

Example 13.2. On $\mathbb{R}^{2n}$, a two form $\omega_0 := \sum_{j=1}^n dq_j \wedge dp_j$ can be an example of symplectic form where $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ is a coordinate of $\mathbb{R}^{2n}$.

Example 13.3. Let L be an oriented n-dimensional manifold and $\pi : T^\star L \longrightarrow L$ a cotangent bundle. More generally, any $T^\star L$ carries a canonical symplectic form ω_0 given by

$$\omega_0 := \sum_{j=1}^n dq_j \wedge dp_j$$

in a local canonical coordiate $(q_1, \cdots, q_n, p_1, \cdots, p_n)$. We explain what a canonical coordinate is in this situation. Let (U, φ) be a chart on the base L where φ has coordinate functions $(x_1, \cdots, x_n)$. Then, $T^\star L|_U$ can be given a coordinate as follows: $\{dx_1|_U, \cdots, dx_n|_U\}$ is a local basis for $T^\star L|_U$. Thus, for any $T^\star L|_U$, it can be expressed of the form

$$\alpha|_U = \sum_{j=1}^n p_j dx_j|_U.$$

in a unique way. Then, a canonical coordinate on $T^\star L|_U$ is defined as

$$q_j := x_j \circ \pi$$
$$p_j := \alpha|_U \left(\frac{\partial}{\partial x_j} \right)$$

associated to $(x_1, \cdots, x_n)$ on U. Thus, we obatin a local canonical symplectic form ω_0.

Proposition 13.4. A canonical symplectic form ω_0 on U does not depend on the choice of coordinate. Indeed, ω_0 is globally defined.

We can prove it by direct calcaluation. Alternatively, we give a coordinate-free description of ω_0.

Definition 13.5. A **Lioville one-form** Θ on $T^\star L$ is defined by

$$\Theta_\alpha(\xi) := \alpha(d\pi(\xi))$$

where $\alpha \in T^\star L$ amd $\xi \in T(T^\star L)$.

Then, one can check that $\omega_o = -d\Theta$.

In this example, we note that ω_0 is not only closed, but exact. Yet, this is actually a special case.

Definition 13.6. A symplectic manifold (M, ω) is called exact if ω is an exact two form, i.e., $\omega = d\alpha$ for some one-form α.

Here are some examples where a symplectic form is closed, but not exact.

Example 13.7. One simple example is a sphere S^2 or any 2-dimensional surface Σ with an area form. Also, every complex projective space $\mathbb{C}P^N$ has the Fubini-Study form, which is symplectic. More generally, any complex algebraic manifold carries a symplectic form, which is the pullback of the Fubini-Study on $\mathbb{C}P^N$ form via $M \hookrightarrow (\mathbb{C}P^N, \omega_{FS})$.

Example 13.8. Gompf proved that any finitely presented group can be realized as a fundamental group π_1 of some symplectic 4-manifold.

Theorem 13.9 (Darboux Theorem)**.** Let (M, ω) be a $2n$-dimensional symplectic manifold. Then, for any $x \in M$, there exists a coordinate chart (U, ϕ) around x such that $\omega = \phi^\star \omega_0$, where ω_0 is the canonical symplectic form on $\mathbb{R}^{2n} \cong T^\star \mathbb{R}^n$. ($\omega_0 = \sum_{i=1}^{i=n} dq_i \wedge dp_i$, where $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ is a coordinate of $\mathbb{R}^n$.

We have some consequences of the definition and the above theorem.

(1) A symplectic manifold is even dimensional.
(2) There is no local invariant for a symplectic manifold.

The following is an easy consequence of the definition of symplectic form.

Proposition 13.10. Prove that any exact Lagrangian submanifold cannot be closed, i.e., either it must be noncompact or has a boundary.

Proof. Suppose to the contrary that M is closed. Then the top exterior power ω^n is closed but not exact because if we assume ω is exact and M carries no boundary and is compact, then Stoke's theorem implies

$$\int_M \omega^n = 0$$

On the other hand, nondegeneracy of ω implies ω^n is nowhere vanishing i.e., it defines a volume form, and hence $\int_M \omega^n > 0$ if M is compact and is equipped with the orientation induced by the volume form ω^n, which gives rise to a contradiction. This finishes the proof. □

On the other hand, if M is closed, $[\omega]^n = [\omega^n] \neq 0$ in $H^{2n}(M, \mathbb{R})$ implies $[\omega] \neq 0$ in $H^2(M, \mathbb{R})$.

Corollary 13.11. A compact manifold without boundary cannot be a symplectic manifold if its 2-nd De Rham cohomology group is trivial. In particular S^{2n} has a symplectic structure if and only if $n = 1$.

14. Lagrangian submanifolds and Hamiltonian flows

Definition 14.1. Let (M,ω) be a symplectic manifold. $L \subset M$ is a **Lagrangian** submanifold if $dimL = n$ and $\omega|_L = i^\star\omega = 0$, where $dimM = 2n$ and $i : L \longrightarrow M$ is the inclusion.

Example 14.2. Consider $(T^\star N, \omega_0)$, where $\omega_0 = -d\Theta$ and Θ is a Liouville 1-form. Then o_N, the zero section in $T^\star N$, is Lagrangian.

Another important example is the following.

Proposition 14.3. Let $S \subset N$ be a submanifold. Then the conormal bundle $\nu^\star S = \{\alpha \in T^\star N | \ \alpha|_{T_{\pi(\alpha)}S} = 0\}$ is a Lagrangian submanifold of $T^\star N$, where $\pi : T^\star N \longrightarrow N$ is the projection.

Proof. Let $\alpha \in \nu^\star S$ and $\xi \in T_\alpha(\nu^\star S)$. Note that $d\pi(\xi) \in T_{\pi(\alpha)}S$. Therefore,

$$\Theta_\alpha(\xi) = \alpha(d\pi(\xi)) = 0,$$

and this proves the proposition. □

Remark 14.4. (1) Locally any Lagrangian submanifold can be written as a conormal bundle in some coordinate.
(2) Any curve in a symplectic surface is Lagrangian.
(3) Let $f : N \longrightarrow \mathbb{R}$ be smooth. Then df is a 1-form. Then $Graph(df)$ is Lagrangian in $T^\star N$. Note that $i^\star_{defn}\Theta = df$. This property holds for every 1-form.

Proposition 14.5. (Funtorial property of Liouville 1-form) For any 1-form β on N, denote the associate section map by $\tilde{\beta} : N \longrightarrow T^\star N$. Then $\tilde{\beta}^\star\Theta = \beta$. Conversely, this property completely determines Θ.

Proposition 14.6. Let β be a 1-form on N. Then β is closed if and only if $Graph(\tilde{\beta})$ is Lagrangian in $T^\star N$.

According to Morse theory we have that if $f : N \longrightarrow \mathbb{R}$ is a Morse function, then $\#(Crit(f)) \geq \sum_{k=0}^n b_k(N)$, where b_k is the kth Betti number of N. Note that the number of critical points of f is equal to the number of intersection points of zero section in $T^\star N$ and $Graph(df)$.

Definition 14.7. Let (M,ω) be a symplectic manifold and $h : M \longrightarrow \mathbb{R}$ be smooth. A **Hamiltonian** vector field associated to h, denoted by X_h, is determined by $X_h \lrcorner \omega = dh$. When the Hamiltonian H is time dependent, we denote its Hamiltonian vector field by

$$X_H = X_H(t,x) := X_{H_t}(x)$$

The associate ordinary differential equation $\dot{x} = X_H(t,x)$ is called a Hamiltonian equation.

Example 14.8. Consider $\mathbb{R}^{2n}$ with standard symplectic form. Then the Hamiltonian equation associate to H is the following system ODE.

$$\dot{q} = \frac{\partial H}{\partial p_i}, \quad \dot{p} = -\frac{\partial H}{\partial q_i}$$

Definition 14.9. A diffeomorphism $\phi : (M,\omega) \longrightarrow (M,\omega)$ is called symplectic if $\phi^\star\omega = \omega$. We denote the set of symplectic diffeomorphisms of (M,ω) by $Symp(M,\omega)$.

Proposition 14.10. X_h,the Hamiltonian vector field associated to h, is symplectic in that $\mathcal{L}_{X_h}\omega = 0$.

Proof. We recall Cartan's magic formula.

$$\mathcal{L}_X\Omega = d(X \lrcorner \Omega) + X \lrcorner \Omega$$

Noting that ω is closed and $d^2 = 0$, the proposition is proved. □

Corollary 14.11. The flow $\phi^t_{X_h}$ of X_h is a symplectic diffeomorphism.

Proposition 14.12. Let ϕ^t be the flow for a vector field X. Then $\{\phi^t\}$ is symplectic if and only if ϕ^0 is symplectic and $X \lrcorner \omega$ is closed.

Proof. Consider

$$\frac{d}{dt}(\phi^t)^\star\omega = (\phi^t)^\star\mathcal{L}_X\omega,$$

and apply Cartan's formula. □

Definition 14.13. A vector field X is called **symplectic**(locally Hamiltonian) if $X \lrcorner \omega$ is closed. If $X \lrcorner \omega$ is exact, we call X **Hamiltonian**.

Definition 14.14. We call $\phi \in Symp(M, \omega_)$ a **Hamiltonian** diffeomorhphism if it can be connected to $id \in Symp(M, \omega)$ by a time dependent Hamiltonian flow. That is, $\phi = \phi^1_H$ for some time dependent $H : [0, 1] \times M \longrightarrow \mathbb{R}$.

Proposition 14.15. We denote by $Ham(M, \omega)$ the set of Hamiltonian homeomorphisms. Then $Ham(M, \omega)$ is a subgroup of $Symp(M, \omega)$.

Proof. It is clear that $id \in Ham(M, \omega)$. Now let $\phi, \psi \in Ham(M, \omega)$. We need to show that $\phi \circ \psi \in Ham(M, \omega)$. Let $\phi = \phi^1_H$ and $\psi = \phi^1_K$ for some time dependent Hamiltonian H, K. We need the following lemma.

Lemma 14.16. Define $L = L(t, x)$ by

$$L(t, x) = H(t, x) + K(t, (\phi^t_H)^{-1}(x))$$

Then L generates the flow $t \mapsto \phi^t_H \circ \phi^t_K$.

Proof. (lemma) Note that for a flow ϕ^t,

$$X_t(x) = \frac{d\phi^t}{dt}((\phi^t)^{-1}(x))$$

is the generating vector field. We compute

$$\begin{aligned}
&\frac{d}{dt}(\phi^t_H \circ \phi^t_K)(x) \\
&= X_{H_t}((\phi^t_H \circ \phi^t_K)(x)) + d\phi^t_H(X_{K_t}(\phi^t_K)(x)) \\
&= X_{H_t}((\phi^t_H \circ \phi^t_K)(x)) + (d\phi^t_H X_{K_t}(d\phi^t_H)^{-1})((\phi^t_H \circ \phi^t_K)(x)) \\
&= (X_{H_t} + (\phi^t_H)_\star X_{K_t})((\phi^t_H \circ \phi^t_K)(x))
\end{aligned}$$

Therefore, $Y(t, x) = (X_{H_t} + (\phi^t_H)_\star X_{K_t})(x)$ is the vector field generating $t \mapsto \phi^t_H \circ \phi^t_K$.

We need another lemma.

Lemma 14.17.

(1) $X_{g+h} = X_g + X_h$.
(2) If $\phi \in Symp(M, \omega)$, then $\phi^\star X_h = X_{h\circ\phi}$.

Using this lemma, we conclude that $X_{H_t} + (\phi_H^t)_\star X_{K_t} = X_{H_t + K_t \circ (\phi_H^t)^{-1}}$.

Hence, $L(t,x) = H(t,x) + K(t,(\phi_H^t)^{-1}(x))$ is the associate Hamiltonian generating $\phi_H^t \circ \phi_K^t$. □

This implies that $\phi \circ \psi \in Ham(M,\omega)$. Similarly, we can prove that if $\phi \in Ham(M,\omega)$, then $\phi^{-1} \in Ham(M,\omega)$ with associate Hamiltonian

$$\bar{H}(t,x) = -H(t,\phi_H^t(x))$$

□

Now consider $Symp(M,\omega)$. It is easy to see that $Symp(M,\omega)$ is C^1-closed in $Diff(M)$ since the characterization $\phi^\star\omega = \omega$ depends on the first derivative. We also have the following C^0 **symplectic rigidity**.

Theorem 14.18. (Eliashberg)
Suppose that (M,ω) is a compact symplectic manifold. Then $Symp(M,\omega)$ is C^0-closed in $Diff(M)$.

15. Compatible almost complex structures

We start with the general definition of almost complex structrues.

Definition 15.1. J is called an **almost complex structure** on a manifold M if $J : TM \longrightarrow TM$ is a bundle map such that $J^2 = -\text{id}$.

Theorem 15.2. Any symplectic manifold (M, ω) carries an almost complex structure J such that
(1) (J-positivity) $\omega(X, JX) \geq 0$ and the equality holds if and only if $X = 0$.
(2) (J-Hermitian) $\omega(JX, JY) = \omega(X, Y)$
That is, $g_J(\cdot,\cdot) = \omega(\cdot, J\cdot)$ becomes a Riemannian metric. g_J is positive definite and symmetric.

(In the literature J-positivity is also called ω-tameness.) Gromov proved that the set of such compatible J is a contractible infinite dimensional manifold.

Remark 15.3. In general the (M, g, J) is called an almost Hermitian manifold if $g(J\cdot,\cdot) = g(\cdot,\cdot)$ and an almost Kähler manifold if the two form $g(J\cdot,\cdot)$, called the fundamental two form, is closed in addition. In this sense, a symplectic manifold (M, ω) with a compatible almost complex structure canonically defines an almost Kähler structure by considering the associated Riemannian metric $g = \omega(\cdot, J\cdot)$.

We note that a complex manifold has a natural almost complex structure induced from holomorphic coordinate charts. In this case, we call the almost complex structure **integrable**. However, not every almost complex structure comes from holomorphic coordinate charts. The obstruction for the integrability is the **Nijenhuis** tensor

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y].$$

Theorem 15.4. (**Newlander-Nirenberg**)
An almost complex structure J is integrable if and only if $N_J = 0$.

Theorem 15.5. Any almost complex structure on a 2-dimensional surface is integrable.

Proof. Let (M, J) be a 2 dimensional almost complex manifold. We need to prove that $N_J = 0$. Let $p \in M$ and X be a nonzero vector field such that $X_p \neq 0$. We compute

$$N_J(X, JX) = -[JX, X] - J[JX, JX] + J[X, X] - [X, JX] = 0$$

since the bracket is skew symmetric. Note that X_p and $(JX)_p = J_p X_p$ are basis of T_pM, and N_J is a tensor. Since p is arbitrary, and thus this proves the theorem. ☐

Definition 15.6. Let (M_1, J_1) and (M_2, J_2) be almost complex manifolds. A map $\phi : M_1 \longrightarrow M_2$ is called **almost complex** if

$$J_2 \circ d\phi = d\phi \circ J_1$$

A natural question is
"Does there exist such a map if (M_1, J_1) and (M_2, J_2) are given?"

To answer this question we examine a kind of symmetry of the equation $J_2 \circ d\phi = d\phi \circ J_1$, which is equivalent to $J_2 \circ d\phi \circ J_1 = -d\phi$. In general,

$$d\phi = \partial_{(J_1,J_2)}\phi + \bar{\partial}_{(J_1,J_2)}\phi,$$

where

$$\partial_{(J_1,J_2)}\phi = \frac{1}{2}(d\phi - J_2 \circ d\phi \circ J_1)$$
$$\bar{\partial}_{(J_1,J_2)}\phi = \frac{1}{2}(d\phi + J_2 \circ d\phi \circ J_1)$$

Therefore, $J_2 \circ d\phi = d\phi \circ J_1$ if and only if $\bar{\partial}_{(J_1,J_2)}\phi = 0$. Then we consider the assignment $\phi \mapsto \bar{\partial}_{(J_1,J_2)}\phi$, and count the number of equations. Then we have $2m_2$ inputs and $2m_1m_2$ outputs, where m_1 and m_2 are dimension of M_1 and M_2, respectively. So, when $m_1 = 1$, $J_2 \circ d\phi = d\phi \circ J_1$ becomes a well-posed system in that the numbers of equations and unknowns match.
Also, we can regard the equation $J_2 \circ d\phi = d\phi \circ J_1$ as generalization of classical Cauchy-Riemann equation to an almost complex manifold. More precisely, let (Σ, j) be a Riemann surface and $z = x+iy$ be its local complex coordinate. Consider a map $\phi : (\Sigma, j) \longrightarrow (M, J)$ such that $J \circ d\phi = d\phi \circ j$, where (M, J) is a complex manifold. Then $\bar{\partial}_{(j,J)}\phi = 0$ holds if and only if $\frac{\partial w_i}{\partial \bar{z}} = 0$ for $i = 1, \cdots, n$, where $(w_1, \cdots, w_n)$ is a complex coordinate for (M, J).
Moreover, $\bar{\partial}_{(j,J)}\phi = 0$ is an elliptic equation.

Definition 15.7. A map $\phi : (\Sigma, j) \longrightarrow (M, J)$ is called J-holomorphic if $\bar{\partial}_{(j,J)}\phi = 0$.

Gromov exploited the deformation theory of the set of J-holomorphic maps $w : (\Sigma, j) \longrightarrow (M, J, \omega)$. We have the following local existence theorem.

Theorem 15.8. (**Nijenhuis-Woolf**)
Let (M, J) be an almost complex manifold and $v \in T_xM$. Then there exists a J-holomorphic map $w : D^2(\delta) \longrightarrow M$ such that $w(0) = x$ and $Im(dw)_0$ is contained in span(v, Jv), where $D^2(\delta)$ is a disc in $\mathbb{C}$ with radius δ.

We can regard the assignment $w \mapsto \bar{\partial}_J w$ as a section of a bundle over $C^\infty(\Sigma, M)$. More precisely, let

$$\mathrm{Hom}(T_xM, T_{w(x)}M) = \mathrm{Hom}'(T_xM, T_{w(x)}M) \oplus \mathrm{Hom}''(T_xM, T_{w(x)}M)$$

be the decompositon of $\mathrm{Hom}(T_xM, T_{w(x)}M)$ with complex linear and anticomplex linear parts. Then

$$\bar{\partial}_J w(x) \in \mathrm{Hom}''(T_xM, T_{w(x)}M) = \mathrm{Hom}''(T_xM, w^\star TM|_x) = \Lambda_J^{(0,1)}(w^\star TM)|_x,$$

where $\Lambda_J^{(0,1)}(w^\star TM)$ is $w^\star TM$-valued $(0,1)$-forms. Let $E \longrightarrow C^\infty(\Sigma, M)$ be the vector bundle such that the fiber over w is $\Lambda_J^{(0,1)}(w^\star TM)$. Then, the assignment $w \mapsto \bar{\partial}_J w$ is a section of this vector bundle.

16. Definition of pseudoholomorphic curves

There are two kinds of Lagrangian submanifolds: one is nondisplaceable and the other is dispaceable by a Hamiltonian diffeomorphism.

Example 16.1. The zero section of a contagent bundle of a compact manifold is an example of nondisplaceable Lagrangian submanifold.

Theorem 16.2. (Floer, Hofer) Let N be a compact manifold and o_N the zero section of a cotangent bundle $T^\star N$. Then, we have a lower bound of intersection as follows:

$$\#(o_N \cap \phi(o_N)) \geq \operatorname{rank}_{\mathbb{Z}} H^\star(N;\mathbb{Z})$$

for any Hamiltonian diffeomorphism ϕ. In particular, the zero section and its Hamiltonian perturbation $\phi(o_N)$ always intersects.

Such a Lagrangian submanifold is called **nondisplaceable**.

Definition 16.3. A symplectic manifold (M,ω) is called **exact** if $\omega = d\alpha$ for some 1-form α. A Lagrangian submanifold L of $(M, d\alpha)$ is called **exact** if $\alpha|_L$ is exact.

More generally, the same kind intersection result holds for any compact exact Lagrangian submanifold in exact symplectic manifold. We observe that a cotangent bundle $T^\star N$ with a standard symplectic structure $\omega_0 = -d\Theta$ where Θ is a Lioville one-form. The following proposition shows that $\phi(o_N)$ in $T^\star N$ is exact for a Hamiltonian diffeomorphism ϕ.

Proposition 16.4. A Lagrangian submanifold $\phi(o_N)$ in $T^\star N$ is exact.

Proof. For the zero section o_N, we know $\Theta|_{o_N} = 0$ so that o_N is exact. Let $H = H(t,x)$ be a Hamiltonian function generating ϕ (i.e., $\phi = \phi_H^1$). Then, we are given a Hamiltonian isotopy $\phi_H^t(o_N)$ from o_N to $\phi(o_N)$. By examining the derivative $i_t^\star\Theta$ on N where $i_t = \phi_H^t \circ i_{o_N}$, we prove exactness. □

Example 16.5. A compact Lagrangian submanifold in $\mathbb{C}^n = T^\star\mathbb{R}^n$ is an example of displaceable Lagrangian submanifold. The following proposition asserts that any translation on $\mathbb{C}^n$ is a Hamiltonian isotopy.

Proposition 16.6. Consider a translation $x \mapsto x + t\mathbf{v}$ for $\mathbf{v} \in \mathbb{C}^n$. This is a Hamiltonian flow generated by $H(t,x) = \omega_0(\mathbf{v}, x)$.

Proof. Let

$$\mathbf{v} = \sum_{j=1}^n v_{j,q}\frac{\partial}{\partial q_j} + \sum_{k=1}^n v_{k,p}\frac{\partial}{\partial p_k}.$$

Setting $\iota_{\mathbf{v}}\omega_0 = dH$ (ι denotes the interior product), we will find a suitable expression of a Hamiltonian H. The left hand side becomes

$$\begin{aligned}\iota_{\mathbf{v}}\omega_0 &= \iota_{\mathbf{v}}\left(\sum_{i=1}^n dq_i \wedge dp_i\right)\\ &= \sum_{j,k=1}^n \left(v_{j,q}dp_j - v_{k,p}dq_k\right).\end{aligned}$$

The right hand side becomes

$$dH = \sum_{k=1}^n \frac{\partial H}{\partial q_k}dq_k + \sum_{j=1}^n \frac{\partial H}{\partial p_j}dp_j.$$

Comparing them, we get

$$\frac{\partial H}{\partial q_j} = v_{j,q}, \quad \frac{\partial H}{\partial p_i} = -v_{i,p}.$$

Thus, a Hamtilonian H can be chosen

$$H = \sum_{j=1}^{n} (v_{i,q} p_j - v_{j,p} q_j) = \omega_0(v, x)$$

where $x = (q_1, \cdots, q_n, p_1, \cdots, p_n)$. □

Any compact Lagrangian submanifold (indeed any compact submanifold) can be displaced away from itself by a translation. Such a Lagrangian submanifold is called **displaceable**.

Now, we move into Gromov's pseudoholomorphic curves. Our main task is to construct as many surfaces $\Sigma \subset (M, \omega)$ with positive symplectic density $(\omega|_\Sigma \geq 0)$ compared to a chosen area form dA on Σ.

Definition 16.7. Let (M, ω) be a symplectic manifold. Let Σ be a 2-dimensional surface with an area form dA in M. A 2-dimensional surface $(\Sigma, dA) \subset (M, \omega)$ is called **nonnegative** if $\omega|_\Sigma = f dA$ for $f \geq 0$.

Example 16.8. Let (M, g, J) be a Kähler manifold where g is a Hermitian metric. By definition, it satisfies

(i) J is integrable
(ii) $\Phi := g(J\cdot, \cdot)$ is closed.

Then, any holomorphic curve is nonnegative for Φ.

Let (M, ω) be a symplectic manifold with ω-compatible almost complex structure J. Then, (M, ω, J) is called an almost Kähler manifold without integrability condition.

Definition 16.9. A smooth map $u : (\Sigma, j) \longrightarrow (M, J)$ is called **J-holomorphic** (or **(j, J)-holomorphic**) if

$$J \circ du = du \circ j$$

This condition is equivalent to $\bar{\partial}_{(j,J)} u = 0$ where

$$\bar{\partial}_{(j,J)} u = \frac{du + J \circ du \circ j}{2}.$$

Definition 16.10. Fix a metric g determined by J on M and ω, and fix a Kähler metric h on Σ. For a smooth map $u : \Sigma \longrightarrow M$, we define a **harmonic energy density** by

$$e(u)(z) = |\, du(z)|^2$$

as the norm square of the linear map

$$du(z) : (T_z\Sigma, h_z) \longrightarrow (T_{u(z)}M, g_{u(z)}).$$

Here, the norm $|\cdot|$ is defined by

$$|du(z)|^2 = |du(z)(e_1)|^2_{g_{u(z)}} + |du(z)(e_2)|^2_{g_{u(z)}}$$

for an orthonormal bases $\{e_1, e_2\}$ of $T_z\Sigma$. We set

$$E(u) := \frac{1}{2} \int_\Sigma |du|^2 dA$$

which is called a **harmornic energy** of u. (This is $W^{1,2}$-norm of u).

A straightforward computation gives the following proposition.

Proposition 16.11. For a smooth map $u : \Sigma \longrightarrow M$,

(i) $du = \bar{\partial}_J u + \partial_J u$ is an orthogonal decomposition in the sense that
$$|du(z)|^2 = |\bar{\partial}_J u(z)|^2 + |\partial_J u(z)|^2$$

(ii) $\frac{1}{2}(|\partial_J u|^2 - |\bar{\partial}_J u|^2)dA = u^\star\omega$.

Corollary 16.12. Suppose that a smooth map $u : \Sigma \longrightarrow M$ is (j, J)-holomorphic. (i.e., $\bar{\partial}_J u = 0$). Then,
$$u^\star\omega = \frac{1}{2}|du|^2 dA.$$
In particular, we have $u^\star\omega \geq 0$. Also, a harmonic energy becomes a *topological* invariant as follows:
$$E(u) = \int_\Sigma u^\star\omega.$$

Remark 16.13. Here are two reasons why 2 dimensional domain Σ is interesting in the definition of harmonic energy.

(i) When $\dim \Sigma = 2$, $E(u)$ is invariant under conformal transformation. Let $u : D^2 \longrightarrow M$ be a smooth map and $\phi : D' \longrightarrow D$ be a holomorphic map on the disc. Then
$$\int_D |du|^2 dA_D = \int_{D'} |d(u \circ \phi)|^2 dA_{D'}$$
For dialation $R_\delta(z) := \delta z : D^2(1) \longrightarrow D^2(\delta)$ as an example, we have
$$\int_{D^2(1)} |du|^2 dA = \int_{D^2(\delta)} |d(u \circ R_\delta)|^2 dA.$$
This happens only when Σ is 2 dimensional.

(ii) Recall that if $k - \frac{n}{p} > 0$, there is a compact embedding
$$W^{k,p}(\mathbb{R}^n) \hookrightarrow C^0(\mathbb{R}^n),$$
which is called a Sobolev embedding. In the point of view, our energy function is a borderline case because a domain Σ is 2-dimensional and a harmonic energy is defined as a $W^{1,2}$ type norm. In general, $W^{1,2}(\Sigma, M)$ is not continuous but very close.

Remark 16.14. For J-holomorphic map u, we know
$$\frac{1}{2}\int_\Sigma |du|^2 dA = \int_\Sigma u^\star\omega$$
which provides automatic $W^{1,2}$ bounds for u if we fix a homology class $[u] \in H_2(M, \mathbb{Z})$. In other word, the moduli space of a map $u : \Sigma \longrightarrow M$ satisfying

(i) $\bar{\partial}_J u = 0$

(ii) For $A \in H_2(M, \mathbb{Z})$, $[u] = A$.

satisfies an energy bound
$$E(u) \leq \delta(A)$$
where $\delta(A)$ is a constant independent to u. The moduli space is denoted by $\widetilde{\mathcal{M}}(\Sigma, M, J, A)$.

Let $\mathcal{M}(\Sigma, M, J, A)$ be the moduli space of isomorphism class modulo a reparametrization group. Natually, the following questions arise

(i) Is $\mathcal{M}(\Sigma, M, J, A)$ compact in C^∞-topology?
(ii) Does $\mathcal{M}(\Sigma, M, J, A)$ (or its compactification $\overline{\mathcal{M}}(\Sigma, M, J, A)$) have a manifold structure?

In general, the answers of above questions are no. For (i), it is essential to obtain a bound for derivative du. Once such a bound is achieved, the Ascoli-Arzela theorem can be applied. To see why the compactness fails, we need to study what makes $|du_i|_{C^0} \to \infty$ for a given sequence u_i with $E(u_i) \leq C$. This is related to the Gromov's compactness theorem. Roughly speaking, $\mathcal{M}(\Sigma, M, J, A)$ can be *nicely* compactifed by including nodal singular curves. Next time, we are goint to look at detailed description of the compactifed moduli space

ADD: For (ii),?

17. Genus 0 bordered stable maps

Let (M, ω, J) be an almost Kähler manifold. For a (j, J)-holomorphic map $u : \Sigma \longrightarrow M$ (i.e., $\bar{\partial}_{(j,J)} u = 0$), we have

$$E(u) := \frac{1}{2}\int_\Sigma |du|^2 DA = \int_\Sigma u^*\omega.$$

(The same holds for the case when $\partial\Sigma \neq \varnothing$).

Corollary 17.1. Let $u : \Sigma \to M$ be a J-holomorphic map. If $\partial\Sigma = \varnothing$, then the only J-holomorphic map u with $[u] = 0$ in $H_2(M)$ is a constant map.

Proof. Since $[u] = 0$, there exists a map $U : C \longrightarrow M$ such that $\partial U = u$. By Stokes' formula, we have

$$\begin{aligned}\int_\Sigma u^*\omega = \int_{u(\Sigma)} \omega = \int_{\partial U} \omega \\ = \int_U d\omega = 0\end{aligned}$$

Then,

$$\frac{1}{2}\int_\Sigma |du|^2_{(j,J)} = 0.$$

Thus, $du = 0$ almost everywhere. By continuity, du is identically zero so that u has to be constant. □

We look at the case with boundary.

Definition 17.2. A submanifold R in (M, J) is called **totally real** if it satisfies

(i) $TR \cap J \cdot TR = \{0\}$
(ii) $\dim R = \frac{1}{2}\dim M$.

For any totally real submanifold R, it is a fact that

$$\begin{cases} \bar{\partial} u = 0, \\ u(\partial\Sigma) \subset R \end{cases}$$

is an elliptic boundary value problem. Moreover, all apriori estimates hold as long as u is of Hölder classes C^ε for $\varepsilon > 0$.

Any Lagrangian submanifold L in (M, ω) is totally real for any compatible almost complex structure. Hence

$$\begin{cases} \bar{\partial} u = 0, \\ u(\partial\Sigma) \subset L \end{cases}$$

is a nonlinear elliptic boundary value problem for any J. In addition to that, taking a Lagrangian boundary, we achieve following identities.

Lemma 17.3. Suppose that L is a Lagrangian submanifold of M. For $w : (\Sigma, \partial\Sigma) \longrightarrow (M, L)$ and $w' : (\Sigma', \partial\Sigma') \longrightarrow (M, L)$, we have

$$\int_\Sigma w^*\omega = \int_{\Sigma'} (w')^*\omega$$

if $[w] = [w']$ in $H_2(M, L)$.

Proof. For simplicity, we additionally assume that w is homotopic to w' relative to L. We may take

$$W : [0,1] \times \Sigma \longrightarrow M$$

such that $W|_{s=0} = w$, $W|_{s=1} = w'$ and $W([0,1] \times \partial\Sigma) \subset L$. Let $C := [0,1] \times \Sigma$. By Stokes' formula, we have

$$\begin{aligned} 0 &= \int_C W^*(d\omega) \\ &= \int_{\{0,1\}\times\Sigma} W^*\omega + \int_{[0,1]\times\partial\Sigma} W^*\omega \\ &= \int_\Sigma (w')^*\omega - \int_\Sigma w'\omega. \end{aligned}$$

Here, $\int_{[0,1]\times\partial\Sigma} W^*\omega = 0$ because of the Lagrangian boundary condition. This completes the proof. □

Corollary 17.4. Suppose that $w : (\Sigma, \partial\Sigma) \longrightarrow (M, L)$ satisfies

$$\begin{cases} \bar\partial_J w = 0 \\ w(\partial\Sigma) \subset L. \end{cases}$$

Then, w must be constant if $[w] = 0$ in $H_2(M, L)$.

Remark 17.5. *Totally real* condition is open, but *Lagrangian* condition is closed.

Corollary 17.6. For a J-holomorphic map $w : (\Sigma, \partial\Sigma) \longrightarrow (M, L)$ with $[w] = \beta$ in $H_2(M, L)$, we have

$$E(u) \le \delta(\beta)$$

where $\delta(\beta)$ is a constant which is independent to w.

We set up some notations as follows:

$$\widetilde{\mathcal{M}}(J;\beta) = \left\{ (w, \mathbf{z}) : \begin{array}{c} w : (D^2, \partial D^2) \longrightarrow (M, L) \text{ satisfying} \\ \bar\partial_J w = 0, w(\partial\Sigma) \subset L, [w] = \beta \text{ in } \pi_2(M, L) \end{array} \right\}$$

$$\widetilde{\mathcal{M}}_{k+1}(J;\beta) = \left\{ (w, \mathbf{z}) : \begin{array}{c} w \in \widetilde{\mathcal{M}}(J;\beta) \text{ and } w \text{ satisfies stability condition.} \\ \mathbf{z} = (z_0, \cdots, z_k) \text{ where } z_i \text{ are all distinct and in } \partial\Sigma \end{array} \right\}.$$

Definition 17.7. A **genus 0 stable map** from a pre-stable curve Σ with $(k+1)$ marked points on $\partial\Sigma$ is a pair $((\Sigma, \mathbf{z}), w)$ satisfying following conditions:

(i) $(\Sigma, \mathbf{z})$ is a genus 0 pre-stable curve with $(k+1)$ marked points on $\partial\Sigma$.
(ii) $w : \Sigma \longrightarrow M$ is a component-wise smooth map whose restriction to each irreducible component is a J-holomorphic map.
(iii) (Stability Condition) $\#\mathrm{Aut}((\Sigma, \mathbf{z}), w) < \infty$.

When $\Sigma = D^2$, we recall that $\mathrm{Aut}(D^2) = PSL(2, \mathbb{R})$ which acts on $\widetilde{\mathcal{M}}_{k+1}(J, \beta)$ by

$$\phi * ((\Sigma, \mathbf{z}), w) = ((\Sigma, \phi(\mathbf{z})), w \circ \phi^{-1})$$

where $\phi : \Sigma \longrightarrow \Sigma$ is biholomorphic and $\phi(\mathbf{z}) = \{\phi(z_1), \cdots, \phi(z_k)\} \subset \partial\Sigma$.

Definition 17.8. $((\Sigma, \mathbf{z}), w) \sim ((\Sigma, \mathbf{z}'), w')$ if there is $\phi \in \mathrm{Aut}(\Sigma)$ such that

$$\phi * ((\Sigma, \mathbf{z}), w) = ((\Sigma, \mathbf{z}'), w').$$

Definition 17.9. We call $\phi \in \mathrm{Aut}(\Sigma)$ an **automorphism** of $((\Sigma, \mathbf{z}), w)$ if $\phi(\mathbf{z}) = \mathbf{z}$ and $w \circ \phi^{-1} = w$. We call $((\Sigma, \mathbf{z}), w)$ **stable** if $\#\mathrm{Aut}((\Sigma, \mathbf{z}), w)$ is finite.

Example 17.10. (i) If w is not constant, then $((\Sigma, \mathbf{z}), w)$ is always stable whether the domain curve is stable or not. By unique continuation of holomorphic property, there cannot be any continuous nontrivial family ϕ_t such that $w \circ \phi_t = w$.

(ii) If w is constant, the domain curve has to be stable.

(iii) A stable map whose domain looks like Figure 18 should obey
  (a) w_3 and w_4 must not be constant.
  (b) w_1 and w_2 may be constant.

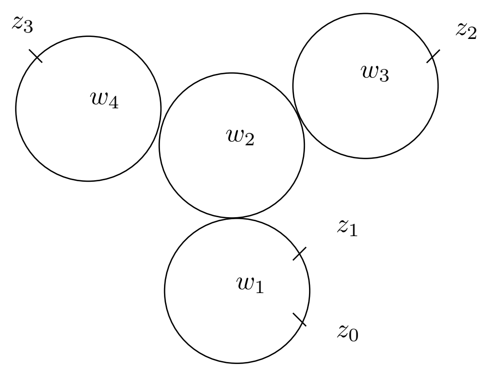

FIGURE 18. Stable map

Remark 17.11. A stable curve is a stable map with its target being a point.

The moduli space of smooth stable maps for (J, β) is denoted by

$$\overline{\mathcal{M}}_{k+1}(J;\beta) = \widetilde{\mathcal{M}}_{k+1}(J;\beta)/\sim .$$

So, $\overline{\mathcal{M}}_{k+1}(J;\beta)$ is the set of isomorphism classes of stable maps. We denote by $[(\Sigma, \mathbf{z}), w]$ the isomorphism class of $((\Sigma, \mathbf{z}), w)$.

Definition 17.12. An **evaluation map** $ev_i : \overline{\mathcal{M}}_{k+1}(J;\beta) \longrightarrow L$ is given by

$$ev_i([(\Sigma, \mathbf{z}), w]) = w(z_i).$$

By definition of action, it is obvious that an evaluation map ev_i is well-defined.

18. Lagrangian submanifolds and filtered A_∞ structure

We recall the notion of stable map.

Definition 18.1. A genus 0 stable map from Σ to a symplectic manifold M (The genus of Σ is 0, and the number of boundary component is 1.) is a pair $((\Sigma, \vec{z}), w)$ such that

(1) $(\Sigma, \vec{z})$ is a genus 0 prestable curve with $\vec{z} \subset \partial\Sigma$. (Recall that Σ is a connected union of discs and spheres with ordinary double point at worst as singularities: See figure 19)

(2) Each irreducible component of (Σ, w) is stable. (Note that the restriction to an irreducible component could be a constant map.)

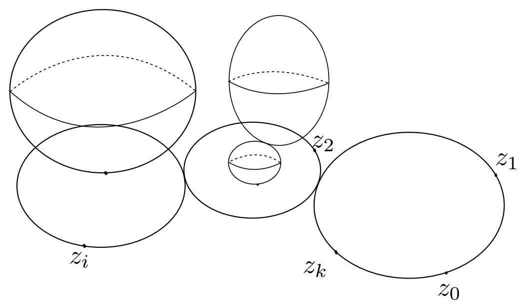

Figure 19. A stable map from a bordered Riemann surface with $k+1$ marked points

We recall that each irreducible component of Σ is a sphere or disc, and the restriction of stable map to each irreducible component satisfies the following:

$$\bar{\partial}_J u = 0 \ \text{ for } u : S^2 \longrightarrow M$$

$$\begin{cases} \bar{\partial}_J w = 0 \\ w(\partial D^2) \subset L \end{cases} \quad \text{for } w : (D^2, \partial D^2) \longrightarrow (M, L),$$

where L is a Lagrangian submanifold of M.

Let $Aut(\Sigma)$ be an automorphism group of a prestable curve Σ. $\phi \in Aut(\Sigma)$ acts on $((\Sigma, \vec{z}), w)$ by

$$\phi \cdot ((\Sigma, \vec{z}), w) = ((\Sigma, \overrightarrow{\phi(z_i)}), w \circ \phi^{-1})$$

We define an equivalence relation on the set of stable maps $\Sigma \longrightarrow M$ by

$$((\Sigma, \vec{z}), w) \sim ((\Sigma', \vec{z'}), w')$$

if and only if there exists $\phi \in Aut(\Sigma)$ such that

$$\phi \cdot ((\Sigma, \vec{z}), w) = ((\Sigma', \vec{z'}), w').$$

Also, we call $\phi \in Aut(\Sigma)$ an **automorphism of** $((\Sigma, \vec{z}), w)$ if

$$\phi \cdot ((\Sigma, \vec{z}), w) = ((\Sigma, \vec{z}), w).$$

We denote the set of automorphisms of $((\Sigma, \vec{z}), w)$ by $Aut((\Sigma, \vec{z}), w)$, and the cardinality of $Aut((\Sigma, \vec{z}), w)$ is finite due to the stability condition.

Definition 18.2. Let $\beta \in \pi_2(M, L)$. We denote by $\overline{\mathcal{M}}_{k+1}(L, \beta)$ the set of equivalence classes of $((\Sigma, (z_0, \cdots, z_k)), w)$ with homotopy class β. We also define the evaluation maps $ev_i : \overline{\mathcal{M}}_{k+1}(L, \beta) \longrightarrow L$ by

$$ev_i([((\Sigma, (z_0, \cdots, z_k)), w)]) = w(z_i).$$

We note that the evaluation maps are well defined.

Theorem 18.3. When L is spin, $\overline{\mathcal{M}}_{k+1}(L, \beta)$ can be oriented in a way that

$$\partial o_{k+1}(\beta) = o_{k_1+1}(\beta_1) \# o_{k_2+1}(\beta_2),$$

where $k = k_1 + k_2$, and $\beta = \beta_1 + \beta_2 \in \pi_2(M, L)$.

This theorem implies that we can orient each moduli space $\overline{\mathcal{M}}_{k+1}(L, \beta)$ so that the orientation of the boundary of each moduli space is compatible with the orientation induced by the gluing map.

Theorem 18.4.

$$\begin{aligned} & dim \overline{\mathcal{M}}_{k+1}(L, \beta) \\ = & \mu(\beta) + dim L - dim PSL(2, \mathbb{R}) + k + 1 \\ = & \mu(\beta) + dim L + k - 2, \end{aligned}$$

where $\mu(\beta)$ is the Maslov index of $\beta \in \pi_2(M, L)$.

Now, we construct an A_∞-algebra associated to a Lagrangian submanifold L in a symplectic manifold M. First, we need a graded module over a specific ring.

Definition 18.5. (**Novikov ring**) Let R be a commutative ring with identity. Then the Novikov ring Λ^R_{nov} is

$$\Lambda^R_{nov} = \{\sum_{i=0}^{\infty} a_i T^{\lambda_i} | a_i \in R, \lim_{i \to \infty} \lambda_i = \infty\}.$$

We also consider

$$\Lambda^R_{0,nov} = \{\sum_{i=0}^{\infty} a_i T^{\lambda_i} | a_i \in R, \lim_{i \to \infty} \lambda_i = \infty, \lambda_i \geq 0\}.$$

Here, R could be $\mathbb{Z}_2, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, or $\mathbb{Q}[e]$. For the A_∞-algebra associated to a Lagrangian submanifold, we usually use

$$\Lambda^{\mathbb{Q}[e]}_{0,nov} = \{\sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{\frac{d_i}{2}} | a_i \in R, \lim_{i \to \infty} \lambda_i = \infty, \lambda_i \geq 0\}.$$

For $\Lambda^{\mathbb{Q}[e]}_{0,nov}$, we set $deg\ e = 2$, and $deg\, T = 0$.

By definition, Λ^R_{nov} is filtered by $\mathbb{R}_{\geq 0}$ which defines a non-Archimedean topology induced by the valuation $\nu : \Lambda^R_{nov} \longrightarrow \mathbb{R}$ such that

$$\nu(\textstyle\sum_{i=0}^{\infty} a_i T^{\lambda_i}) = \min\{\lambda_i\}$$
$$\nu(y_1 + y_2) \geq \min\{\nu(y_1), \nu(y_2)\}$$

Note that the strict inequality holds only when the "initial" terms of y_1 and y_2 are cancelled out. This valuation defines a norm $e^{-\nu} : \Lambda^R_{nov} \longrightarrow \mathbb{R}_{\geq 0}$, and so induces a topology on Λ^R_{nov}.

Next, we consider a graded module over $\Lambda_{0,nov}^{\mathbb{Q}[e]}$. We take a countably generated chain complex $C(L,\mathbb{Q})$ whose cohomology group is isomorphic to the singular cohomology group of L over $\mathbb{Q}$. Here, we regard $C(L,\mathbb{Q})$ as a cochain complex: A chain $P \in C(L,\mathbb{Q})$ of $dim P$ has cohomological degree. That is, $deg P = codim P$. We denote by deg' the shifted degree. That is, $deg' P = deg P - 1$. Then we denote $C(L,\mathbb{Q})\widehat{\otimes}\Lambda_{0,nov}^{\mathbb{Q}[e]}$ by $C(L,\Lambda_{0,nov})$. Here, the grading is given as follows:

$$deg(Pe^dT^\lambda) = deg(P) + 2d$$

Moreover, we can give $C(L,\Lambda_{0,nov})$ a topology as follows. Define $val : C(L,\Lambda_{0,nov}) \longrightarrow \mathbb{R}_{\geq 0}$ by

$$val(\sum c_i P_i T^{\lambda_i} e^{q_i}) = min\{\lambda_i\},$$

and then e^{-val} defines a norm and thus topology on $C(L,\Lambda_{0,nov})$. Also, we define the filtration on $C(L,\Lambda_{0,nov})$ and $C(L,\Lambda_{0,nov})^{\otimes k}$ as follows:

$$F^\lambda C(L,\Lambda_{0,nov}) = \{x \in C(L,\Lambda_{0,nov}) | val(x) \geq \lambda\}$$

$$F^\lambda(C(L,\Lambda_{0,nov})^{\otimes k}) = \bigcup_{\lambda_1+\cdots+\lambda_k \geq \lambda} F^{\lambda_1}C(L,\Lambda_{0,nov}) \otimes \cdots \otimes F^{\lambda_k}C(L,\Lambda_{0,nov})$$

For the operations $m_k : C(L,\Lambda_{0,nov})^{\otimes k} \longrightarrow C(L,\Lambda_{0,nov})$, we first define $m_{k,\beta}$ for each given $\beta \in \pi_2(M,L)$.

$$m_{k,\beta}(P_1,\cdots,P_k) = [\overline{\mathcal{M}}_{k+1}(L,\beta)_{ev+} \times (P_1 \times \cdots \times P_k), ev_0],$$

where $ev+ = (ev_1,\cdots,ev_k)$.(See figure 20) We define $m_{0,0} = 0$ and $m_{1,0} = (-1)^n\partial$, where ∂ is the classical boundary operator. Then define

$$m_k(P_1,\cdots,P_k) = \sum_{\beta \in \pi_2(M,L)} m_{k,\beta}(P_1,\cdots,P_k)T^{\omega(\beta)}e^{\frac{\mu(\beta)}{2}}$$

Here, we remark that for the right hand side to make sense or lie in $C(L,\Lambda_{0,nov})$, we need to show that the right hand side satisfies the following Novikov finiteness condition: For each $\lambda > 0$, there are only finite number of β such that $0 \leq \omega(\beta) \leq \lambda$. This fact is a consequence of Gromov compactness theorem.

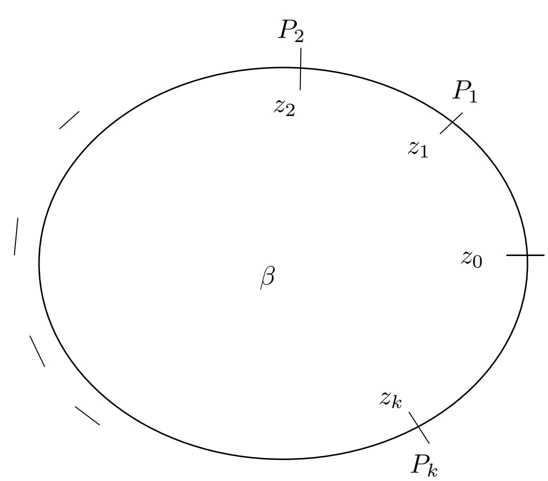

FIGURE 20. $m_{k,\beta}(P_1,\cdots,P_k)$

We check the operations m_k has degree 1 for all k in the shifted complex $C(L, \Lambda_{0,nov})[1]$. Note that

$$dim(\overline{\mathcal{M}}_{k+1}(L,\beta)_{ev+} \times (P_1 \times \cdots \times P_k)) = \mu(\beta) + dimL + k + 2 - \sum_{i=1}^{k}(n - dimP_i),$$

and

$$deg(m_{k,\beta}(P_1, \cdots, P_k)) = \sum_{i=1}^{k} degP_i - k + 2 - \mu(\beta).$$

Hence,

$$deg'(m_{k,\beta}(P_1, \cdots, P_k)) = \sum_{i=1}^{k} deg'P_i + 1 - \mu(\beta),$$

and

$$deg'(m_{k,\beta}(P_1, \cdots, P_k)T^{\omega(\beta)}e^{\frac{\mu(\beta)}{2}}) = \sum_{i=1}^{k} deg'P_i + 1,$$

which implies that m_k is of degree 1.

19. Construction of A_∞ structure

Recall that we define an A_∞-algebra associated to a Lagrangian submanifold L in a symplectic manifold M in last lecture. It remains to show that the operations $m_k : C(L,\Lambda_{0,nov})^{\otimes k} \longrightarrow C(L,\Lambda_{0,nov})$ satisfy the A_∞-relation. For this purpose, we need to describe the boundary of each moduli space $\mathcal{M}_{k+1}(L,\beta)$. This is the reason why we consider the gluing map.

Consider evaluation maps $ev_0 : \overline{\mathcal{M}}_{k+1}(L,\beta_1) \longrightarrow L$ and $ev_i : \overline{\mathcal{M}}_{k+1}(L,\beta_2) \longrightarrow L$ and $ev_0 \times ev_i$. Let $\Delta \subset L \times L$ be the diagonal. Here, we **assume** that $ev_0 \times ev_i$ is transverse to Δ. We denote

$$\overline{\mathcal{M}}_{k_1+1}(L,\beta_1) \times_i \mathcal{M}_{k_2+1}(L,\beta_2) := (ev_0 \times ev_i)^{-1}(\Delta).$$

(See Figure 21)

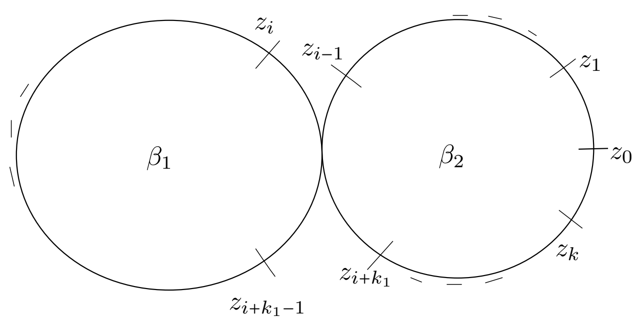

Figure 21. $\overline{\mathcal{M}}_{k_1+1}(L,\beta_1) \times_i \overline{\mathcal{M}}_{k_2+1}(L,\beta_2)$

Theorem 19.1.

$$\partial\overline{\mathcal{M}}_{k+1}(L,\beta) = \bigcup_{k_1+k_2=k} \bigcup_{i=0}^{k_2} \bigcup_{\beta_1+\beta_2=\beta} \overline{\mathcal{M}}_{k_1+1}(L,\beta_1) \times_i \overline{\mathcal{M}}_{k_2+1}(L,\beta_2)$$

Corollary 19.2. Let $P_1,\cdots,P_k \in C(L)$. Then

$$\begin{aligned}&\partial[\overline{\mathcal{M}}_{k+1}(L,\beta)_{ev+} \times (P_1 \times \cdots \times P_k), ev_0] \\ &= [\partial\overline{\mathcal{M}}_{k+1}(L,\beta)_{ev+} \times (P_1 \times \cdots \times P_k), ev_0] + \bigcup_{i=1}^{k} [\overline{\mathcal{M}}_{k+1}(L,\beta) \times (P_1 \times \cdots \times \partial P_i \times \cdots \times P_k), ev_o]\end{aligned}$$

Now, we consider the operation $\widehat{d} = \sum_{k=0}^{\infty} \widehat{m}_k$. We note that

$$\begin{aligned}
&(\widehat{d}\circ\widehat{d})_{k,\beta}\\
&= \sum_{\beta_1+\beta_2=\beta}\ \sum_{k_1+k_2=k+1}\ \sum_i (-1)^{degP_1+\cdots+degP_{i-1}+i-1}\\
&\qquad\qquad m_{k_1,\beta_1}(P_1,\cdots,m_{k_2,\beta_2}(P_i,\cdots,P_{i+k_2-1}),\cdots,P_k)\\
&= m_{1,0}m_{k,\beta}(P_1,\cdots,P_k)\\
&+\sum_i(-1)^{degP_1+\cdots+degP_{i-1}+i-1}m_{k,\beta}(P_1,\cdots,m_{1,0}(P_i),\cdots,P_k)\\
&+\sum\sum_i(-1)^{degP_1+\cdots+degP_{i-1}+i-1}m_{k_1,\beta_1}(P_1,\cdots,m_{k_2,\beta_2}(P_i,\cdots,P_{i+k_2-1}),\cdots,P_k)\\
&= (-1)^n\partial[\overline{\mathcal{M}}_{k+1}(L,\beta)_{ev+}\times(P_1\times\cdots\times P_k),ev_0]\\
&+\sum_i(-1)^{n+degP_1+\cdots+degP_{i-1}+i-1}[\overline{\mathcal{M}}_{k+1}(L,\beta)\times(P_1\times\cdots\times\partial P_i\times\cdots\times P_k),ev_o]\\
&+\sum(-1)^{degP_1+\cdots+degP_{i-1}+i-1}(\text{terms described in Figure 21})\\
&= 0
\end{aligned}$$

Hence, we have the following proposition.

Proposition 19.3. $(C(L;\Lambda_{0,nov}),m_k)$ is an A_∞-algebra.

Moreover, $(C(L;\Lambda_{0,nov}),m_k)$ defines a curved, gapped, and filtered A_∞-algebra. Here, we explain the words curved, gapped, and filtered.

(1) *Curved*
An A_∞-algebra is **curved** if m_0 is nonzero. We recall that

$$m_{0,\beta}(1) = [\overline{\mathcal{M}}_1(L;\beta),ev_0]$$

for $\beta \neq 0$ and $m_{0,0} = 0$.

(2) *Filtered*
'Filtered' means the A_∞-algebra $(C(L;\Lambda_{0,nov}),m_k)$ has the following filtration.

$$F^\lambda(C(L;\Lambda_{0,nov})) = \{x \in C(L;\Lambda_{0,nov}) | \nu(x) \geq \lambda\},$$

where ν is the valuation.

(3) *Gapped*
Note that $m_k = m_{k,0} + m'_k$,where

$$m'_k = \sum_{\beta\neq 0,\beta\in\pi_2(M,L)} m_{k,\beta}T^{\omega(\beta)}e^{\frac{\mu(\beta)}{x}}.$$

Proposition 19.4. Let (M,ω,J) be a compact almost Kahler manifold, and $L \subset M$ be a Lagrangian submanifold. Then there exists a positive constant $A = A(M,\omega,J)$ such that $\omega(w) \geq A$ for any nonconstant J-holomorphic disc $w : (D^2,\partial D^2) \longrightarrow (M,L)$.

This proposition implies that the power of T in m'_k has some gap from 0 for $\beta \neq 0$. This is what 'gapped' means. Precise definition will be given later. Also, we have the following corollary.

Corollary 19.5. $\nu(m_{k,\beta}) \geq A(M,\omega,J)$ for all $\beta \neq 0$ and $k = 0$.

Example 19.6. Let $M = \mathbb{C}P^1$, and let $L \subset M$ be a circle. We denote by D_+ and D_- discs with boundary in L such that $\text{Area}(D_+) \leq \text{Area}(D_-)$. Also, let $\beta_\pm = [D_\pm]$ and $\omega(\beta_\pm) = A_\pm$. Then $\pi_2(M, L)$ is a free abelian group with basis $\beta_\pm$. (See Figure 22.)

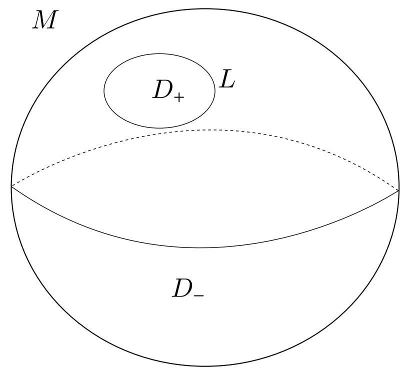

FIGURE 22. A circle L in $M = \mathbb{C}P^1$

We regard $p \in L$ be a 0-chain. Then $m_{1,0}(p) = -\partial p = 0$. Also, note that

$$dim\overline{\mathcal{M}}_{k+1}(L, \beta_\pm) = k + 1$$

since the Maslov index of $\beta_\pm$ $\mu(\beta_\pm) = 2$. In this case,

$$\overline{\mathcal{M}}_1(L, \beta_\pm) = \mathcal{M}_1(L, \beta_\pm)$$

because $\beta_\pm$ is primitive. Hence,

$$[\overline{\mathcal{M}}_1(L, \beta_\pm), ev_0] = [L],$$

where $[L] \in H_1(M, \mathbb{Z})$ is the fundamental class of L. Therefore, $m_{0,\beta_\pm}(1) = [L]$.

20. Example: on S^2

We continute the example of last time. Let $M := S^2$ be a symplectic manifold with a symplectic form ω satisfying $\int_M \omega = 4\pi$. Let L be a round circle on M, which is a Lagrangian submanifold of M. Then, M is separated into two pieces D_+, D_- of surfaces with boundary L. (See Figure 23)

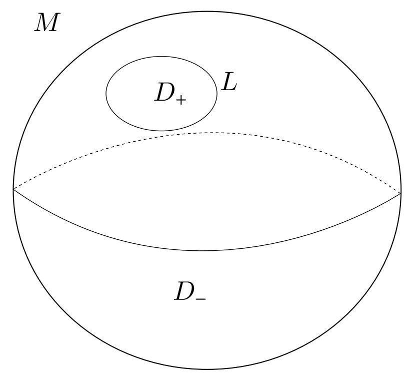

Figure 23. A circle L in $M = S^2$

Note that $\{D_+, D_-\}$ generate a relative homotopy class group $\pi_2(M.L) \simeq \mathbb{Z}\times\mathbb{Z}$. Let

$$\beta_+ := [D_+],\ \beta_- := [D_-] \quad \text{in } \pi_2(M, L)$$

and

$$A := \text{Area } (D_+), \quad B := \text{Area } (D_-).$$

Here, without loss of generality, $A \le B$ will be assumed.

We recall the definition of m_k:

$$m_k := \sum_{\beta\in\pi_2(M,L)} m_{k,\beta} T^{\omega(\beta)} e^{\mu(\beta)/2}$$

where $m_{k,\beta}$ is given by

$$m_{k,\beta}(P_1, \cdots, P_k) := [\overline{\mathcal{M}}^{\text{main}}_{k+1}(\beta)_{\text{ev}_+} \times_{L^k} (P_1 \times \cdots \times P_k), \text{ev}_0]$$

for smooth singular simplices $P_1, \cdots, P_k$ on L.

In order to see what m_k is in this example, we collect facts

(i) A Maslov index $\mu_L(D^+) = 2$

(ii) The Riemann mapping theorem asserts that there exists a holomorphic disc $u : D \longrightarrow M$ such that $u(D) = D_+$ and $[u] = \beta_+$. The same result holds for D_-.

(iii) We have (virtual) dimension formulas:

$$\dim(\overline{\mathcal{M}}^{\text{main}}_{k+1}(\beta)) = \mu_L(\beta) + \dim L + k - 2.$$

$$\dim([\overline{\mathcal{M}}^{\text{main}}_{k+1}(\beta)_{\text{ev}_+} \times_{L^k} (P_1 \times \cdots \times P_k)] = \mu_L(\beta) + \dim L + k - 2 - \sum_{i=1}^{k}(n - \dim P_i),$$

(iv) If $\mu_L(\beta) < 0$, then $\mathcal{M}(L, \beta) = \emptyset$. If $\mu_L(\beta) = 0$, then every holomorphic disc representing a relative homotopy class β is constant.

Now, we start calculation for m_0. Since $\mu_L(\beta_\pm) = 2$, $\dim L = 1$, from the dimension formula, it follows

$$\dim(\overline{\mathcal{M}}_{k+1}^{\text{main}}(\beta_\pm)) = k+1.$$

When $k = 0$, we get

$$m_{0,\beta_\pm}(1) = [\overline{\mathcal{M}}_1^{\text{main}}(\beta_\pm), \text{ev}_0]$$

where 1 is a unit of base ring R. As we observed last time,

$$[\overline{\mathcal{M}}_1^{\text{main}}(\beta_\pm), \text{ev}_0] = \pm[L].$$

Therefore, we get

$$\begin{aligned} m_{0,\beta_\pm}(1) &= [L]T^A e^1 - [L]T^B e^1 \\ &= [L]e^1(T^A - T^B). \end{aligned}$$

We observe that

$$m_0(1) \text{ iff } A = B \text{ iff } m_1^2 = 0.$$

Remark 20.1. If L bisects the area of S^2 (i.e., $A = B$), the Floer cohomology is defined:

$$HF^*(S^2; L) = \frac{\ker m_1}{\text{im } m_1}.$$

In fact,

$$HF^*(S^2; L) \simeq H^*(L) \otimes \Lambda_{0,\text{nov}}.$$

Exercise 20.2. Prove the above remark.

Let us move on to m_1, which is given by

$$m_1 := \sum_{\beta \in \pi_2(M,L)} m_{1,\beta} T^{\omega(\beta)} e^{\mu(\beta)/2}$$

where

$$m_{1,\beta}(P) := [\overline{\mathcal{M}}_1^{\text{main}}(\beta)_{\text{ev}_1} \times_L P, \text{ ev}_0].$$

When P is represented by a point q on L (i.e., $P = [q]$), the dimension is

$$\dim[\overline{\mathcal{M}}_2^{\text{main}}(\beta)_{\text{ev}_1} \times_L [q], \text{ ev}_0] = \mu_L(\beta) - 1.$$

Thus, $m_{1.\beta}([q]) \neq 0$ only when $\mu_L(\beta) = 2$ in $\pi_2(M, L)$. Furthermore, $\beta = \beta_\pm$ is the only case where it can be realized by a holomorphic disk with Maslov index 2. Therefore, it suffices to consider $m_{1,0}([q])$ and $m_{1,\beta_\pm}([q])$ in order to calculate $m_1([q])$. Therefore, we deduce

$$\begin{aligned} m_1([q]) &= m_{1,0}([q]) + \sum_{\beta \neq 0} m_{1,\beta}([q]) T^{\omega(\beta)} e^{\mu(\beta)/2} \\ &= (-1)^{\dim L} \partial([q]) + \sum_{\beta \neq 0} m_{1,\beta}([q]) T^{\omega(\beta)} e^{\mu(\beta)/2} \\ &= m_{1,\beta_+}([q]) T^{\omega(\beta_+)} e + m_{1,\beta_-}([q]) T^{\omega(\beta_-)} e \\ &= [L]e\left(T^A - T^B\right) \end{aligned}$$

Similarly, we calculate $m_1([L])$. The dimension is

$$\dim[\overline{\mathcal{M}}_2^{\text{main}}(\beta)_{\text{ev}_1} \times_L [L], \text{ ev}_0] = \mu_L(\beta).$$

so that $m_{1,\beta}([L]) = 0$ as long as $\beta \neq 0$ in $\pi_2(M, L)$. Thus, we have

$$\begin{aligned} m_1([L]) &= m_{1,0}([L]) + \sum_{\beta \neq 0} m_{1,\beta}([L]) T^{\omega(\beta)} e^{\mu(\beta)/2} \\ &= (-1)^{\dim L} \partial([L]) + \sum_{\beta \neq 0} m_{1,\beta}([L]) T^{\omega(\beta)} e^{\mu(\beta)/2} \\ &= m_{1,\beta_+}([L]) T^{\omega(\beta_+)} e + m_{1,\beta_-}([L]) T^{\omega(\beta_-)} e \\ &= 0. \end{aligned}$$

Now, we compute m_2. As we've seen in the calculation of m_1, it is easy to check that all $m_{2,\beta}(P_1, P_2)$ vanish except $m_{2,\beta_\pm}([p],[q])$ where p, q are distinct points on L. We see that

$$m_{2,\beta_\pm}([p],[q]) = [\overline{\mathcal{M}}_3^{\text{main}}(\beta_\pm)_{\text{ev}_+} \times_{L^2} ([p] \times [q]), \text{ev}_0]$$

has dimension 1. We consider a holomorphic disk $w : D^2 \longrightarrow M$ such that $[w(D)] = \beta_+$. If p and q are given in Figure 24 where $p = w(z_1), q = w(z_2)$, then the image of z_0 can be taken in I_+. Similarly, in this time, we consider a holomorphic disk

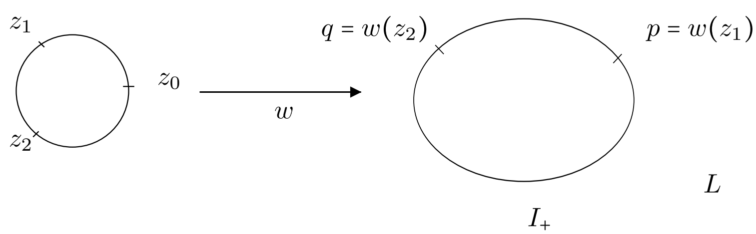

FIGURE 24. w representing β_+

$w : D^2 \longrightarrow M$ such that $[w(D)] = \beta_-$. If p and q are given in Figure 25 where $p = w(z_2), q = w(z_1)$, then the image of z_0 can be taken in I_-. Hence, we obtain

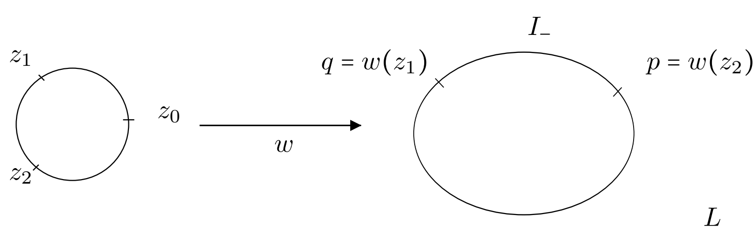

FIGURE 25. w representing β_-

$$\begin{aligned} m_{2,\beta_+}([p],[q]) &= I_+ T^A e^1 \\ m_{2,\beta_-}([p],[q]) &= I_- T^B e^1 \end{aligned}$$

and

$$m_2([p],[q]) = (I_+ T^A + I_- T^B) e^1.$$

Especially, if $A = B$, then

$$m_2([p],[q]) = [L] e^1 T^{2\pi}.$$

Remark 20.3. In this example, we observe that

$$m_0(1) = [L](T^A - T^B)e^1.$$

Note that $m_0(1)$ is a multiple of fundamental class $[L]$. If such a special case happens, then we have

$$m_1^2 = 0.$$

Note that $e = [L]$ is the unit of A_∞-algebra. i.e.,

$$m_2(e.x) = x$$

$$m_2(x.e) = (-1)^{\deg'(x)+1}x$$

and then one can see that the right hand side of the following equation vanishes

$$m_1^2(x) = m_2(m_0(1), x) + (-1)^{\deg'(x)}m_2(x, m_0(1)).$$

We achieve that $m_1^2 = 0$.

Definition 20.4. An A_∞-algebra (A, m) is called **unobstructed** if $m_0(1) = 0$. An A_∞-algebra (A, m) is called **weakly unobstructed** if $m_0(1)$ is a multiple of the unit.

21. Gapped filtered A_∞ structure

Let R be a ring (for example, $R = \mathbb{Q}$ or $\mathbb{C}$). We set up some notations as follows:

$$\Lambda_{0,\mathrm{nov}} := \left\{ \sum_i a_i T^{\lambda_i} e^{\mu_i/2} \;:\; \begin{array}{l} 0 \le \lambda_1 < \lambda_2 < \cdots \to \infty \\ \mu_i \in 2\mathbb{Z},\, a_i \in R \end{array} \right\}$$
$$:= \left\{ \sum_i b_i T^{\lambda_i} \;:\; \begin{array}{l} 0 \le \lambda_1 < \lambda_2 < \cdots \to \infty \\ b_i \in R[e, e^{-1}] \end{array} \right\}$$

and

$$\Lambda^R_{0,\mathrm{nov}} := \left\{ \sum_i b_i T^{\lambda_i} \;:\; \begin{array}{l} 0 \le \lambda_1 < \lambda_2 < \cdots \to \infty \\ b_i \in R \end{array} \right\}.$$

Let C be a graded filtered R-module. Its a bar complex $B_k C$ is defined as

$$B_0 C := R$$
$$B_k C := C[1]^{\otimes k} \text{ for } k \ge 1.$$

We consider a family of maps $m = \{m_k\}_{k \ge 0}$ where

$$m_k : B_k C \longrightarrow C[1]$$

satisfying

(i) $m_k(F^{\lambda_1} C^{m_1} \otimes \cdots \otimes F^{\lambda_k} C^{m_k}) \subset F^{\lambda_1 + \cdots + \lambda_k} C^{m_1 + \cdots + m_k - k + 2}$. (In the shifted degree, m_k has degree 1).
(ii) $m_0(1) \in F^{\lambda'} C[1]$ for some $\lambda' > 0$ and $m_{0,0}(1) = 0$.

We observe that $\Lambda_{0,\mathrm{nov}}$ has the unique maximal ideal

$$\Lambda_{+,\mathrm{nov}} = \{a \in \Lambda_{0,\mathrm{nov}} : \nu(a) > 0\}$$

with

$$\Lambda_{0,\mathrm{nov}} / \Lambda_{+,\mathrm{nov}} \simeq R.$$

Similary, we define

$$\overline{C} := F^{\lambda \ge 0} C / F^{\lambda > 0} C$$

which carries naturally induced

$$\overline{m}_k : B_k \overline{C} \longrightarrow \overline{C}[1].$$

for $k \ge 1$ with $\overline{m}_0 = 0$. Hence, $(\overline{C}, \overline{m})$ defineds a classical A_∞-algebra over R.

Proposition 21.1. We have a natural embedding $\overline{C} \hookrightarrow C$ as a level 0 part and $C \simeq \overline{C} \otimes \lambda_{0,\mathrm{nov}}$.

We call the topology on $\Lambda_{0,\mathrm{nov}}$ and C induced by ν on $\Lambda_{0,\mathrm{nov}}$ and the level $l : C \to \mathbb{R}_{\ge 0}$ a T-adic topology.

Recalling formal parameters of $\Lambda_{0,\mathrm{nov}}$, it contains two pieces of information: $\lambda \in \mathbb{R}_{\ge 0}$ and $\mu \in 2\mathbb{Z}$. Let $G \in \mathbb{R}_{\ge 0} \times 2\mathbb{Z}$ be an (additive) submonoid with unit $(0,0)$. For $\beta \in G$, we will denote its component by $\beta = (\lambda(\beta), \mu(\beta)) \in \mathbb{R}_{\ge 0} \times 2\mathbb{Z}$.

Definition 21.2. A submonoind G in $\mathbb{R}_{\ge 0} \times 2\mathbb{Z}$ is called a **Novikov monoid** if it satisfies

(i) $\lambda(G)$ is discrete,
(ii) $G \cap \{\{0\} \times 2\mathbb{Z}\} = \{(0,0)\}$,
(iii) $G \cap \{\{\lambda\} \times 2\mathbb{Z}\}$ is finite for all λ.

In the case of (M, L), we first introduce

$$G(L,J)_0 := \{(\omega(\beta), \mu(\beta)) : \beta \in \pi_2(M,L),\ \mathcal{M}_J(L,\beta) \neq \varnothing\}$$

which may not be a monoid. Then, we consider the monoid $G(L)$ generated by $G(L,J)_0$.

Definition 21.3. Let G be a Novikov submonoid of $\mathbb{R}_{\geq 0} \times 2\mathbb{Z}$. We say

$$m_k : B_k \longrightarrow C[1]$$

G-gapped if m_k has a decomposition

$$m_k = \sum_{\beta \in G} T^{\lambda(\beta)} e^{\mu(\beta)/2} m_{k,\beta}$$

where

$$m_{k,\beta} : B_k \overline{C} \longrightarrow \overline{C}[1].$$

Remark 21.4. Let C be a filtered complex with T-adic topology, which is complete. In general, $C \otimes C$ may not be complete and so we define

$$B_k C = C[1] \widehat{\otimes} \cdots \widehat{\otimes} C[1].$$

and then $\widehat{BC}$ is defined as the completion of $\oplus B_k C$. Next time(??), we will clarify this.

Definition 21.5. A structure of a filtered A_∞-algebra on a filtered $\Lambda_{0,\text{nov}}$-module C is a sequence of $\{m_k\}_{k=0,1,\cdots}$

$$m_k : B_k C \longrightarrow C[1] \ \text{ and } \ m_0(1) \in F^{\lambda > 0} C[1]$$

of degree 1 such that

$$\delta \circ \delta = 0$$

where $\delta := \sum_{k=0}^{\infty} \widehat{m}_k$. Here, $\widehat{m}_k$ is the coderivation induced by m_k:

$$\widehat{m}_k(x_1 \otimes \cdots \otimes x_n) := \sum_{i=1}^{n} \pm x_1 \otimes \cdots \otimes x_{i-1} \otimes m_k(x_i, \cdots, x_{i+k-1}) \otimes x_{i+k} \otimes \cdots \otimes x_n$$

.

Let (C_1, m_1) and (C_2, m_2) be filtered A_∞-algebras.

Definition 21.6. A sequence of maps $f_k : B_k C_1 \longrightarrow C_2[1]$ of degree 0 with

(1) $f_k(F^\lambda B_k C_1) \subset F^\lambda C_2[1]$
(2) $f_0(1) \in F^\lambda C_2[1]$ for some $\lambda > 0$

is called a filtered A_∞ homomorphism if its associated coalgebra map $\hat{f} : BC_1 \longrightarrow BC_2$ satisfies $\hat{f} \circ \widehat{d_1} = \widehat{d_2} \circ \hat{f}$.

Remark 21.7. We do not assume that $f_0(1) =$, but (2) implies that $f_{0,0}(1) = 0$. Any such map induces a coalgebra map $\hat{f} : \widehat{B}C_1 \longrightarrow \widehat{B}C_2$, which is continuous. In particular,

$$\hat{f}(1) = f_0(1) + f_0(1) \otimes f_0(1) + f_0(1) \otimes f_0(1) \otimes f_0(1) + \cdots$$

is well defined.

Definition 21.8. We say an A_∞ homomorphism $\{f_k : B_k C_1 \longrightarrow C_2[1]\}_{k=0}^{\infty}$ is G-gapped for a given submonoid $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ if f_k has decomposition

$$f_k = \sum_{\beta \in G} f_{k,\beta} T^{\lambda(\beta)} e^{\frac{\mu(\beta)}{2}}$$

for $f_{k,\beta} : B_k \overline{C}_1[1] \longrightarrow \overline{C}_2[1]$

Example 21.9. Consider A_∞-algebras $C_i = C(L_i, \Lambda_{0,nov})$ associated to Lagrangian submanifold L_i in a symplectic manifold $M.(i = 1,2)$. If an A_∞ homomorphism $f_k : B_k C_1 \longrightarrow C_2[1]$ ia G-gapped, then each $f_{k,\beta} : C(L_1)^{\otimes k} \longrightarrow C(L_2)$ is a K-linear map, where K is the base field.

22. A_K-MODULES AND A_K-HOMOMORPHISMS

Let (A, m) be a strict A_∞-algebra. Let M be a (right) A_∞-module over A_∞-algebra A along with a sequence of structure maps $\eta = \{\eta_k\}_{k=0}^\infty$ each of which

$$\eta_k : M \otimes B_k A = M \otimes A[1]^{\otimes k} \longrightarrow M$$

is of (shifted) degree 1 and obeys

$$\begin{aligned} 0 = &\sum_{i=0}^{k} \eta_{k-i}(\eta_i(v, a_1, \cdots, a_i), a_{i+1}, \cdots, a_k) \\ &+ \sum_{j=1}^{k} \sum_{i=1}^{k-j+1} (-1)^\star \eta_{k-j+1}(v, a_1, \cdots, a_{i-1}, m_j(a_i, \cdots, a_{i+j-1}), a_{i+j}, \cdots, a_k) \end{aligned}$$

for any $v \in M$ and $a_i \in A[1]$. Here, $\star = \deg' v + \deg' a_1 + \cdots + \deg' a_{i-1}$. We recall that a sequence of structure maps $\{\eta_k\}_{k=0}^\infty$ can be extended to a sequence of BA-comodule homomorphisms $\{\widehat{\eta}_k\}_{k=0}^\infty$ where

$$\widehat{\eta}_k : M \otimes BA \longrightarrow M \otimes BA$$

can be obtained by extending linealy from

$$\begin{aligned} \widehat{\eta}_k(v, a_1, \cdots, a_n) = &\ \eta_k(v, a_1, \cdots, a_k) \otimes a_{k+1} \otimes \cdots \otimes a_n \\ &+ \sum_{i=1}^{n-k+1} (-1)^\star v \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes m_k(a_i, \cdots, a_{i+k-1}) \otimes a_{i+k} \otimes \cdots \otimes a_n \end{aligned}$$

for $v \in M$ and $a_i \in A[1]$. Again, $\star = \deg' v + \deg' a_1 + \cdots + \deg' a_{i-1}$. Letting

$$\widehat{\eta} := \sum_{k=0}^{\infty} \widehat{\eta}_k,$$

we've seen that $\widehat{\eta} \circ \widehat{\eta} = 0$ if and only if M is a (right) A_∞-module with respect to η.

Let $\psi = \{\psi_k\}_{k=0}^\infty$ be a prehomomorphism between two A_∞-modules (M, η^M) and (N, η^N). That is, $\psi = \{\psi_k\}_{k=0}^\infty$ is a sequence of R-module homomorphisms

$$\psi_k : M \otimes B_k A = M \otimes A[1]^{\otimes k} \longrightarrow N$$

such that the degree of each ψ_k is always 0. Similar to a sequence of structure maps, we can extend $\psi = \{\psi_k\}_{k=0}^\infty$ into a sequence $\widehat{\psi} = \{\widehat{\psi}_k\}_{k=0}^\infty$ where

$$\widehat{\psi}_k : M \otimes BA \longrightarrow N \otimes BA$$

is obtained from

$$\widehat{\psi}_k(v, a_1, \cdots, a_n) = \psi_k(v, a_1, \cdots, a_k) \otimes a_{k+1} \otimes \cdots \otimes a_n$$

for $v \in M$ and $a_i \in A$. Letting

$$\widehat{\psi} := \sum_{k=0}^{\infty} \widehat{\psi}_k,$$

we've seen that $\widehat{\eta}^N \circ \widehat{\psi} = \widehat{\psi} \circ \widehat{\eta}^M$ on $M \otimes BA$ if and only if ψ is an A_∞-module homomorphism from M to N over A. Equivalently, ψ is a coboundary in Hochschild complex, i.e., $\delta(\psi) = 0$ where

$$\delta(\psi) := \widehat{\eta}^N \circ \widehat{\psi} - (-1)^{\deg(\psi)} \widehat{\psi} \circ \widehat{\eta}^M$$

is defined under the identification

$$CH_A(M, N) \simeq \mathrm{Co}Mod_{BA}(M, N).$$

To prove the Whitehead theorem for A_∞-module homomorphisms, we've introduced notions of A_K-module homomorphism and A_K-homotopy.

Definition 22.1. Let (M,η^M) and (N,η^N)be two A_∞-modules over A_∞-algebra A. An A_K**-module homomorphism** $\psi = \{\psi_k\}_{k=0}^K$ from M to N over A is a finite sequence of R-module homomorphisms

$$\psi_k : M \otimes B_k A = M \otimes A[1]^{\otimes k} \longrightarrow N$$

of the degree 0 and satisfying

$$\widehat{\eta}^N_{\le K} \circ \widehat{\psi}_{\le K} = \widehat{\psi}_{\le K} \circ \widehat{\eta}^M_{\le K}.$$

where

$$\widehat{\eta}^M_{\le K} = \sum_{k=0}^K \widehat{\eta}^M_k : \prod_{k=0}^K M \otimes B_k A \longrightarrow \prod_{k=0}^K M \otimes B_k A$$

$$\widehat{\psi}_{\le K} = \sum_{k=0}^K \widehat{\psi}_k : \prod_{k=0}^K M \otimes B_k A \longrightarrow \prod_{k=0}^K N \otimes B_k A.$$

It is equivalent to say that

$$\delta(\psi)\Big|_{\prod_{k=0}^K M\otimes B_k A} = 0.$$

We note any A_∞-module homomorphism induces an A_K-module homomorphism for any $K \ge 0$. We inverstigate some basic property of A_K-module homomorphism. For $m_1, m_2 \in \mathbb{N}$ satisfying $m_1 < m_2$, we set

$$M \otimes B_{m_1,\cdots,m_2} := \frac{\prod_{k=0}^{m_2} M \otimes B_k A}{\prod_{k=0}^{m_1-1} M \otimes B_k A} \simeq \prod_{k=m_1}^{m_2} M \otimes B_k A.$$

Note we can restrict any module homomorphisms on $M \otimes BA$ to $\prod_{k=0}^m M \otimes B_k A$ and on $M \otimes B_{m_1,\cdots,m_2}$

Lemma 22.2. Let (M,η^M) and (N,η^N)be two A_∞-modules over A_∞-algebra A. Let $\psi = \{\psi_k\}_{k=0}^K$ be a A_K-module homomorphism. Then, for any $m_1, m_2 \in \mathbb{N}$ satisfying $0 < m_2 - m_1 < K$, ψ induces a module homomorphism. Namely,

$$\widehat{\psi}_{m_1,\cdots,m_2} := \sum_{k=m_1}^{m_2} \widehat{\psi}_k : M \otimes B_{m_1,\cdots,m_2} \longrightarrow M \otimes B_{m_1,\cdots,m_2}.$$

satisfies

$$\widehat{\psi}_{m_1,\cdots,m_2} \circ \widehat{\eta}_{m_1,\cdots,m_2} = \widehat{\eta}_{m_1,\cdots,m_2} \circ \widehat{\psi}_{m_1,\cdots,m_2}.$$

Definition 22.3. Let ψ^1 and ψ^2 be A_K-module homomorphisms over A_∞-algebra A. An A_K**-homotopy** $T = \{T_k\}_{k=0}^K$ is a (finite) sequence of

$$T_k : M \otimes B_k A = M \otimes A[1]^{\otimes k} \longrightarrow N$$

of the degree -1 and satisfying

$$\widehat{\psi}^1 - \widehat{\psi}^2 = \delta(T)$$

on $\prod_{k=0}^K M \otimes B_k A$.

For induction argument for the Whitehead theorem, we would like to extend A_K-module homomorphism to A_{K+1}-module homomorphisms up to homotopy. Unfortunately, it turns out that the extension is not always possible. We now investigate what an obstruction is.

Definition 22.4. Let $\psi = \{\psi_k\}_{k=0}^K$ be an A_K-module homomorphism. Setting $\psi_{K+1} = 0$, we let $\psi_{\le K+1} = \{\psi_k\}_{k=0}^{K+1}$. We define the **associated A_{K+1}-obstruction** $\widehat{O}_{K+1}$ of ψ as follows:

$$\widehat{O}_{K+1}(\psi) : \prod_{k=0}^{K+1} M \otimes B_k A \longrightarrow \prod_{k=0}^{\infty} N \otimes B_k A.$$

is given by

$$\widehat{O}_{K+1}(\psi) = \widehat{\eta}^N_{\le K+1} \circ \widehat{\psi}_{\le K+1} - \widehat{\psi}_{\le K+1} \circ \widehat{\eta}^M_{\le K+1}.$$

Lemma 22.5. Let $\psi = \{\psi_k\}_{k=0}^K$ be as above. Let $\widehat{O}_{K+1}(\psi)$ be the associated A_{K+1}-obstruction. Then,

$$\widehat{O}_{K+1}(\psi)\Big|_{M\otimes B_k A} = 0$$

on $M \otimes B_k A$ for $k < K+1$ and $O_{K+1}(\psi)$ on $M \otimes B_{K+1} A$ has values in $N \subset \prod_{k=0}^\infty N \otimes B_k A$.

The above lemma says that the obstruction $\widehat{O}_{K+1}(\psi)$ is meaningful on $M \otimes B_{K+1} A$, which leads to the following definition

Definition 22.6. The **associated A_{K+1}-obstruction chain** $O_{K+1}(\psi)$ is defined by

$$O_{K+1}(\psi) := \widehat{O}_{K+1}(\psi)\Big|_{M\otimes B_{K+1} A} : M \otimes B_{K+1} A \longrightarrow N.$$

Equivalently, the obstruction chain can be defined by

$$O_{K+1}(\psi) = \delta(\psi)\Big|_{M\otimes B_{K+1} A}$$

where δ is a Hochschild differential. We now explain why O_{K+1} is called a chain. We define a smaller complex

$$\delta_1 : Hom(M \otimes B_{K+1} A, N) \longrightarrow Hom(M \otimes B_{K+1} A, N)$$

by

$$\delta_1(B) = \pi_0 \circ \delta(B)\Big|_{M\otimes B_{K+1} A}.$$

In other words, for $(v, a_1, \cdots, a_{K+1})$, its value is given by

$$\delta_1(B)(v, a_1, \cdots, a_{K+1}) = \eta_0(B(v, a_1, \cdots, a_{K+1})) - (-1)^{\deg B} B(\eta_0(v), a_1, \cdots, a_{K+1})$$
$$- (-1)^{\deg B} \sum_{i=1}^{K+1} B(v, a_1, \cdots, a_{i-1}, m_1(a_i), a_{i+1}, \cdots, a_{K+1}).$$

Next time, we will prove that $\delta_1(O_{K+1}(\psi)) = 0$ so that it actually defines a cohomology class on the complex (
$Hom(M \otimes B_{K+1} A, N), \delta_1)$.

23. A_{K+1}-OBSTRUCTION CLASS $o_{K+1}(\psi)$

We recall the Hochschild complex

$$CH_A(M,N) = \prod_{i=0}^{\infty} Hom(M \otimes B_i A, N),$$

with $\delta : CH_A(M,N) \longrightarrow CH_A(M,N)$ defined by

$$\delta(\psi) = \widehat{\eta} \circ \widehat{\psi} - (-1)^{\deg(\psi)} \widehat{\psi} \circ \widehat{\eta},$$

which makes sense under the identification

$$CH_A(M,N) \simeq \mathrm{Co}Mod_{BA}(M,N).$$

Remark 23.1. Hereafter, using the above identification, we sometimes write down

$$\delta(\psi) = \delta(\widehat{\psi}).$$

For given A_K-module homomorphism $\psi = \{\psi_k\}_{k=1}^{K}$, due to the last lemma of previous lecture, the A_{K+1}-obstruction chain of ψ can be considered as a map

$$O_{K+1}(\psi) : M \otimes B_{K+1} A \longrightarrow N.$$

given by

$$O_{K+1}(\psi) = \delta(\psi)\Big|_{M \otimes B_{K+1} A}$$

We define a smaller complex

$$\delta_1 : Hom(M \otimes B_{K+1} A, N) \longrightarrow Hom(M \otimes B_{K+1} A, N)$$

by

$$\delta_1(B) = \pi_0 \circ \delta(B)\Big|_{M \otimes B_{K+1} A}.$$

In other words, for $(v, a_1, \cdots, a_{K+1}) \in M \otimes B_{K+1} A$, its value is given by

$$\begin{aligned}\delta_1(B)(v, a_1, \cdots, a_{K+1}) &= \eta_0(B(v, a_1, \cdots, a_{K+1})) - (-1)^{\deg B} B(\eta_0(v), a_1, \cdots, a_{K+1}) \\ &\quad - (-1)^{\deg B} \sum_{i=1}^{K+1} B(v, a_1, \cdots, a_{i-1}, m_1(a_i), a_{i+1}, \cdots, a_{K+1}).\end{aligned}$$

Proposition 23.2. Let $\psi = \{\psi_k\}_{k=0}^{K}$ be an A_K-module homomorphism. Let $O_{K+1}(\psi)$ be the associated A_{K+1}-obstruction class of ψ. Then,

(i) $\delta_1(O_{K+1}(\psi)) = 0$ and so defines a cohomology class on the complex

$$(Hom(M \otimes B_{K+1} A, M), \delta_1)$$

(ii) If $[O_{K+1}(\psi)] = 0$ as a δ_1-cohomology class, then there exists an A_{K+1}-module homomorphism $\psi_{\leq K+1} = \{\psi_k\}_{k=0}^{K+1}$ extending the given A_K-homomorphism ψ.

(iii) If ψ' is A_K-homotopic to ψ, then $[O_{K+1}(\psi)] = [O_{K+1}(\psi')]$ as a δ_1-cohomology class.

Proof. We start with (i). By definition and a lemma from previous lecture, we have

$$\delta_1(O_{K+1}(\psi)) = \delta(\widehat{O}_{K+1}(\psi))\Big|_{M \otimes B_{K+1} A}.$$

Note that

$$\widehat{O}_{K+1}(\psi) = 0$$

on $M \otimes B_k A$ unless $k = K+1$. For $(v, a_1, \cdots, a_{K+1}) \in M \otimes B_{K+1}A$, we have

$$\widehat{O}_{K+1}(\psi)(v, a_1, \cdots, a_{K+1}) = O_{K+1}(\psi)(v, a_1, \cdots, a_{K+1}) = \delta(\psi)(v, a_1, \cdots, a_{K+1}).$$

Therefore we derive

$$\delta_1(O_{K+1}(\psi))(v, a_1, \cdots, a_{K+1}) = \delta\delta(\psi)(v, a_1, \cdots, a_{K+1}) = 0.$$

which establishes $\delta_1(O_{K+1}(\psi)) = 0$.

For (ii), we consider the A_K-module homomorphism

$$\widehat{\psi}_{\leq K} : \prod_{k=0}^{K} M \otimes B_k A \longrightarrow \prod_{k=0}^{\infty} N \otimes B_k A.$$

given by $\widehat{\psi}_{\leq K} = \sum_{k=0}^{K} \widehat{\psi}_k$. By the given hypothesis, we have $\psi_{K+1} \in Hom(M \otimes B_{K+1}A, N)$ such that $\delta_1(\psi_{K+1}) + O_{K+1}(\psi) = 0$, i.e., $\{\psi_0, \ldots, \psi_K, \psi_{K+1}\}$ defines an A_{K+1}-module homomorphism.

Finally to prove (iii), suppose $\widehat{\psi}' - \widehat{\psi} = \delta(T)$. We compute

$$\begin{aligned} O_{K+1}(\psi') - O_{K+1}(\psi) &= O_{K+1}(\psi' - \psi) \\ &= \delta(\psi' - \psi)\big|_{M \otimes B_{\leq K+1}A}. \end{aligned}$$

By the property of $O_{K+1}(\psi' - \psi)$, we have

$$\delta(\psi' - \psi)\big|_{M \otimes B_{\leq K+1}A} = \delta(\psi' - \psi)\big|_{M \otimes B_{K+1}A}$$

and its image lies in N.

On the other hand, by the hypothesis, we have $\widehat{\psi}' - \widehat{\psi} = \delta(T)$ on $M \otimes B_{\leq K}A$ and so $\delta(T) = 0$ thereon and has its values in N. This implies

$$\delta(T) = O_{K+1}(T).$$

Then we obtain

$$O_{K+1}(\psi') - O_{K+1}(\psi) = \delta(\psi' - \psi)\big|_{M \otimes B_{K+1}A} = \delta(\psi' - \psi)\big|_{M \otimes B_{\leq K}A} + \delta(\psi' - \psi)\big|_{M \otimes B_{K+1}A}.$$

For the first term, we have

$$\delta(\psi' - \psi)\big|_{M \otimes B_{\leq K}A} = \delta(\widehat{\psi}' - \widehat{\psi})\big|_{M \otimes B_{\leq K}A} = \delta(\delta(T))\big|_{M \otimes B_{\leq K}A} = 0$$

where we use the fact that $\widehat{\psi' - \psi} = \widehat{\psi}' - \widehat{\psi}$ for the first equality and δ respects the length filtration for the second equality. Therefore, we obtain

$$O_{K+1}(\psi') - O_{K+1}(\psi) = \delta(\psi' - \psi)\big|_{M \otimes B_{K+1}A} = \delta_1(O_{K+1}(T)).$$

In particular $[O_{K+1}(\psi')] = [O_{K+1}(\psi)]$. □

Proposition 23.3. Let $\rho : M \to N$ be an A_∞-quasi-isomorphism and $\psi : N \to M$ an A_K-module homomorphism such that $\psi \circ \rho \cong id$ in A_K-homotopy. Then ψ can be extended to an A_{K+1}-module homomorphism $\psi_{\leq K+1}$ such that $\psi_{\leq K+1} \circ \rho \cong id$ in A_{K+1}-homotopy.

Proof. We start with the following lemmata

Lemma 23.4. If ψ' and ψ are A_0-homotopic, then $\psi_* = \psi'_*$ in H^*.

By definition, there exists an A_0-homomorphism T such that

$$\widehat{\psi}' - \widehat{\psi} = \delta(T)|_M.$$

But this equation on the right is nothing but

$$\delta(T)(v) = \widehat{\eta} \circ \widehat{T}(v) - (-1)^{\deg' T} \widehat{T} \circ \widehat{\eta}(v) = \eta_0 \circ T_0(v) + T_0 \circ \eta_0(v)$$

for all $v \in M$. This finishes the proof since T_0 provides a homotopy map between ψ_0 and ψ_0'. □

We go back to the proof of the proposition. By hypothesis,

$$[O_{K+1}(\psi \circ \rho)] = [O_{K+1}(id)] = 0$$

where the first follows from Proposition 23.2 (3) and the second is obvious since id is an A_∞-module homomorphism.

We also have

Lemma 23.5.

$$[O_{K+1}(\psi \circ \rho)] = [O_{K+1}(\psi) \circ (\rho_0 \circ id_{B_{K+1}A})]$$

Proof. By definition, we have

$$O_{K+1}(\psi \circ \rho) - O_{K+1}(\psi) \circ (\rho_0 \circ id_{B_{K+1}A}) = \delta_1((\psi \circ \rho)) - \delta_1(\psi) \circ (\rho_0 \circ id_{B_{K+1}A}).$$

We evaluate this for $(v, a_1, \cdots, a_{K+1})$ and derive

$$\begin{aligned}
&\delta_1((\psi \circ \rho))(v, a_1, \cdots, a_{K+1}) \\
&= \eta_0((\psi \circ \rho)_{K+1}(v, a_1, \cdots, a_{K+1}) - (-1)\eta_{K+1}((\psi \circ \rho)_0(v), a_1, \cdots, a_{K+1}) \\
&+ \sum_{1 \le j \le K} (-1)^* \eta_i((\psi \circ \rho)_j(v, a_1, \cdots, a_j), a_{j+1}, \cdots, a_{K+1}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& (O_{K+1}(\psi) \circ (\rho_0 \circ id_{B_{K+1}A})(v, a_1, \cdots, a_j), a_{j+1}, \cdots, a_{K+1}) \\
= \;& O_{K+1}(\psi)(\rho_0 \circ id_{B_{K+1}A}(v, a_1, \cdots, a_j), a_{j+1}, \cdots, a_{K+1})) \\
= \;& O_{K+1}(\psi)(\rho_0(v), a_1, \cdots, a_{K+1}) \\
= \;& \delta_1(\psi)(\rho_0(v), a_1, \cdots, a_{K+1}) \\
= \;& \eta_0(\psi_{K+1}(\rho_0(v), a_1, \cdots, a_{K+1}) - (-1)\eta_{K+1}(\psi_0(\rho_0(v), a_1, \cdots, a_{K+1})
\end{aligned}$$

Subtracting the second from the first, we derive

$$\begin{aligned}
& \delta_1((\psi \circ \rho))(v, a_1, \cdots, a_{K+1}) - (O_{K+1}(\psi) \circ (\rho_0 \circ id_{B_{K+1}A})(v, a_1, \cdots, a_j), a_{j+1}, \cdots, a_{K+1}) \\
= \;& \eta_0(\psi_0((\rho_{K+1}(v, a_1, \cdots, a_{K+1}))) \\
& + \sum_{1 \le j \le K} (-1)^* \eta_i((\psi \circ \rho)_j(v, a_1, \cdots, a_j), a_{j+1}, \cdots, a_{K+1}) \\
= \;& \delta_1(\psi_{\le K} \circ \rho).
\end{aligned}$$

This finishes the proof. □

We note that

$$[O_{K+1}(\psi) \circ (\rho_0 \circ id_{B_{K+1}A})] = (\rho)_*[O_{K+1}(\psi)].$$

Since ρ is A_∞-quasi-isomorphism (and so A_0-quasi-isomorphism),

$$[O_{K+1}(\psi) \circ (\rho_0 \circ id_{B_{K+1}A})] = 0 \text{ if and only if } [O_{K+1}(\psi)] = 0.$$

But we know $[O_{K+1}(\psi)\circ(\rho_0 \circ id_{B_{K+1}A})] = [O_{K+1}(\psi\circ\rho)] = 0$ and hence $[O_{K+1}(\psi)] = 0$.

Therefore we can pick $\widetilde{\psi}_{K+1}$ and h_{K+1} so that they extend ψ and $\psi\circ\rho$ respectively to A_{K+1}-homomorphisms. The first property implies

$$\delta_1(\widetilde{\psi}\circ\rho) + O_{K+1}(\widehat{\psi\circ\rho_{\le K}}) = 0$$
$$\delta_1(h_{\le K+1}) + O_{K+1}(h_{\le K}) = 0.$$

But recall that $O_{K+1}(\psi)$ depends only on $\psi_{\le K}$ by definition. Since, by construction, both $h_{\le K+1}$ and

$$\widetilde{\psi}\circ\rho = \widetilde{\psi}_{K+1}\circ(\rho_0\otimes id) + \psi_{\le K}\circ\rho)$$

are A_{K+1}-homomorphisms that extend $\psi\circ\rho$ and hence their O_{K+1} coincide.

This implies

$$0 = \delta_1(\widetilde{\psi}_{K+1}\circ\rho_0\otimes id + \psi_{\le K}\circ\rho - h_{\le K+1})$$

Since $\rho_0\otimes id$ induces an isomorphism in $(H^*(\cdot,\delta_1))$, we can find a δ_1-cycle $g_{K+1}\in \mathrm{Hom}(N\otimes B_{K+1}A, M)$ such that

$$[g_{K+1}\circ\rho_0\otimes id] = [\widetilde{\psi}_{K+1}\circ\rho_0\otimes id + \psi_{\le K}\circ\rho - h_{\le K+1}].$$

Finally, we define

$$\psi_{K+1} := \widetilde{\psi}_{K+1} - g_{K+1}.$$

Exercise 23.6. Prove ψ_{K+1} satisfies the requirements:

(1) $\psi_{\le K+1}$ extends ψ,
(2) $(\psi\circ\rho)_{\le K+1}$ extends $\psi\circ\rho$,
(3) $\psi_{\le K+1}\circ\rho \cong id$ in A_{K+1}-homotopy.

In conclusion, we have shown that if ρ is an A_∞ quasi-isomorphism, then there exists an A_∞-homomorphism ψ such that $\psi\circ\rho\cong id$. In particular, ψ itself is an A_∞-quasi-isomorphism. Finally to show ψ is also homotopy left-inverse, we construct the A_∞-homomorphism φ for ψ by the same construction used for ψ so that $\varphi\circ\psi\cong id$. Then we obtain the chain of homotopy equivalence

$$\rho\cong\varphi\circ\psi\circ\rho\cong\varphi.$$

Therefore, we derive $\rho\circ\psi\cong\varphi\circ\psi\cong id$ which finishes the proof of the theorem, finally.

24. Canonical model(unfiltered case): statement

Recall that we defined A_∞-algebra associated to a Lagrangian submanifold L in a symplectic manifold M in Lecture 19 and 20. But this A_∞-algebra is defined on chain level, which makes computation hard. So we need to reduce a given A_∞-algebra to carry out computation. This is why we introduce the notion of a canonical model of A_∞-algebra. Let (A, m) be an A_∞-algebra. We assume the base ring R is a field.

Definition 24.1.

(1) An unfiltered A_∞-algebra with $m_1 = 0$ is called a **canonical model**.
(2) A gapped filtered A_∞-algebra is called a **canonical model** if $m_{1,0} = 0$.

Then we can reduce an A_∞-algebra to a canonical model.

Theorem 24.2.

(1) Any unfiltered A_∞-algebra (A, m) is homotopy equivalent to a canonical model.
(2) Any gapped filtered A_∞-algebra is homotopy equivalent to a canonical model. Moreover, the homotopy equivalence can be taken as a gapped A_∞-homomorphism.

We will prove this theorem through several lectures. From now on, we focus on the strict case, that is $m_0 = 0$, which implies $m_1 \circ m_1 = 0$.

First, we note that we can pick a subspace H^ℓ which satisfies assumptions of the following lemma since R is a field. Then we consider the associated idempotent $\Pi : A^\ell \longrightarrow A^\ell$ such that $i(H^\ell) = \text{Image}(\Pi)$ and $\Pi \circ \Pi = \Pi$. In other words, Π is the compostion of the projection $p : A^\ell \to i(H^\ell)$ and the inclusion $i(H^\ell) \hookrightarrow A^\ell$.

Lemma 24.3. Fix an embedding $i : H^\ell \to A^\ell$ that satisfies

(1) $i(H^\ell) \subset \ker m_1$ and
(2) the composition $H^\ell \to \ker m_1 \cap A^\ell \to H^\ell(A, m_1)$ induces an isomorphism.

Then there exist a sequence of maps $G^\ell : A^\ell \to A^{\ell-1}$ such that

(1) $\Pi - id = m_1 \circ G^\ell + G^{\ell+1} \circ m_1$,
(2) $G^\ell \circ G^{\ell+1} = 0$

Proof. We denote $Z^\ell = \ker m_1^\ell \cap A^\ell$ and $B^\ell = \mathfrak{I} m_1^{\ell-1} \cap A^\ell$. We recall the basic exact sequences

$$0 \to B^\ell \to Z^\ell \to H^\ell(A, m_1) \to 0$$
$$0 \to Z^\ell \to A^\ell \to B^{\ell+1} \to 0$$

where the left maps are inclusion maps and the second map on the top is the quotient π and the second map on the bottom is the map induced by m_1. Then we have the natural splitting

$$Z^\ell = i(H^\ell) \oplus B^\ell$$

such that the restriction of the quotient map π to $i(H^\ell)$ induces an isomorphism $i(H^\ell) \cong H^\ell(A, m_1)$ given above. We then take a splitting map of the second exact sequence so that

$$A^\ell = Z^\ell \oplus B^{\ell+1,\prime}$$

and so we have the decomposition

$$A^\ell = i(H^\ell) \oplus B^\ell \oplus B^{\ell+1,\prime}.$$

Then we define $G^\ell : A^\ell \to A^{\ell-1}$ by

$$G^\ell(a) = \begin{cases} -m_1^{-1}(a) & a \in B^\ell \\ 0 & a \in i(H^\ell) \oplus B^{\ell+1,\prime}. \end{cases}$$

Here, we note that m_1 induces an isomorphism $B^{\ell,\prime} \cong B^\ell$. Therefore, $m_1^{-1}(a)$ is well defined. Then it follows $G^\ell \circ G^{\ell+1} = 0$ by construction. On the other hand, we compute

$$m_1 \circ G^\ell(a) + G^{\ell+1} \circ m_1(a)$$

If $a \in B^\ell$, we obtain $(m_1 \circ G^\ell)(a) + (G^{\ell+1} \circ m_1)(a) = m_1 \circ (-m_1^{-1}(a)) = -a$. On the other hand, we have $\Pi(a) = 0$. Therefore (2) holds for this case. If $a \in i(H^\ell)$, then $a - \Pi(a) = 0$. On the other hand we also obtain $m_1 \circ G^\ell(a) + G^{\ell+1} \circ m_1(a) = 0$. Finally when $a \in B^{\ell+1,\prime}$, we obtain $a - \Pi(a) = a$, while

$$m_1 \circ G^\ell(a) + G^{\ell+1} \circ m_1(a) = 0 + G^{\ell+1}(m_1(a)) = -a$$

because m_1 restricts to an isomorphism $B^{\ell+1,\prime} \to B^{\ell+1}$ by construction and then $G^{\ell+1}$ inverts back by definition. This finishes the proof. □

Theorem 24.4. Let $i : H \to A$ be the inclusion. Then
(1) There exists an A_∞ structure on H such that $m_1^H = p \circ m_1 \circ i$
(2) The inclusion i extends to an A_∞-homomorphism $f = \{f_k\}_{k=1}^\infty$ such that $f_1 = i$.

Now, we overview the construction of A_∞ structure on H. We consider planar rooted trees T with $k+1$ vertices, v_0 the root vertex, and an embedding $i : T \to d^2$ so that $i^{-1}(\partial D^2) = C^0_{ext}(T)$, where $C^0_{ext}(T)$ is the set of vertices with valence 1. We assume T is stable. We denote by G_{k+1} the set of stable planar rooted trees with $C^0_{ext}(T) = k+1$. For each $\Gamma \in G_{k+1}$, we construct $m_\Gamma : B_k H \to H[1]$ of degree 1 and $f_\Gamma : B_k H \to A[1]$ of degree 0. Define

$$m_k^H = \sum_{\Gamma \in G_{k+1}} m_\Gamma : B_k H \longrightarrow H[1]$$
$$f_k = \sum_{\Gamma \in G_{k+1}} f_\Gamma : B_k H \longrightarrow A[1]$$

Then we show that (H, m^H) defines an A_∞-algebra and $f = \{f_k\}_{k=1}^\infty$ is an A_∞-homomorphism.

Now we explain the definition of m_Γ and f_Γ. First we define

$$m_1^H = p \circ m_1 \circ i$$
$$f_1 = i$$

We have to define m_1^H separately since there does not exist a stable planar rooted tree with two exterior vertices. Let $k \geq 2$. For given $\Gamma \in G_{k+1}$, we associate another tree $\overline{\Gamma}$ by inserting a vertex to each interior edge. Then $\overline{\Gamma}$ has three kinds of vertices, leaves(exterior vertices), "old" interior vertices, and "new" interior vertices. We assign maps to each vertex.
(1) To every leaf vertex, assign the inclusion i.
(2) To every "old" interior vertices, assign m_l if the valence is $l+1$.
(3) To every "new" interior vertices, assign G.
(4) To the root vertex, assign p for m_Γ and G for f_Γ.

Then, as we move down the tree down to the root vertex, we reads off maps when passing through vertices.

Example 24.5. Let Γ be given as the left picture of Figure 26. The right picture of Figure 26 is $\overline{\Gamma}$.

$$m_\Gamma(a_1,\cdots,a_6) = (p\circ m_2)((G\circ m_2)(a_1,a_2), G(m_2((G\circ m_3)(a_3,a_4,a_5),a_6)))$$
$$f_\Gamma(a_1,\cdots,a_6) = (G\circ m_2)((G\circ m_2)(a_1,a_2), G(m_2((G\circ m_3)(a_3,a_4,a_5),a_6)))$$

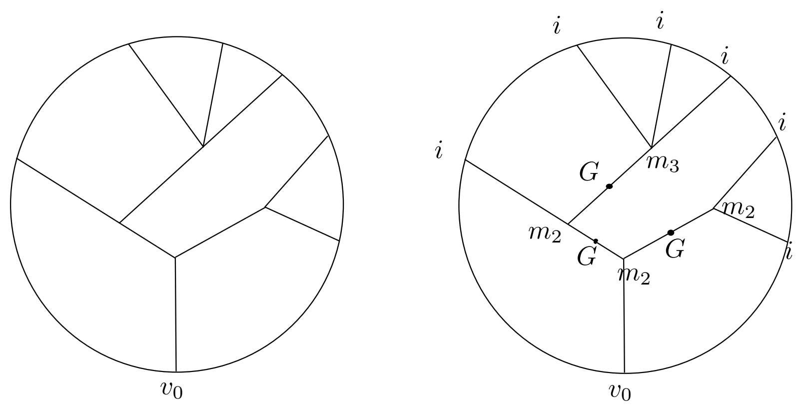

FIGURE 26. Γ and $\overline{\Gamma}$

25. Canonical model(unfiltered case): proof

We have defined $m_\Gamma : B_k H \to H[1]$ of degree 1 and $f_\Gamma : B_k H \to A[1]$ of degree 0 for a given $\Gamma \in G_{k+1}$ in the previous lecture. We need to express m_Γ and f_Γ in another way. Suppose that $\Gamma \in G_{k+1}$ is given. Let v_0 be the root vertex of Γ and v_1 the vertex closest to v_0. Cut Γ at v_0. Then Γ is decomposed into stable rooted trees $\Gamma^{(1)}, \cdots, \Gamma^{(l)}$ and an interval toward v_0 in counterclockwise order. (See Figure 27.)For $k \geq 2$, we have

$$m_\Gamma = \sum_{l \neq 1} p \circ m_l(f_{\Gamma^{(1)}} \otimes \cdots \otimes f_{\Gamma^{(l)}})$$
$$f_\Gamma = \sum_{l \neq 1} G \circ m_l(f_{\Gamma^{(1)}} \otimes \cdots \otimes f_{\Gamma^{(l)}}),$$

by definition. In other words,

$$m_\Gamma(\mathbf{x}) = \sum_{l \neq 1} (p \circ m_l)(f_{\Gamma^{(1)}}(\mathbf{x}_a^{(1)}) \otimes \cdots \otimes (f_{\Gamma^{(1)}}(\mathbf{x}_a^{(l)})))$$
$$f_\Gamma(\mathbf{x}) = \sum_{l \neq 1} (G \circ m_l)(f_{\Gamma^{(1)}}(\mathbf{x}_a^{(1)}) \otimes \cdots \otimes (f_{\Gamma^{(1)}}(\mathbf{x}_a^{(l)}))),$$

where $\mathbf{x} \in B_k H$ $(k \geq 2)$ and

$$\Delta^{l-1}\mathbf{x} = \sum_a \mathbf{x}_a^{(1)} \otimes \cdots \otimes \mathbf{x}_a^{(l)}$$

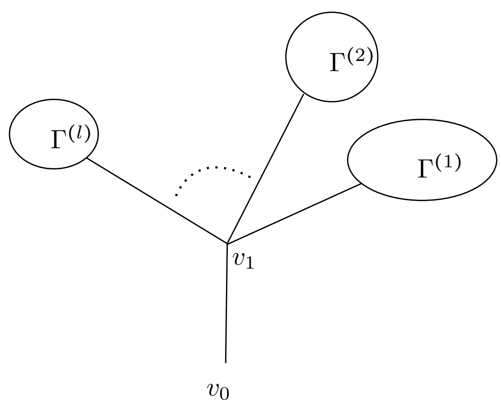

Figure 27. Decomposition of Γ

Then we consider the associated graded coderivation $\widehat{m}^H : BH \longrightarrow BH$ and the associated coalgebra map $\hat{f} : BH \longrightarrow BA$. We note that $\Gamma \in G_{k+1}$ is determined by the subtrees $\Gamma^{(1)}, \cdots, \Gamma^{(l)}$ if we fix the counterclockwise order. From this observation, we have

Lemma 25.1.

$$f_k = \sum_{l \neq 1} G \circ m_l \circ \hat{f}$$
$$\widehat{m}_k^H = \sum_{l \neq 1} p \circ m_l \circ \hat{f}$$

on $B_k H$ for $k \geq 2$.

Proposition 25.2. $\hat{f} \circ \widehat{m}^H = \widehat{m} \circ \hat{f}$.

Before proving the proposition, we discuss the result of the proposition.

Corollary 25.3. $\widehat{m}^H \circ \widehat{m}^H = 0$.

Proof. First, we note that $f_1 = i$ is the inclusion. Hence, using number filtration, we can easily prove that $\hat{f}$ is injective. However,

$$\hat{f} \circ \widehat{m}^H \circ \widehat{m}^H = \widehat{m} \circ \hat{f} \circ \widehat{m}^H = \widehat{m} \circ \widehat{m} \circ \hat{f} = 0$$

as a map $BH \longrightarrow BA$. Hence, $\widehat{m}^H \circ \widehat{m}^H = 0$. □

This corollary implies that (H, m^H) is an A_∞-algebra and $f = \{f_k\}_{k=1}^\infty$ is an A_∞-homomorphism. Moreover, we know that $m_1^H = 0$. In other words, (H, m^H) is a canonical model. Also, $f_1 = i$ induces an isomorphism between m_1-cohomology of A and m_1^H-cohomology of H by construction. Therefore, $f = \{f_k\}_{k=1}^\infty$ is a homotopy equivalence due to the following Whitehead type theorem.

Theorem 25.4. A weak homotopy equivalence of A_∞-algebras is a homotopy equivalence.

This proves the first part of Theorem 27.2. It remains to prove Proposition 28.2.

Proof. (Proposition 28.2) First, note that $\hat{f} \circ \widehat{m}^H = \widehat{m} \circ \hat{f}$ if and only if $f \circ \widehat{m}^H = m \circ \hat{f}$. We prove by induction over k by proving

$$f \circ \widehat{m}^H = m \circ \hat{f}$$

on $B_{\le k}H$. Let us denote this equation by $(*)_{\le k}$.

Suppose that $k = 1$. Note that $\widehat{m}^H|_{B_1H=H[1]} = m_1^H = p \circ m_1 \circ i$, and $\hat{f}|_{B_1H=H[1]} = f_1 = i$ and so by construction of H, both sides of $(*)_1$ is 0.

Now we assume $(*)_{\le k}$ holds. We want to prove $(*)_{\le k+1}$, and so we evaluate $m \circ \hat{f}$ on $B_{k+1}H$.

$$\begin{aligned}&(m \circ \hat{f})(x_1, \cdots, x_{k+1})\\ =&m_1(f_{k+1}(x_1, \cdots, x_{k+1})) + \sum_{l \neq 1}(m_l \circ \hat{f})(x_1, \cdots, x_{k+1})\end{aligned}$$

Therefore, on $B_{k+1}H$

$$\begin{aligned}&m \circ \hat{f}\\ =&m_1(f_{k+1}) + \sum_{l\neq 1}(m_l \circ \hat{f})\\ =&m_1(\sum_{l\neq 1} G \circ m_l \circ \hat{f}) + \sum_{l \neq 1}(m_l \circ \hat{f})\\ =&\sum_{l\neq 1}(m_1 \circ G \circ m_l \circ \hat{f} + m_l \circ \hat{f})\\ =&\sum_{l \neq 1}(-G \circ m_1 \circ m_l \circ \hat{f} + i \circ p \circ m_l \circ \hat{f})\end{aligned}$$

For the first term, note that

$$\sum_{l\neq 1}(-m_1 \circ m_l) = \sum_{l \neq 1} m_l \circ \widehat{m}$$

by the A_∞ relation $m \circ \widehat{m} = 0$. Hence, the first term equals

$$\sum_{l\neq 1}(G \circ m_l \circ \widehat{m} \circ \hat{f}).$$

Since $\widehat{m}$ does not increase the length and m_1 is removed in the sum, nontrivial contribution $\hat{f}(x_1, \cdots, x_k)$ does not involve $f_{k+1}(x_1, \cdots, x_k)$. That is, the sum involves only $f_{\leq k}$. Hence, by induction hypothesis,

$$\widehat{m} \circ \hat{f} \equiv \hat{f} \circ \widehat{m}^H \mod H[1] = B_1 H,$$

which implies that

$$\sum_{l\neq 1}(G \circ m_l \circ \widehat{m} \circ \hat{f}) = \sum_{l\neq 1}(G \circ m_l \circ \hat{f} \circ \widehat{m}^H)$$

on $B_{k+1}H$. For the second term,

$$\begin{aligned}
&\sum_{l\neq 1}(i \circ p \circ m_l \circ \hat{f}) \\
&= i \circ \sum_{l\neq 1}(p \circ m_l \circ \hat{f}) \\
&= i \circ m_{k+1}^H \\
&= f_1 \circ m_{k+1}^H
\end{aligned}$$

on $B_{k+1}H$ by Lemma 28.1. Therefore,

$$\begin{aligned}
&m \circ \hat{f} \\
&= (\sum_{l\neq 1}(G \circ m_l \circ \hat{f} + f_1)) \circ \widehat{m}^H + f_1 \circ mk + 1^H \\
&= (f_2 + f_3 + \cdots) \circ \widehat{m}^H + f_1 \circ mk + 1^H \\
&= f \circ \widehat{m}^H
\end{aligned}$$

on $B_{k+1}H$ by Lemma 28.1, which finishes the proof. □

26. Canonical model:filtered case

In the previous lecture, we proved that there exists a canonical model which is homotopy equivalent to a given unfiltered strict A_∞-algebra. Now we consider the filtered case. Let (C,m) be a G-gapped filtered A_∞-algebra. Here, G is a submonoid of $\mathbb{R}_{\geq 0}\times\mathbb{Z}$. We write $\beta\in G$ as $\beta=(\lambda(\beta),\mu(\beta))$. Recall that we assume $\lambda^{-1}(0)=\{(0,0)\}$ and Novikov finiteness condition, that is, $\lambda^{-1}([0,c])$ is a finite set for any $c\geq 0$. Also, $C=\overline{C}\otimes\lambda_{0,nov}$ and

$$m_k=\sum_{\beta\in G} m_{k,\beta}T^{\lambda(\beta)}e^{\frac{\mu(\beta)}{2}},$$

where $m_{k,\beta}:B_k\overline{C}\longrightarrow\overline{C}[1]$ and $m_{0,0}=0$. Due to Novikov finiteness condition, we can enumerate $\lambda(\beta)$ for $\beta\in G$ and denote them by $\lambda_{(i)}$.

$$0=\lambda_{(0)}<\lambda_{(1)}<\lambda_{(2)}<\cdots$$

Now we explain the construction of a canonical model for the filtered case. As we did for the unfiltered case, we fix an embedding $i:H^\ell\to\overline{C}^\ell$ such that $i(H^\ell)\subset\ker m_1$ and the composition $H^\ell\to\ker m_1\cap\overline{C}^\ell\to H^\ell(\overline{C},\overline{m_1})$ induces an isomorphism. Then we identify $i(H^\ell)$ with H^ℓ and fix a projection $p:\overline{C}^\ell\longrightarrow H^\ell$. Define $\Pi=i\circ p:\overline{C}^\ell\longrightarrow\overline{C}^\ell$. We write the operations

$$m_k=\sum_{i=0}^{\infty} m_{k,i}T^{\lambda_{(i)}}$$

where $m_{k,i}:B_kH\otimes R[e,e^{-1}]\longrightarrow H[1]\otimes R[e,e^{-1}]$. Here,

$$m_{k,i}=\sum_{\lambda_\beta=\lambda_{(i)}} m_{k,\beta}e^{\frac{\mu(\beta)}{2}}$$

Definition 26.1. A **decorated rooted tree** is a quintuple (T,i,v_0,V_{tad},η) such that

(1) (T,i,v_0) is a rooted tree, not necessarily stable.
(2) $V_{tad}=$\{vertices of valence 1\}$-\ C^0_{ext}$.
(3) $\eta:C^0_{int}(T)\longrightarrow\{0,1,2,\cdots\}$ is a function such that $\eta(v)>0$ if the valence of v is 1 or 2.

We denote by G^+_{k+1} the set of decorated rooted trees (T,i,v_0,V_{tad},η) with $\#(C^0_{ext}(T))=k$.

We remark that we regard V_{tad} as a subset of $C^0_{int}(T)$. For given $\Gamma=(T,i,v_0,V_{tad},\eta)\in G^+_{k+1}$, we define energy of Γ by

$$E(\Gamma)=\sum_{v\in C^0_{int}(T)}\lambda_{\eta(v)}$$

Remark 26.2. G^+_{k+1} is the dual graph of stable maps of open Riemann surface with genus 0 and Lagrangian boundary condition. This is the reason why we assign a positive number to v with valence 1 or 2. The restriction of a stable map to an irreducible component of prestable curve with 1 or 2 special points should be nonconstant, and thus has positive energy.

Definition 26.3. Let Γ and Γ' be elements of G^+_{k+1}. We say $\Gamma' > \Gamma$ if either $E(\Gamma') > E(\Gamma)$ or $E(\Gamma') = E(\Gamma)$ and $k' > k$.

The construction of m_Γ and f_Γ now is in order.

step 1. The case that $C^0_{int}(T) = 0$.

Such T consists of two exterior vertices and an edge joining them. Therefore, there is a unique element $\Gamma_0 \in G^+_{k+1}$.(See Figure 28.) We define

$$m_{\Gamma_0} = \overline{m}|_{H[1]}$$

and

$$f_{\Gamma_0} : H[1] \otimes R[e, e^{-1}] \longrightarrow \overline{C}[1] \otimes R[e, e^{-1}]$$

to be the inclusion i.

step 2. The case that $C^0_{int}(T) = 1$.

For any $k = 0, 1, 2, \cdots$, there is a unique element with $C^0_{ext}(T) = k+1$ and $C^0_{int}(T) = 1$ in G^+_{k+1}. Let Γ_{k+1} be a decorated rooted tree with one interior vertex v in G^+_{k+1}.(See Figure 28.) We define

$$m_{\Gamma_{k+1}} = p \circ m_{k,\eta(v)} : B_k H \otimes R[e, e^{-1}] \longrightarrow H[1] \otimes R[e, e^{-1}]$$
$$f_{\Gamma_{k+1}} = G \circ m_{k,\eta(v)} : B_k H \otimes R[e, e^{-1}] \longrightarrow \overline{C}[1] \otimes R[e, e^{-1}].$$

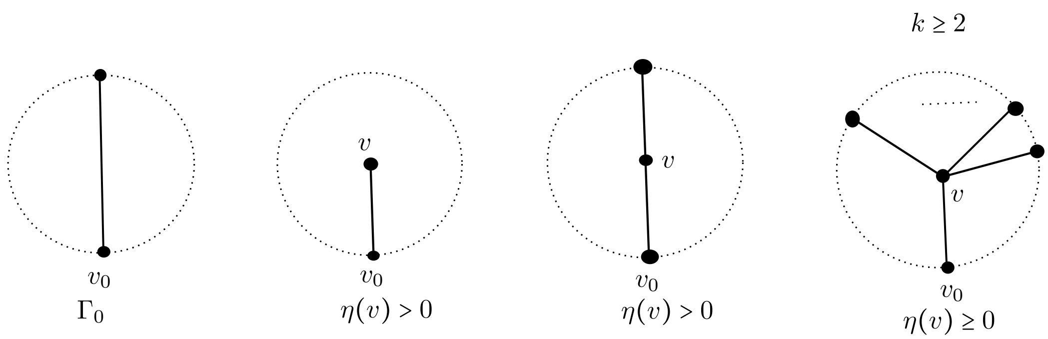

FIGURE 28. Γ_0 and Γ_{k+1}

step 3. General case

Suppose that $\Gamma \in G^+_{k+1}$ is given. Let v_0 be the root vertex of Γ and v_1 the vertex closest to v_0. Cut Γ at v_0. Then Γ is decomposed into decorated rooted trees $\Gamma^{(1)}, \cdots, \Gamma^{(l)}$ and an interval toward v_0 in counterclockwise order. (See Figure 29.) Then we define

$$m_\Gamma = p \circ m_{l,\eta(v_1)} \circ (f_{\Gamma^{(1)}} \otimes \cdots \otimes f_{\Gamma^{(l)}})$$
$$f_\Gamma = G \circ m_{l,\eta(v_1)} \circ (f_{\Gamma^{(1)}} \otimes \cdots \otimes f_{\Gamma^{(l)}})$$

Finally, $m^H_k : B_k H \longrightarrow H[1]$ and $f_k : B_k H \longrightarrow C[1]$ are defined by

$$m^H_k = \sum_{\Gamma \in G^+_{k+1}} T^{E(\Gamma)} m_\Gamma$$
$$f_k = \sum_{\Gamma \in G^+_{k+1}} T^{E(\Gamma)} f_\Gamma$$

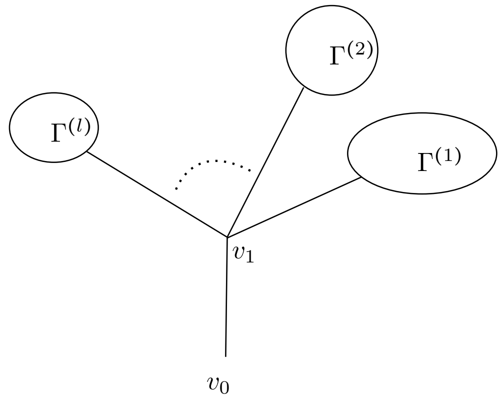

FIGURE 29. Decomposition of Γ

Theorem 26.4. $(BH \otimes \lambda_{0,nov}, m^H)$ defines a filtered A_∞-algebra and $f = \{f_k\}_{k=0}^\infty$ is a gapped filtered A_∞-quasi-isomorphism.

This proves the second part of Theorem 27.2 combining with the following Whitehead type theorem for the filtered case.

Theorem 26.5. Any gapped weak A_∞ homotopy equivalence between gapped filtered A_∞-algebras is a homotopy equivalence.

27. Bounding cochains and potential function

Let (C, m) be a unital filtered gapped A_∞-algebra. Let $\mathbf{e}$ be the unit. We look at first two A_∞ relations.

$$m_1(m_0(1)) = 0$$

$$m_2(m_0(1), x) + (-1)^{|x|'} m_2(x, m_0(1)) + m_1(m_1(x)) = 0$$

If $m_0(1) = \lambda \mathbf{e}$ for some $\lambda \in \Lambda_{0,nov}$, then we easily find that $m_1 \circ m_1 = 0$. In other words, we can define m_1-cohomology in the case that $m_0(1)$ is a constant multiple of the unit. This is why we want to deform the operations m_k by some cochain in C so that we can define cohomology using these new operations. Hence, we need to pick out some specific class of cochains.
First we introduce notation. Recall that BC has the filtration.

Definition 27.1. For $b \in F^\lambda BC$ with $\lambda > 0$, we define

$$e^b = 1 + b + b \otimes b + b \otimes b \otimes b + \cdots$$

Note that this is an infinite sum and the condition $\lambda > 0$ guarantees that e^b is well-defined.

Definition 27.2. A cochain $b \in F^\lambda C[1]^0$ with $\lambda > 0$ is called a **bounding cochain** if $\widehat{m}(e^b) = 0$.

Lemma 27.3. For any $b \in F^\lambda C[1]^0$ with $\lambda > 0$, the map $\Phi^b : BC \longrightarrow BC$ defined by

$$\Phi^b(x_1 \otimes \cdots \otimes x_k) = e^b \otimes x_1 \otimes e^b \otimes x_2 \otimes e^b \otimes \cdots \otimes e^b \otimes x_k \otimes e^b$$

is a coalgebra homomorphism.

Proof. Using $\Delta(e^b) = e^b \otimes e^b$, it is easy to check that $\Delta \circ \Phi^b = (\Phi^b \otimes \Phi^b) \circ \Delta$. □

Definition 27.4. Let $(C, \{m_k\}_{k=0}^\infty)$ be a gapped filtered A_∞-algebra and $b \in F^\lambda C[1]^0$ with $\lambda > 0$. We define new operations $\{m_k^b\}_{k=0}^\infty$ by

$$m_k^b(x_1 \otimes \cdots \otimes x_k) = (m \circ \Phi^b)(x_1 \otimes \cdots \otimes x_k),$$

where the operation $m : BC \longrightarrow C[1]$ is defined $m|_{B_k C[1]} = m_k$.

Note that $m_0^b(1) = m(e^b)$.

Theorem 27.5. $(C, \{m_k^b\}_{k=0}^\infty)$ defines a new A_∞-algebra.

Proposition 27.6. Consider the new A_∞-algebra $(C, \{m_k^b\}_{k=0}^\infty)$ in the previous theorem. Then b is a bounding cochain if and only if $m_0^b(1) = 0$.

Proof. We compute

$$\begin{aligned} &\widehat{m}(e^b) \\ =& \sum_{k,l \geq 0} \widehat{m}_k(b^{\otimes l}) \\ =& \sum_{k_0,k_1,k \geq 0} b^{\otimes k_0} \otimes m_k(b^{\otimes k}) \otimes b^{\otimes k_1} \\ =& e^b m(e^b) e^b \end{aligned}$$

Therefore, $\widehat{m}(e^b) = 0$ if and only $m(e^b) = 0$. Also, we note that $m_0^b(1) = m(e^b)$. This finishes the proof. □

This proposition implies that if we deform the operations m_k using a bounding cochain, then we can define cohomology since $m_1^b \circ m_1^b = 0$. However, we can still define cohomology in the case that $m_0^b(1)$ is not 0, but a constant multiple of unit. Now we want to pick out that kind of cochains.

Definition 27.7. Let $(C, \{m_k\}_{k=0}^\infty)$ be a unital gapped filtered A_∞-algebra and $b \in F^\lambda C[1]^0$ with $\lambda > 0$. We call b a weak bounding cochain if

$$m(e^b) = ce\mathbf{e},$$

for some $c \in \Lambda_{0,nov}^{(0)}$, where $\Lambda_{0,nov}^{(0)}$ is the degree 0 part of $\Lambda_{0,nov}^+$. e is the formal parameter of $\Lambda_{0,nov}$. We denote by $\widehat{\mathcal{M}}_{weak}(C)$ the set of all weak bounding cochains.

Since $m_0^b(1) = m(e^b)$, $m_1^b \circ m_1^b = 0$. Therefore, the deformed operation m_1^b defines cohomology when b is a weak bounding cochain.

Definition 27.8. We say two weak bounding cochains b and b' are **gauge equivalent** if there exists $c \in C[1]^{-1}$ such that $b' - b = m(e^b c e^{b'})$. We denote by $\mathcal{M}_{weak}(C)$ the set of gauge equivalence classes of weak bounding cochains.

Theorem 27.9. Let $b, b' \in \widehat{\mathcal{M}}_{weak}(C)$. Then $H^*(C, m_1^b)$ is isomorphic to $H^*(C, m_1^{b'})$ if b is gauge equivalent to b'.

Definition 27.10. We define a function $\mathcal{PO} : \widehat{\mathcal{M}}_{weak}(C) \longrightarrow \Lambda_{0,nov}^{(0)}$ by the equation

$$m(e^b) = \mathcal{PO}(b)e\mathbf{e}$$

We call this function a **potential function**.

Theorem 27.11.

(1) If b is gauge equivalent to b', then $\mathcal{PO}(b) = \mathcal{PO}(b')$. Hence $\mathcal{PO}$ descends to $\mathcal{M}_{weak}(C)$.

(2) $H^*(C, m_1^b) \neq 0$ if and only if the differential $d(\mathcal{PO}(b)) = 0$.

Center for Geometry and Physics, Institute for Basic Sciences (IBS), Pohang, Korea & Department of Mathematics, POSTECH, Pohang, Korea

E-mail address: yongoh1@postech.ac.kr