

# Floer homology in symplectic geometry and mirror symmetry

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**Abstract.** In this article, the authors review what the Floer homology is and what it does in symplectic geometry both in the closed string and in the open string context. In the first case, the authors will explain how the chain level Floer theory leads to the  $C^0$  symplectic invariants of Hamiltonian flows and to the study of topological Hamiltonian dynamics. In the second case, the authors explain how Floer's original construction of Lagrangian intersection Floer homology is obstructed in general as soon as one leaves the category of exact Lagrangian submanifolds. They will survey construction, obstruction and promotion of the Floer complex to the  $A_\infty$  category of symplectic manifolds. Some applications of this general machinery to the study of the topology of Lagrangian embeddings in relation to symplectic topology and to mirror symmetry are also reviewed.

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## 1. Prologue

The Darboux theorem in symplectic geometry manifests *flexibility* of the group of symplectic transformations. On the other hand, the following celebrated theorem of Eliashberg [E11] revealed subtle *rigidity* of symplectic transformations : *The subgroup  $Symp(M, \omega)$  consisting of symplectomorphisms is closed in  $Diff(M)$  with respect to the  $C^0$ -topology.*

This demonstrates that the study of symplectic topology is subtle and interesting. Eliashberg's theorem relies on a version of non-squeezing theorem as proven by Gromov [Gr]. Gromov [Gr] uses the machinery of pseudo-holomorphic curves to prove his theorem. There is also a different proof by Ekeland and Hofer [EH] of the classical variational approach to Hamiltonian systems. The interplay between these two facets of symplectic geometry has been the main locomotive in the development of symplectic topology since Floer's pioneering work on his 'semi-infinite' dimensional homology theory, now called the *Floer homology theory*.

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As in classical mechanics, there are two most important boundary conditions in relation to Hamilton's equation  $\dot{x} = X_H(t, x)$  on a general symplectic manifold : one is the *periodic* boundary condition  $\gamma(0) = \gamma(1)$ , and the other is the *Lagrangian* boundary condition  $\gamma(0) \in L_0, \gamma(1) \in L_1$  for a given pair  $(L_0, L_1)$  of two Lagrangian submanifolds : A submanifold  $i : L \hookrightarrow (M, \omega)$  is called Lagrangian if  $i^*\omega = 0$  and  $\dim L = \frac{1}{2} \dim M$ . The latter replaces the *two-point* boundary condition in classical mechanics.

In either of the above two boundary conditions, we have a version of the *least action principle* : a solution of Hamilton's equation corresponds to a critical point of the action functional on a suitable path space with the corresponding boundary condition. For the periodic boundary condition, we consider the free loop space

$$\mathcal{LM} = \{\gamma : S^1 \rightarrow M\}$$

and for the Lagrangian boundary condition, we consider the space of paths connecting

$$\Omega(L_0, L_1) = \{\gamma : [0, 1] \rightarrow M \mid \gamma(0) \in L_0, \gamma(1) \in L_1\}.$$

Both  $\mathcal{LM}$  and  $\Omega(L_0, L_1)$  have countable number of connected components. For the case of  $\mathcal{LM}$ , it has a distinguished component consisting of the contractible loops. On the other hand, for the case of  $\Omega(L_0, L_1)$  there is no such distinguished component in general.

**Daunting Questions.** For a given time dependent Hamiltonian  $H = H(t, x)$  on  $(M, \omega)$ , does there exist a solution of the Hamilton equation  $\dot{x} = X_H(t, x)$  with the corresponding boundary conditions? If so, how many different ones can exist?

One crucial tool for the study of these questions is the least action principle. Another seemingly trivial but crucial observation is that when  $H \equiv 0$  for the closed case and when  $L_1 = L_0$  (and  $H \equiv 0$ ) for the open case, there are "many" solutions given by *constant* solutions. It turns out that these two ingredients, combined with Gromov's machinery of pseudo-holomorphic curves, can be utilized to study each of the above questions, culminating in Floer's proof of Arnold's conjecture for the fixed points [F12], and for the intersection points of  $L$  with its Hamiltonian deformation  $\phi_H^1(L)$  [F11] for the exact case respectively.

We divide the rest of our exposition into two categories, one in the closed string and the other in the open string context.

## 2. Floer homology of Hamiltonian fixed points

**2.1. Construction.** On a symplectic manifold  $(M, \omega)$ , for each given time-periodic Hamiltonian  $H$  i.e.,  $H$  with  $H(t, x) = H(t + 1, x)$ , there exists an analog  $\mathcal{A}_H$  to the classical action functional defined on a suitable covering space of the free loop space. To exploit the fact that in the vacuum, i.e., when  $H \equiv 0$  we have many constant solutions all lying in the distinguished component of the free loop space  $\mathcal{L}(M)$

$$\mathcal{L}_0(M) = \{\gamma : [0, 1] \rightarrow M \mid \gamma(0) = \gamma(1), \gamma \text{ contractible}\},$$

one studies the contractible periodic solutions and so the action functional on  $\mathcal{L}_0(M)$ . The covering space, denoted by  $\tilde{\mathcal{L}}_0(M)$ , is realized by the set of suitable equivalence classes  $[z, w]$  of the pair  $(z, w)$  where  $z : S^1 \rightarrow M$  is a loop and  $w : D^2 \rightarrow M$  is a disc bounding  $z$ . Then  $\mathcal{A}_H$  is defined by

$$\mathcal{A}_H([z, w]) = - \int w^* \omega - \int_0^1 H(t, \gamma(t)) dt. \quad (2.1)$$

This reduces to the classical action  $\int pdq - H dt$  if we define the canonical symplectic form as  $\omega_0 = \sum_j dq^j \wedge dp_j$  on the phase space  $\mathbb{R}^{2n} \cong T^*\mathbb{R}^n$ .

To do Morse theory, one needs to introduce a metric on  $\Omega(M)$ , which is done by introducing an almost complex structure  $J$  that is compatible to  $\omega$  (in that the bilinear form  $g_J := \omega(\cdot, J\cdot)$  defines a Riemannian metric on  $M$ ) and integrating the norm of the tangent vectors of the loop  $\gamma$ . To make the Floer theory a more flexible tool to use, one should allow this  $J$  to be time-dependent.

A computation shows that the negative  $L^2$ -gradient flow equation of the action functional for the path  $u : \mathbb{R} \times S^1 \rightarrow M$  is the following *nonlinear* first order partial differential equation

$$\frac{\partial u}{\partial \tau} + J \left( \frac{\partial u}{\partial t} - X_H(t, u) \right) = 0. \quad (2.2)$$

The rest points of this gradient flow are the periodic orbits of  $\dot{x} = X_H(t, x)$ . Note that when  $H = 0$ , this equation becomes the pseudo-holomorphic equation

$$\bar{\partial}_J(u) = \frac{\partial u}{\partial \tau} + J \frac{\partial u}{\partial t} = 0 \quad (2.3)$$

which has many constant solutions. Following Floer [Fl2], for each given *nondegenerate*  $H$ , i.e., one whose time-one map  $\phi_H^1$  has the linearization with no eigenvalue 1, we consider a vector space  $CF(H)$  consisting of *Novikov Floer chains*

**Definition 2.1.** For each formal sum

$$\beta = \sum_{[z, w] \in \text{Crit } \mathcal{A}_H} a_{[z, w]} [z, w], \quad a_{[z, w]} \in \mathbb{Q} \quad (2.4)$$

we define the *support* of  $\beta$  by the set

$$\text{supp } \beta = \{[z, w] \in \text{Crit } \mathcal{A}_H \mid a_{[z, w]} \neq 0 \text{ in (2.4)}\}.$$

We call  $\beta$  a Novikov Floer chain or (simply a Floer chain) if it satisfies the condition

$$\#\{[z, w] \in \text{supp } \beta \mid \mathcal{A}_H([z, w]) \geq \lambda\} < \infty$$

for all  $\lambda \in \mathbb{R}$  and define  $CF(H)$  to be the set of Novikov Floer chains.

$CF(H)$  can be considered either as a  $\mathbb{Q}$ -vector space or a module over the Novikov ring  $\Lambda_\omega$  of  $(M, \omega)$ . Each Floer chain  $\beta$  as a  $\mathbb{Q}$ -chain can be regarded as the union of ‘‘unstable manifolds’’ of the generators  $[z, w]$  of  $\beta$ , which has a ‘peak’. There is the natural Floer boundary map  $\partial = \partial_{(H, J)} : CF(H) \rightarrow CF(H)$  i.e., a

linear map satisfying  $\partial\partial = 0$ . The pair  $(CF(H), \partial_{(H,J)})$  is the *Floer complex* and the quotient

$$HF_*(H, J; M) := \ker \partial_{(H,J)} / \text{im } \partial_{(H,J)}$$

the *Floer homology*. By now the general construction of this Floer homology has been carried out by Fukaya-Ono [FOn], Liu-Tian [LT1], and Ruan [Ru] in its complete generality, after the construction had been previously carried out by Floer [F12], Hofer-Salamon [HS] and by Ono [On] in some special cases.

The Floer homology  $HF_*(H, J; M)$  also has the ring structure arising from the pants product, which becomes the *quantum product* on  $H^*(M)$  in “vacuum” i.e., when  $H \equiv 0$ . The module  $H^*(M) \otimes \Lambda_\omega$  with this ring structure is the *quantum cohomology ring* denoted by  $QH^*(M)$ . We denote by  $a \cdot b$  the quantum product of two quantum cohomology classes  $a$  and  $b$ .

**2.2. Spectral invariants and spectral norm.** Knowing the existence of periodic orbits of a given Hamiltonian flow, the next task is to organize the collection of the actions of different periodic orbits and to study their relationships.

We first collect the actions of all possible periodic orbits, *including their quantum contributions*, and define the *action spectrum* of  $H$  by

$$\text{Spec}(H) := \{\mathcal{A}_H([z, w]) \in \mathbb{R} \mid [z, w] \in \tilde{\Omega}_0(M), d\mathcal{A}_H([z, w]) = 0\} \quad (2.5)$$

i.e., the set of critical values of  $\mathcal{A}_H : \tilde{\mathcal{L}}_0(M) \rightarrow \mathbb{R}$ . In general this set is a countable subset of  $\mathbb{R}$  on which the (spherical) period group  $\Gamma_\omega$  acts. Motivated by classical Morse theory and mini-max theory, one would like to consider a sub-collection of critical values that are *homologically essential*: each non-trivial cohomology class gives rise to a mini-max value, which cannot be pushed further down by the gradient flow. One crucial ingredient in the classical mini-max theory is a choice of semi-infinite cycles that are linked under the gradient flow.

Applying this idea in the context of chain level Floer theory, Oh generalized his previous construction [Oh4, Oh5] to the non-exact case in [Oh8, Oh10]. We define the *level* of a Floer chain  $\beta$  by the maximum value

$$\lambda_H(\beta) := \max_{[z, w]} \{\mathcal{A}_H([z, w]) \mid [z, w] \in \text{supp } \beta\}. \quad (2.6)$$

Now for each  $a \in QH^k(M)$  and a generic  $J$ , Oh considers the mini-max values

$$\rho(H, J; a) = \inf_{\alpha} \{\lambda_H(\alpha) \mid \alpha \in CF_{n-k}(H), \partial_{(H,J)}\alpha = 0, [\alpha] = a^b\} \quad (2.7)$$

where  $2n = \dim M$  and proves that this number is independent of  $J$  [Oh8]. The common number denoted by  $\rho(H; a)$  is called the *spectral invariant* associated to the Hamiltonian  $H$  relative to the class  $a \in QH^*(M)$ . The collection of the values  $\rho(H; a)$  extend to arbitrary smooth Hamiltonian function  $H$ , *whether  $H$  is nondegenerate or not*, and satisfy the following basic properties.

**Theorem 2.2** ([Oh8, Oh10]). *Let  $(M, \omega)$  be an arbitrary closed symplectic manifold. For any given quantum cohomology class  $0 \neq a \in QH^*(M)$ , we have a continuous function denoted by  $\rho = \rho(H; a) : C_m^\infty([0, 1] \times M) \times (QH^*(M) \setminus \{0\}) \rightarrow \mathbb{R}$*

which satisfies the following axioms: Let  $H, F \in C_m^\infty([0, 1] \times M)$  be smooth Hamiltonian functions and  $a \neq 0 \in QH^*(M)$ . Then we have :

1. **(Projective invariance)**  $\rho(H; \lambda a) = \rho(H; a)$  for any  $0 \neq \lambda \in \mathbb{Q}$ .
2. **(Normalization)** For a quantum cohomology class  $a$ , we have  $\rho(\underline{0}; a) = v(a)$  where  $\underline{0}$  is the zero function and  $v(a)$  is the valuation of  $a$  on  $QH^*(M)$ .
3. **(Symplectic invariance)**  $\rho(\eta^* H; a) = \rho(H; a)$  for any  $\eta \in \text{Symp}(M, \omega)$
4. **(Homotopy invariance)** For any  $H, K$  with  $[H] = [K]$ ,  $\rho(H; a) = \rho(K; a)$ .
5. **(Multiplicative triangle inequality)**  $\rho(H \# F; a \cdot b) \leq \rho(H; a) + \rho(F; b)$
6. **( $C^0$ -continuity)**  $|\rho(H; a) - \rho(F; a)| \leq \|H - F\|$ . In particular, the function  $\rho_a : H \mapsto \rho(H; a)$  is  $C^0$ -continuous.
7. **(Additive triangle inequality)**  $\rho(H; a + b) \leq \max\{\rho(H; a), \rho(H; b)\}$

Under the canonical one-one correspondence between (smooth)  $H$  (satisfying  $\int_M H_t = 0$ ) and its Hamiltonian path  $\phi_H : t \mapsto \phi_H^t$ , we denote by  $[H]$  the path-homotopy class of the Hamiltonian path  $\phi_H : [0, 1] \rightarrow \text{Ham}(M, \omega); \phi_H(t) = \phi_H^t$  with fixed end points, and by  $\widetilde{\text{Ham}}(M, \omega)$  the set of  $[H]$ 's which represents the universal covering space of  $\text{Ham}(M, \omega)$ .

This theorem generalizes the results on the exact case by Viterbo [V2], Oh [Oh4, Oh5] and Schwarz's [Sc] to the non-exact case. The axioms 1 and 7 already hold at the level of *cycles* or for  $\lambda_H$ , and follow immediately from its definition. All other axioms are proved in [Oh8] except the homotopy invariance for the *irrational* symplectic manifolds which is proven in [Oh10]. The additive triangle inequality was explicitly used by Entov and Polterovich in their construction of some quasi-morphisms on  $\text{Ham}(M, \omega)$  [EnP]. The axiom of homotopy invariance implies that  $\rho(\cdot; a)$  projects down to  $\widetilde{\text{Ham}}(M, \omega)$ . It is a consequence of the following spectrality axiom, which is proved for *any*  $H$  on *rational*  $(M, \omega)$  in [Oh8] and just for *nondegenerate*  $H$  on *irrational*  $(M, \omega)$  [Oh10] :

8. **(Nondegenerate spectrality)** For nondegenerate  $H$ , the mini-max values  $\rho(H; a)$  lie in  $\text{Spec}(H)$  i.e., are critical values of  $\mathcal{A}_H$  for all  $a \in QH^*(M) \setminus \{0\}$ .

The following is still an open problem.

**Question 2.3.** Let  $(M, \omega)$  be a irrational symplectic manifold, i.e., the period group  $\Gamma_\omega = \{\omega(A) \mid A \in \pi_2(M)\}$  be a dense subgroup of  $\mathbb{R}$ . Does  $\rho(H; a)$  still lie in  $\text{Spec}(H)$  for all  $a \neq 0$  for *degenerate* Hamiltonian  $H$ ?

It turns out that the invariant  $\rho(H; 1)$  can be used to construct a canonical invariant norm on  $\text{Ham}(M, \omega)$  of the Viterbo type which is called the *spectral norm*. To describe this construction, we start by reviewing the definition of the Hofer norm  $\|\phi\|$  of a Hamiltonian diffeomorphism  $\phi$ .

There are two natural operations on the space of Hamiltonians  $H$  : one the *inverse*  $H \mapsto \bar{H}$  where  $\bar{H}$  is the Hamiltonian generating the inverse flow  $(\phi_H^t)^{-1}$  and the *product*  $(H, F) \mapsto H \# F$  where  $H \# F$  is the one generating the composition flow  $\phi_H^t \circ \phi_F^t$ . Hofer [H] introduced an invariant norm on  $\text{Ham}(M, \omega)$ . Hofer also considered its  $L^{(1, \infty)}$ -version  $\|\phi\|$  defined by

$$\|\phi\| = \inf_{H \mapsto \phi} \|H\|; \quad \|H\| = \int_0^1 (\max H_t - \min H_t) dt$$

where  $H \mapsto \phi$  stands for  $\phi = \phi_H^1$ . We call  $\|H\|$  the  $L^{(1,\infty)}$ -norm of  $H$  and  $\|\phi\|$  the  $L^{(1,\infty)}$  Hofer norm of  $\phi$ .

Using the spectral invariant  $\rho(H; 1)$ , Oh [Oh9] defined a function  $\gamma : C_m^\infty([0, 1] \times M) \rightarrow \mathbb{R}$  by

$$\gamma(H) = \rho(H; 1) + \rho(\overline{H}; 1)$$

on  $C_m^\infty([0, 1] \times M)$ , whose definition is more *topological* than  $\|H\|$ . For example,  $\gamma$  canonically projects down to a function on  $\widetilde{Ham}(M, \omega)$  by the homotopy invariance axiom while  $\|H\|$  does not. Obviously  $\gamma(H) = \gamma(\overline{H})$ . The inequality  $\gamma(H) \leq \|H\|$  was also shown in [Oh4, Oh9] and the inequality  $\gamma(H) \geq 0$  follows from the triangle inequality applied to  $a = b = 1$  and from the normalization axiom  $\rho(0; 1) = 0$ .

Now we define a non-negative function  $\gamma : Ham(M, \omega) \rightarrow \mathbb{R}_+$  by  $\gamma(\phi) := \inf_{H \mapsto \phi} \gamma(H)$ . Then the following theorem is proved in [Oh9].

**Theorem 2.4** ([Oh9]). *Let  $(M, \omega)$  be any closed symplectic manifold. Then  $\gamma : Ham(M, \omega) \rightarrow \mathbb{R}_+$  defines a (non-degenerate) norm on  $Ham(M, \omega)$  which satisfies the following additional properties :*

1.  $\gamma(\eta^{-1}\phi\eta) = \gamma(\phi)$  for any symplectic diffeomorphism  $\eta$
2.  $\gamma(\phi^{-1}) = \gamma(\phi)$ ,  $\gamma(\phi) \leq \|\phi\|$ .

Oh then applied the function  $\gamma = \gamma(H)$  to the study of the geodesic property of Hamiltonian flows [Oh7, Oh9].

As another interesting application of spectral invariants is a new construction of *quasi-morphisms* on  $Ham(M, \omega)$  carried out by Entov and Polterovich [EnP]. Recall that for a closed  $(M, \omega)$ , there exists no non-trivial homomorphism to  $\mathbb{R}$  because  $Ham(M, \omega)$  is a simple group [Ba]. However for a certain class of *semi-simple* symplectic manifolds, e.g. for  $(S^2, \omega)$ ,  $(S^2 \times S^2, \omega \oplus \omega)$ ,  $(\mathbb{C}P^n, \omega_{FS})$ , Entov and Polterovich [EnP] produced non-trivial quasi-morphisms, exploiting the spectral invariants  $\rho(e, \cdot)$  corresponding to a certain idempotent element  $e$  of the quantum cohomology ring  $QH^*(M)$ .

It would be an important problem to unravel what the true meaning of Gromov's pseudo-holomorphic curves or of the Floer homology in general is in regard to the underlying symplectic structure.

**2.3. Towards topological Hamiltonian dynamics.** We note that construction of spectral invariants largely depends on the smoothness (or at least differentiability) of Hamiltonians  $H$  because it involves the study of Hamilton's equation  $\dot{x} = X_H(t, x)$ . If  $H$  is smooth, there is a one-one correspondence between  $H$  and its flow  $\phi_H^t$ . However this correspondence breaks down when  $H$  does not have enough regularity, e.g., if  $H$  is only continuous or even  $C^1$  because the *fundamental existence and uniqueness theorems of ODE fail*.

However the final outcome  $\rho(H; a)$  still makes sense for and can be extended to a certain natural class of  $C^0$ -functions  $H$ . Now a natural questions to ask is

**Question 2.5.** Can we define the notion of topological Hamiltonian dynamical systems? If so, what is the dynamical meaning of the numbers  $\rho(H; a)$  when  $H$  is just continuous but not differentiable?

These questions led to the notions of *topological Hamiltonian paths* and *Hamiltonian homeomorphisms* in [OM].

**Definition 2.6.** A continuous path  $\lambda : [0, 1] \rightarrow \text{Homeo}(M)$  with  $\lambda(0) = \text{id}$  is called a topological Hamiltonian path if there exists a sequence of smooth Hamiltonians  $H_i : [0, 1] \times M \rightarrow \mathbb{R}$  such that

1.  $H_i$  converges in the  $L^{(1,\infty)}$ -topology (or Hofer topology) of Hamiltonians and
2.  $\phi_{H_i}^t \rightarrow \lambda(t)$  uniformly converges on  $[0, 1]$ .

We say that the  $L^{(1,\infty)}$ -limit of any such sequence  $H_i$  is a *Hamiltonian* of the topological Hamiltonian flow  $\lambda$ . The following uniqueness result is proved in [Oh12].

**Theorem 2.7** ([Oh12]). *Let  $\lambda$  be a topological Hamiltonian path. Suppose that there exist two sequences  $H_i$  and  $H'_i$  satisfying the conditions in Definition 2.6. Then their limits coincide as an  $L^{(1,\infty)}$ -function.*

The proof of this theorem is a modification of Viterbo's proof [V3] of a similar uniqueness result for the  $C^0$  Hamiltonians, combined with a structure theorem of topological Hamiltonians which is also proven in [Oh12]. An immediate corollary is the following extension of the spectral invariants to the space of topological Hamiltonian paths.

**Definition 2.8.** Suppose  $\lambda$  is a topological Hamiltonian path and let  $H_i$  be the sequence of smooth Hamiltonians that converges in  $L^{(1,\infty)}$ -topology and whose associated Hamiltonian paths  $\phi_{H_i}$  converges to  $\lambda$  uniformly. We define

$$\rho(\lambda; a) = \lim_{i \rightarrow \infty} \rho(H_i; a).$$

The uniqueness theorem of topological Hamiltonians and the  $L^{(1,\infty)}$  continuity property of  $\rho$  imply that this definition is well-defined.

**Definition 2.9.** A homeomorphism  $h$  of  $M$  is a Hamiltonian homeomorphism if there exists a sequence of smooth Hamiltonians  $H_i : [0, 1] \times M \rightarrow \mathbb{R}$  such that

1.  $H_i$  converges in the  $L^{(1,\infty)}$ -topology of Hamiltonians and
2. the Hamiltonian path  $\phi_{H_i} : t \mapsto \phi_{H_i}^t$  uniformly converges on  $[0, 1]$  in the  $C^0$ -topology of  $\text{Homeo}(M)$ , and  $\phi_{H_i}^1 \rightarrow h$ .

We denote by  $\text{Hameo}(M, \omega)$  the set of such homeomorphisms.

Motivated by Eliashberg's rigidity theorem, we also define the group  $\text{Sympeo}(M, \omega)$  as the subgroup of  $\text{Homeo}(M)$  consisting of the  $C^0$ -limits of symplectic diffeomorphisms. Then Oh and Müller [OM] proved the following theorem

**Theorem 2.10** ([OM]).  *$\text{Hameo}(M, \omega)$  is a path-connected normal subgroup of  $\text{Sympeo}_0(M, \omega)$ , the identity component of  $\text{Sympeo}(M, \omega)$ .*

One can easily derive that  $\text{Hameo}(M, \omega)$  is a proper subgroup of  $\text{Sympeo}_0(M, \omega)$  whenever the so called mass flow homomorphism [Fa] is non-trivial or there exists a symplectic diffeomorphism that has no fixed point, e.g.,  $T^{2n}$  [OM]. In fact, we conjecture that this is always the case.

**Conjecture 2.11.** *The group  $\text{Hameo}(M, \omega)$  is a proper subgroup of  $\text{Sympeo}_0(M, \omega)$  for any closed symplectic manifold  $(M, \omega)$ .*

A case of particular interest is the case  $(M, \omega) = (S^2, \omega)$ . In this case, together with the smoothing result proven in [Oh11], the affirmative answer to this conjecture would negatively answer to the following open question in the area preserving dynamical systems. See [Fa] for the basic theorems on the measure preserving homeomorphisms in dimension greater than equal to 3.

**Question 2.12.** Is the identity component of the group of area preserving homeomorphisms on  $S^2$  a simple group?

### 3. Floer theory of Lagrangian intersections

Floer's original definition [F11] of the homology  $HF(L_0, L_1)$  of Lagrangian submanifolds meets many obstacles when one attempts to generalize his definition beyond the exact cases i.e., the case

$$L_0 = L, \quad L_1 = \phi(L) \quad \text{with } \pi_2(M, L) = \{0\}.$$

In this exposition, we will consider the cases of Lagrangian submanifolds that are not necessarily exact. In the open string case of Lagrangian submanifolds, one has to deal with the phenomenon of bubbling-off discs besides bubbling-off spheres. One crucial difference between the former and the latter is that the former is a phenomenon of codimension one while the latter is that of codimension two. This difference makes the general Lagrangian intersection Floer theory display very different perspective compared to the Floer theory of Hamiltonian fixed points. For example, for the intersection case in general, one has to study the theory *in the chain level*, which forces one to consider the chain complexes. Then the meaning of invariance of the resulting objects is much more non-trivial to define compared to that of Gromov-Witten invariants for which one can work with in the level of homology.

There is one particular case that Oh singled out in [Oh1] for which the original version of Floer cohomology is well-defined and invariant just under the change of almost complex structures and under the Hamiltonian isotopy. This is the case of *monotone* Lagrangian submanifolds with *minimal Maslov number*  $\Sigma_L \geq 3$  :

**Definition 3.1.** A Lagrangian submanifold  $L \subset (M, \omega)$  is *monotone* if there exists a constant  $\lambda \geq 0$  such that  $\omega(A) = \lambda\mu(A)$  for all elements  $A \in \pi_2(M, L)$ . The minimal Maslov number is defined by the integer

$$\Sigma_L = \min\{\mu(\beta) \mid \beta \in \pi_2(M, L), \mu(\beta) > 0\}.$$

We will postpone further discussion on this particular case until later in this survey but proceed with describing the general story now.

To obtain the maximal possible generalization of Floer's construction, it is crucial to develop a proper *off-shell* formulation of the relevant Floer moduli spaces.

**3.1. Off-shell formulation.** We consider the space of paths

$$\Omega = \Omega(L_0, L_1) = \{\ell : [0, 1] \rightarrow P \mid \ell(0) \in L_0, \ell(1) \in L_1\}.$$

On this space, we are given the *action one-form*  $\alpha$  defined by

$$\alpha(\ell)(\xi) = \int_0^1 \omega(\dot{\ell}(t), \xi(t)) dt$$

for each tangent vector  $\xi \in T_\ell\Omega$ . From this expression, it follows that

$$\text{Zero}(\alpha) = \{\ell_p : [0, 1] \rightarrow M \mid p \in L_0 \cap L_1, \ell_p \equiv p\}.$$

Using the Lagrangian property of  $(L_0, L_1)$ , a straightforward calculation shows that this form is *closed*. Note that  $\Omega(L_0, L_1)$  is not connected but has countably many connected components. We will work on a particular fixed connected component of  $\Omega(L_0, L_1)$ . We pick up a based path  $\ell_0 \in \Omega(L_0, L_1)$  and consider the corresponding component  $\Omega(L_0, L_1; \ell_0)$ , and then define a covering space

$$\pi : \tilde{\Omega}(L_0, L_1; \ell_0) \rightarrow \Omega(L_0, L_1; \ell_0)$$

on which we have a single valued action functional such that  $d\mathcal{A} = -\pi^*\alpha$ . One can repeat Floer's construction similarly as in the closed case replacing  $\mathcal{L}_0(M)$  by the chosen component of the path space  $\Omega(L_0, L_1)$ . We refer to [FOOO] for the details of this construction. We then denote by  $\Pi(L_0, L_1; \ell_0)$  the group of deck transformations. We define the associated Novikov ring  $\Lambda(L_0, L_1; \ell_0)$  as a completion of the group ring  $\mathbb{Q}[\Pi(L_0, L_1; \ell_0)]$ .

**Definition 3.2.**  $\Lambda_k(L_0, L_1; \ell_0)$  denotes the set of all (infinite) sums

$$\sum_{\substack{g \in \Pi(L_0, L_1; \ell_0) \\ \mu(g) = k}} a_g [g]$$

such that  $a_g \in \mathbb{Q}$  and that for each  $C \in \mathbb{R}$ , the set

$$\#\{g \in \Pi(L_0, L_1; \ell_0) \mid E(g) \leq C, a_g \neq 0\} < \infty.$$

We put  $\Lambda(L_0, L_1; \ell_0) = \bigoplus_k \Lambda_k(L_0, L_1; \ell_0)$ .

We call this graded ring the *Novikov ring* of the pair  $(L_0, L_1)$  relative to the path  $\ell_0$ . Note that this ring depends on  $L$  and  $\ell_0$ . In relation to mirror symmetry, one needs to consider a family of Lagrangian submanifolds and to use a universal form of this ring. The following ring was introduced in [FOOO] which plays an important role in the rigorous formulation of homological mirror symmetry conjecture.

**Definition 3.3** (Universal Novikov ring). We define

$$\Lambda_{nov} = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{Q}, \lambda_i \in \mathbb{R}, \lambda_i \leq \lambda_{i+1}, \lim_{i \rightarrow \infty} \lambda_i = \infty \right\} \quad (3.1)$$

$$\Lambda_{0,nov} = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \in \Lambda_{nov} \mid \lambda_i \geq 0 \right\}. \quad (3.2)$$

In the above definitions of Novikov rings, one can replace  $\mathbb{Q}$  by other commutative ring with unit, e.g.,  $\mathbb{Z}$ ,  $\mathbb{Z}_2$  or  $\mathbb{Q}[e]$  with a formal variable  $e$ .

There is a natural filtration on these rings provided by the valuation  $v : \Lambda_{nov}, \Lambda_{0,nov} \rightarrow \mathbb{R}$  defined by

$$v \left( \sum_{i=1}^{\infty} a_i T^{\lambda_i} \right) := \lambda_1. \quad (3.3)$$

This is well-defined by the definition of the Novikov ring and induces a filtration  $F^\lambda \Lambda_{nov} := v^{-1}([\lambda, \infty))$  on  $\Lambda_{nov}$ . The function  $e^{-v} : \Lambda_{nov} \rightarrow \mathbb{R}_+$  also provides a natural non-Archimedean norm on  $\Lambda_{nov}$ . We call the induced topology on  $\Lambda_{nov}$  a *non-Archimedean topology*.

We now assume that  $L_0$  intersects  $L_1$  transversely and form the  $\mathbb{Q}$ -vector space  $CF(L_0, L_1)$  over the set  $\text{span}_{\mathbb{Q}} \text{Crit } \mathcal{A}$  similarly as  $CF(H)$ . Now let  $p, q \in L_0 \cap L_1$ . We denote by  $\pi_2(p, q) = \pi_2(p, q; L_0, L_1)$  the set of homotopy classes of smooth maps  $u : [0, 1] \times [0, 1] \rightarrow M$  relative to the boundary

$$u(0, t) \equiv p, \quad u(1, t) = q; \quad u(s, 0) \in L_0, \quad u(s, 1) \in L_1$$

and by  $[u] \in \pi_2(p, q)$  the homotopy class of  $u$  and by  $B$  a general element in  $\pi_2(p, q)$ . For given  $B \in \pi_2(p, q)$ , we denote by  $Map(p, q; B)$  the set of such  $w$ 's in class  $B$ . Each element  $B \in \pi_2(p, q)$  induces a map given by the obvious gluing map  $[p, w] \mapsto [q, w \# u]$  for  $u \in Map(p, q; B)$ . There is also the natural gluing map

$$\pi_2(p, q) \times \pi_2(q, r) \rightarrow \pi_2(p, r)$$

induced by the concatenation  $(u_1, u_2) \mapsto u_1 \# u_2$ .

**3.2. Floer moduli spaces and Floer operators.** Now for each given  $J = \{J_t\}_{0 \leq t \leq 1}$  and  $B \in \pi_2(p, q)$ , we define the moduli space  $\widetilde{\mathcal{M}}(p, q; B)$  consisting of finite energy solutions of the Cauchy-Riemann equation

$$\begin{cases} \frac{du}{d\tau} + J_t \frac{du}{dt} = 0 \\ u(\tau, 0) \in L_0, \quad u(\tau, 1) \in L_1, \quad \int u^* \omega < \infty \end{cases}$$

with the asymptotic condition and the homotopy condition

$$u(-\infty, \cdot) \equiv p, \quad u(\infty, \cdot) \equiv q; \quad [u] = B.$$

We then define  $\mathcal{M}(p, q; B) = \widetilde{\mathcal{M}}(p, q; B)/\mathbb{R}$  the quotient by the  $\tau$ -translations and a collection of *rational* numbers  $n(p, q; J, B) = \#(\mathcal{M}(p, q; J, B))$  whenever the expected dimension of  $\mathcal{M}(p, q; B)$  is zero. Finally we define the Floer 'boundary' map  $\partial : CF(L_0, L_1; \ell_0) \rightarrow CF(L_0, L_1; \ell_0)$  by the sum

$$\partial([p, w]) = \sum_{q \in L_0 \cap L_1} \sum_{B \in \pi_2(p, q)} n(p, q; J, B) [q, w \# B]. \quad (3.4)$$

When a Hamiltonian isotopy  $\{L'_s\}_{0 \leq s \leq 1}$  is given one also considers the non-autonomous version of the Floer equation

$$\begin{cases} \frac{du}{d\tau} + J_{t,\rho(\tau)} \frac{du}{dt} = 0 \\ u(\tau, 0) \in L, \quad u(\tau, 1) \in L'_{\rho(\tau)} \end{cases}$$

as done in [Oh1] where  $\rho : \mathbb{R} \rightarrow [0, 1]$  is a smooth function with  $\rho(-\infty) = 0$ ,  $\rho(\infty) = 1$  such that  $\rho'$  is compactly supported and define the Floer ‘chain’ map

$$h : CF^*(L_0, L'_0) \rightarrow CF^*(L_1, L'_1).$$

However unlike the closed case or the exact case, many things go wrong when one asks for the property  $\partial \circ \partial = 0$  or  $\partial h + h\partial = 0$  especially over the rational coefficients, and even when  $HF^*(L, \phi_H^1(L))$  is defined, it is not isomorphic to the classical cohomology  $H^*(L)$ .

In the next 3 subsections, we explain how to overcome these troubles and describe the spectral sequence relating  $HF^*(L, \phi_H^1(L))$  to  $H^*(L)$  when the former is defined. All the results in these subsections are joint works with H. Ohta and K. Ono that appeared in [FOOO], unless otherwise said. We refer to Ohta’s article [Ot] for a more detailed survey on the work from [FOOO].

**3.3. Orientation.** We first recall the following definition from [FOOO].

**Definition 3.4.** A submanifold  $L \subset M$  is called *relatively spin* if it is orientable and there exists a class  $st \in H^2(M, \mathbb{Z}_2)$  such that  $st|_L = w_2(TL)$  for the Stiefel-Whitney class  $w_2(TL)$  of  $TL$ . A pair  $(L_0, L_1)$  is relatively spin, if there exists a class  $st \in H^2(M, \mathbb{Z}_2)$  satisfying  $st|_{L_i} = w_2(TL_i)$  for each  $i = 0, 1$ .

We fix such a class  $st \in H^2(M, \mathbb{Z}_2)$  and a triangulation of  $M$ . Denote by  $M^{(k)}$  its  $k$ -skeleton. There exists a unique rank 2 vector bundle  $V(st)$  on  $M^{(3)}$  with  $w^1(V(st)) = 0$ ,  $w^2(V(st)) = st$ . Now suppose that  $L$  is relatively spin and  $L^{(2)}$  be the 2-skeleton of  $L$ . Then  $V \oplus TL$  is trivial on the 2-skeleton of  $L$ . We define

**Definition 3.5.** We define a  $(M, st)$ -relative spin structure of  $L$  to be a spin structure of the restriction of the vector bundle  $V \oplus TL$  to  $L^{(2)}$ .

The following theorem was proved by de Silva [Si] and in [FOOO] independently.

**Theorem 3.6.** *The moduli space of pseudo-holomorphic discs is orientable, if  $L \subset (M, \omega)$  is relatively spin Lagrangian submanifold. Furthermore the choice of relative spin structure on  $L$  canonically determines an orientation on the moduli space  $\mathcal{M}(L; \beta)$  of holomorphic discs for all  $\beta \in \pi_2(M, L)$ .*

For the orientations on the Floer moduli spaces, the following theorem was proved in [FOOO].

**Theorem 3.7.** *Let  $J = \{J_t\}_{0 \leq t \leq 1}$  and suppose that a pair of Lagrangian submanifolds  $(L_0, L_1)$  are  $(M, st)$ -relatively spin. Then for any  $p, q \in L_0 \cap L_1$  and  $B \in \pi_2(p, q)$ , the Floer moduli space  $\mathcal{M}(p, q; B)$  is orientable. Furthermore a choice of relative spin structures for the pair  $(L_0, L_1)$  determines an orientation on  $\mathcal{M}(p, q; B)$ .*

One can amplify the orientation to the moduli space of pseudo-holomorphic polygons  $\mathcal{M}(\mathcal{L}, \vec{p}; B)$  where  $\mathcal{L} = (L_0, L_1, \dots, L_k)$  and  $\vec{p} = (p_{01}, p_{12}, \dots, p_{k0})$  with  $p_{ij} \in L_i \cap L_j$  and extend the construction to the setting of  $A_\infty$  category [Fu1]. We refer to [Fu2] for more detailed discussion on this.

**3.4. Obstruction and  $A_\infty$  structure.** Let  $(L_0, L_1)$  be a relatively spin pair with  $L_0 \pitchfork L_1$  and fix a  $(M, st)$ -relatively spin structure on each  $L_i$ . To convey the appearance of obstruction to the boundary property  $\partial\partial = 0$  in a coherent way, we assume in this survey, for the simplicity, that all the Floer moduli spaces involved in the construction are transverse and so the expected dimension is the same as the actual dimension. For example, this is the case for monotone Lagrangian submanifolds at least for the Floer moduli spaces of dimension 0, 1 and 2. However, we would like to emphasize that we have to use the machinery of *Kuranishi structure* introduced in [FOn] in the level of *chain* to properly treat the transversality problem for the general case, whose detailed study we refer to [FOOO].

We compute  $\partial\partial([p, w])$ . According to the definition (3.4) of the map  $\partial$ , we have the formula for its matrix coefficients

$$\langle \partial\partial[p, w], [r, w\#B] \rangle = \sum_{q \in L_0 \cap L_1} \sum_{B = B_1 \# B_2 \in \pi_2(p, r)} n(p, q; B_1) n(q, r; B_2) \quad (3.5)$$

where  $B_1 \in \pi_2(p, q)$  and  $B_2 \in \pi_2(q, r)$ . To prove,  $\partial\partial = 0$ , one needs to prove  $\langle \partial\partial[p, w], [r, w\#B] \rangle = 0$  for all pairs  $[p, w], [r, w\#B]$ . On the other hand it follows from definition that each summand  $n(p, q; B_1) n(q, r; B_2)$  is nothing but the number of broken trajectories lying in  $\mathcal{M}(p, q; B_1) \# \mathcal{M}(q, r; B_2)$ . The way how Floer [F11] proved the vanishing of (3.5) under the assumption

$$L_0 = L, L_1 = \phi_H^1(L); \pi_2(M, L_i) = 0 \quad (3.6)$$

is to construct a suitable compactification of the one-dimensional (smooth) moduli space  $\mathcal{M}(p, r; B) = \widetilde{\mathcal{M}}(p, r; B)/\mathbb{R}$  in which the broken trajectories of the form  $u_1 \# u_2$  comprise *all* the boundary components of the compactified moduli space. By definition, the expected dimension of  $\mathcal{M}(p, r; B)$  is one and so compactified moduli space becomes a compact one-dimensional manifold. Then  $\partial\partial = 0$  follows.

As soon as one goes beyond Floer's case (3.6), one must concern the problems of *a priori energy bound* and *bubbling-off discs*. As in the closed case, the Novikov ring is introduced to solve the problem of energy bounds. On the other hand, bubbling-off-discs is a new phenomenon which is that of codimension one and can indeed occur in the boundary of the compactification of Floer moduli spaces.

To handle the problem of bubbling-off discs, Fukaya-Oh-Ohta-Ono [FOOO], associated a structure of filtered  $A_\infty$  algebra  $(C, \mathfrak{m})$  with *non-zero  $\mathfrak{m}_0$ -term* in general. The notion of  $A_\infty$  structure was first introduced by Stasheff [St]. We refer to [GJ] for an exposition close to ours with different sign conventions. The above mentioned obstruction is closely related to non-vanishing of  $\mathfrak{m}_0$  in this  $A_\infty$  structure. Description of this obstruction is now in order.

Let  $C$  be a graded  $R$ -module where  $R$  is the coefficient ring. In our case,  $R$  will be  $\Lambda_{0, \text{nov}}$ . We denote by  $C[1]$  its suspension defined by  $C[1]^k = C^{k+1}$ . We denote by  $\text{deg}(x) = |x|$  the degree of  $x \in C$  before the shift and  $\text{deg}'(x) = |x|'$  that after the degree shifting, i.e.,  $|x|' = |x| - 1$ . Define the *bar complex*  $B(C[1])$  by

$$B_k(C[1]) = (C[1])^{k\otimes}, \quad B(C[1]) = \bigoplus_{k=0}^{\infty} B_k(C[1]).$$

Here  $B_0(C[1]) = R$  by definition. We provide the degree of elements of  $B(C[1])$  by the rule

$$|x_1 \otimes \cdots \otimes x_k|' := \sum_{i=1}^k |x_i|' = \sum_{i=1}^k |x_i| - k \quad (3.7)$$

where  $|\cdot|'$  is the shifted degree. The ring  $B(C[1])$  has the structure of *graded coalgebra*.

**Definition 3.8.** The structure of (strong)  $A_\infty$  algebra is a sequence of  $R$  module homomorphisms

$$\mathfrak{m}_k : B_k(C[1]) \rightarrow C[1], \quad k = 1, 2, \dots,$$

of degree  $+1$  such that the coderivation  $d = \sum_{k=1}^{\infty} \widehat{\mathfrak{m}}_k$  satisfies  $dd = 0$ , which is called the  $A_\infty$ -relation. Here we denote by  $\widehat{\mathfrak{m}}_k : B(C[1]) \rightarrow B(C[1])$  the unique extension of  $\mathfrak{m}_k$  as a coderivation on  $B(C[1])$ . A *filtered  $A_\infty$  algebra* is an  $A_\infty$  algebra with a filtration for which  $\mathfrak{m}_k$  are continuous with respect to the induce non-Archimedean topology.

In particular, we have  $\mathfrak{m}_1 \mathfrak{m}_1 = 0$  and so it defines a complex  $(C, \mathfrak{m}_1)$ . We define the  $\mathfrak{m}_1$ -cohomology by

$$H(C, \mathfrak{m}_1) = \ker \mathfrak{m}_1 / \text{im } \mathfrak{m}_1. \quad (3.8)$$

A *weak  $A_\infty$  algebra* is defined in the same way, except that it also includes

$$\mathfrak{m}_0 : R \rightarrow B(C[1]).$$

The first two terms of the  $A_\infty$  relation for a weak  $L_\infty$  algebra are given as

$$\mathfrak{m}_1(\mathfrak{m}_0(1)) = 0 \quad (3.9)$$

$$\mathfrak{m}_1 \mathfrak{m}_1(x) + (-1)^{|x|'} \mathfrak{m}_2(x, \mathfrak{m}_0(1)) + \mathfrak{m}_2(\mathfrak{m}_0(1), x) = 0. \quad (3.10)$$

In particular, for the case of weak  $A_\infty$  algebras,  $\mathfrak{m}_1$  will not necessarily satisfy the boundary property, i.e.,  $\mathfrak{m}_1 \mathfrak{m}_1 \neq 0$  in general.

The way how a weak  $A_\infty$  algebra is attached to a Lagrangian submanifold  $L \subset (M, \omega)$  arises as an  $A_\infty$  deformation of the classical singular cochain complex including the instanton contributions. In particular, when there is no instanton contribution as in the case  $\pi_2(M, L) = 0$ , it will reduce to an  $A_\infty$  deformation of the singular cohomology in the chain level including all possible higher Massey product. One outstanding circumstances arise in relation to the *quantization* of rational homotopy theory on the cotangent bundle  $T^*N$  of a compact manifold  $N$ .

In this case, the authors proved in [FOh] that the  $A_\infty$  sub-category ‘generated’ by such graphs is literally isomorphic to a certain  $A_\infty$  category constructed by the Morse theory of *graph flows*.

We now describe the basic  $A_\infty$  operators  $\mathfrak{m}_k$  in the context of  $A_\infty$  algebra of Lagrangian submanifolds. For a given compatible almost complex structure  $J$ , consider the moduli space of stable maps of genus zero

$$\mathcal{M}_{k+1}(\beta; L) = \{(w, (z_0, z_1, \dots, z_k)) \mid \bar{\partial}_J w = 0, z_i \in \partial D^2, [w] = \beta \text{ in } \pi_2(M, L)\} / \sim$$

where  $\sim$  is the conformal reparameterization of the disc  $D^2$ . The expected dimension of this space is given by

$$n + \mu(\beta) - 3 + (k + 1) = n + \mu(\beta) + k - 2. \quad (3.11)$$

Now given  $k$  chains

$$[P_1, f_1], \dots, [P_k, f_k] \in C_*(L)$$

of  $L$  considered as *currents* on  $L$ , we put the cohomological grading  $\deg P_i = n - \dim P_i$  and consider the fiber product

$$ev_0 : \mathcal{M}_{k+1}(\beta; L) \times_{(ev_1, \dots, ev_k)} (P_1 \times \dots \times P_k) \rightarrow L.$$

A simple calculation shows that the expected dimension of this chain is given by

$$n + \mu(\beta) - 2 + \sum_{j=1}^k (\dim P_j + 1 - n)$$

or equivalently we have the expected degree

$$\deg [\mathcal{M}_{k+1}(\beta; L) \times_{(ev_1, \dots, ev_k)} (P_1 \times \dots \times P_k), ev_0] = \sum_{j=1}^k (\deg P_j - 1) + 2 - \mu(\beta).$$

For each given  $\beta \in \pi_2(M, L)$  and  $k = 0, \dots$ , we define

$$\mathfrak{m}_{k,\beta}(P_1, \dots, P_k) = [\mathcal{M}_{k+1}(\beta; L) \times_{(ev_1, \dots, ev_k)} (P_1 \times \dots \times P_k), ev_0]$$

and  $\mathfrak{m}_k = \sum_{\beta \in \pi_2(M, L)} \mathfrak{m}_{k,\beta} \cdot q^\beta$  where  $q^\beta = T^{\omega(\beta)} e^{\mu(\beta)/2}$  with  $T, e$  formal parameters encoding the area and the Maslov index of  $\beta$ . We provide  $T$  with degree 0 and  $e$  with 2. Now we denote by  $C[1]$  the completion of a *suitably chosen countably generated* cochain complex with  $\Lambda_{0, nov}$  as its coefficients with respect to the non-Archimedean topology. Then it follows that the map  $\mathfrak{m}_k : C[1]^{\otimes k} \rightarrow C[1]$  is well-defined, has degree 1 and continuous with respect to non-Archimedean topology. We extend  $\mathfrak{m}_k$  as a coderivation  $\widehat{\mathfrak{m}}_k : BC[1] \rightarrow BC[1]$  where  $BC[1]$  is the completion of the direct sum  $\bigoplus_{k=0}^{\infty} B^k C[1]$  where  $B^k C[1]$  itself is the completion of  $C[1]^{\otimes k}$ .  $BC[1]$  has a natural filtration defined similarly as 3.3 Finally we take the sum

$$\widehat{d} = \sum_{k=0}^{\infty} \widehat{\mathfrak{m}}_k : BC[1] \rightarrow BC[1].$$

A main theorem then is the following coboundary property

**Theorem 3.9.** *Let  $L$  be an arbitrary compact relatively spin Lagrangian submanifold of an arbitrary tame symplectic manifold  $(M, \omega)$ . The coderivation  $\widehat{d}$  is a continuous map that satisfies the  $A_\infty$  relation  $\widehat{d}\widehat{d} = 0$ , and so  $(C, \mathfrak{m})$  is a filtered weak  $A_\infty$  algebra.*

One might want to consider the homology of this huge complex but if one naively takes homology of this complex itself, it will end up with getting a trivial group, which is isomorphic to the ground ring  $\Lambda_{0, nov}$ . This is because the  $A_\infty$  algebra associated to  $L$  in [FOOO] has the (homotopy) unit : if an  $A_\infty$  algebra has a unit, the homology of  $\widehat{d}$  is isomorphic to its ground ring.

A more geometrically useful homology relevant to the Floer homology is the  $\mathfrak{m}_1$ -homology (3.8) in this context, which is the Bott-Morse version of the Floer cohomology for the pair  $(L, L)$ . However in the presence of  $\mathfrak{m}_0$ ,  $\widehat{\mathfrak{m}}_1\widehat{\mathfrak{m}}_1 = 0$  no longer holds in general. Motivated by Kontsevich's suggestion [K2], this led Fukaya-Oh-Ohta-Ono to consider deforming Floer's original definition by a bounding chain of the obstruction cycle arising from bubbling-off discs. One can always deform the given (filtered)  $A_\infty$  algebra  $(C, \mathfrak{m})$  by an element  $b \in C[1]^0$  by re-defining the  $A_\infty$  operators as

$$\mathfrak{m}_k^b(x_1, \dots, x_k) = \mathfrak{m}(e^b, x_1, e^b, x_2, e^b, x_3, \dots, x_k, e^b)$$

and taking the sum  $\widehat{d}^b = \sum_{k=0}^\infty \widehat{\mathfrak{m}}_k^b$ . This defines a new weak  $A_\infty$  algebra in general. Here we simplify notations by writing

$$e^b = 1 + b + b \otimes b + \dots + b \otimes \dots \otimes b + \dots .$$

Note that each summand in this infinite sum has degree 0 in  $C[1]$  and converges in the non-Archimedean topology if  $b$  has positive valuation, i.e.,  $v(b) > 0$ .

**Proposition 3.10.** *For the  $A_\infty$  algebra  $(C, \mathfrak{m}_k^b)$ ,  $\mathfrak{m}_0^b = 0$  if and only if  $b$  satisfies*

$$\sum_{k=0}^\infty \mathfrak{m}_k(b, \dots, b) = 0. \tag{3.12}$$

*This equation is a version of Maurer-Cartan equation for the filtered  $A_\infty$  algebra.*

**Definition 3.11.** Let  $(C, \mathfrak{m})$  be a filtered weak  $A_\infty$  algebra in general and  $BC[1]$  be its bar complex. An element  $b \in C[1]^0 = C^1$  is called a *bounding cochain* if it satisfies the equation (3.12) and  $v(b) > 0$ .

In general a given  $A_\infty$  algebra may or may not have a solution to (3.12).

**Definition 3.12.** A filtered weak  $A_\infty$  algebra is called *unobstructed* if the equation (3.12) has a solution  $b \in C[1]^0 = C^1$  with  $v(b) > 0$ .

One can define a notion of homotopy equivalence between two bounding cochains and et al as described in [FOOO]. We denote by  $\mathcal{M}(L)$  the set of equivalence classes of bounding cochains of  $L$ .

Once the  $A_\infty$  algebra is attached to each Lagrangian submanifold  $L$ , we then construct an  $A_\infty$  bimodule  $C(L, L')$  for the pair by considering operators

$$\mathbf{n}_{k_1, k_2} : C(L, L') \rightarrow C(L, L')$$

defined similarly to  $\mathbf{m}_k$  : A typical generator of  $C(L, L')$  has the form

$$P_{1,1} \otimes \cdots \otimes P_{1,k_1} \otimes [p, w] \otimes P_{2,1} \otimes \cdots \otimes P_{2,k_2}$$

and then the image  $\mathbf{n}_{k_1, k_2}$  thereof is given by

$$\sum_{[q, w']} [(\mathcal{M}([p, w], [q, w']); P_{1,1}, \dots, P_{1,k_1}; P_{2,1}, \dots, P_{2,k_2}), ev_\infty] [q, w'].$$

Here  $\mathcal{M}([p, w], [q, w']); P_{1,1}, \dots, P_{1,k_1}; P_{2,1}, \dots, P_{2,k_2}$  is the Floer moduli space

$$\mathcal{M}([p, w], [q, w']) = \bigcup_{[q, w']=[q, w\#B]} \mathcal{M}(p, q; B)$$

cut-down by intersecting with the given chains  $P_{1,i} \subset L$  and  $P_{2,j} \subset L'$ , and the evaluation map

$$ev_\infty : \mathcal{M}([p, w], [q, w']); P_{1,1}, \dots, P_{1,k_1}; P_{2,1}, \dots, P_{2,k_2} \rightarrow \text{Crit} \mathcal{A}$$

is defined by  $ev_\infty(u) = u(+\infty)$ .

**Theorem 3.13.** *Let  $(L, L')$  be an arbitrary relatively spin pair of compact Lagrangian submanifolds. Then the family  $\{\mathbf{n}_{k_1, k_2}\}$  define a left  $(C(L), \mathbf{m})$  and right  $(C(L'), \mathbf{m}')$  filtered  $A_\infty$  bimodule structure on  $C(L, L')$ .*

In other words, each of the map  $\mathbf{n}_{k_1, k_2}$  extends to a  $A_\infty$  bimodule homomorphism  $\widehat{\mathbf{n}}_{k_1, k_2}$  and if we take the sum

$$\widehat{d} := \sum_{k_1, k_2} \widehat{\mathbf{n}}_{k_1, k_2} : C(L, L') \rightarrow C(L, L'),$$

$\widehat{d}$  satisfies the following coboundary property

**Proposition 3.14.** *The map  $\widehat{d}$  is a continuous map and satisfies  $\widehat{d}\widehat{d} = 0$ .*

Again this complex is too big for the computational purpose and we would like to consider the Floer homology by restricting the  $A_\infty$  bimodule to a much smaller complex, an ordinary  $\Lambda_{nov}$  module  $CF(L, L')$ . However Floer's original definition again meets obstruction coming from the obstruction cycles of either  $L_0, L_1$  or of both. We need to deform Floer's 'boundary' map  $\delta$  using suitable bounding cochains of  $L, L'$ . The bimodule  $C(L, L')$  is introduced to perform this deformation coherently.

In the case where both  $L, L'$  are unobstructed, we can carry out this deformation of  $\mathbf{n}$  by bounding chains  $b_1 \in \mathcal{M}(L)$  and  $b_2 \in \mathcal{M}(L')$  similarly as  $\mathbf{m}^b$  above. Symbolically we can write the new operator as

$$\delta^{b_1, b_2}(x) = \widehat{\mathbf{n}}(e^{b_1}, x, e^{b_2}).$$

**Theorem 3.15.** *For each  $b_1 \in \mathcal{M}(L)$  and  $b_2 \in \mathcal{M}(L')$ , the map  $\delta^{b_1, b_2}$  defines a continuous map  $\delta^{b_1, b_2} : CF(L, L') \rightarrow CF(L, L')$  that satisfies  $\delta^{b_1, b_2} \delta^{b_1, b_2} = 0$ .*

This theorem enables us to define the *deformed Floer cohomology*

**Definition 3.16.** For each  $b \in \mathcal{M}(L)$  and  $b' \in \mathcal{M}(L')$ , we define the  $(b, b')$ -Floer cohomology of the pair  $(L, L')$  by

$$HF((L, b), (L', b'); \Lambda_{nov}) = \frac{\ker \delta^{b_1, b_2}}{\text{im } \delta^{b_1, b_2}}.$$

**Theorem 3.17.** *The above cohomology remains isomorphic under the Hamiltonian isotopy of  $L, L'$  and under the homotopy of bounding cochains  $b, b'$ .*

We refer to [FOOO] and its revised version for all the details of algebraic languages needed to make the statements in the above theorems precise.

**3.5. Spectral sequence.** The idea of spectral sequence is quite simple to describe. One can follow more or less the standard construction of spectral sequence on the filtered complex e.g. in [Mc]. One trouble to overcome in the construction of spectral sequence on  $(C(L), \delta)$  or  $(C(L, L'), \delta)$  is that the general Novikov ring, in particular  $\Lambda_{0, nov}$  is *not* Noetherian and so the standard theorems on the Noetherian modules cannot be applied. In addition, the Floer complex is not bounded above which also makes the proof of convergence of the spectral sequence somewhat tricky. We refer to [FOOO] for complete discussions on the construction of the spectral sequence and the study of their convergences.

However for the case of monotone Lagrangian submanifolds, the Novikov ring becomes a field and the corresponding spectral sequence is much simplified as originally carried out by Oh [Oh3] by a crude analysis of thick-thin decomposition of Floer moduli spaces as two Lagrangian submanifolds collapse to one. Then the geometric origin of the spectral sequence is the decomposition of the Floer boundary map  $\delta$  into  $\delta = \delta_0 + \delta_1 + \delta_2 + \dots$  where each  $\delta_i$  is the contribution coming from the Floer trajectories of a given symplectic area in a way that the corresponding area is increasing as  $i \rightarrow \infty$ . Here  $\delta_0$  is the contribution from the classical cohomology. In general this sequence may not stop at a finite stage but it does for monotone Lagrangian submanifolds. In this regard, we can roughly state the following general theorem :

- There exists a spectral sequence whose  $E^2$ -term is isomorphic to the singular cohomology  $H^*(L)$  and which converges to the Floer cohomology  $HF^*(L, L)$ .

See [Oh3] and [FOOO] for the details of the monotone case and of the general case respectively. The above decomposition also provides an algorithm to utilize the spectral sequence in examples, especially when the Floer cohomology is known as for the case of Lagrangian submanifolds in  $\mathbb{C}^n$ . Here are some sample results.

**Theorem 3.18** (Theorem II [Oh3]). *Let  $(M, \omega)$  be a tame symplectic manifold with  $\dim M \geq 4$ . Let  $L$  be a compact monotone Lagrangian submanifold of  $M$*

and  $\phi$  be a Hamiltonian diffeomorphism of  $(M, \omega)$  such that  $L \pitchfork \phi(L)$ . Then the followings hold :

1. If  $\Sigma_L \geq n + 2$ ,  $HF^k(L, \phi(L); \mathbb{Z}_2) \cong H^k(L; \mathbb{Z}_2)$  for all  $k \pmod{\Sigma_L}$ .
2. If  $\Sigma_L = n + 1$ , the same is true for  $k \neq 0, n \pmod{n + 1}$ .

**Theorem 3.19** (Theorem III [Oh3]). *Let  $L \subset \mathbb{C}^n$  be a compact monotone Lagrangian torus. Then we have  $\Sigma_L = 2$  provided  $1 \leq n \leq 24$ .*

A similar consideration, using a more precise form of the spectral sequence from [FOOO], proves

**Theorem 3.20.** *Let  $L \subset \mathbb{C}^n$  be a compact Lagrangian embedding with  $H^2(L; \mathbb{Z}_2) = 0$ . Then its Maslov class  $\mu_L$  is not zero.*

The following theorem can be derived from Theorem E [FOOO] which should be useful for the study of intersection properties of special Lagrangian submanifolds on Calabi-Yau manifold.

**Theorem 3.21.** *Let  $M$  be a Calabi-Yau manifold and  $L$  be an unobstructed Lagrangian submanifold with its Maslov class  $\mu_L = 0$  in  $H^1(L; \mathbb{Z})$ . Then we have  $HF^i(L; \Lambda_{0, nov}) \neq 0$  for  $i = 0, \dim L$ .*

For example, any special Lagrangian homology sphere satisfies all the hypotheses required in this theorem. Using this result combined with some Morse theory argument, Thomas and Yau [TY] proved the following uniqueness result of special Lagrangian homology sphere in its Hamiltonian isotopy class

**Theorem 3.22** (Thomas-Yau). *For any Hamiltonian isotopy class of embedded Lagrangian submanifold  $L$  with  $H^*(L) \cong H^*(S^n)$ , there exists at most one smooth special Lagrangian representative.*

Biran [Bi] also used this spectral sequence for the study of geometry of Lagrangian skeletons and polarizations of Kähler manifolds.

## 4. Displaceable Lagrangian submanifolds

**Definition 4.1.** We call a compact Lagrangian submanifold  $L \subset (M, \omega)$  displaceable if there exists a Hamiltonian isotopy  $\phi_H$  such that  $L \cap \phi_H^1(L) = \emptyset$ .

One motivating question for studying such Lagrangian submanifolds is the following well-known folklore conjecture in symplectic geometry.

**Conjecture 4.2** (Maslov Class Conjecture). *Any compact Lagrangian embedding in  $\mathbb{C}^n$  has non-zero Maslov class.*

Polterovich [P] proved the conjecture in dimension  $n = 2$  whose proof uses a loop  $\gamma$  realized by the boundary of Gromov's holomorphic disc constructed in [Gr]. Viterbo proved this conjecture for any Lagrangian torus in  $\mathbb{C}^n$  by a different

method using the critical point theory on the free loop spaces of  $\mathbb{C}^n$  [V1]. Also see Theorem 3.20 in the previous section for  $L$  with  $H^2(L; \mathbb{Z}_2) \neq 0$ .

It follows from definition that  $HF^*(L, \phi_H^1(L)) = 0$  for a displaceable Lagrangian submanifold  $L$  whenever  $HF^*(L, \phi_H^1(L))$  is defined. An obvious class of displaceable Lagrangian submanifolds are those in  $\mathbb{C}^n$ . This simple observation, when combined with the spectral sequence described in the previous section, provides many interesting consequences on the symplectic topology of such Lagrangian submanifolds as illustrated by Theorem 3.18 and 3.19.

Some further amplification of this line of reasoning was made by Biran and Cieliebak [BC] for the study of topology of Lagrangian submanifolds in (complete) *sub-critical Stein manifolds*  $(V, J)$  or a symplectic manifold  $M$  with such  $V$  as a factor. They cooked up some class of Lagrangian submanifolds in such symplectic manifolds with suitable condition on the first Chern class of  $M$  under which the Lagrangian submanifolds become monotone and satisfy the hypotheses in Theorem 3.18. Then applying this theorem, they derived restrictions on the topology of such Lagrangian submanifolds, e.g., some *cohomological sphericity* of such Lagrangian submanifolds (see Theorem 1.1 [BC]).

Recently Fukaya [Fu4] gave a new construction of the  $A_\infty$ -structure described in the previous section as a deformation of the differential graded algebra of de Rham complex of  $L$  associated to a natural solution to the Maurer-Cartan equation of the Batalin-Vilkovisky structure discovered by Chas and Sullivan [CS] on the loop space. In this way, Fukaya combined Gromov and Polterovich's pseudo-holomorphic curve approach and Viterbo's loop space approach [V1], and proved several new results on the structure of Lagrangian embeddings in  $\mathbb{C}^n$ . The following are some sample results proven by this method [Fu4] :

1. If  $L$  is spin and aspherical in  $\mathbb{C}^n$  then a finite cover  $\tilde{L}$  of  $L$  is homotopy equivalent to a product  $S^1 \times \tilde{L}'$ . Moreover the Maslov index of  $[S^1] \times [point]$  is 2.
2. If  $S^1 \times S^{2n}$  is embedded as a Lagrangian submanifold of  $\mathbb{C}^{2n+1}$ , then the Maslov index of  $[S^1] \times [point]$  is 2.

There is also the symplectic field theory approach to the proof of the result 1 above for the case of torus  $L = T^n$  as Eliashberg explained to the authors [E12]. Eliashberg's scheme has been further detailed by Cieliebak and Mohnke [CM]. The result 1 for  $T^n$  answers affirmatively to Audin's question [Au] on the minimal Maslov number of the embedded Lagrangian torus in  $\mathbb{C}^n$  for general  $n$ . Previously this was known only for  $n = 2$  [P], [V1] and for monotone Lagrangian tori [Oh3] (see Theorem 3.19).

## 5. Applications to mirror symmetry

Mirror symmetry discovered in the super string theory attracted much attention from many (algebraic) geometers since it made a remarkable prediction on the relation between the number of rational curves on a Calabi-Yau 3-fold  $M$  and the deformation theory of complex structures of another Calabi-Yau manifold  $M^\dagger$ .

**5.1. Homological mirror symmetry.** Based on Fukaya's construction of the  $A_\infty$  category of symplectic manifolds [Fu1], Kontsevich [K1] proposed a conjecture on the relation between the category  $\text{Fuk}(M)$  of  $(M, \omega)$  and the derived category of coherent sheaves  $\text{Coh}(M^\dagger)$  of  $M^\dagger$ , and extended the mirror conjecture in a more conceptual way. This extended version is called the *homological mirror symmetry*, which is closely related to the D-brane duality studied much in physics. Due to the obstruction phenomenon we described in §3.4, the original construction in [Fu1] requires some clarification of the definition of  $\text{Fuk}(M)$ . The necessary modification has been completed in [FOOO, Fu2].

For the rest of this subsection, we will formulate a precise mathematical conjecture of homological mirror symmetry. Let  $(M, \omega)$  be an integral symplectic manifold i.e., one with  $[\omega] \in H^2(M; \mathbb{Z})$ . For such  $(M, \omega)$ , we consider a family of complexified symplectic structures  $M_\tau = (M, -\sqrt{-1}\tau\omega)$  parameterized by  $\tau \in \mathfrak{h}$  where  $\mathfrak{h}$  is the upper half plane. The mirror of this family is expected to be a family of complex manifolds  $M_q^\dagger$  parameterized by  $q = e^{\sqrt{-1}\tau} \in D^2 \setminus \{0\}$ , the punctured disc. Suitably 'formalizing' this family at 0, we obtain a scheme  $\mathfrak{M}^\dagger$  defined over the ring  $\mathbb{Q}[[q]][q^{-1}]$ . We identify  $\mathbb{Q}[[q]][q^{-1}]$  with a sub-ring of the universal Novikov ring  $\Lambda_{\text{nov}}$  defined in subsection 3.1. The ext group  $\text{Ext}(\mathcal{E}_0, \mathcal{E}_1)$  between the coherent sheaves  $\mathcal{E}_i$  on  $\mathfrak{M}^\dagger$  is a module over  $\mathbb{Q}[[q]][q^{-1}]$ .

We consider the quadruple  $\mathcal{L} = (L, s, d, [b])$ , which we call a *Lagrangian brane*, that satisfies the following data :

1.  $L$  a Lagrangian submanifold of  $M$  such that the Maslov index of  $L$  is zero and  $[\omega] \in H^2(M, L; \mathbb{Z})$ . We also enhance  $L$  with flat complex line bundle on it.
2.  $s$  is a spin structure of  $L$ .
3.  $d$  is a grading in the sense of [K1], [Se1].
4.  $[b] \in \mathcal{M}(\mathcal{L})$  is a bounding cochain described in subsection 3.4.

**Conjecture 5.1.** *To each Lagrangian brane  $\mathcal{L}$  as above, we can associate an object  $\mathcal{E}(\mathcal{L})$  of the derived category of coherent sheaves on the scheme  $\mathfrak{M}^\dagger$  so that the following holds :*

1. *There exists a canonical isomorphism.*

$$HF(\mathcal{L}_1, \mathcal{L}_2) \cong \text{Ext}(\mathcal{E}(\mathcal{L}_1), \mathcal{E}(\mathcal{L}_2)) \otimes_{\mathbb{Q}[[q]][q^{-1}]} \Lambda_{\text{nov}}$$

2. *The isomorphism in 1 is functorial : Namely the product of Floer cohomology is mapped to the Yoneda product of Ext group by the isomorphism 1.*

The correct Floer cohomology  $HF(\mathcal{L}_1, \mathcal{L}_2)$  used in this formulation of the conjecture is given in [FOOO] (see §3.4 for a brief description). The spin structure in  $\mathcal{L}$  is needed to define orientations on the various moduli spaces involved in the definition of Floer cohomology and the grading  $d$  is used to define an absolute integer grading on  $HF(\mathcal{L}_1, \mathcal{L}_2)$ . We refer readers to [FOOO] §1.4, [Fu3] for the details of construction and for more references.

We now provide some evidences for this conjecture. A conjecture of this kind was first observed by Kontsevich in [K1] for the case of an elliptic curve  $M$ , which is further explored by Polischchuk-Zaslow [PZ], and by Fukaya in [Fu3] for the

case when  $M$  is a torus (and so  $M^\dagger$  is also a torus) and  $L \subset M$  is an affine subtorus. In fact, in these cases one can use the convergent power series for the formal power series or the Novikov ring. Kontsevich-Soibelman [KS] gave an alternative proof, based on the adiabatic degeneration result of the authors [FOh], for the case where  $L$  is an étalé covering of the base torus of the Lagrangian torus fibration  $M = T^{2n} \rightarrow T^n$ . Seidel proved Conjecture 5.1 for the quartic surface  $M$  [Se2].

**5.2. Toric Fano and Landau-Ginzburg correspondence.** So far we have discussed the case of Calabi-Yau manifolds (or a symplectic manifold  $(M, \omega)$  with  $c_1(M) = 0$ ). The other important case that physicists studied much is the case of toric Fano manifolds, which physicists call the correspondence between the  $\sigma$ -model and the Landau-Ginzburg model. Referring readers to [Ho] and [HV] for detailed physical description of this correspondence, we briefly describe an application of machinery developed in [FOOO] for an explicit calculation of Floer cohomology of Lagrangian torus orbits of toric Fano manifolds. We will focus on the correspondence of the  $A$ -model of a toric Fano manifold and the  $B$ -model of Landau-Ginzburg model of its mirror. We refer to [HIV] for the other side of the correspondence between the toric Fano  $B$ -model and the Landau-Ginzburg  $A$ -model.

According to [FOOO], the obstruction cycles of the filtered  $A_\infty$  algebra associated to a Lagrangian submanifold is closely related to  $\mathfrak{m}_0$ . This  $\mathfrak{m}_0$ , by definition, is defined by a collection of the (co)chains  $[\mathcal{M}_1(\beta), ev_0]$  for all  $\beta \in \pi_2(M, L)$ . More precisely, we have

$$\mathfrak{m}_0(1) = \sum_{\beta \in \pi_2(M, L)} [\mathcal{M}_1(\beta), ev_0] \cdot T^{\omega(\beta)} q^{\mu(\beta)/2} \in C^*(L) \otimes \Lambda_{0, nov}. \quad (5.1)$$

This is the sum of all genus zero instanton contributions with one marked point.

On the other hand, based on a  $B$ -model calculations, Hori [Ho], Hori-Vafa [HV] proposed some correspondence between the instanton contributions of the  $A$ -model of toric Fano manifolds and the Landau-Ginzburg potential of the  $B$ -model of its mirror. This correspondence was made precise by Cho and Oh [CO]. A description of this correspondence is now in order.

First they proved the following

**Theorem 5.2.**  $[\mathcal{M}_1(\beta), ev_0] = [L]$  as a chain, for every  $\beta \in \pi_2(M, L)$  with  $\mu(\beta) = 2$  and so  $\mathfrak{m}_0(1) = \lambda[L]$  for some  $\lambda \in \Lambda_{0, nov}$ .

It had been previously observed in Addenda of [Oh1] for the monotone case that Floer cohomology  $HF^*(L, L)$  is defined even when the minimal Maslov number  $\Sigma_L = 2$ . Using the same argument Cho and Oh proved that  $HF^*(L, L; \Lambda_{0, nov})$  is well-defined for the torus fibers of toric Fano manifolds *without deforming* Floer's 'boundary' map, at least for the *convex* case. We believe this convexity condition can be removed. More specifically, they proved  $\mathfrak{m}_1 \mathfrak{m}_1 = 0$ . This is because in (3.10), the last two terms cancel each other if  $\mathfrak{m}_0(1) = \lambda e$  is a multiple of the unit  $e = [L]$  and then (3.9) implies that  $\mathfrak{m}_0(1)$  is a  $\mathfrak{m}_1$ -cycle. We refer readers to the revision of [FOOO] for more discussion on this case, in which any filtered  $A_\infty$  algebra deformable to such a case is called *weakly unobstructed*.

In fact, Cho and Oh obtained the explicit formula

$$\mathbf{m}_0(1) = \sum_{i=1}^N h^{v_j} e^{\sqrt{-1}\langle \nu, v_j \rangle} T^{\omega(\beta_j)}[L] \cdot q \quad (5.2)$$

after including flat line bundles attached to  $L$  and computing precise formulae for the area  $\omega(\beta_j)$ 's. Here  $h^{v_j} = e^{\sqrt{-1}\langle \nu, v_j \rangle}$  is the holonomy of the flat line bundle and  $\omega(\beta_j)$  was calculated explicitly in [CO]. Denote by  $\nu = (\nu_1, \dots, \nu_N)$  the holonomy vector of the line bundle appearing in the description of linear  $\sigma$ -model [HV].

On the other hand, the Landau-Ginzburg potential is given by the formula

$$\sum_{i=1}^N \exp(-y_i - \langle \Theta, v_i \rangle) =: W(\Theta)$$

for the mirror of the given toric manifold (see [HV] for example).

**Theorem 5.3** ([CO]). *Let  $A \in \mathfrak{t}^*$  and denote  $\Theta = A - \sqrt{-1}\nu$ . We denote by  $\mathbf{m}^\Theta$  the  $A_\infty$  operators associated to the torus fiber  $T_A = \pi^{-1}(A)$  coupled with the flat line bundle whose holonomy vector is given by  $\nu \in (S^1)^N$  over the toric fibration  $\pi : X \rightarrow \mathfrak{t}^*$ . Under the substitution of  $T^{2\pi} = e^{-1}$  and ignoring the harmless grading parameter  $q$ , we have the exact correspondence*

$$\mathbf{m}_0^\Theta \longleftrightarrow W(\Theta) \quad (5.3)$$

$$\mathbf{m}_1^\Theta(pt) \longleftrightarrow dW(\Theta) = \sum_{j=1}^n \frac{\partial W}{\partial \Theta_j}(\Theta) d\Theta_j \quad (5.4)$$

under the mirror map given in [HV].

Combined with a theorem from [CO] which states that  $HF^*(\mathcal{L}, \mathcal{L}) \cong H^*(L; \mathbb{C}) \otimes \Lambda_{0, nov}$  whenever  $\mathbf{m}_1^\Theta(pt) = 0$ , this theorem confirms the prediction made by Hori [Ho], Hori-Vafa [HV] about the Floer cohomology of Lagrangian torus fibers. This theorem has been further enhanced by Cho [Cho] who relates the higher derivatives of  $W$  with the higher Massey products  $\mathbf{m}_k^\Theta$ . For example, Cho proved that the natural product structure on  $HF^*(L, L)$  is not isomorphic to the cohomology ring  $H^*(T^n) \otimes \Lambda_{0, nov}$  but isomorphic to the Clifford algebra associated to the quadratic form given by the Hessian of the potential  $W$  under the mirror map. This was also predicted by physicists (see [Ho]).

## References

- [Au] Audin, M., Fibres normaux d'immersions en dimension double, points doubles d'immersions Lagrangiennes et plongements totalement réels, *Comment. Math. Helv.* **63** (1988), 593–623.
- [Ba] Banyaga, A., Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique, *Comm. Math. Helv.* **53** (1978), 174–227.
- [Bi] Biran, P., Lagrangian non-intersections, preprint 2004, arXiv:math.SG/0412110.

- [BC] Biran, P., Cieliebak, K., Lagrangian embeddings into subcritical Stein manifolds, *Israel J. Math.* **127** (2002), 221–244.
- [CS] Chas, M., Sullivan D., String topology, preprint 1999, math.GT/9911159.
- [Cho] Cho, C.-H., Products of Floer cohomology of Lagrangian torus fibers in toric Fano manifolds, *Comm. Math. Phys.* (to appear).
- [CO] Cho, C.-H., Oh, Y.-G. Floer cohomology and disc instantons of Lagrangian torus fibers in toric Fano manifolds, submitted, arXiv:math.SG/0308225.
- [CM] Cieliebak, K., Mohnke, K. a talk by Cieliebak in Banach Center, Warsaw, 2004.
- [EH] Ekeland, I., Hofer, H., Symplectic topology and Hamiltonian dynamics & II, *Math. Z.* **200** (1990), 355–378, & **203** (1990), 553–567.
- [El1] Eliashberg, Y., A theorem on the structure of wave fronts and applications in symplectic topology, *Funct. Anal. and its Appl.* **21** (1987), 227–232.
- [El2] Eliashberg, Y., private communication, 2001.
- [EnP] Entov, M., Polterovich, L., Calabi quasimorphism and quantum homology, *Internat. Math. Res. Notices* no **30** (2003), 1635–1676.
- [Fa] Fathi, A., Structure of the group of homeomorphisms preserving a good measure on a compact manifold, *Ann. Scient. Éc. Norm. Sup.* **13** (1980), 45–93.
- [Fl1] Floer, A., Morse theory for Lagrangian intersections, *J. Differ. Geom.* **28** (1988), 513–547.
- [Fl2] Floer, A., Symplectic fixed points and holomorphic spheres, *Commun. Math. Phys.* **120** (1989), 575–611.
- [Fu1] Fukaya, K., Morse homotopy,  $A^\infty$ -category, and Floer homologies, *Proceedings of GARC Workshop on Geometry and Topology '93*, 1–102, Lecture Notes Ser., **18**, Seoul Nat. Univ., Seoul, (1993).
- [Fu2] Fukaya, K., Floer homology and mirror symmetry. II, 31–127, *Adv. Stud. Pure Math.*, **34**, Math. Soc. Japan, Tokyo, (2002).
- [Fu3] Fukaya, K., Mirror symmetry of abelian varieties and multi-theta functions, *J. Algebraic Geom.* **11** (2002), 393–512.
- [Fu4] Fukaya, K., Application of Floer Homology of Lagrangian Submanifolds to Symplectic Topology, *Morse Theoretic Methods in Nonlinear Analysis and in Symplectic Topology* (2006).
- [FOh] Fukaya, K., Oh, Y.-G., Zero-loop open strings in the cotangent bundle and Morse homotopy, *Asian J. Math.* **1** (1997), 96–180.
- [FOOO] Fukaya, K., Oh, Y.-G., Ohta, H., Ono, K. *Lagrangian intersection Floer homology - anomaly and obstruction*, preprint 2000, <http://www.math.kyoto-u.ac.jp/fukaya/> ; a revision 2005.
- [FOn] Fukaya, K., Ono, K., Arnold conjecture and Gromov-Witten invariants, *Topology* **38** (1999), 933–1048.
- [GJ] Getzler, E., Jones, D.S.,  $A_\infty$  algebra and cyclic bar complex, *Illinois J. Math.* **34** (1990), 256–283.
- [Gr] Gromov, M., Pseudo-holomorphic curves in symplectic manifolds, *Invent. Math.* **82** (1985), 307–347.

- [H] Hofer, H. On the topological properties of symplectic maps, *Proc. Royal Soc. Edinburgh* **115** (1990), 25–38.
- [HS] Hofer, H., Salamon, D., Floer homology and Novikov rings, in the *Floer Memorial Volume*, Hofer, H. et al, Birkhäuser, 1995, pp 483–524.
- [Ho] K. Hori, Linear models in supersymmetric D-branes, Proceedings of the KIAS conference, *Mirror Symmetry and Symplectic Geometry*, 111–186, (Seoul 2000), eds by K. Fukaya, Y.-G. Oh, K. Ono and G. Tian, World Sci. Publishing, River Edge, New Jersey, 2001.
- [HIV] Hori, K., Iqbal, A., Vafa, C., D-branes and mirror symmetry, preprint, hep-th/0005247.
- [HV] K. Hori and C. Vafa, Mirror symmetry, preprint, 2000, hep-th/0002222.
- [K1] Kontsevich, M., Homological algebra of mirror symmetry. *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, 120–139, Birkhäuser, Basel, 1995.
- [K2] Kontsevich, M., private communication, 1997.
- [KS] Kontsevich, M., Soibelman, Y., Homological mirror symmetry and torus fibrations. *Symplectic geometry and Mirror Symmetry*, (Seoul 2000), 203–263, World Sci. Publishing, River Edge, New Jersey, 2001.
- [LT1] Liu, G., Tian, G., Floer homology and Arnold conjecture, *J. Differ. Geom.* **49** (1998), 1–74.
- [Mc] McCleary, J., *User's Guide to Spectral Sequences*, Math. Lec. Series **12**, Publish or Perish, Wilmington, 1985.
- [N] Novikov, S.P., Multivalued functions and functionals. An analogue of the Morse theory, (Russian) *Dokl. Akad. Nauk SSSR* **260** (1981), no. 1, 31–35.
- [Oh1] Oh, Y.-G., Floer cohomology of Lagrangian intersections and pseudo-holomorphic discs, I & II, *Comm. Pure and Applied Math.* **46** (1993), 949 – 994 & 995 – 1012 ; Addenda, *ibid*, **48** (1995), 1299 - 1302
- [Oh2] Oh, Y.-G., Relative Floer and quantum cohomology and the symplectic topology of Lagrangian submanifolds, *Contact and Symplectic Geometry*, eds. by C. B. Thomas, 201–267, Cambridge University Press, Cambridge, England, 1996.
- [Oh3] Oh, Y.-G., Floer cohomology, spectral sequences, and the Maslov class of Lagrangian embeddings, *Internat. Math. Res. Notices*, no 7 (1996), 305–346.
- [Oh4] Oh, Y.-G., Symplectic topology as the geometry of action functional, I, *J. Differ. Geom.* **46** (1997), 499–577.
- [Oh5] Oh, Y.-G., Symplectic topology as the geometry of action functional, II, *Commun. Anal. Geom.* **7** (1999), 1–55.
- [Oh6] Oh, Y.-G., Chain level Floer theory and Hofer's geometry of the Hamiltonian diffeomorphism group, *Asian J. Math.* **6** (2002), 579–624 ; *Erratum* **7** (2003), 447–448.
- [Oh7] Oh, Y.-G., Spectral invariants and length minimizing property of Hamiltonian paths, *Asian J. Math.* **9** (2005), 1–18.
- [Oh8] Oh, Y.-G., Construction of spectral invariants of Hamiltonian paths on closed symplectic manifolds, in *The Breadth of Symplectic and Poisson Geometry*, Prog. Math. **232**, 525–570, Birkhäuser, Boston, 2005.

- [Oh9] Oh, Y.-G., Spectral invariants, analysis of the Floer moduli space and geometry of Hamiltonian diffeomorphisms, *Duke Math. J.* (to appear), math.SG/0403083.
- [Oh10] Oh, Y.-G., Floer mini-max theory, the Cerf diagram, and the spectral invariants, 2004 preprint, math.SG/0406449.
- [Oh11] Oh, Y.-G.,  $C^0$ -coerciveness of Moser's problem and smoothing of area preserving diffeomorphisms, submitted, 2005.
- [Oh12] Oh, Y.-G., The group of Hamiltonian homeomorphisms and topological Hamiltonian flows, preprint, December 2005.
- [OM] Oh, Y.-G., Müller, S., The group of Hamiltonian homeomorphisms and  $C^0$ -symplectic topology, submitted, a revision, December 2005, math.SG/0402210.
- [Ot] Ohta, H., Obstruction to and deformation of Lagrangian intersection Floer cohomology, *Mirror Symmetry and Symplectic Geometry*, (Seoul 2000), 281-309, World Sci. Publishing, River Edge, New Jersey, 2001.
- [On] Ono, K., On the Arnold conjecture for weakly monotone symplectic manifolds, *Invent. Math.* **119** (1995), 519–537.
- [P] Polterovich, L., The Maslov class of Lagrange surfaces and Gromov's pseudo-holomorphic curves, *Trans. Amer. Soc.* **325** (1991), 241–248
- [PZ] Polishchuk, A and Zaslow, E., Categorical mirror symmetry: the elliptic curve, *Adv. Theor. Math. Phys.* **2** (1998), no. 2, 443–470.
- [Ru] Ruan, Y., Virtual neighborhood and pseudo-holomorphic curves, *Turkish J. Math.* **23** (1999), 161–231
- [Sc] Schwarz, M., On the action spectrum for closed symplectically aspherical manifolds, *Pacific J. Math.* **193** (2000), 419–461
- [Se1] Seidel, P., Graded Lagrangian submanifolds, *Bull. Soc. Math. France* **128** (2000), 103–149.
- [Se2] Seidel, P., Homological mirror symmetry for the quartic surface, preprint, 2003, math.SG/0310414.
- [Si] De Silva, V. Products on symplectic Floer homology, Thesis, Oxford University, 1997.
- [St] Stasheff, J., Homotopy associativity of H-Spaces I & II, *Trans. Amer. Math. Soc.* **108** (1963), 275–312 & 293–312
- [TY] Thomas, R., S.-T. Yau, Special Lagrangians, stable bundles and mean curvature flow, *Comm. Anal. Geom.* **10** (2002), no. 5, 1075–1113.
- [V1] Viterbo, C., A new obstruction to embedding Lagrangian tori, *Invent. Math.* **100** (1990), 301–320.
- [V2] Viterbo, C., Symplectic topology as the geometry of generating functions, *Math. Ann.* **292** (1992), 685–710.
- [V3] Viterbo, C., On the uniqueness of generating Hamiltonian for continuous limits of Hamiltonian flows, preprint, 2005, math.SG/0509179

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