5TH ANSWER.

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1. Introduction

This is an answer to question 5 below.

5.) How is equality of Floer and Morse differential for the Arnold conjecture proven?

(1) Is there an abstract construction along the following lines: Given a compact topological space X with continuous, proper, free S^1-action, and a Kuranishi structure for X/S^1 of virtual dimension -1, there is a Kuranishi structure for X with $[X]^{\rm vir} = 0$.
(2) How would such an abstract construction proceed?
(3) Let X be a space of Hamiltonian Floer trajectories between critical points of index difference 1, in which breaking occurs (due to lack of transversality). How is a Kuranishi structure for X/S^1 constructed?
(4) If the Floer differential is constructed by these means, why is it chain homotopy equivalent to the Floer differential for a non-autonomous Hamiltonian?

Also some more in the post of March 23.

Q5 Could you write a little longer proof of this point in [FOn1]? Reading it there, it sounds to me as if there is an abstract notion. In your answer now you again talk about S^1-equivariant Kuranishi structure, and I honestly don't quite know what that is .... or, more to the point, what an S^1-equivariant good coordinate system is, and how it is constructed.

Answer (copied from [Fu2].)

(1)(2) We do not think it is possible in completely abstract setting. At least I do not know how to do it. In a geometric setting such as one appearing in [FOn1, page

1036], Kuranishi structure is obtained by specifying the choice of the obstruction space E_p for each p. We can take E_p in an S^1 equivariant way so the Kuranishi structure on the quotient X/S^1 is obtained. And it is a quotient of one on X. S^1 equivariant multisection can be constructed in an abstract setting so if the quotient has virtual dimension -1 the zero set is empty.

(3) We can take a direct sum of the obstruction bundles, the support of which is disjoint from the points where two trajectories are glued. In the situation of (1) the obstruction bundle is $S^1 \times S^1$ equivariant. The symmetry is compatible with the diagonal S^1 action nearby.

(4) It is [FOn2, Theorem 20.5].

In this note, we explain the proof of [FOn1] in more detail. Section 2 contains abstract theory of S^1-equivariant Kuranishi structure. We define the notion of Kuranishi structure which admits a locally free S^1 action and its good coordinate system. We explain how the construction of [FOn2] (that is basically the same as [FOn1] and [FOOO1, Sectin A1]) can be modified so that all the constructions are S^1-equivariant. (So this part contains a detailed answer to the additional question in the post of March 23.)

In Section 3 we review the moduli space of Floer's equation (the Cauchy-Riemann equation perturbed by a Hamiltonian vector field).

In Section 4 we study the case of time independent Hamiltonian and prove in detail that the Floer's moduli space has an S^1 equivariant Kuranishi structure in that case. Thus this section completes the detail of the answer to (1)-(3).

In Section 5, we prove in detail that the Floer homology of periodic Hamiltonian system is isomorphic to the singular homology. Namely it provides the detail of the proof of [FOn2, Theorem 20.5] and is the answer to (4).

We also remark that, at the stage of the year 2012 (when this note is written), there are two proofs of isomorphism between Floer homology of periodic Hamiltonian system and ordinary homology of M. One is in [FOn1] and uses identification with Morse complex in the case Hamiltonian is small and time independent. This proof is the same as the one taken in this note. (Namely the proof given in this note coincides with the one in [FOn1] except some technical detail.) The other uses Bott-Morse and de Rham theory and is in [FOOO3, Section 26]. (Several other proofs are written in 1996 by Ruan [Ru], Liu-Tian [LT] also.) This second proof has its origin in (the proof of) [Fu1, Theorem 1.2].

Actually there is a third method using the Lagrangian Floer homology of the diagonal. In this third method we do not need to study S^1 equivariant Kuranishi structure at all. See Remark 5.18.

2. Definition of S^1 equivariant Kuranishi structure, its good coordinate system and perturbation.

We define an S^1 equivariant Kuranishi structure below. A notion of T^n equivariant Kuranishi structure (in the strong sense) is defined in [FOOO2, Definition B.4]. However that definition applies to the case when T^n acts on the target. The S^1 equivariant Kuranishi structure we use to study time independent Hamiltonian is different therefrom since our S^1 action comes from the automorphism of the source. Definition 2.1 gives a definition in the current case. We use the notation of [FOn2, Section 1]. Let X be a Hausdorff metrizable space on which S^1 acts. We assume that the isotropy group of every element is finite.

Definition 2.1. Let $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ be a Kuranishi neighborhood of $p \in X$ as in [FOn2] Definition 1.1, except we do *not* assume $\gamma o_p = o_p$ for $\gamma \in \Gamma_p$.[1] We define a *locally free* S^1 *action* on this chart as follows.

1. There exists a group G_p acting effectively on V_p and E_p.
2. $G_p \supset \Gamma_p$ and the Γ_p action extends to the G_p action.
3. The identity component $G_{p,0}$ of G_p is isomorphic to S^1. We fix an isomorphism $\mathfrak{h}_p : S^1 \to G_{p,0}$.
4. G_p is generated by Γ_p and $G_{p,0}$.
5. $G_{p,0}$ commutes with the action of Γ_p.
6. The isotropy group at every point of G_p action on V_p is finite.
7. s_p and ψ_p are G_p equivariant.

Remark 2.2. Note the Conditions (4), (5) imply that G_p is isomorphic to the direct product $\Gamma_p \times S^1$.

The next example shows a reason why we remove the assumption $\gamma o_p = o_p$.

Example 2.3. We take $V_p = S^1 \times D^2$ and $\Gamma_p = \mathbb{Z}_2$ such that the nontrivial element of Γ_p acts by $(t, z) \mapsto (t + 1/2, -z)$. (Here $S^1 = \mathbb{R}/\mathbb{Z}$.) $G_{p,0} = S^1$ acts on V_p by rotating the first factor S^1. The action of Γ_p is free. The quotient space V_p/Γ_p is a manifold. The induced S^1 action on V_p/Γ_p is locally free but is not free. The quotient space V_p/G_p is an orbifold $D^2/\mathbb{Z}_2$. See Example 4.26.

Definition 2.4. Let $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ and $(V_q, E_q, \Gamma_q, \psi_q, s_q)$ be Kuranishi neighborhoods of $p \in X$ and $q \in \psi_p(s_p^{-1}(0)/\Gamma_p)$, respectively. We assume that they carry locally free S^1 actions. (G_p and G_q.) Let a triple $(\hat\phi_{pq}, \phi_{pq}, h_{pq})$ be coordinate change in the sense of [FOn2, Definition 1.2]. We say it is S^1 equivariant if the following holds.

1. h_{pq} extends to a group homomorphism $G_q \to G_p$, which we denote by $\mathfrak{h}_{pq}$.
2. V_{pq} is G_q invariant.
3. $\phi_{pq} : V_{pq} \to V_p$ is $\mathfrak{h}_{pq}$-equivariant.
4. $\hat\phi_{pq}$ is $\mathfrak{h}_{pq}$-equivariant.
5. $\mathfrak{h}_{pq} \circ \mathfrak{h}_q = \mathfrak{h}_p$.

Definition 2.5. Let $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ and $(\hat\phi_{pq}, \phi_{pq}, h_{pq})$ define Kuranishi structure in the sense of [FOn2, Definition 1.3] on X. A *locally free* S^1 *action* is assigned by S^1 actions in the sense of Definition 2.1 on each chart $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ so that the coordinate change is S^1 equivariant.

Remark 2.6. Let $r \in \psi_q((V_{pq} \cap s_q^{-1}(0))/\Gamma_q)$, $q \in \psi_p(s_p^{-1}(0)/\Gamma_p)$. There exists $\gamma^\alpha_{pqr} \in \Gamma_p$ for each connected component $(\phi_{qr}^{-1}(V_{pq}) \cap V_{qr} \cap V_{pr})_\alpha$ of $\phi_{qr}^{-1}(V_{pq}) \cap V_{qr} \cap V_{pr}$ by [FOn2, Definition 1.3 (2)]. We automatically have

$$\mathfrak{h}_{pq} \circ \mathfrak{h}_{qr} = \gamma^\alpha_{pqr} \cdot \mathfrak{h}_{pr} \cdot (\gamma^\alpha_{pqr})^{-1}, \tag{2.1}$$

because S^1 lies in the center and this formula is already assumed for Γ_r.

Lemma-Definition 2.7. *If X has a Kuranishi structure with a locally free S^1 action then X/S^1 has an induced Kuranishi structure.*

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[1] $o_p \in V_p$ is a point such that $\psi_p([o_p]) = p$.

Proof. Let $p \in X$. We take $o_p \in V_p$ and choose a local transversal $\overline{V}_p$ to the S^1 orbit $G_{p,0}o_p$. We put

$$\Gamma_p^+ = \{\gamma \in G_p \mid \gamma o_p = o_p\}.$$

Γ_p^+ is a finite group. We may choose $\overline{V}_p$ so that it is invariant under Γ_p^+. We restrict E_p to $\overline{V}_p$ to obtain $\overline{E}_p$. The Kuranishi map s_p induces $\overline{s}_p$.

We may shrink our Kuranishi neighborhood and may assume that

$$V_p = G_{p,0} \cdot \overline{V}_p \tag{2.2}$$

for all p.

For $x \in \overline{V}_p$, satisfying $s_p(x) = 0$, we define $\overline{\psi}_p(x)$ to be the equivalence class of $\psi_p(x)$ in X/S^1. It is easy to see that $(\overline{V}_p, \Gamma_p^+, \overline{E}_p, \overline{s}_p, \overline{\psi}_p)$ is a Kuranishi chart of X/S^1 at $[p]$.

Let $[q] \in \overline{\psi}_p(\tilde{q})$, where $\tilde{q} \in \overline{V}_p$. Choose $q \in X$ such that $q = \psi_p(\tilde{q})$. We have a coordinate transformation $(V_{pq}, \phi_{pq}, \hat{\phi}_{pq})$ and a group homomorphism $\mathfrak{h}_{pq} : G_q \to G_p$ such that

$$\tilde{q} = g_{pq}(o_q) \cdot \phi_{pq}(o_q)$$

holds for some $g_{pq}(o_q) \in G_{p,0}$. Moreover there exists a smooth map

$$g_{pq} : \overline{V}_{pq} \to G_{p,0}$$

such that it coincides with $g_{pq}(o_q)$ at o_q and

$$g_{pq}(x) \cdot \phi_{pq}(x) \in \overline{V}_p.$$

Here $\overline{V}_{pq}$ is a neighborhood of o_q in $\overline{V}_q$. We define

$$\overline{\phi}_{pq}(x) = g_{pq}(x) \cdot \phi_{pq}(x).$$

We shrink V_{pq} and may assume

$$V_{pq} = G_{p,0} \cdot \overline{V}_{pq} \tag{2.3}$$

By definition

$$\Gamma_q^+ = \{\gamma \in G_q \mid \gamma o_q = o_q\}.$$

Using the fact that

$$\{\gamma \in \Gamma_p \mid \gamma \cdot \phi_{pq}(o_q) = \phi_{pq}(o_q)\} = h_{pq}(\Gamma_q)$$

and $G_{p,0}$ is contained in the center, we find that

$$\{\gamma \in G_p \mid \gamma\overline{\phi}_{pq}(o_q) = \overline{\phi}_{pq}(o_q)\} = \mathfrak{h}_{pq}(\Gamma_q^+) \subset \Gamma_p^+.$$

We denote by $\overline{h}_{pq}$ the restriction of $\mathfrak{h}_{pq}$ to Γ_q^+. It is easy to see that $\overline{\phi}_{pq}$ is $\overline{h}_{pq}$ equivariant. We can lift $\overline{\phi}_{pq}$ to $\hat{\overline{\phi}}_{pq}$ using ϕ_{pq} and G_p action on E_p.

We have thus constructed a coordinate change of our Kuranishi structure on X/S^1. It is straightforward to check the compatibility among the coordinate changes. □

We next define a good coordinate system. We remark that in [FOn2] we defined a chart of good coordinate system as an orbifold that is not necessarily a global quotient. So we define a notion of locally free S^1 action on orbifold.

Definition 2.8. Let U be an orbifold on which S^1 acts effectively as a topological group. We assume that the isotropy group of this S^1 action is always finite. We say that the action is a *smooth action on orbifold* if the following holds for each $p \in U$.

There exists an S^1 equivariant neighborhood U_p of p in U and V_p a manifold on which G_p acts. (V_p, Γ_p, ψ_p) is a chart of U as an orbifold. The conditions (1)-(6) in Definition 2.1 hold and ψ_p is G_p equivariant. Moreover the S^1 action on $V_p/S^1 \subset U$ induced by $\mathfrak{h}_p : S^1 \to G_{p,0}$ coincides with the given S^1 action.

Let S^1 act effectively on X and assume that that its isotropy group is finite.

Definition 2.9. Suppose X has a locally free S^1 equivariant Kuranishi structure. An S^1 *equivariant good coordinate system* on it is $(U_{\mathfrak{p}}, E_{\mathfrak{p}}, \psi_{\mathfrak{p}}, s_{\mathfrak{p}})$, $(U_{\mathfrak{pq}}, \hat\phi_{\mathfrak{pq}}, \phi_{\mathfrak{pq}})$ as in [FOn2, Definition 2.2]. We require furthermore the following in addition.

(1) There exists a smooth S^1 action on $U_{\mathfrak{p}}$ and $E_{\mathfrak{p}}$.
(2) $\psi_{\mathfrak{p}}$, $s_{\mathfrak{p}}$ are S^1 equivariant.
(3) $U_{\mathfrak{pq}}$ is S^1 invariant and $\hat\phi_{\mathfrak{pq}}, \phi_{\mathfrak{pq}}$ are S^1 equivariant.

Note the notion of S^1-equivariance of maps or subsets are defined set theoretically.

Lemma 2.10. *If $(U_{\mathfrak{p}}, E_{\mathfrak{p}}, \psi_{\mathfrak{p}}, s_{\mathfrak{p}})$, $(U_{\mathfrak{pq}}, \hat\phi_{\mathfrak{pq}}, \phi_{\mathfrak{pq}})$ is an S^1 equivariant good coordinate system then it induces a good coordinate system of X/S^1, that is $(\overline{U}_{\mathfrak{p}}, \overline{E}_{\mathfrak{p}}, \overline{\psi}_{\mathfrak{p}}, \overline{s}_{\mathfrak{p}})$, $(\overline{U}_{\mathfrak{pq}}, \overline{\hat\phi}_{\mathfrak{pq}}, \overline{\phi}_{\mathfrak{pq}})$, where $\overline{U}_{\mathfrak{p}} = U_{\mathfrak{p}}/S^1$ etc..*

Proof. Apply the construction of Lemma-Definition 2.7 locally. □

Proposition 2.11. *For any locally free S^1 equivariant Kuranishi structure we can find an S^1 equivariant good coordinate system.*

Proof. The proof uses the construction of good coordinate system in [FOn2, Section 4]. We defined and used the notion of pure and mixed orbifold neighborhood there. We constructed them for Kuranishi structure. We will use pure and mixed orbifold neighborhood of the Kuranishi structure on X/S^1 and extend them to ones on X. The detail follows.

We stratify $\overline{X} = X/S^1 = \bigcup_{\mathfrak{d}} \overline{X}(\mathfrak{d})$ where $[p] \in \overline{X}(\mathfrak{d})$ if $\dim \overline{U}_{[p]} = \mathfrak{d}$. So $X(\mathfrak{d}+1)/S^1 = \overline{X}(\mathfrak{d})$. Let $\overline{\mathcal{K}}_*$ be a compact subset of $\overline{X}(\mathfrak{d})$. Let $\mathcal{K}_*$ be an S^1-invariant compact subset of $X(\mathfrak{d})$ such that $\overline{\mathcal{K}}_* = \mathcal{K}_*/S^1$. In [FOn2, Proposition 4.4] we constructed a pure orbifold neighborhood $\overline{U}_*$ of $\mathcal{K}_*/S^1$.

Lemma 2.12. *There exists a pure orbifold neighborhood U_* of $\mathcal{K}_*$ on which S^1 acts and $U_*/S^1 = \overline{U}_*$.*

Remark 2.13. This lemma is somewhat loosely stated, since we did not define the notion of S^1 action on pure orbifold neighborhood. The definition is: U_* has locally free effective smooth S^1 action and all the structure maps commute with S^1 action.

Proof. We can prove this lemma by examining the proof of [FOn2, Proposition 4.4]. Namely $\overline{U}_*$ is obtained by gluing various Kuranishi charts and restricting it to suitable open subsets. We take the inverse image of $U_{\mathfrak{p}} \to \overline{U}_{\mathfrak{p}}$ of those charts. We can then glue and restrict them in the same way to obtain U_*. We omit the detail. □

In [FOn2, Section 4] we then proceed to define mixed orbifold neighborhood of $\overline{X}(D)$ for an ideal $D \subset \mathfrak{D}$. For an ideal $D \subset \mathfrak{D}$ we put $D^{+1} = \{\mathfrak{d}+1 \mid \mathfrak{d} \in D\}$.

Lemma 2.14. *We assume that $\{\overline{\mathcal{U}}_{\mathfrak{d}}\}$ together with other data provide the mixed orbifold neighborhood of $\overline{X}(D)$ obtained in* [FOn2, Proposition 4.13].

Then we can take S^1 equivariant mixed orbifold neighborhood $\{\mathcal{U}_{\mathfrak{d}+1}\}$ (plus other data) on $X(D^{+1})$ such that $\overline{\mathcal{U}}_{\mathfrak{d}} = \mathcal{U}_{\mathfrak{d}+1}/S^1$.

Proof. This is proved again by examing the proof of [FOn2, Proposition 4.13] and checking that the gluing process there can be lifted. This is actully fairly obvious. □

We remark that the chart of good coordinat system on $\overline{X}$ constructed in [FOn2] is $\overline{\mathcal{U}}_{\mathfrak{d}}$ and other data of good coordinate system is obtained by the structure maps etc. of mixed orbifold neighborhood. Therefore $\mathcal{U}_{\mathfrak{d}+1}$ becomes the required S^1 equivariant good coordinate system of X. The proof of Proposition 2.11 is complete. □

Lemma 2.15. *If the dimension of X/S^1 in the sense of Kuranishi structure is -1 then there exists an S^1 equivariant multisection on the good coordinate system $\mathcal{U}_{\mathfrak{p}}$ of X whose zero set is empty.*

Proof. It suffices to define an appropriate notion of pull back of the multisection of $\overline{\mathcal{U}}_{\mathfrak{p}}$ to ones of $\mathcal{U}_{\mathfrak{p}}$. This is routine. □

3. Floer's equation and its moduli space

In this section we concern with the moduli space of solutions of Floer's perturbed Cauchy-Riemann equation. Such a moduli space appears in the proof of Arnold's conjecture of various kinds. In the next section, we prove existence of S^1 equivariant Kuranishi structure of such a moduli space in the case when our Morse function is time independent.

Let $H : X \times S^1 \to \mathbb{R}$ be a smooth function on a symplectic manifold X. We put $H_t(x) = H(t,x)$ where $t \in S^1$ and $x \in X$. The function H_t generates the Hamiltonian vector field $\mathfrak{X}_{H_t}$ by

$$i_{\mathfrak{X}_{H_t}}\omega = dH_t.$$

We denote it by $\mathfrak{P}(H)$ the set of the all 1-periodic orbit of the time dependent vector field $\mathfrak{X}_{H_t}$. We put

$$\tilde{\mathfrak{P}}(H) = \{(\gamma, w) \mid \gamma \in \mathfrak{P}(H),\ u : D^2 \to X,\ u(e^{2\pi i t}) = \gamma(t)\}/\sim,$$

where $(\gamma, w) \sim (\gamma', w')$ if and only if $\gamma = \gamma'$ and

$$\omega([w]-[w']) = 0, \quad c^1([w]-[w']) = 0.$$

(Here ω is the symplectic form and c^1 is the first Chern class.)

Assumption 3.1. All the 1-periodic orbits of the time dependent vector field $\mathfrak{X}_{H_t}$ are non-degenerate.

Following [Fl2], we consider the maps $h : \mathbb{R} \times S^1 \to X$ that satisfiy

$$\frac{\partial h}{\partial \tau} + J\left(\frac{\partial h}{\partial t} - \mathfrak{X}_{H_t}\right) = 0. \tag{3.4}$$

Here τ and t are the coordinates of $\mathbb{R}$ and $S^1 = \mathbb{R}/\mathbb{Z}$, respectively. For $\tilde\gamma^{\pm} = (\gamma^{\pm}, w^{\pm}) \in \tilde{\mathfrak{P}}(H)$ we consider the boundary condition

$$\lim_{\tau\to\pm\infty} h(\tau,t) = \gamma^{\pm}(t). \tag{3.5}$$

The following result due to Floer [Fl2] is by now well established.

Proposition 3.2. *We assume Assumption 3.1. Then for any solution h of (3.4) with*

$$\int_{\mathbb{R}\times S^1} \left\Vert \frac{\partial h}{\partial \tau}\right\Vert^2 d\tau dt < \infty$$

there exists $\gamma^{\pm} \in \mathfrak{P}(H)$ such that (3.5) is satisfied.

Let $\tilde\gamma^{\pm} = (\gamma^{\pm}, w^{\pm}) \in \tilde{\mathfrak{P}}(H)$.

Definition 3.3. We denote by $\widetilde{\mathcal{M}}^{\rm reg}(X,H;\tilde\gamma^-,\tilde\gamma^+)$ the set of all maps $h : \mathbb{R}\times S^1 \to X$ that satisfy (3.4), (3.5) and

$$w^- \# h \sim w^+.$$

Here # is an obvious concatenation.

The translation along $\tau \in \mathbb{R}$ defines an $\mathbb{R}$ action on $\widetilde{\mathcal{M}}^{\rm reg}(X,H;\tilde\gamma^-,\tilde\gamma^+)$. This $\mathbb{R}$ action is free unless $\tilde\gamma^- = \tilde\gamma^+$. We denote by $\mathcal{M}^{\rm reg}(X,H;\tilde\gamma^-,\tilde\gamma^+)$ the quotient space of this action.

Theorem 3.4. ([FOn1, Theorem 19.14]) *We assume Assumption 3.1.*

1. *The space $\mathcal{M}^{\rm reg}(X,H;\tilde\gamma^-,\tilde\gamma^+)$ has a compactification $\mathcal{M}(X,H;\tilde\gamma^-,\tilde\gamma^+)$.*
2. *The compact space $\mathcal{M}(X,H;\tilde\gamma^-,\tilde\gamma^+)$ has an oriented Kuranishi structure with corners.*
3. *The codimension k corner of $\mathcal{M}(X,H;\tilde\gamma^-,\tilde\gamma^+)$ is identified with the union of*
$$\prod_{i=0}^{k} \mathcal{M}(X,H;\tilde\gamma^i,\tilde\gamma^{i+1})$$
over the $k+1$-tuples $(\tilde\gamma^0,\dots,\tilde\gamma^{k+1})$ such that $\tilde\gamma^0 = \tilde\gamma^-$, $\tilde\gamma^{k+1} = \tilde\gamma^+$ and $\tilde\gamma^i \in \tilde{\mathfrak{P}}(H)$.

The proof is in Section 5.

The main purpose of this section is to explain the proof of the next result. We consider the case when H is time independent. In this case, Assumption 3.1 implies that $H : X \to \mathbb{R}$ is a Morse function. We also assume the following:

Assumption 3.5. (1) The gradient vector field of H is Morse-Smale.
(2) Any 1-periodic orbit of $\mathfrak{X}_H$ is a constant loop. (Namely it corresponds to a critical point of H.)

Condition (1) is satisfied for generic H. We can replace H by ϵH for small ϵ so that (2) is also satisfied.

By assumption, elements of $\mathfrak{P}(H)$ are constant loops. We write $\mathfrak{x} \in X$ to denote its element. We put

$$\Pi = \frac{{\rm Im}\pi_2(X) \to H^2(X;\mathbb{Z})}{{\rm Ker}(c^1)\cap{\rm Ker}(\omega)\cap{\rm Im}(\pi_2(X)\to H^2(X;\mathbb{Z}))}.$$

(Here we regard $c^1 : H^2(X;\mathbb{Z}) \to \mathbb{Z}$, $\omega : H^2(X;\mathbb{Z}) \to \mathbb{R}$.) An element of $\tilde{\mathfrak{P}}(H)$ is regarded as a pair $(\mathfrak{z},\alpha)$, where $\mathfrak{z}$ is a critical point of H and $\alpha \in \Pi$.

We put

$$\mathcal{M}^{\rm reg}(X,H;\mathfrak{z}^-,\mathfrak{z}^+;\alpha) = \mathcal{M}^{\rm reg}(X,H;(\mathfrak{z}^-,\alpha^-),(\mathfrak{z}^+,\alpha^- + \alpha)).$$

It is easy to see that the right hand is independent of $\alpha^- \in \Pi$.

Let $\mathcal{M}(X,H;\mathfrak{z}^-,\mathfrak{z}^+;\alpha)$ be its compactification as in Theorem 3.4.

Let $\mathcal{M}^{\rm reg}(X,H;\mathfrak{z}^-,\mathfrak{z}^+;\alpha)^{S^1}$ be the fixed point set of the S^1 action obtained by $t_0 h(\tau,t) = h(\tau,t+t_0)$. It is easy to see that this set is empty unless $\alpha = 0$ and in the case $\alpha = 0$ the fixed point set $\mathcal{M}^{\rm reg}(X,H;\mathfrak{z}^-,\mathfrak{z}^+;0)^{S^1}$ can be identified with the set of gradient lines of H joining $\mathfrak{z}^-$ to $\mathfrak{z}^+$. This identification can be extended to their compactifications.

Assumption 3.6.
(1) $\mathcal{M}(X,H;\mathfrak{z}^-,\mathfrak{z}^+;0)^{S^1}$ is an open subset of $\mathcal{M}(X,H;\mathfrak{z}^-,\mathfrak{z}^+;0)$. Namely any solution of (3.4) which is sufficiently close to an S^1 equivariant solution is S^1 equivariant.
(2) The moduli space $\mathcal{M}(X,H;\mathfrak{z}^-,\mathfrak{z}^+;0)$ is Fredholm regular at each point of $\mathcal{M}(X,H;\mathfrak{z}^-,\mathfrak{z}^+;0)^{S^1}$.

Lemma 3.7. *Assumption 3.6 is satisfied if we replace H by ϵH for a sufficiently small ϵ.*

Proof. (2) is proved in [FOn1, page 1038]. More precisely it is proved there that for sufficiently small ϵ the following holds. Let ℓ be a gradient line joining $\mathfrak{z}^-$ to $\mathfrak{z}^+$ and h_ℓ be the corresponding element of $\mathcal{M}(X,\epsilon H;\mathfrak{z}^-,\mathfrak{z}^+;0)^{S^1}$. We consider the deformation complexes of the gradient line equation at ℓ and of the equation (3.4) at h_ℓ. The kernel and the cokernel of the former are contained in the kernel and the cokernel of the later, respectively. It is proved in [FOn1, page 1038] that they actually coincide each other if ϵ is sufficiently small.

Since H is Morse-Smale the element ℓ is Fredholm regular in the moduli space of gradient lines. Therefore by the above mentioned result the moduli space $\mathcal{M}^{\rm reg}(X,\epsilon H;\mathfrak{z}^-,\mathfrak{z}^+;0)$ is Fredholm regular at h_ℓ. This implies (2). (1) is a consequence of the same result and the implicite function theorem. (We remark that we can prove the same result at the point $\mathcal{M}(X,\epsilon H;\mathfrak{z}^-,\mathfrak{z}^+;\alpha)^{S^1} \setminus \mathcal{M}^{\rm reg}(X,\epsilon H;\mathfrak{z}^-,\mathfrak{z}^+;\alpha)^{S^1}$ in the same way.) □

We put

$$\mathcal{M}_0(X,H;\mathfrak{x}^-,\tilde{\mathfrak{x}}^+;0) = \mathcal{M}(X,H;\mathfrak{x}^-,\tilde{\mathfrak{x}}^+;0) \setminus \mathcal{M}(X,H;\mathfrak{x}^-,\tilde{\mathfrak{x}}^+;0)^{S^1}. \qquad (3.6)$$

Lemma 3.7 implies that $\mathcal{M}_0(X,H;\mathfrak{x}^-,\tilde{\mathfrak{x}}^+;0)$ is open and closed in $\mathcal{M}(X,H;\mathfrak{x}^-,\tilde{\mathfrak{x}}^+;0)$.

Theorem 3.8. ([FOn1, page 1036]) *If we assume Assumptions 3.1, 3.5 and 3.6, then the following holds.*
(1) *In case $\alpha \ne 0$ the Kuranishi structure on $\mathcal{M}(X,H;\mathfrak{x}^-,\tilde{\mathfrak{x}}^+;\alpha)$ can be taken to be S^1 equivariant.*
(2) *In case $\alpha = 0$ the same conclusion holds for $\mathcal{M}_0(X,H;\mathfrak{x}^-,\tilde{\mathfrak{x}}^+;0)$.*

4. S^1 equivariant Kuranishi structure for the Floer homology of time independent Hamiltonian.

In this section we prove Theorem 3.8 in detail. We begin with describing the compactification $\mathcal{M}(X,H;\mathfrak{z}^-,\mathfrak{z}^+;\alpha)$ of the moduli space $\mathcal{M}^{\rm reg}(X,H;\mathfrak{z}^-,\mathfrak{z}^+;\alpha)$. Here we include the case $\alpha = 0$ and the S^1 fixed point since it will appear in the fiber product factor of the compactification.

We consider (Σ, z_-, z_+), a genus zero semistable curve with two marked points.

Definition 4.1. Let Σ_0 be the union of the irreducible components of Σ such that

(1) $z_-, z_+ \in \Sigma_0$.
(2) Σ_0 is connected.
(3) Σ_0 is smallest among those satisfying (1),(2) above.

We call Σ_0 the *mainstream* of Σ. An irreducible component of Σ that is not contained in Σ_0 is called a *bubble component*.

Let $\Sigma_a \subset \Sigma$ be an irreducible component of the mainstream. If $z_- \notin \Sigma_a$ then there exists a unique singular point $z_{a,-}$ of Σ contained in Σ_a such that

(1) z_- and $\Sigma_a \setminus \{z_{a,-}\}$ belong to the different connected components of $\Sigma \setminus \{z_{a,-}\}$.
(2) z_+ and $\Sigma_a \setminus \{z_{a,-}\}$ belong to the same connected components of $\Sigma \setminus \{z_{a,-}\}$.

In case $z_- \in \Sigma_a$ we set $z_- = z_{a,-}$.

We define $z_{a,+}$ in the same way.

A *parametrization of the mainstream* of (Σ, z_-, z_+) is $\varphi = \{\varphi_a\}$, where $\varphi_a : \mathbb{R} \times S^1 \to \Sigma_a$ for each irreducible component Σ_a of the mainstream such that:

(1) φ_a is a biholomorphic map $\varphi_a : \mathbb{R} \times S^1 \cong \Sigma_a \setminus \{z_{a,-}, z_{a,+}\}$.
(2) $\lim_{\tau \to \pm\infty} \varphi_a(\tau, t) = z_{a,\pm}$.

Definition 4.2. We consider the triple $((\Sigma, z_-, z_+), u, \varphi)$ where:

(1) (Σ, z_-, z_+) is a genus zero semistable curve with two marked points.
(2) φ is a parametrization of the mainstream.
(3) $u : \Sigma \to X$ a continuous map from Σ to X.
(4) If Σ_a is an irreducible component of the mainstream and $\varphi_a : \mathbb{R} \times S^1 \to \Sigma_a$ is as above then the composition $h_a = u \circ \varphi_a$ satisfies the equation (3.4).
(5) If Σ_a is a bubble component then u is pseudo-holomorphic on it.
(6) $u(z_-) = \mathfrak{z}^-$, $u(z_+) = \mathfrak{z}^+$.
(7) $[u_*[\Sigma]] = \alpha$. Here $\alpha \in \Pi$.

We denote by $\widehat{\mathcal{M}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ the set of all such $((\Sigma, z_-, z_+), u, \varphi)$.

Definition 4.3. On the set $\widehat{\mathcal{M}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ we define three equivalence relations $\sim_1$, $\sim_2$, $\sim_3$ as follows.

$((\Sigma, z_-, z_+), u, \varphi) \sim_1 ((\Sigma', z'_-, z'_+), u', \varphi')$ if and only if there exists a biholomorphic map $v : \Sigma \to \Sigma'$ with the following properties:

(1) $u' = u \circ v$.
(2) $v(z_-) = z'_-$ and $v(z_+) = z'_+$. In particular v sends the mainstream of Σ to the mainstream of Σ'.
(3) If Σ_a is an irreducible component of the mainstream of Σ and $v(\Sigma_a) = \Sigma'_a$, then we have
$$v \circ \varphi_a = \varphi'_a. \tag{4.7}$$

The equivalence relation $\sim_2$ is defined replacing (4.7) by the existence of τ_a such that
$$(v \circ \varphi_a)(\tau, t) = \varphi'_a(\tau + \tau_a, t). \tag{4.8}$$

The equivalence relation $\sim_3$ is defined by requiring only (1), (2) above. (Namely by removing condition (3).

Remark 4.4. After taking the $\sim_3$ equivalence class, the data φ does not remain. Namely $((\Sigma, z_-, z_+), u, \varphi) \sim_3 ((\Sigma, z_-, z_+), u, \varphi')$ for any φ, φ'.

Definition 4.5. We put

$$\begin{aligned}
\widetilde{\mathcal{M}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha) &= \widehat{\mathcal{M}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)/\sim_1, \\
\mathcal{M}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha) &= \widehat{\mathcal{M}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)/\sim_2, \\
\overline{\overline{\mathcal{M}}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha) &= \widehat{\mathcal{M}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)/\sim_3 .
\end{aligned}$$

We use $\mathfrak{p}$ etc. to denote an element of $\mathcal{M}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$. We denote by $[\mathfrak{p}]$ its equivalence class in $\overline{\overline{\mathcal{M}}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$.

Let $((\Sigma, z_-, z_+), u, \varphi)$ be an element of $\widehat{\mathcal{M}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$. Suppose the mainstream of Σ has k irreducible components. We add the bubble tree to the irreducible component of the mainstream where it is rooted. We thus have obtained a decomposition

$$\Sigma = \sum_{i=1}^{k} \Sigma_i. \tag{4.9}$$

Here $z_- \in \Sigma_1$, $z_+ \in \Sigma_k$ and $\#(\Sigma_i \cap \Sigma_{i+1}) = 1$. We call each summand of (4.9) a *mainstream component*. We put $z_{i+1} = \Sigma_i \cap \Sigma_{i+1}$ and call it $(i+1)$-th *transit point*. We put $\mathfrak{z}_i = u(z_i)$ and call it a *transit image*.

Let $\mathfrak{p} = ((\Sigma, z_-, z_+), u, \varphi) \in \widehat{\mathcal{M}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ and Σ_i be one of its mainstream component. We restrict u, φ to Σ_i and obtain $\mathfrak{p}_i$. We say that Σ_i is a *gradient line component* if $\mathfrak{p}_i$ is a fixed point of the S^1 action.

In the definition of $\widehat{\mathcal{M}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ we forget the map u but remember only the homology classes of $u|_{\Sigma_v}$ of each irreducible components and the image $u(z_i)$ of the transit points (that are critical points of H). We then obtain a decorated moduli space of domain curves, $\widehat{\mathcal{M}}(\mathfrak{z}^-, \mathfrak{z}^+, \alpha)$. We define equivalence relation $\sim_j$ $(j = 1, 2, 3)$ on it in the same way and obtain $\widetilde{\mathcal{M}}(\mathfrak{z}^-, \mathfrak{z}^+, \alpha)$, $\mathcal{M}(\mathfrak{z}^-, \mathfrak{z}^+, \alpha)$, and $\overline{\overline{\mathcal{M}}}(\mathfrak{z}^-, \mathfrak{z}^+, \alpha)$.

For each element $\mathfrak{p}$ of $\widehat{\mathcal{M}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ etc., we obtain an element $\mathfrak{x}_{\mathfrak{p}}$ of $\widehat{\mathcal{M}}(\mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ etc. by forgetting u.

For each $\mathfrak{p} \in \widehat{\mathcal{M}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ etc. or $\mathfrak{x} \in \widehat{\mathcal{M}}(\mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ etc., we define a graph $\mathcal{G}_{\mathfrak{p}}$ or $\mathcal{G}_{\mathfrak{x}}$ in the same way as [FOOO5, Section 2.1]. (We include the data of the homology class of each component and the image of the transit point in $\mathcal{G}_{\mathfrak{p}}$ (resp. $\mathcal{G}_{\mathfrak{x}}$). We call $\mathcal{G}_{\mathfrak{p}}$ (resp. $\mathcal{G}_{\mathfrak{x}}$) the combinatorial type of $\mathfrak{p}$ (resp. $\mathfrak{x}$). We denote by $\widehat{\mathcal{M}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathcal{G})$ etc. or $\widehat{\mathcal{M}}(\mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathcal{G})$ etc. the subset of the objects with combinatorial type $\mathcal{G}$.

We consider the subset $\widehat{\mathcal{M}}^{\rm reg}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ of $\widehat{\mathcal{M}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ consisting of the elements $((\Sigma, z_-, z_+), u, \varphi)$ such that $\Sigma = S^2$. Let $\widetilde{\mathcal{M}}^{\rm reg}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$, $\mathcal{M}^{\rm reg}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$, $\overline{\overline{\mathcal{M}}}^{\rm reg}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ be the $\sim_1$, $\sim_2$, $\sim_3$ equivalence classes of $\widehat{\mathcal{M}}^{\rm reg}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$, respectively.

It is easy to see that $\widetilde{\mathcal{M}}^{\rm reg}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$, $\mathcal{M}^{\rm reg}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ coincide with the ones in Definition 3.3. In particular

$$\mathcal{M}^{\rm reg}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha) \cong \widetilde{\mathcal{M}}^{\rm reg}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)/\mathbb{R}. \tag{4.10}$$

Here the $\mathbb{R}$ action is obtained by the translation along the $\mathbb{R}$ direction of the source and is free.

Moreover we have

$$\overline{\overline{\mathcal{M}}}^{\rm reg}(X,H;\mathfrak{z}^-,\mathfrak{z}^+,\alpha) \cong \mathcal{M}^{\rm reg}(X,H;\mathfrak{z}^-,\mathfrak{z}^+,\alpha)/S^1. \tag{4.11}$$

Here the S^1 action is obtained by the S^1 action of the source.

We however remark that the fiber of the canonical map

$$\mathcal{M}(X,H;\mathfrak{z}^-,\mathfrak{z}^+,\alpha) \to \overline{\overline{\mathcal{M}}}(X,H;\mathfrak{z}^-,\mathfrak{z}^+,\alpha) \tag{4.12}$$

between the compactified moduli spaces may be bigger than S^1. In fact if (Σ, z_-, z_+) has k mainstream components then the fiber of $[(\Sigma,z_-,z_+),u,\varphi]$ in $\mathcal{M}(X,H;\mathfrak{z}^-,\mathfrak{z}^+,\alpha)$ is $(S^1)^k$ for the generic points.

On the other hand there exists an S^1 action on $\mathcal{M}(X,H;\mathfrak{z}^-,\mathfrak{z}^+,\alpha)$ obtained by

$$t_0\cdot[(\Sigma,z_-,z_+),u,\varphi] = [(\Sigma,z_-,z_+),u,t_0\cdot\varphi]$$

where

$$(t_0\cdot\varphi)_a(\tau,t) = \varphi_a(\tau,t+t_0).$$

Definition 4.6.

$$\overline{\mathcal{M}}(X,H;\mathfrak{z}^-,\mathfrak{z}^+,\alpha) = \mathcal{M}(X,H;\mathfrak{z}^-,\mathfrak{z}^+,\alpha)/S^1.$$

The map (4.12) factors through $\overline{\mathcal{M}}(X,H;\mathfrak{z}^-,\mathfrak{z}^+,\alpha)$.

We can prove that $\overline{\mathcal{M}}(X,H;\mathfrak{z}^-,\mathfrak{z}^+,\alpha)$ is compact in the same way as [FOn1, Theorem 11.1].

In place of taking the quotient by the $\mathbb{R}$ action in (4.10) we can require the following balancing condition. (In other words we can take a global section of this $\mathbb{R}$ action.)

Definition 4.7. Let $((\Sigma,z_-,z_+),u,\varphi) \in \widetilde{\mathcal{M}}(X,H;\mathfrak{z}^-,\mathfrak{z}^+,\alpha)$. Suppose that it has only one mainstream component. We define a function $\mathcal{A} : \mathbb{R} \setminus \text{a finite set} \to \mathbb{R}$ as follows.

Let $\tau_0 \in \mathbb{R}$. We assume $\varphi(\{\tau_0\}\times S^1)$ does not contain a root of the bubble tree. (This is the way how we remove a finite set from the domain of $\mathcal{A}$.) Let Σ_{v_i}, $i = 1,\dots,m$ be the set of the irreducible components that is in a bubble tree rooted on $\mathbb{R}_{\le\tau_0}\times S^1$. We define

$$\mathcal{A}(\tau_0) = \sum_{i=1}^m \int_{\Sigma_{\mathrm{v}_i}} u^*\omega + \int_{\tau=-\infty}^{\tau_0}\int_{t\in S^1}(u\circ\varphi)^*\omega + \int_{t\in S^1} H(u(\varphi(\tau_0,t)))dt. \tag{4.13}$$

This is a nondecreasing function and satsifies

$$\lim_{\tau\to-\infty}\mathcal{A}(\tau) = H(\mathfrak{z}_-), \qquad \lim_{\tau\to+\infty}\mathcal{A}(\tau) = H(\mathfrak{z}_+) + \alpha\cap\omega.$$

We say φ satisfies the *balancing condition* if

$$\lim_{\substack{\tau<0\\ \tau\to0}}\mathcal{A}(\tau) \le \frac{1}{2}\left(H(\mathfrak{z}_-)+H(\mathfrak{z}_+)+\alpha\cap\omega\right) \le \lim_{\substack{\tau>0\\ \tau\to0}}\mathcal{A}(\tau). \tag{4.14}$$

In case $((\Sigma,z_-,z_+),u,\varphi) \in \widetilde{\mathcal{M}}(X,H;\mathfrak{z}^-,\mathfrak{z}^+,\alpha)$ we require the balancing condition in mainstream-component-wise.

Remark 4.8. We remark that there exists unique φ satisfying the balancing condition in each of the $\mathbb{R}$ orbits, except the following case: u is constant on the (unique) irreducible component in the mainstream. (In this case there must occur a non-trivial bubble.) The uniqueness breaks down in the case when there exists τ_0 such that

$$\sum_{i=1}^{m} \int_{\Sigma_{\mathrm{v}_i}} u^*\omega = \frac{1}{2}\omega \cap \alpha$$

in addition. (Here $\{\mathrm{v}_i \mid i = 1, \dots, m\}$ is the bubbles associated to τ_0 as in Definition 4.7.) In such a case we replace $\mathcal{A}$ by the following reguralized version

$$\mathcal{A}'(\tau_0) = \frac{2}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-(\tau-\tau_0)^2} \mathcal{A}(\tau) d\tau.$$

The first derivative of $\mathcal{A}'$ is always strictly positive. So there exists a unique φ satisfying the modified balanced codition (using $\mathcal{A}'$) in each $\mathbb{R}$ orbit.

Note however the balancing condition will be mainly used later to define canonical marked point. In the case there is a sphere bubble we will not take a canonical marked point. So this remark is only for consistency of the terminology.

We have thus defined a compactification of $\mathcal{M}^{\mathrm{reg}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$. We will construct a Kuranishi structure with corner on it. The construction is mostly the same as the proof of [FOOO5, Theorem 2.3], which is a detailed version of the proof of [FOn1, Theorem 7.10]. Here we explain this proof in more detail than [FOn1].

We first remark that $\overline{\overline{\mathcal{M}}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ does *not* have Kuranishi structure in general. (Even in the case $\alpha \neq 0$.) This is because there is an element in this moduli space whose isotropy group is of positive dimension. Namely if Σ_i is a gradient line component, then the biholomorphic S^1 action on the component Σ_i is in the group of automorphisms of this element of $\overline{\overline{\mathcal{M}}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$. So a neighborhood of this element may not be a manifold with corner.

On the other hand, the S^1 action on $\mathcal{M}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ is always locally free (namely its isotropy group is a finite group) in case $\alpha \neq 0$. In the case of $\alpha = 0$ the S^1 action on $\mathcal{M}_0(X, H; \mathfrak{z}^-, \mathfrak{z}^+, 0)$ is locally free.

To define obstruction bundle on the compactification we need to take an obstruction bundle data in the same way as [FOOO5, Definition 2.33]. To keep consistency with the fiber product description of the boundary or corner of $\overline{\mathcal{M}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ we will define it in a way invariant not only under the S^1 action but also under the $(S^1)^k$ action on the part where there are k irreducible components in the mainstream.

We also need to consider the case of gradient line component at the same time. Note we assumed that the map u is an immersion at the additional marked points in [FOOO5, Definition 2.31 (3)] (the definition of symmetric stabilization). In case of gradient line component, there is no such point. However since we assumed that the gradient flow of our Hamiltonian H is Morse-Smale and satisfying Assumption 3.6 (2), we actually do *not* need to perturb the equation on such a component. So our obstruction bundle there is, by definition, a trivial bundle. (And we do not need to stabilize such a component to define obstruction bundle.)

Taking this into account we define the obstruction bundle data in our situation as the following Definition 4.12.

Definition 4.9. A symmetric stabilization of $((\Sigma, z_-, z_+), u, \varphi)$ is $\vec{w}$ such that $\vec{w} \cap \Sigma_i$ is a symmetric stabilization of Σ_i in the sense of [FOOO5, Definition 2.31] if Σ_i is not a gradient line component, and $\vec{w} \cap \Sigma_i = \emptyset$ if Σ_i is a gradient line component.

Definition 4.10. Let $\mathfrak{p} = ((\Sigma, z_-, z_+), u, \varphi)$ be as above. We assume Σ_i is a gradient line component. Note, then $\ell(\tau) = u(\varphi_i(\tau, t))$ is a gradient line joining transit images $\mathfrak{z}_i$ and $\mathfrak{z}_{i+1}$. There exists a unique τ_0 such that

$$H(\ell(\tau_0)) = \frac{1}{2}\left(H(\mathfrak{z}_i) + H(\mathfrak{z}_{i+1})\right).$$

We put $w_i = \varphi_i(\tau_0, 0)$, which we call the *canonical marked point* of our gradient line component.

Remark 4.11. We remark that the pair $(\mathfrak{p}, w_i)$ where w_i is the canonical marked point depends only on the $\sim_3$ equivalece class of $\mathfrak{p}$ in the following sense. Suppose $\mathfrak{p} \sim_3 \mathfrak{p}'$. We define w'_i in the i-th mainstream component of $\Sigma_{\mathfrak{p}'}$ as above. Then there exists an isomorphism $v : \Sigma_{\mathfrak{p}} \to \Sigma_{\mathfrak{p}'}$ satisfying Definition 4.3 (1), (2) and such that $v(w_i) = w'_i$. This is because u is S^1 equivariant on this irreducible component.

On the other hand, if the homology class $u_*([\Sigma_i])$ is nonzero, the pair $(\mathfrak{p}, w_i)$ is not $\sim_3$ equiviant to $(\mathfrak{p}, w'_i)$ in the above sense, where $w'_i = \varphi_i(\tau_0, t_0)$.

We remark that, if the i-th mainstream component Σ_i consists of a gradient line and sphere bubbles, then we put $\vec{w}$ only on the part of the sphere bubble. Note the irreducble component that is the intersection of this mainsteam component and the mainstream is (source) stable since the root of the bubble is the third marked point.

Definition 4.12. An *obstruction bundle data* $\mathfrak{E}_{\mathfrak{p}}$ *centered at*

$$[\mathfrak{p}] = [(\Sigma, z_-, z_+), u, \varphi] \in \overline{\overline{\mathcal{M}}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$$

is the data satisfying the conditions described below. We put $\mathfrak{x} = (\Sigma, z_-, z_+)$. Let $\mathfrak{x}_i$ be the i-th mainstream component. (It has two marked points.) We put $\alpha_i = u_*[\Sigma_i] \in \Pi$.

(1) A symmetric stabilization $\vec{w}$ of $\mathfrak{p}$. We put $\vec{w}^{(i)} = \vec{w} \cap \mathfrak{x}_i$.
(2) The same as [FOOO5, Definition 2.33 (2)].
(3) A universal family with coordinate at infinity of $\mathfrak{x}_{\mathfrak{p}} \cup \vec{w} \cup \vec{w}^{\rm can}$. Here we put canonical marked points (Definition 4.10) for each gradient line components and denote it by $\vec{w}^{\rm can}$. We require some additional condition (Condition 4.13 below) for the coordinate at infinity.
(4) The same as [FOOO5, Definition 2.33 (4)]. Namely compact subsets $K_{\rm v}^{\rm obst}$ of $\Sigma_{\rm v}$. (The support of the obstruction bundle.) We do not put $K_{\rm v}^{\rm obst}$ on the gradient line components. In case the i-th mainstream component Σ_i consists of a gradient line and sphere bubbles, then we put $K_{\rm v}^{\rm obst}$ only on the bubble.
(5) The same as [FOOO5, Definition 2.33 (5)]. Namely, finite dimensional complex linear subspaces $E_{\mathfrak{p},{\rm v}}(\mathfrak{y}, u)$. We do not put them on the gradient line components. In case i-th mainstream component Σ_i consists of the gradient line and a sphere bubble, then we put them only on the bubble.

(6) The same as [FOOO5, Definition 2.33 (6)] except the differential operator there
$$\begin{aligned}\overline{D}_u\overline{\partial} :&L^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_{\rm v}}, \partial\Sigma_{\mathfrak{y}_{\rm v}}); u^*TX, u^*TL)\\ &\to L^2_{m,\delta}(\Sigma_{\mathfrak{y}_{\rm v}}; u^*TX \otimes \Lambda^{01})/E_{\mathfrak{p},{\rm v}}(\mathfrak{y}, u)\end{aligned} \tag{4.15}$$
is replaced by the linearization of the equation (3.4)

(7) The same as [FOOO5, Definition 2.33 (7)].

(8) We take a codimension 2 submanifold $\mathcal{D}_j$ for each of $w_j \in \vec{w}$ in the same way as [FOOO5, Definition 2.33 (8)]. We remark that we do *not* take such submanifolds for the canonical marked points $\in \vec{w}^{\rm can}$. (In fact since u is not an immersion at the canonical marked points we can not choose such submanifolds.)

We require that the deta $K^{\rm obst}_{\rm v}$, $E_{\mathfrak{p},{\rm v}}(\mathfrak{y}, u)$ depend only on the mainstream component $\mathfrak{p}_i = [(\Sigma_i, z_{i-1}, z_i), u, \varphi]$ (Here z_i is the i-th transit point.) that contains the v-th irreducible component. We call this condition *mainstream-component-wise.*

The additional condition we assume in item (3) above is as follows.

Condition 4.13. (1) Let z_{i+1} be the $(i+1)$-th transit point, which is contained in Σ_i and in Σ_{i+1}. Then the coordinate at infinity near z_{i+1} coincides with the parametrization φ_i or φ_{i+1} up to the $\mathbb{R} \times S^1$ action. Namely it is $(\tau, t) \mapsto \varphi_i(\tau + \tau_0, t + t_0)$ (resp. $\varphi_{i+1}(\tau + \tau_0, t + t_0)$) for some τ_0 and t_0.

(2) Let z be a singular point that is not a transit point and $\Sigma_{\rm v}$ an irreducible component containing z. Since $\Sigma_{\rm v}$ is a sphere there exists a biholomorphic map
$$\phi : \Sigma_{\rm v} \cong \mathbb{C} \cup \{\infty\}$$
such that $\phi(z) = 0$.

Then a coordinate at infinity around z is given as
$$(\tau, t) \mapsto \phi^{-1}(e^{\pm 2\pi(\tau+\sqrt{-1}t)}),$$
for some choice of ϕ. Here $\pm$ depends on the orientation of the edge corresponding to z.

Note the choice of coordinate at infinity satisfying the above condition is not unique.

Remark 4.14. In (2) above we make full use of the fact that our curve is of genus 0. The construction of [FOOO5] is designed so that it works in the case of arbitrary genus without change. So we did not put this condition in [FOOO5]. Condition 4.13 (2) will be used to simplify the discussion on how to handle the Hamiltonian perturbation in the gluing analysis. See Lemma 4.24.

We can prove the existence of an obstruction bundle data in the same way as [FOOO5, Lemma 2.37]. For example we can choose the marked points $\vec{w}$ as follows: We remark that the restriction of u to the irreducible component $\Sigma_{\rm v}$ is not homologous to zero except the following two cases. So we can find a point of $\Sigma_{\rm v}$ at which u is an immersion and take it as an additional marked point.

(1) $\Sigma_{\rm v}$ is in the mainstream and is not a root of the sphere bubble.
(2) $\Sigma_{\rm v}$ is in the mainstream and is a root of the sphere bubble.

In Case (1), we take only the canonical marked point on this component. In Case (2), this irreducible component is stable. So we do not take additional marked point on this component. Thus we can define $\vec{w}$.

We take and fix obstruction bundle data for each element of $\overline{\overline{\mathcal{M}}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$.

We defined the moduli spaces $\widehat{\mathcal{M}}(\mathfrak{z}^-, \mathfrak{z}^+, \alpha)$, $\mathcal{M}(\mathfrak{z}^-, \mathfrak{z}^+, \alpha)$, and $\overline{\overline{\mathcal{M}}}(\mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ in Definition 4.5. We add ℓ additional marked points on it and denote the moduli space of such objects as $\widehat{\mathcal{M}}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha)$, $\mathcal{M}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha)$, and $\overline{\overline{\mathcal{M}}}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha)$. We denote by $\widehat{\mathcal{M}}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathcal{G})$, $\mathcal{M}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathcal{G})$, and $\overline{\overline{\mathcal{M}}}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathcal{G})$ its subset so that its combinatorial type is $\mathcal{G}$. (We include the datum on how the additional marked points $\vec{w}$ in $\mathcal{G}$ are distributed over the irreducible components.)

Let $((\Sigma, z_-, z_+), u, \varphi) \cup \vec{w} \cup \vec{w}^{\rm can} = \mathfrak{p} \cup \vec{w} \cup \vec{w}^{\rm can}$ be as in Definition 4.9 with decomposition (4.9). We put $\ell = \#(\vec{w} \cup \vec{w}^{\rm can})$ and denote by $\overline{\overline{\mathfrak{V}}}(\mathfrak{p} \cup \vec{w} \cup \vec{w}^{\rm can})$ a neighborhood of $\mathfrak{x}_{\mathfrak{p}} \cup \vec{w} \cup \vec{w}^{\rm can}$ in $\overline{\overline{\mathcal{M}}}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathcal{G}_{\mathfrak{p} \cup \vec{w} \cup \vec{w}^{\rm can}})$.

In the same way as [FOOO5, Definition 2.14] we define a map

$$\overline{\overline{\Phi}} : \overline{\overline{\mathfrak{V}}}(\mathfrak{p} \cup \vec{w} \cup \vec{w}^{\rm can}) \times ((\vec{T}, \infty] \times S^1) \to \overline{\overline{\mathcal{M}}}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha), \tag{4.16}$$

that is an isomorphism onto an open neighborhood of $[\mathfrak{x}_{\mathfrak{p}} \cup \vec{w} \cup \vec{w}^{\rm can}]$. Here the notation $((\vec{T}, \infty] \times S^1)$ is similar to [FOOO5, Definition 2.13].

We next add the parametrization φ of the mainstream to the map (4.16) and define its $\mathcal{M}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha)$-version below.

We define a manifold with corner $\widetilde{D}(k; \vec{T}_0)$ as follows. We put

$$\widetilde{\overset{\circ}{D}}(k; T_0) = \{(T_1, \dots, T_k) \in \mathbb{R}^k \mid T_{i+1} - T_i \geq T_{0,i}\}. \tag{4.17}$$

We (partially) compactify $\widetilde{\overset{\circ}{D}}(k; \vec{T}_0)$ to $\widetilde{D}(k; \vec{T}_0)$ by admitting $T_{i+1} - T_i = \infty$ as follows. We take $s'_i = 1/(T_{i+1} - T_i)$ then T_1 and $s'_1, \dots, s'_{k-1}$ define another parameters. So (4.17) is identified with $\mathbb{R} \times \prod_{i=1}^k (0, 1/T_{0,i}]$. We (partially) compactify it to $\mathbb{R} \times \prod_{i=1}^k [0, 1/T_{0,i}]$.

By taking the quotient of the $\mathbb{R}$ action $T(T_1, \dots, T_k) = (T_1 + T, \dots, T_k + T)$, we obtain $\overset{\circ}{D}(k; \vec{T}_0)$ and $D(k; \vec{T}_0)$.

Let $\mathfrak{V}(\mathfrak{p} \cup \vec{w} \cup \vec{w}^{\rm can}) \subset \mathcal{M}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathcal{G}_{\mathfrak{p} \cup \vec{w} \cup \vec{w}^{\rm can}})$ be the inverse image of $\overline{\overline{\mathfrak{V}}}(\mathfrak{p} \cup \vec{w} \cup \vec{w}^{\rm can})$ under the projection

$$\mathcal{M}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathcal{G}_{\mathfrak{p} \cup \vec{w} \cup \vec{w}^{\rm can}}) \to \overline{\overline{\mathcal{M}}}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha). \tag{4.18}$$

Remark 4.15. Note for an elements $((\Sigma', z'_-, z'_+), \varphi') \cup \vec{w}'$, the marked points w'_i that correspond to canonical marked points $\in \vec{w}^{\rm can}$ may not be canonical. (Namely it may not be of the form $\varphi'(\tau_0, 0)$ where τ_0 is an in Definition 4.10.)

$\mathcal{M}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathcal{G}_{\mathfrak{p} \cup \vec{w} \cup \vec{w}^{\rm can}})$ carries an $(S^1)^k$ action on it given by

$$(t_1, \dots, t_k)(((\Sigma, z_-, z_+), u, \varphi) \cup \vec{w} \cup \vec{w}^{\rm can}) = ((\Sigma, z_-, z_+), u, \varphi') \cup \vec{w} \cup \vec{w}^{\rm can}$$

where $\varphi = (\varphi_i)_{i=1}^k$ and $\varphi' = (\varphi'_i)_{i=1}^k$ such that

$$\varphi'_i(\tau, t) = \varphi_i(\tau, t + t_i).$$

This action is locally free and the map (4.18) can be identified with the canonical projection:

$$\mathcal{M}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathcal{G}_{\mathfrak{p} \cup \vec{w} \cup \vec{w}^{\rm can}}) \to \mathcal{M}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathcal{G}_{\mathfrak{p} \cup \vec{w} \cup \vec{w}^{\rm can}})/(S^1)^k.$$

It follows from this fact that $\mathfrak{V}(\mathfrak{p} \cup \vec{w} \cup \vec{w}^{\rm can})$ is an open neighborhood of the inverse image of $[\mathfrak{p}]$ in $\mathcal{M}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathcal{G}_{\mathfrak{p}\cup\vec{w}\cup\vec{w}^{\rm can}})$.

We now define:

$$\overline{\Phi} : \mathfrak{V}(\mathfrak{p} \cup \vec{w} \cup \vec{w}^{\rm can}) \times D(k;\vec{T}_0) \times (\prod_{j=1}^m (T_{0,j},\infty] \times S^1)/\sim) \to \mathcal{M}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha) \tag{4.19}$$

that is a homeomorphism onto an open neighborhood of the inverse image of $[\mathfrak{p}]$ in $\mathcal{M}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha)$. (Here $\sim$ is as in [FOOO5, Remark 2.9].)

Let $\mathfrak{y} \in \mathfrak{V}(\mathfrak{p} \cup \vec{w} \cup \vec{w}^{\rm can})$ and $(\vec{T}, \vec{\theta}) \in \widetilde{D}(k;\vec{T}_0) \times (\prod_{j=1}^m (T_{0,j},\infty] \times S^1)/\sim)$. Note $\mathfrak{V}(\mathfrak{p} \cup \vec{w} \cup \vec{w}^{\rm can})$ is the quotient space of the $\mathbb{R}$ action. We represent the quotient by the slice obtained by requiring the balancing condition (Definition 4.7).

The element $\mathfrak{y}$ comes with the coordinate around the m singular points of Σ that are not transit points. (This is a part of the stabilization data centered at $\mathfrak{p}$.) We use the parameters in $(\prod_{j=1}^m (T_{0,j},\infty] \times S^1)/\sim$ to resolve those singular points.

The rest of the parameter $\vec{T}' = (T_1,\dots,T_k) \in D(k_1,k_2;\vec{T}_0)$ is used to resolve the transit points as follows. We consider the case this parameter $\vec{T}'$ is in $\overset{\circ}{D}(k;\vec{T}_0)$. Let us consider

$$[-5T_i, 5T_i]_i \times S^1_i$$

and regard it as a subset of the domain of $\varphi_i : \mathbb{R} \times S^1 \to \Sigma_i$. We define

$$\varphi_0 : \bigcup_i ([-5T_i, 5T_i]_i \times S^1_i) \to \mathbb{R} \times S^1$$

as follows. If $(\tau, t) \in [-5T_i, 5T_i]_i \times S^1_i$ then

$$\varphi_0(\tau, t) = (\tau + 10T_i, t).$$

We use $\varphi_0 \circ \varphi_i^{-1}$ to identify (a part of) Σ_i with a subset of $\mathbb{R} \times S^1$. We then use this identification to move bubble components (glued) and marked points. So together with $\mathbb{R} \times S^1$ it gives a marked Riemann surface. We thus obtain $\mathfrak{Y}$.

The image of φ_0 are in the *core* of $\mathfrak{Y}$ and the complement of the core in the mainstream is the neck region.

By taking a quotient with respect to the $\mathbb{R}$ action, we obtain:

$$\mathfrak{Y} = \overline{\Phi}(\mathfrak{y}, \vec{T}, \vec{\theta}) \in \mathcal{M}_\ell(\mathfrak{z}^-, \mathfrak{z}^+, \alpha).$$

We have thus defined (4.19).

We next consider $((\Sigma', z'_-, z'_+), u', \varphi') \cup \vec{w}'$ and define the ϵ-close-ness of it from $[\mathfrak{p} \cup \vec{w} \cup \vec{w}^{\rm can}] = [((\Sigma, z_-, z_+), u, \varphi) \cup \vec{w} \cup \vec{w}^{\rm can}]$. Here $((\Sigma', z'_-, z'_+), u', \varphi')$ is assumed to satisfy Definition 4.2 (1)(2)(3)(6)(7) and $\vec{w}'$ is the set of ℓ additional marked points. We decompose

$$\Sigma' = \sum_{j=1}^{k'} \Sigma'_j \tag{4.20}$$

into the mainstream components. Let z'_j be the j-th transit point and $\alpha'_j = u'_*([\Sigma'_j])$. We assume there exists a map $i : \{1,\dots,k'\} \to \{1,\dots,k\}$ such that

(a) $u(z_{i(j)}) = u'(z'_j)$.
(b) $\sum_{i=i(j)}^{i(j+1)-1} \alpha_i = \alpha'_j$.

Here z_i is the the i-th transit point of $\mathfrak{p}$ and $\alpha_i = u_*([\Sigma_i])$.

Definition 4.16. We say $((\Sigma',z'_-,z'_+),u',\varphi')\cup\vec w'$ is ϵ-close to $[\mathfrak p\cup\vec w\cup\vec w^{\rm can}]$ if the following holds.

(1)
$$((\Sigma',z'_-,z'_+),u')\cup\vec w' = \overline{\overline{\Phi}}(\overline{\mathfrak y},\vec T,\vec\theta) \tag{4.21}$$
where $\overline{\mathfrak y}\in\overline{\overline{\mathfrak V}}(\mathfrak p\cup\vec w\cup\vec w^{\rm can})$. And [FOOO5, Definition 2.38 (1)] holds.

(2) [FOOO5, Definition 2.38 (2)] holds.

(3) [FOOO5, Definition 2.38 (3)] holds.

(4) [FOOO5, Definition 2.38 (4)] holds. (Namely each of the component of $\vec T$ is $> 1/\epsilon$.)

(5) If Σ_i, $i=i(j),\dots,i(j+1)-1$ are all gradient line components, then Σ'_j is also a gradient line component that is ϵ-close to the union of the gradient lines $u|_{\Sigma_i}$, $i=i(j),\dots,i(j+1)-1$. We also require that $\vec w'\cap\Sigma'_j$ consists of $i(j+1)-i(j)$ points $z'_{i(j)+1},\dots,z'_{i(j+1)}$ such that
$$\left|H(z_i)-\frac{1}{2}\left(H(\mathfrak z_i)+H(\mathfrak z_{i+1})\right)\right|<\epsilon$$
for $i(j)\le i\le i(j+1)-1$.

If $((\Sigma',z'_-,z'_+),u',\varphi')\cup\vec w'$ is ϵ-close to $[\mathfrak p\cup\vec w\cup\vec w^{\rm can}]$ and ϵ is sufficiently small, we can define an obstruction bundle for $((\Sigma',z'_-,z'_+),u',\varphi')$ in a similar way as [FOOO5, Definition 2.41] as follows.

Definition 4.17. We consider the decomposition (4.20) of Σ' into the mainstream components and define $i(j)$ as in (a),(b) there. We will define an obstruction bundle supported on each of Σ'_j.

If Σ'_j is a gradient line component then we set an obstruction bundle to be trivial on Σ'_j.

Suppose Σ'_j is not a gradient line component. We remove all the marked points $\vec w'\cap\Sigma'_j$ that correspond to $\vec w^{\rm can}$. We denote by $\vec w'_{0,j}\subseteq\vec w'\cap\Sigma'_j$ the remaining marked points on Σ'_j. It is easy to see that $(\Sigma'_j;z'_j,z'_{j+1})\cup\vec w'_{0,j}$ is stable. Let $\vec w_{0,j}\subseteq\vec w$ be the set of the marked poins on Σ corresponding to the marked points $\vec w'_{0,j}$.

In the union $\bigcup_{i=i(j)}^{i(j+1)-1}\Sigma_i$, we shrink each of the gradient line components Σ_i to a point. Let $\Sigma_{0,j}$ be the resulting semi-stable curve. Then $(\Sigma_{0,j};z_{i(j)},z_{i(j+1)})\cup\vec w_{0,j}$ is stable. It has a coordinate at infinity induced by one given in Definition 4.12 (3). We remark that the union of the supports of the obstruction bundles in $\bigcup_{i=i(j)}^{i(j+1)-1}\Sigma_i$ may be regarded as subsets of $\Sigma_{0,j}$.

We observe that $(\Sigma'_j;z'_j,z'_{j+1})\cup\vec w'_{0,j}$ is obtained from $(\Sigma_{0,j};z_{i(j)},z_{i(j+1)})\cup\vec w_{0,j}$ by resolving singular points. Therefore using the above mentioned coordinate at infinity we have a diffeomorphism from the supports of the obstruction bundles in $\bigcup_{i=i(j)}^{i(j+1)-1}\Sigma_i$ onto open subsets of Σ'_j, together with parallel transport to define an obstruction bundle on Σ'_j, in the same way as [FOOO5, Definition 2.41].

Remark 4.18. We remark that by construction the obstruction bundle that we defined on $((\Sigma',z'_-,z'_+),u',\varphi')\cup\vec w'$ is independent of the marked points $\in\vec w'$ corresponding to the canonical marked points $\vec w^{\rm can}$. This is important to see that our construction is S^1 equivariant.

In other words, the canonical marked points $\vec{w}^{\rm can}$ and the corresponding marked points in $\vec{w}'$ do *not* play an important role in our construction. We introduce them so that our terminology is as close to the one in [FOOO5] as possible.

In the same way as [FOOO5, Corollary 2.43], we can show this obstruction bundle is independent of the equivalence relation $\sim_i$ ($i = 1, 2, 3$) that are defined in the same way as Definition 4.3. We can define Fredholm regurality and evaluation-map-transversality of such obstruction bundle in the same way as [FOOO5, Definition 2.44], [FOOO5, Definition 2.46], respectively. Then an obvious analogue of [FOOO5, Proposition 2.48] is proved in the same way.

Definition 4.19. Suppose $((\Sigma', z'_-, z'_+), u', \varphi') \cup \vec{w}'$ is as in Defintion 4.16. We say that it satisfies the *transversal constraint* if the following holds.

(1) Let w'_i be one of the elements of $\vec{w}'$. If the corresponding $w_i \in \vec{w} \cup \vec{w}^{\rm can}$ is contained in $\vec{w}$ then $u'(w'_i)$ in contained in the codimension 2 submanifold $\mathcal{D}_i$ that are given as a part of Definition 4.12 (8).
(2) Suppose Σ'_j is a gradient line component and $w'_{i(j)+1}, \dots, w'_{i(j+1)} = \vec{w}' \cap \Sigma'_j$. Then
$$H(u(w'_i)) = \frac{1}{2}\left(H(\mathfrak{z}_i) + H(\mathfrak{z}_{i+1})\right)$$
for $i(j) + 1 \le i \le i(j+1)$.
(3) Let $w'_1, \dots, w'_n$ be the points in $\vec{w}'$ corresponding to $\vec{w}^{\rm can}$. We may assume that they are all contained in the mainstream (by taking ϵ small.) Then we require that the S^1 coordinate thereof are all $[0] \in \mathbb{R}/\mathbb{Z} = S^1$.

Then for each $\mathfrak{p} \in \overline{\overline{\mathcal{M}}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ we fix an obstruction bundle data $\mathfrak{E}_{\mathfrak{p}}$ centered at $[\mathfrak{p}]$. In particular we have $\vec{w}_{\mathfrak{p}}$. We choose $\epsilon_{\mathfrak{p}}$ so that the conclusion of [FOOO5, Lemma 2.50] holds.

For each $[\mathfrak{p}] \in \overline{\overline{\mathcal{M}}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ we denote by $\mathfrak{W}_{\mathfrak{p}}$ the subset of $\overline{\overline{\mathcal{M}}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ consisting of $[\mathfrak{p}']$ satisfying the following conditions. There exists $\vec{w}'$ such that $\mathfrak{p}' \cup \vec{w}'$ is $\epsilon_{\mathfrak{p}}$-close to $[\mathfrak{p} \cup \vec{w}^{\mathfrak{p}} \cup \vec{w}^{\rm can}]$.

We then find a finite set $\mathfrak{C} = \{\mathfrak{p}_c\}$ such that
$$\bigcup_c {\rm Int}\, \mathfrak{W}_{\mathfrak{p}_c} = \overline{\overline{\mathcal{M}}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha).$$
where $\mathfrak{W}_{\mathfrak{p}_c}$ consists of elements $\mathfrak{p}$ with the following property: there exists $\vec{w}_{\mathfrak{p}}$ such that $\mathfrak{p} \cup \vec{w}_{\mathfrak{p}}$ is $\epsilon_{\mathfrak{p}_c}$-close to $\mathfrak{p}_c \cup \vec{w}_{\mathfrak{p}_c} \cup \vec{w}^{\rm can}$ and $\vec{w}_{\mathfrak{p}}$ satisfies the transversal constraint.

We can construct such $\mathfrak{C} = \mathfrak{C}(\alpha)$ mainstream-component-wise in the following sense. We decompose $\mathfrak{p} = \cup \mathfrak{p}_i$ into its mainstream components. Then $\mathfrak{p} \in \mathfrak{C}(\alpha)$ if and only if $\mathfrak{p}_i \in \mathfrak{C}(\alpha_i)$[2] for each i. As long as we consider a finite number of α's, we can construct such $\mathfrak{C}(\alpha)$ inductively over the energy of α.

For each $[\mathfrak{p}] \in \overline{\overline{\mathcal{M}}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ we define the notion of stabilization data in the same way as [FOOO5, Definition 2.57]. (In other words we require Definition 4.12 (1)(2)(3)(8).)

We put
$$\mathfrak{C}_{[\mathfrak{p}]} = \{c \in \mathfrak{C}(\alpha) \mid [\mathfrak{p}] \in {\rm Int}\, \mathfrak{W}_{\mathfrak{p}_c}\}. \tag{4.22}$$

[2]For the component $\alpha_i = 0$ we take a sufficiently dense subset of the moduli space of gradient lines and use it.

We remark that that if $c \in \mathfrak{C}_{[\mathfrak{p}]}$ there exists $\vec{w}_c^{\mathfrak{p}}$ such that $\mathfrak{p} \cup \vec{w}_c^{\mathfrak{p}}$ is ϵ_c-close to $\mathfrak{p}_c \cup \vec{w}_c \cup \vec{w}^{\rm can}$ and $\vec{w}_c^{\mathfrak{p}}$ satisfies the transversal constraint. We take such $\vec{w}_c^{\mathfrak{p}}$ and fix it.

Let

$$[\mathfrak{p}] = [(\Sigma_{\mathfrak{p}}, z_{\mathfrak{p},-}, z_{\mathfrak{p},+}), u_{\mathfrak{p}}, \varphi_{\mathfrak{p}})] \in \overline{\overline{\mathcal{M}}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha).$$

Definition 4.20. We define a *thickened moduli space*

$$\mathcal{M}_{(\ell_{\mathfrak{p}},(\ell_c))}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathfrak{p}; \mathfrak{A}; \mathfrak{B})_{\epsilon_0, \vec{T}_0} \tag{4.23}$$

as follows. (Here $\mathfrak{A} \subset \mathfrak{B} \subset \mathfrak{C}_{[\mathfrak{p}]}$.)

(4.23) is the set of $\sim_2$ equivalence classes of $((\mathfrak{Y}, u', \varphi'), \vec{w}'_{\mathfrak{p}}, (\vec{w}'_c))$ with the following properties.

(1) $(\mathfrak{Y}, u', \varphi') \cup \vec{w}'_{\mathfrak{p}}$ is ϵ_0-close to $\mathfrak{p} \cup \vec{w}_{\mathfrak{p}} \cup \vec{w}^{\rm can}$. Here $\ell_{\mathfrak{p}} = \#\vec{w}'_{\mathfrak{p}}$.
(2) $(\mathfrak{Y}, u', \varphi') \cup \vec{w}'_c$ is ϵ_0-close to $\mathfrak{p} \cup \vec{w}_c^{\mathfrak{p}}$. Here $\ell_c = \#\vec{w}'_c$.
(3) On the bubble we have

$$\overline{\partial} u' \equiv 0 \mod \mathcal{E}_{\mathfrak{B}}.$$

Here $\mathcal{E}_{\mathfrak{B}}$ is the obstruction bundle defined from Definition 4.17 in the same way as [FOOO5, Definition 2.55].
(4) On the i-th irreducible component of the mainstream we consider $h'_i = u' \circ \varphi'_i$. Then it satisfies

$$\frac{\partial h'_i}{\partial \tau} + J\left(\frac{\partial h'_i}{\partial t} - \mathfrak{X}_{H_t}\right) \equiv 0 \mod \mathcal{E}_{\mathfrak{B}}.$$

Here $\mathcal{E}_{\mathfrak{B}}$ is as in (3).

The next lemma says that $\mathcal{M}_{(\ell_{\mathfrak{p}},(\ell_c))}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathfrak{p}; \mathfrak{A}; \mathfrak{B})_{\epsilon_0, \vec{T}_0}$ carries the following S^1 action.

Lemma 4.21. *Suppose that* $((\mathfrak{Y}, u', \varphi'), \vec{w}'_{\mathfrak{p}}, (\vec{w}'_c))$ *is an element of our moduli space* $\mathcal{M}_{(\ell_{\mathfrak{p}},(\ell_c))}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathfrak{p}; \mathfrak{A}; \mathfrak{B})_{\epsilon_0, \vec{T}_0}$ *and* $t_0 \in S^1$. *Then* $((\mathfrak{Y}, u', t_0\varphi'), t_0\vec{w}'_{\mathfrak{p}}, (t_0\vec{w}'_c))$ *is an element of* $\mathcal{M}_{(\ell_{\mathfrak{p}},(\ell_c))}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathfrak{p}; \mathfrak{A}; \mathfrak{B})_{\epsilon_0, \vec{T}_0}$. *Here* $(t_0 \cdot \varphi')(\tau, t) = \varphi'(\tau, t + t_0)$. *We define* $t_0\vec{w}'_c$ *as follows.* $(t_0\vec{w}'_c)_i = (\vec{w}'_c)_i$ *if it corrresponds to a marked point in* $\vec{w}_{\mathfrak{p}_c}$. *If* $(\vec{w}'_c)_i$ *corresponds to a canonical marked point then* $(t_0\vec{w}'_c)_i = ((t_0 \cdot \varphi') \circ (\varphi)^{-1})((\vec{w}'_c)_i)$.

Proof. The only part of our construction which potentially breaks S^1 symmetry is Definition 4.19 (3). However as we remarked in Remark 4.18, the marked points that correspond to the canonical marked points do not affect the obstruction bundle. Therefore S^1 symmetry is not broken. □

Definition 4.22. We denote by $V_{(\ell_{\mathfrak{p}},(\ell_c))}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathfrak{p}; \mathfrak{A}; \mathfrak{B})_{\epsilon_0}$ the subset of $\mathcal{M}_{(\ell_{\mathfrak{p}},(\ell_c))}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathfrak{p}; \mathfrak{A}; \mathfrak{B})_{\epsilon_0, \vec{T}_0}$ consisting of the elements with the same combinatorial type as $\mathfrak{p}$. (Compare [FOOO5, (2.212)].)

In the same way as [FOOO5, Lemma 2.68] $V_{(\ell_{\mathfrak{p}},(\ell_c))}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathfrak{p}; \mathfrak{A}; \mathfrak{B})_{\epsilon_0}$ is a smooth manifold. In the same way as Lemma 4.21 we can show that our space $V_{(\ell_{\mathfrak{p}},(\ell_c))}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha; \mathfrak{p}; \mathfrak{A}; \mathfrak{B})_{\epsilon_0}$ has $(S^1)^k$ action. Here k is the number of mainstream components of $\mathfrak{p}$. Let m be the number of singular points of $\Sigma_{\mathfrak{p}}$ which are not transit points.

Now we have the following analogue of [FOOO5, Theorem 2.70].

Proposition 4.23. *There exists a map*

$$\begin{aligned}\text{Glue}: &V_{(\ell_{\mathfrak p},(\ell_c))}(X,H;\mathfrak z^-,\mathfrak z^+,\alpha;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_0}\times\left(\prod_{i=1}^m (T_{0,i},\infty]\times S^1)\right)/\sim\times D(k;\vec T_0')\\ &\to \mathcal M_{(\ell_{\mathfrak p},(\ell_c))}(X,H;\mathfrak z^-,\mathfrak z^+,\alpha;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_2}.\end{aligned}$$

Its image contains $\mathcal M_{(\ell_{\mathfrak p},(\ell_c))}(X,H;\mathfrak z^-,\mathfrak z^+,\alpha;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon_3}$ *for sufficiently small* ϵ_3.

An estimate similar to [FOOO5, Theorem 2.72] *also holds.*

(The notation $\mathcal M_{(\ell_{\mathfrak p},(\ell_c))}(X,H;\mathfrak z^-,\mathfrak z^+,\alpha;\mathfrak p;\mathfrak A;\mathfrak B)_{\epsilon}$ is similar to one in [FOOO5, Theorem 2.70].)

Proof. The proof is mostly the same as the proofs of [FOOO5, Theorems 2.70, 2.72]. The only new point we need to discuss is the following.

Our equation is pseudo-holomorphic curve equation ((3) in Definition 4.20) on the bubble but involves Hamiltonian vector field ((4) in Definition 4.20) on the mainstream. When we resolve the singular points, the bubble becomes the mainstream. So we need to estimate the contribution of the (pull back by appropriate diffeomorphisms of the) Hamiltonian vecotor field on such a part. We need to do so by using the coordinate that is similar to those we used in the proofs of [FOOO5, Theorems 2.70, 2.72]. We will use Lemma 4.24 below for this purpose.

We put

$$\Sigma_{\mathfrak p}=\bigcup_{i=1}^k\Sigma_i\cup\bigcup_{\rm v}\Sigma_{\rm v}$$

where Σ_i are in the mainstream and $\Sigma_{\rm v}$ are the bubbles. We remark $\Sigma_{\rm v}$ is a sphere S^2. Let z be a singular point contained in $\Sigma_{\rm v}$. According to Condition 4.13 we have a disk $D_{z,{\rm v}}\subset\Sigma_{\rm v}$ centered at z on which coordinate at infinity is defiend. In case $z\in\Sigma_0$, we also have a disk $D_{z,i}\subset\Sigma_i$ on which coordinate at infinity is defiend.

We also have

$$\varphi_i(((-\infty,5T_i]\cup[5T_{i+1},\infty))\times S^1)\subset\Sigma_i. \tag{4.24}$$

The union of the images of (4.24) and the disks $D_{z,{\rm v}}$ and $D_{z,i}$ are the neck regions. Its complement is called the core. We write $K_{\rm v}$ or K_i the part of the core in the component $\Sigma_{\rm v}$, Σ_i, respectively.

Using the coordinate at infinity, we have an embedding

$$i_{\rm v}:K_{\rm v}\to\Sigma',\qquad i_i:K_i\to\Sigma'. \tag{4.25}$$

Lemma 4.24 provides an estimate of the maps (4.25). We put the metric on $K_{\rm v}$ regarding them as subsets of the sphere. We put the metric on K_i by regarding them as subsets of $\mathbb R\times S^1$. (They are compact. So actually the choice of the metric does not matter.)

We fix $\vec T$ (the lengths of the neck region when we glue and obtain Σ'.) For each v we define $T_{\rm v}$ as follows. Take a shortest path joining our irreducible component $\Sigma_{\rm v}$ to the mainstream. Let $z_1,\dots,z_r$ be the singular points contained in this path. Let $10T_{z_i}$ be the length of the neck region corresponding to the singular point z_i. We put $T_{\rm v}=\sum T_{z_i}$.

We remark that $i_{\rm v}(K_{\rm v})$ is in the mainstream if and only if $T_{\rm v}$ is finite.

Lemma 4.24. *There exist* $C_\ell,c_\ell>0$ *with the following properties.*

(1) *Suppose $i_{\rm v}(K_{\rm v})$ is in the mainstream. We then regard*

$$i_{\rm v} : K_{\rm v} \to \mathbb{R} \times S^1.$$

The C^ℓ norm of this map is smaller than $C_\ell e^{-c_\ell T_{\rm v}}$. In particular the diameter of its image is smaller than $C_1 e^{-c_1 T_{\rm v}}$.

(2) *We remark that the image of i_i is in certain mainstream component. So we may regard*

$$i_i : K_i \to \mathbb{R} \times S^1.$$

Note $K_i \subset \mathbb{R} \times S^1$. Then i_i extends to a biholomorphic map of the form $(\tau, t) \mapsto (\tau + \tau_0, t + t_0)$.

Proof. For simplicity of the notation we consider the case

$$\Sigma_{\mathfrak p} = (\mathbb{R} \times S^1) \cup S^2 = \Sigma_1 \cup \Sigma_{\rm v}.$$

Let T be the length of the neck region of Σ'. Then we have a canonical isomorphism

$$\Sigma' = ((\mathbb{R} \times S^1) \setminus D^2_0) \cup ([-5T, 5T] \times S^1) \cup (\mathbb{C} \setminus D^2). \tag{4.26}$$

Here D^2_0 is an image of a small disk $\subset \mathbb{R} \times \mathbb{R}$ by the projection $\mathbb{R} \times \mathbb{R} \to \mathbb{R} \times S^1$ and the disk D^2 in the third term is the disk of radius 1 centered at origin. The identification (4.26) is a consequence of Condition 4.13.

We have a biholomorphic map

$$I : \Sigma' \to \mathbb{R} \times S^1$$

that preserves the coordinates at their two ends.

We can take I as follows. Let $I_0 : D^2 \to D^2_0$ be the isomorphism that lifts to a homothetic embedding $D^2 \to \mathbb{R}^2$.

(1) On $(\mathbb{R} \times S^1) \setminus D^2_0$, the map I is the identity map.

(2) If $z \in \mathbb{C} \setminus D^2$ then

$$I(z) = I_0(e^{-20\pi T}/z).$$

(3) If $z = (\tau, t) \in ([-5T, 5T] \times S^1)$ then

$$I(z) = I_0(e^{-2\pi((\tau+5T)+\sqrt{-1}t)}).$$

Lemma 4.24 is immediate from this description.

The general case can be proved by iterating a similar process. □

Remark 4.25. On the part of the neck region $[-4T, 4T] \times S^1$ where we perform the gluing construction, we can prove a similar estimate as Lemma 4.24 (1).

Now we go back to the proof of Proposition 4.23. Lemma 4.24 (2) implies that on the mainstream the equation Definition 4.20 (4) is preserved by gluing. Lemma 4.24 (1) and Remark 4.25 imply that the effect of the Hamitonian vector field is small in the exponential order on the other part. Therefore the presence of the Hamiltonian term does not affect the proof of [FOOO5, Theorems 2.70, 2.72] and we can prove Proposition 4.23 in the same way as [FOOO5, Section 2.5]. □

We are now in the position to complete the proof of Theorem 3.8. We have defined the thickend moduli space and described it by the gluing map. The rest of the construction of the Kuranishi structure is mostly the same as the construction in [FOOO5, Sections 2.6-2.10]. We mention two points below. Except them there are nothing to modify.

(1) We consdier the process to put (transversal) constraint and forget the marked point. This was done in [FOOO5, Sections 2.6] to cut down the thickened moduli space to an orbifold of correct dimension, which will become our Kuranishi neighborhood. Here we use the constraint defined in Definition 4.19. In the case when the marked point w'_i corresponds to one of $\vec{w}_{\mathfrak p}$ or $\vec{w}_{\mathfrak{p}_c}$ that is not a canonical marked point, this process is exactly the same as [FOOO5, Sections 2.6].

In the case of the marked point w'_i corresponds to a canonical marked point of $\mathfrak p$ or $\mathfrak{p}_c$ we use Definition 4.19 (2),(3). Note these conditions determine the position of w'_i on Σ' uniquely. On the hand, as we remarked in Remark 4.18, the marked point w'_i does not affect the obstruction bundle and hence the equations defining our thickened moduli space. So the discussion of the process to put constraint and forget such a marked point is rather trivial.

(2) Our thickened moduli space has an S^1 action. The gluing map we constructed in Proposition 4.23 is obviously S^1 equivariant. (The obstruction bundle is invariant under the S^1 action as we remarked before.) Therefore all the construction of the Kuranishi structure is done in an S^1 equivariant way. Note we define S^1 action on the thickened moduli space. The smoothness of this action is fairly obvious.

We remark that the group $\Gamma_{\mathfrak p}$ for $\mathfrak p$ (Definition 2.1) in our case of $\mathfrak p = ((\Sigma, z_-, z_+), u, \varphi)$ consists of maps $v : \Sigma \to \Sigma$ that satisfies Definition 4.3 (1)(2) for $((\Sigma', z'_-, z'_+), u, \varphi') = ((\Sigma, z_-, z_+), u, \varphi)$ and

$$v \circ \varphi_a(\tau, t) = \varphi_a(\tau, t + t_0)$$

for some $t_0 \in S^1$. The groups $\Gamma_{\mathfrak p}$ and S^1 generate the group $G_{\mathfrak p}$.

The proof of Theorem 3.8 is now complete. □

Example 4.26. Let $h : \mathbb{R} \times S^1 \to X$ be a solution of the equation (3.4) (without bubble). We assume that h is injective. We put

$$h_k(\tau, t) = h(k\tau, kt) : \mathbb{R} \times S^1 \to X.$$

We define $\varphi : \mathbb{R} \times S^1 \to S^2 \setminus \{z_-, z_+\}$ by $\varphi(\tau, t) = e^{2\pi(\tau + \sqrt{-1}t)}$. We put $u_k = h_k \circ \varphi^{-1}$. Then $((S^2, z_-, z_+), u_k, \varphi)$ is an element of $\mathcal{M}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$. Let $t_0\varphi(\tau, t) = \varphi(\tau, t + t_0)$. It is easy to see that

$$((S^2, z_-, z_+), u_k, \varphi) \sim_2 ((S^2, z_-, z_+), u_k, t_0\varphi)$$

if and only if $t_0 = m/k$, $m \in \mathbb{Z}$.

We can take the Kuranishi neighborhood of $\mathfrak p = ((S^2, z_-, z_+), u_k, \varphi)$ of the form $V = S^1 \times V'$, on which the generator of the group $\Gamma_{\mathfrak p} = \mathbb{Z}_k$ acts by $(t, v) \mapsto (t + 1/k, \psi(v))$ where $\psi : V' \to V'$ is not an identity map. The S^1 action is by rotation of the first factor. Thus the quotient $V/\Gamma_{\mathfrak p}$ is a manifold and $V/(\Gamma_{\mathfrak p} \times S^1)$ is an orbifold. See Example 2.3.

Remark 4.27. In this section we studied the moduli space $\mathcal{M}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ in the case when H is a time independent Morse function. In an alternative approach, (such as those [FOOO3, Section 26]) we studied the case $H \equiv 0$ using Bott-Morse gluing. Actually the discussion corresponding to this section is easier in the case of $H \equiv 0$. In fact in case of $H \equiv 0$, we do not need to study the moduli space of its gradient lines. (Some argument was necessary to discuss the moduli space of gradient lines since its element has S^1 as an isotropy group. The main part of this seciton is devoted to this point.)

In [FOn1] we used the case H is a time independent Morse function rather than studying the case $H \equiv 0$ by Bott-Morse theory. The reason is that the chain level argument that we need to use for the case $H \equiv 0$ was not written in detail at the time when [FOn1] was written in 1996. Now full detail of the chain level argument was written in [FOOO1]. So at the stage of 2012 (16 years after [FOn1] was written) using the case $H \equiv 0$ to calculate Floer homology of periodic Hamiltonian system is somewhat simpler to write up in detail rather than using the case when H is a time independent Morse function. In this section however we focused on the case of time independent Morse function and written up as much detail as possible to convince the readers

5. Calculation of Floer homology of periodic Hamiltonian system.

We first prove Theorem 3.4. The proof is similar to the proof of Theorem 3.8. We indicate below the points where proofs are different.

Let (Σ, z_-, z_+) be as in Definition 4.1. We define the notion of mainstream, mainstream component, and transit point in the same way as in Section 4.

Let $\tilde\gamma^\pm = (\gamma^\pm, w^\pm) \in \tilde{\mathfrak{P}}(H)$. (Here H is a time *dependent* periodic Hamiltonian.) Let $((\Sigma, z_-, z_+), u, \varphi)$ be as in Definition 4.2, such that it satisfies (1)(2)(4)(5) of Definition 4.2 and the following 3 conditions:

- (3)′ $u : \Sigma \setminus \{\text{transit points}\} \to X$ is a continuous map.
- (6)′ There exists $\tilde\gamma_i = (\gamma_i, w_i) \in \tilde{\mathfrak{P}}(H)$ for $i = 1, \dots, k+1$ with $\tilde\gamma_1 = \tilde\gamma^-$, $\tilde\gamma_{k+1} = \tilde\gamma^+$ such that the following holds. Let $\varphi_i : \mathbb{R} \times S^1 \to \Sigma_i$ be the i-th component of φ. Then
$$\lim_{\tau \to -\infty} u(\varphi_i(\tau, t)) = \gamma_i(t), \qquad \lim_{\tau \to +\infty} u(\varphi_i(\tau, t)) = \gamma_{i+1}(t).$$
- (7)′ $w_i \# u(\Sigma_i) \sim w_{i+1}$.

We denote by $\widehat{\mathcal{M}}(X, H; \tilde\gamma^-, \tilde\gamma^+)$ the set of such $((\Sigma, z_-, z_+), u, \varphi)$. We define equivalence relations $\sim_1$ and $\sim_2$ on it in the same way as Definition 4.3. (We do not use $\sim_3$ here.) We then put

$$\widetilde{\mathcal{M}}(X, H; \tilde\gamma^-, \tilde\gamma^+) = \widehat{\mathcal{M}}(X, H; \tilde\gamma^-, \tilde\gamma^+)/\sim_1,$$
$$\mathcal{M}(X, H; \tilde\gamma^-, \tilde\gamma^+) = \widehat{\mathcal{M}}(X, H; \tilde\gamma^-, \tilde\gamma^+)/\sim_2 .$$

We next define balancing condition. Let $((\Sigma, z_-, z_+), u, \varphi) \in \widehat{\mathcal{M}}(X, H; \tilde\gamma^-, \tilde\gamma^+)$ be an element with one mainstream component. We define a function $\mathcal{A} : \mathbb{R} \setminus$ a finite set $\to \mathbb{R}$ as follows.

Let $\tau_0 \in \mathbb{R}$. We assume $\varphi(\{\tau_0\} \times S^1)$ does not contain a root of the bubble tree. (This is the way how we remove a finite set from the domain of $\mathcal{A}$.) Let Σ_{v_i}, $i = 1, \dots, m$ be the set of the irreducible components that is in a bubble tree rooted on $\mathbb{R}_{\le \tau_0} \times S^1$. We define

$$\begin{aligned}\mathcal{A}(\tau_0) = \sum_{i=1}^m \int_{\Sigma_{\mathrm{v}_i}} u^*\omega + \int_{\tau=-\infty}^{\tau_0} \int_{t \in S^1} (u \circ \varphi)^*\omega \\ + \int_{t \in S^1} H(t, u(\varphi(\tau_0, t)))dt + \int_{D^2} (w^-)^*\omega.\end{aligned} \tag{5.27}$$

Note the action functional $\mathcal{A}_H$ is defined by

$$\mathcal{A}_H(\tilde\gamma) = \int_{t\in S^1} H(t, u(\gamma(t)))dt + \int_{D^2} (w)^*\omega$$

The function $\mathcal{A}(\tau_0)$ is nondecreasing and satsifies

$$\lim_{\tau\to-\infty} \mathcal{A}(\tau) = \mathcal{A}_H(\tilde\gamma^-), \qquad \lim_{\tau\to+\infty} \mathcal{A}(\tau) = \mathcal{A}_H(\tilde\gamma^+).$$

We say φ satisfies the *balancing condition* if

$$\lim_{\substack{\tau<0\\ \tau\to 0}} \mathcal{A}(\tau) \le \frac{1}{2}\left(\mathcal{A}_H(\tilde\gamma^-) + \mathcal{A}_H(\tilde\gamma^+)\right) \le \lim_{\substack{\tau>0\\ \tau\to 0}} \mathcal{A}(\tau). \tag{5.28}$$

In case of genearal $((\Sigma, z_-, z_+), u, \varphi) \in \widehat{\mathcal{M}}(X, H; \tilde\gamma^-, \tilde\gamma^+)$ we consider the balancing condition in mainstream-component-wise.

In the case there is a mainstream component Σ_i such that $\partial(u\circ\varphi_i)/\partial\tau = 0$ we can apply the method of Remark 4.8.

We next define the notion of canonical marked point. Let $\mathfrak{p} = ((\Sigma, z_-, z_+), u, \varphi) \in \mathcal{M}(X, H; \tilde\gamma^-, \tilde\gamma^+)$. Let Σ_i be its mainstream component. We assume that there is no sphere bubble rooted on it. We are given a biholomorphic map $\varphi_i : \mathbb{R}\times S^1 \to \Sigma_i \setminus \{z_i, z_{i+1}\}$ where z_i, z_{i+1} are transit points on Σ_i. We require φ_i to satisfy the balancing condition. Now we define the canonical marked point $w_i^{\rm can}$ on Σ_i by $w_i = \varphi_i(0,0)$. Let $\vec w^{\rm can}$ be the totality of all the canonical marked points on Σ.

A symmetric stabilization of $\mathfrak{p} = ((\Sigma, z_-, z_+), u, \varphi) \in \mathcal{M}(X, H; \tilde\gamma^-, \tilde\gamma^+)$ is $\vec w$ such that $\vec w \cap \Sigma_0 = \emptyset$ where Σ_0 is the mainstream of Σ and $\vec w \cup \vec w^{\rm can}$ is the symmetric stabilization of (Σ, z_-, z_+).

Definition 5.1. An *obstruction bundle data* $\mathfrak{E}_{\mathfrak{p}}$ *centered at*

$$\mathfrak{p} = ((\Sigma, z_-, z_+), u, \varphi) \in \mathcal{M}(X, H; \tilde\gamma^-, \tilde\gamma^+)$$

is the data satisfying the conditions described below. We put $\mathfrak{x} = (\Sigma, z_-, z_+)$. Let $\mathfrak{x}_i$ be the i-th mainstream component. (It has two marked points.)

1. A symmetric stabilization $\vec w$ of $\mathfrak{p}$. We put $\vec w^{(i)} = \vec w \cap \mathfrak{x}_i$.
2. The same as [FOOO5, Definition 2.33 (2)].
3. A universal family with coordinate at infinity of $\mathfrak{x}_{\mathfrak{p}} \cup \vec w \cup \vec w^{\rm can}$. Here $\vec w^{\rm can}$ is the canonical marked points as above. We require Condition 4.13 for the coordinate at infinity.
4. The same as [FOOO5, Definition 2.33 (4)]. Namely compact subsets $K_{\rm v}^{\rm obst}$ of $\Sigma_{\rm v}$. (We put it also on the mainstream.)
5. The same as [FOOO5, Definition 2.33 (5)]. Namely, finite dimensional complex linear subspaces $E_{\mathfrak{p},{\rm v}}(\mathfrak{y}, u)$. (We put it also on the mainstream.)
6. The same as [FOOO5, Definition 2.33 (6)] except the differential operator there
$$\begin{aligned}\overline{D}_u\overline\partial :&L^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_{\rm v}}, \partial\Sigma_{\mathfrak{y}_{\rm v}}); u^*TX, u^*TL)\\ &\to L^2_{m,\delta}(\Sigma_{\mathfrak{y}_{\rm v}}; u^*TX\otimes\Lambda^{01})/E_{\mathfrak{p},{\rm v}}(\mathfrak{y}, u)\end{aligned} \tag{5.29}$$
is replaced by the linearization of the equation (3.4)
7. The same as [FOOO5, Definition 2.33 (7)].
8. We take a codimension 2 submanifold $\mathcal{D}_j$ for each of $w_j \in \vec w$ in the same way as [FOOO5, Definition 2.33 (8)].

We require that the deta $K_{\rm v}^{\rm obst}$, $E_{\mathfrak p,{\rm v}}(\mathfrak y,u)$ depend only on $\mathfrak p_i = [(\Sigma_i,z_{i-1},z_i),u,\varphi]$ (Here z_i is the i-th transit point.) that contains v-th irreducible component. We call this condition *mainstream-component-wise*.

We define $\mathcal M_\ell(\tilde\gamma^-,\tilde\gamma^+)$ in the same way as $\mathcal M_\ell(\mathfrak z^-,\mathfrak z^+,\alpha)$. (Namely its element is $((\Sigma_i,z_i,z_{i+1}),\varphi)$ together with ℓ additional markd points on Σ, elements $\tilde\gamma_i \in \tilde{\mathfrak P}(H)$ assigned to each of the transit point, and homology classes of each of the bubbles.) We denote by $\mathcal M_\ell(\tilde\gamma^-,\tilde\gamma^+;\mathcal G)$ its subset consisting of elements with given combinatorial type $\mathcal G$.

Let $\mathfrak p = ((\Sigma,z_-,z_+),u,\varphi) \in \mathcal M(X,H;\tilde\gamma^-,\tilde\gamma^+)$ and $\vec w \cup \vec w^{\rm can}$ be its symmetric stabilization. We denote by $\mathfrak V(\mathfrak p\cup\vec w\cup\vec w^{\rm can})$ a neighborhood of $\mathfrak p\cup\vec w\cup\vec w^{\rm can}$ in $\mathcal M_\ell(\tilde\gamma^-,\tilde\gamma^+;\mathcal G_{\mathfrak p\cup\vec w\cup\vec w^{\rm can}})$.

We can define

$$\overline\Phi : \mathfrak V(\mathfrak p\cup\vec w\cup\vec w^{\rm can}) \times D(k;\vec T_0) \times (\prod_{j=1}^m (T_{0,j},\infty]\times S^1)/\sim) \to \mathcal M_\ell(\tilde\gamma^-,\tilde\gamma^+) \quad (5.30)$$

in the same way as (4.19).

We say $((\Sigma',z'_-,z'_+),u',\varphi')\cup\vec w'$ is ϵ-close to $\mathfrak p\cup\vec w\cup\vec w^{\rm can}$, if (1)(2)(3)(4) of Definition 4.16 hold and if

(5)′ Let $w'_j = \varphi'_j(\tau_j,t_j) \in \vec w'$ be the marked point corresponding to the canonical marked point $\in \vec w^{\rm can}\cap\Sigma_i$. Then

$$\left|\mathcal A(\tau_j) - \frac{1}{2}\left(\mathcal A_H(\tilde\gamma_i)+\mathcal A_H(\tilde\gamma_{i+1})\right)\right| < \epsilon$$

and $|t_j - 0| < \epsilon$.

If $((\Sigma',z'_-,z'_+),u',\varphi')\cup\vec w'$ is ϵ-close to $\mathfrak p\cup\vec w\cup\vec w^{\rm can}$ and the obstruction bundle data at $\mathfrak p$ is given then they induce an obstruction bundle at $((\Sigma',z'_-,z'_+),u',\varphi')\cup\vec w'$ in the same way as Definition 4.17.

Definition 5.2. We say that $((\Sigma',z'_-,z'_+),u',\varphi')\cup\vec w'$ satisfies the transversal constraint if the following holds.

1. The same as Definition 4.19 (1).
2. Let $w'_j = \varphi'_j(\tau_j,t_j) \in \vec w'$ be the marked point corresponding to the canonical marked point $\in \vec w^{\rm can}\cap\Sigma_i$. Then
$$\mathcal A(\tau_j) = \frac{1}{2}\left(\mathcal A_H(\tilde\gamma_i)+\mathcal A_H(\tilde\gamma_{i+1})\right)$$
3. In the situation of (2) we have $t_j = [0]$.

For each $\mathfrak p \in \mathcal M(X,H;\tilde\gamma^-,\tilde\gamma^+)$ we take a stabilization data $\mathfrak C_{\mathfrak p}$ centered at $\mathfrak p$. We also take $\epsilon_{\mathfrak p}$ so that if $((\Sigma',z'_-,z'_+),u',\varphi')\cup\vec w'$ is $\epsilon_{\mathfrak p}$-close to $\mathfrak p\cup\vec w\cup\vec w^{\rm can}$ then Fredholm regurality (See [FOOO5, Definition 2.44].) and evaluation map transversality (See [FOOO5, Definition 2.46].) hold for $((\Sigma',z'_-,z'_+),u',\varphi')\cup\vec w'$.

Let $\mathfrak W_{\mathfrak p}$ be the set of elements $\mathfrak p'$ with the following property: there exists $\vec w'$ such that $\mathfrak p'\cup\vec w'$ is $\epsilon_{\mathfrak p}$-close to $\mathfrak p\cup\vec w^{\mathfrak p}\cup\vec w^{\rm can}$ and $\vec w'$ satisies the transversal constraint in the sense of Definition 5.2.

We use it to find a finite set $\mathfrak C = \{\mathfrak p_c\}$ such that

$$\bigcup_c {\rm Int}\,\mathfrak W_{\mathfrak p_c} = \mathcal M(X,H;\tilde\gamma^-,\tilde\gamma^+).$$

We then define

$$\mathfrak{C}_{\mathfrak{p}} = \{c \in \mathfrak{C} \mid \mathfrak{p} \in \mathfrak{W}_{\mathfrak{p}_c}\}.$$

Definition 5.3. We define a *thickened moduli space*

$$\mathcal{M}_{(\ell_{\mathfrak{p}},(\ell_c))}(X, H; \tilde{\gamma}^-, \tilde{\gamma}^+; \mathfrak{p}; \mathfrak{A}; \mathfrak{B})_{\epsilon_0,\vec{T}_0} \tag{5.31}$$

as follows. (Here $\mathfrak{A} \subset \mathfrak{B} \subset \mathfrak{C}_{\mathfrak{p}}$.)

(5.31) is the set of $\sim_2$ equivalence classes of $((\mathfrak{Y}, u', \varphi'), \vec{w}'_{\mathfrak{p}}, (\vec{w}'_c))$ with the following properties.

(1) $(\mathfrak{Y}, u', \varphi') \cup \vec{w}'_{\mathfrak{p}}$ is ϵ_0-close to $\mathfrak{p} \cup \vec{w}_{\mathfrak{p}} \cup \vec{w}^{\rm can}$. Here $\ell_{\mathfrak{p}} = \#\vec{w}'_{\mathfrak{p}}$.
(2) $(\mathfrak{Y}, u', \varphi') \cup \vec{w}'_c$ is ϵ_0-close to $\mathfrak{p} \cup \vec{w}^{\mathfrak{p}}_c$. Here $\ell_c = \#\vec{w}'_c$.
(3) On the bubble we have

$$\overline{\partial} u' \equiv 0 \mod \mathcal{E}_{\mathfrak{B}}.$$

Here $\mathcal{E}_{\mathfrak{B}}$ is the obstruction bundle defined from Definition 4.17 in the same way as [FOOO5, Definition 2.55].
(4) On the i-th irreducible component of the mainstream we consider $h'_i = u' \circ \varphi'_i$. Then it satisfies

$$\frac{\partial h'_i}{\partial \tau} + J\left(\frac{\partial h'_i}{\partial t} - \mathfrak{X}_{H_t}\right) \equiv 0 \mod \mathcal{E}_{\mathfrak{B}}.$$

Here $\mathcal{E}_{\mathfrak{B}}$ is as in (3).

Definition 5.4. We denote by $V_{(\ell_{\mathfrak{p}},(\ell_c))}(X, H; \tilde{\gamma}^-, \tilde{\gamma}^+; \mathfrak{A}; \mathfrak{B})_{\epsilon_0}$ the subset of the thickened moduli space $\mathcal{M}_{(\ell_{\mathfrak{p}},(\ell_c))}(X, H; \tilde{\gamma}^-, \tilde{\gamma}^+; \mathfrak{A}; \mathfrak{B})_{\epsilon_0,\vec{T}_0}$ consisting of elements with the same combinatorial type as $\mathfrak{p}, \vec{w}, \vec{w}_c$. (Compare [FOOO5, (2.212)].)

The Fredholm regurality and evaluation map transversality imply that the space $V_{(\ell_{\mathfrak{p}},(\ell_c))}(X, H; \tilde{\gamma}^-, \tilde{\gamma}^+; \mathfrak{A}; \mathfrak{B})_{\epsilon_0}$ is a smooth manifold.

Proposition 5.5. *There exists a map*

$$\begin{aligned}\text{Glue} :& V_{(\ell_{\mathfrak{p}},(\ell_c))}(X, H; \tilde{\gamma}^-, \tilde{\gamma}^+; \mathfrak{A}; \mathfrak{B})_{\epsilon_0} \times \left(\prod_{i=1}^m (T_{0,i}, \infty] \times S^1)\right) / \sim \times D(k; \vec{T}'_0) \\ &\to \mathcal{M}_{(\ell_{\mathfrak{p}},(\ell_c))}(X, H; \tilde{\gamma}^-, \tilde{\gamma}^+; \mathfrak{A}; \mathfrak{B})_{\epsilon_2}.\end{aligned}$$

Its image contains $\mathcal{M}_{(\ell_{\mathfrak{p}},(\ell_c))}(X, H; \tilde{\gamma}^-, \tilde{\gamma}^+; \mathfrak{A}; \mathfrak{B})_{\epsilon_3}$ for sufficiently small ϵ_3.

An estimate similar to [FOOO5, Theorem 2.72] *also holds.*

The proof is the same as the proof of Proposition 4.23. Using Proposition 5.5 we can prove Theorem 3.4 in the same way as the last step of the proof of Theorem 3.8. □

Remark 5.6. In case H in Theorem 3.4 happens to be time independent, the Kuranishi structure obtained by Theorem 3.4 is different from the one obtained by Theorem 3.8. In fact, during the proof of Theorem 3.8 we chose a (sufficiently dense finite) subset of $\overline{\overline{\mathcal{M}}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ to define the obstruction bundle. During the proof of Theorem 3.8 we chose a (sufficiently dense finite) subset of $\mathcal{M}(X, H; \tilde{\gamma}^-, \tilde{\gamma}^+)$ for the same purpose. The elements of the moduli space $\overline{\overline{\mathcal{M}}}(X, H; \mathfrak{z}^-, \mathfrak{z}^+, \alpha)$ are $\sim_3$ equivalence classes and the elements of the moduli space $\mathcal{M}(X, H; \tilde{\gamma}^-, \tilde{\gamma}^+)$ are $\sim_2$ equivalence classes.

Using Theroem 3.4 we can define Floer homology $HF(X,H)$ for time dependent 1-periodic Hamiltonian H satisfying Assumption 3.1. This construction (going back to Floer [Fl1, Fl2], see also [HS, On]) is well-established. We sketch the construction here for completeness. We use the universal Novikov ring

$$\Lambda_0 = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{Q},\ \lambda_i \in \mathbb{R},\ \lim_{i\to\infty} \lambda_i = +\infty \right\}.$$

Let $CF(X,H)$ be the free Λ_0 module whose basis is identified with the set $\mathfrak{P}(H)$.

We take $E > 0$. By Theorem 3.4, we obtained a system of Kuranishi structures on $\mathcal{M}(X,H;\tilde\gamma^-,\tilde\gamma^+)$ for each pair $\tilde\gamma^-,\tilde\gamma^+$ with $\mathcal{A}_H(\tilde\gamma^+) - \mathcal{A}_H(\tilde\gamma^-) < E$. We take a system of multisections $\mathfrak{s}$ on them that are compatible with the description of its boundary and corner as in Theorem 3.4 (3). Here we use the fact that the obstruction bundle is defined mainstream-component-wise. Note our Kuranishi strucutre is oriented. We define

$$\partial^E[\gamma^-] = \sum_{\substack{\tilde\gamma^+,\mu(\tilde\gamma^+)-\mu(\tilde\gamma^-)=1\\ \mathcal{A}_H(\tilde\gamma^+)-\mathcal{A}_H(\tilde\gamma^-)<E}} \#\mathcal{M}(X,H;\tilde\gamma^-,\tilde\gamma^+)^{\mathfrak{s}} T^{\mathcal{A}_H(\tilde\gamma^+)-\mathcal{A}_H(\tilde\gamma^-)}[\gamma^+]. \tag{5.32}$$

Here we take a lift $\tilde\gamma^-$ of γ^- to define the right hand side. We can show that the right hand side is independent of the choice of the lift. The number $\mu(\tilde\gamma^+)$ is the Maslov index. We have

$$\dim \mathcal{M}(X,H;\tilde\gamma^-,\tilde\gamma^+) = \mu(\tilde\gamma^+) - \mu(\tilde\gamma^-) - 1.$$

Using the moduli space $\mathcal{M}(X,H;\tilde\gamma^-,\tilde\gamma^+)$ with $\dim \mathcal{M}(X,H;\tilde\gamma^-,\tilde\gamma^+) = 2$, there is by now well-established way to prove

$$\partial^E \circ \partial^E \equiv 0 \mod T^E. \tag{5.33}$$

Thus we define

$$HF(X;H;\Lambda_0/T^E\Lambda_0) \cong H(CF(X,H)\otimes_{\Lambda_0} \Lambda_0/T^E\Lambda_0, \partial^E).$$

We can prove that $HF(X;H;\Lambda_0/T^E\Lambda_0)$ is independent of the choice of Kuranishi structure and its multisection. (See the proof of Theorem 5.10 later.) We thus define

Definition 5.7.

$$HF(X;H;\Lambda_0) = \varprojlim HF(X;H;\Lambda_0/T^E\Lambda_0).$$

We also define

$$HF(X;H) = HF(X;H;\Lambda_0)\otimes_{\Lambda_0}\Lambda$$

where Λ is the field of fraction of Λ_0.

In fact using the next lemma we can find a boundary operator ∂ on full module $CF(X,H)$ so that its homology is $HF(X;H;\Lambda_0)$.

Lemma 5.8. *Let C be a finitely generated free Λ_0 module and $E < E'$. Suppose we are given $\partial_E : C\otimes_{\Lambda_0}\Lambda_0/T^E \to C\otimes_{\Lambda_0}\Lambda_0/T^E$, $\partial_{E'} : C\otimes_{\Lambda_0}\Lambda_0/T^{E'} \to C\otimes_{\Lambda_0}\Lambda_0/T^{E'}$ with $\partial_E\circ\partial_E = 0$, $\partial_{E'}\circ\partial_{E'} = 0$. Moreover we assume $(C\otimes_{\Lambda_0}\Lambda_0/T^E, \partial_{E'} \mod T^E)$ is chain homotopy equivalent to $(C\otimes_{\Lambda_0}\Lambda_0/T^E, \partial_E)$. Then we can lift ∂_E to $\partial'_{E'} : C\otimes_{\Lambda_0}\Lambda_0/T^{E'}\Lambda_0 \to C\otimes_{\Lambda_0}\Lambda_0/T^{E'}\Lambda_0$ such that $(C\otimes_{\Lambda_0}\Lambda_0/T^{E'}, \partial_{E'})$ is chain homotopy equivalent to $(C\otimes_{\Lambda_0}\Lambda_0/T^{E'}\Lambda_0, \partial'_{E'})$.*

We omit the proof.

Remark 5.9. The method for taking projective limit $E \to \infty$ that we explained above is a baby version of the one employed in [FOOO1, Section 7]. (In [FOOO1] the filtered A_∞ structure is defined by using a similar method.) In [On], a slightly different way to go to projective limit was taken.

For the main applicatin, that is, to estimate the order of $\mathfrak{P}(H)$ by Betti numer, we actually do not need to go to the projective limit. See Remark 5.17.

Now we use S^1 equivariant Kuranishi structure in Theorem 3.8 to prove the next theorem.

Theorem 5.10. *For any time dependent 1-periodic Hamiltonian H on a compact manifold X satisfying Assumption 3.1, we have*

$$HF(X,H) \cong H(X;\Lambda)$$

where the right hand side is the singular homology group with Λ coefficient.

Proof. Let H' be a time *independent* Hamiltonian satisfying Assumptions 3.5, 3.6. We regard H' as a Morse function and let $\mathrm{Crit}(H')$ be the set of the critical points of H'. We denote by $CF(X,H';\Lambda_0)$ the free Λ_0 module with basis identified with $\mathrm{Crit}(H')$. Let $\mu : \mathrm{Crit}(H') \to \mathbb{Z}$ be the Morse index. If $\mathfrak{x}^+, \mathfrak{x}^- \in \mathrm{Crit}(H')$ with $\mu(\mathfrak{x}^+) - \mu(\mathfrak{x}^-) = 1$ we define

$$\langle \partial \mathfrak{x}^-, \mathfrak{x}^+ \rangle = T^{H'(\mathfrak{x}^+) - H'(\mathfrak{x}^-)} \# \mathcal{M}(X, H', \mathfrak{x}^-, \mathfrak{x}^+; 0), \tag{5.34}$$

where $\#\mathcal{M}(X,H',\mathfrak{x}^-,\mathfrak{x}^+;0)$ is the number counted with orientation. (Here 0 denotes the equivalence class of zero in Π.) By Assumptions 3.5 this moduli space is smooth.) It induces $\partial : CF(X,H';\Lambda_0) \to CF(X,H';\Lambda_0)$. It is by now well established that $\partial \circ \partial = 0$. We put

$$HF(X,H';\Lambda_0) = \frac{\mathrm{Ker}\partial}{\mathrm{Im}\partial}.$$

It is also standard by now that

$$HF(X,H') = HF(X,H';\Lambda_0) \otimes_{\Lambda_0} \Lambda$$

is isomorphic to the singular homology $H(X;\Lambda)$ of Λ coefficient.

We will construct a chain map from $CF(X,H';\Lambda_0)$ to $CF(X,H;\Lambda_0)$. Let $\mathcal{H} : \mathbb{R} \times S^1 \times X \to \mathbb{R}$ be a smooth map such that

$$\mathcal{H}(\tau,t,x) = \begin{cases} H'(x) & \text{if } \tau \le -1, \\ H(t,x) & \text{if } \tau \ge -1. \end{cases} \tag{5.35}$$

For a map $h : \mathbb{R} \times S^1 \to X$ we consider the equation

$$\frac{\partial h}{\partial \tau} + J\left(\frac{\partial h}{\partial t} - \mathfrak{X}_{\mathcal{H}_{\tau,t}}\right) = 0, \tag{5.36}$$

where $\mathcal{H}_{\tau,t}(x) = \mathcal{H}(\tau,t,x)$.

Given $\mathfrak{x} \in \mathrm{Crit}(H')$ and $\tilde{\gamma} = (\gamma, w) \in \tilde{\mathfrak{P}}(H)$ we consider the set of the maps h satisfying (5.36) together with the following boundary conditions.

1. $\lim_{\tau \to -\infty} h(\tau,t) = \mathfrak{x}$.
2. $\lim_{\tau \to +\infty} h(\tau,t) = \gamma(t)$.
3. $[h] \sim [w]$. Here h is regarded as a map from D^2 by identifying $\{-\infty\} \cup ((-\infty,+\infty] \times S^1)$ with D^2.

We denote the totality of such h by $\mathcal{M}^{\mathrm{reg}}(X,\mathcal{H};\mathfrak{x},\tilde{\gamma})$.

Theorem 5.11. *There exists a compactification* $\mathcal{M}(X,\mathcal{H};\mathfrak{r},\tilde\gamma)$ *of* $\mathcal{M}^{\rm reg}(X,\mathcal{H};\mathfrak{r},\tilde\gamma)$*, which is Hausdorff.*

For each $E>0$ *there exists a system of oriented Kuranishi structures with corners on* $\mathcal{M}(X,\mathcal{H};\mathfrak{r},\tilde\gamma)$ *for* $\mathcal{A}_H(\tilde\gamma)\le E$*. Its boundary is identified with the union of the following spaces, together with its Kuranishi structures.*

(1)
$$\mathcal{M}(X,H';\mathfrak{r},\mathfrak{r}',\alpha)\times\mathcal{M}(X,\mathcal{H};\mathfrak{r}',\tilde\gamma-\alpha)$$
where $\mathfrak{r}'\in{\rm Crit}(H')$*,* $\alpha\in\Pi$ *and* $\tilde\gamma-\alpha=(\gamma,w-\alpha)$*.*

(2)
$$\mathcal{M}(X,\mathcal{H};\mathfrak{r},\tilde\gamma')\times\mathcal{M}(X,H,\tilde\gamma',\tilde\gamma)$$
where $\tilde\gamma'\in\tilde{\mathfrak{P}}(H)$*.*

Proof. The proof of Theorem 5.11 is mostly the same as the proof of Theorems 3.4 and 3.8. So we mainly discuss the point where there is a difference in the proof.

Let (Σ,z_-,z_+) be a genus zero semi-stable curve with two marked points. We fix one of the irreducible components in the main stream and denote it by Σ^0_0. We decompose
$$\Sigma=\Sigma^-\cup\Sigma^0\cup\Sigma^+ \tag{5.37}$$
as follows. Σ^0 is the mainstream component containing Σ^0_0. Σ^- (resp. Σ^+) is the connected component of $\Sigma\setminus\Sigma^0$ containing z_- (resp. z_+). We remark Σ^0 and/or Σ^- may be empty.

We consider $((\Sigma,z_-,z_+),\Sigma^0,u,\varphi)$ such that Definition 4.2 (1)(2)(4)(5) are satisfied. We assume moreover the following conditions.

- $(3.1)'$ $\varphi|_{\Sigma^+}:\Sigma^+\to X$ is continuous.
- $(3.2)'$ We put either $\{z_{0,-}\}=\Sigma^0\cap\Sigma^-$ or $z_{0,-}=z_-$. (Here we put $z_{0,-}=z_-$ if $z_-\in\Sigma^0$.) Then $\varphi|_{\Sigma^0\setminus\{z_{0,-}\}}:\Sigma^0\setminus\{z_{0,-}\}\to X$ is continuous.
- $(3.3)'$ The same condition as the condition $(3)'$ stated at the beginning of Section 5 holds for Σ^+.
- $(4.1)'$ $h\circ\varphi_i$ satsifies the equation (3.4) for H' if $\Sigma_i\subset\Sigma^-$.
- $(4.2)'$ $h\circ\varphi_0$ satsifies the equation (5.36). Here φ_0 is a part of φ and is a parametrization of Σ^0_0.
- $(4.3)'$ $h\circ\varphi_i$ satsifies the equation (3.4) for H if $\Sigma_i\subset\Sigma^+$.
- $(6.1)'$ Definition 4.2 (6) is satisfied on Σ^-.
- $(6.2)'$ Let $k-1$ be the number of the mainstream component in Σ^+. Then there exists $\tilde\gamma_j=(\gamma_j,w_j)\in\tilde{\mathfrak{P}}(H)$ for $j=1,\dots,k-1$. We have
$$\lim_{\tau\to\infty}u(\varphi_0(\tau,t))=\gamma_1(t).$$
Here φ_0 is as in $(4.2)'$.
- $(6.3)'$ Let Σ^+_j $(j=1,\dots,k-1)$ be the irreducible components in the mainstream of Σ^+. Let φ^+_j is a part of φ and is a parametrization of Σ^+_j. Then we have
$$\lim_{\tau\to-\infty}u(\varphi^+_j(\tau,t))=\gamma_j(t)\qquad \lim_{\tau\to\infty}u(\varphi^+_j(\tau,t))=\gamma_{j+1}(t).$$
Here $\gamma_k=\gamma$.
- $(7)'$ $u_*([\Sigma])=[w]$.

We denote by $\widehat{\mathcal{M}}(X,\mathcal{H};\mathfrak{r},\tilde\gamma)$ the set of all such $((\Sigma,z_-,z_+),\Sigma^0,u,\varphi)$.

We define three equivalence relations $\sim_1$, $\sim_2$, $\sim_3$ on $\widehat{\mathcal{M}}(X,\mathcal{H};\mathfrak{r},\tilde\gamma)$ as follows.

The definition of $\sim_1$ is the same as Definition 4.3.

We apply $\sim_2$ of Definition 4.3 on Σ^+ and Σ^- and $\sim_1$ of Definition 4.3 on Σ^0. This is $\sim_2$ here.

We apply $\sim_2$ of Definition 4.3 on Σ^+, $\sim_3$ of Definition 4.3 on Σ^- and $\sim_1$ of Definition 4.3 on Σ^0. This is $\sim_3$ here. We put

$$\begin{aligned}
\widetilde{\mathcal{M}}(X,\mathcal{H};\mathfrak{x},\tilde{\gamma}) &= \widehat{\mathcal{M}}(X,\mathcal{H};\mathfrak{x},\tilde{\gamma})/\sim_1,\\
\mathcal{M}(X,\mathcal{H};\mathfrak{x},\tilde{\gamma}) &= \widehat{\mathcal{M}}(X,\mathcal{H};\mathfrak{x},\tilde{\gamma})/\sim_2,\\
\overline{\overline{\mathcal{M}}}(X,\mathcal{H};\mathfrak{x},\tilde{\gamma}) &= \widehat{\mathcal{M}}(X,\mathcal{H};\mathfrak{x},\tilde{\gamma})/\sim_3 .
\end{aligned}$$

We define the notion of balancing condition on the mainstream component in Σ^- and Σ^+ in the same way as before. (We do not define such a notion for Σ^0_0 since the equation (5.36) is not invariant under the $\mathbb{R}$ action.)

We next define the notion of canonical marked points. For the mainstream components in Σ^+ or in Σ^-, the definition is the same as before. If Σ^0 contains a sphere bubble we do not define canonial marked points on it. Otherwise the canonical marked point on this mainstream component is $\varphi_0(0,0)$.

Using this notion of canonical marked points we can define the notion of obstruction bundle data for $[\mathfrak{p}] \in \overline{\overline{\mathcal{M}}}(X,\mathcal{H};\mathfrak{x},\tilde{\gamma})$ in the same way as before. (We put an obstruction bundle also on Σ^0_0.) We take and fix an obstruction bundle data for each of $[\mathfrak{p}] \in \overline{\overline{\mathcal{M}}}(X,\mathcal{H};\mathfrak{x},\tilde{\gamma})$.

We can use it to define a map similar to $\overline{\Phi}$ and $\overline{\overline{\Phi}}$ in the same way.

We then use them to define the notion of ϵ-close-ness in the same way.

We next define transversal constraint. Let $((\Sigma', z'_-, z'_+), \Sigma'^0, u', \varphi') \cup \vec{w}'$ is ϵ close to $((\Sigma, z_-, z_+), \Sigma^0, u, \varphi) \cup \vec{w} \cup \vec{w}^{\rm can}$. We consider $w'_j \in \vec{w}'$. If w'_j is either in Σ'_+ or in Σ'_- then the definition of transversal constraint is the same as Definition 4.19 or Definition 5.2, respectively.

Suppose $w'_j \in \Sigma'_0$. If w'_j corresponds to a marked point in $\vec{w}$ then the definition of transversal costraint is the same as Definition 4.19. We consdier the case where w'_j corresponds to a canonical marked point w_i. There are three cases.

(1) $w_i \in \Sigma_0$. In this case the transversal constraint is $w'_j = \varphi_0(0,0)$.

(2) $w_i \in \Sigma_-$. Let $\Sigma_{-,i}$ be the mainstream component containing w_i. (It is irreducible since w_i is a canonical marked point.) The transversal constraint first requires $w'_j = \varphi'_0(\tau_0, 0)$ with $\tau_0 \le -1$. Moreover it requires

$$\begin{aligned}
&\int_{\Sigma_-} (u')^*\omega + \int_{\tau=-\infty}^{\tau_0} (u')^*\omega + H'(u'(\varphi_0(\tau_0,t)))\\
&= \frac{1}{2}\left(H'(u(z_i)) + H'(u(z_{i+1}))\right) + \int_{\Sigma_{\tau\le\tau(w_i)}} u^*\omega.
\end{aligned}$$

Here z_i and z_{i+1} are transit points contained in Σ_i and $\Sigma_{\tau\le\tau_0}$ is defined as follows. Let $w_i = \varphi_i(\tau_i, 0)$. We consider $\Sigma \setminus \{\varphi_i(\tau_i,t) \mid t \in S^1\}$. $\Sigma_{\tau\le\tau_0}$ is the connected component of it containing z_-.

(3) $w_i \in \Sigma_+$. Let $\Sigma_{+,i}$ be the mainstream component containing w_i. (It is irreducible since w_i is a canonical marked point.) The transversal constraint

first requires $w'_j = \varphi'_0(\tau_0, 0)$ with $\tau_0 \ge +1$. Moreover it requires

$$\int_{\Sigma_-} (u')^*\omega + \int_{\tau=-\infty}^{\tau_0} (u')^*\omega + \int_{t\in S^1} H(t, u'(\varphi_0(\tau_0, t)))dt$$
$$= \frac{1}{2}\left(\mathcal{A}_H(\tilde{\gamma_i}) + \mathcal{A}_H(\tilde{\gamma_{i+1}})\right).$$

Here the restriction of u to $\Sigma_{+,i}$ gives an element of $\mathcal{M}(X, H; \tilde{\gamma}^i, \tilde{\gamma}^{i+1})$.

For a point $[\mathfrak{p}] \in \overline{\overline{\mathcal{M}}}(X, \mathcal{H}; \mathfrak{x}, \tilde{\gamma})$ we define the notion of stabilization data in the same way as before.

Now using this notion of transversal constraint and ϵ-close-ness, we define $\mathfrak{C}_{[\mathfrak{p}]} \subset \overline{\overline{\mathcal{M}}}(X, \mathcal{H}; \mathfrak{x}, \tilde{\gamma})$ for each $[\mathfrak{p}] \in \overline{\overline{\mathcal{M}}}(X, \mathcal{H}; \mathfrak{x}, \tilde{\gamma})$. We then define a finite set $\mathfrak{C}$ such that

$$\bigcup_{[\mathfrak{p}_c]\in\mathfrak{C}} \mathfrak{C}_{[\mathfrak{p}_c]} = \overline{\overline{\mathcal{M}}}(X, \mathcal{H}; \mathfrak{x}, \tilde{\gamma}).$$

We may assume that this choice is mainstream-component-wise in the same sense as before.

We use the choice of $\mathfrak{C}$ together with obstruction bundle data we can define an obstruction bundle in the same way as before. We use it to define a thickened moduli space. The rest of the proof is the same as the proof of Theorems 3.4 and 3.8. □

Lemma 5.12. *There exists a constant $\mathfrak{E}^-(\mathcal{H})$ dependig only on $\mathcal{H}$ with the following properties. If $\mathcal{M}(X, \mathcal{H}; \mathfrak{x}, \tilde{\gamma})$ is nonempty then*

$$\mathcal{A}_H(\tilde{\gamma}) \ge H'(\mathfrak{x}) - \mathfrak{E}^-(\mathcal{H}). \tag{5.38}$$

This lemma is classical. See [On, (2.14)]. (It was written also in [FlH, Lemma 9].)

Remark 5.13. The optimal estimate for $\mathfrak{E}^-(\mathcal{H})$ is $\mathfrak{E}^-(\mathcal{H}) = \mathcal{E}^-(H - H')$, where

$$\mathcal{E}^+(F) = \int_0^1 \max_{p\in X} F(t, p)dt, \quad \mathcal{E}^-(F) = -\int_0^1 \inf_{p\in X} F(t, p)dt.$$

See [Oh, Proposition 3.2]. (See also [Us, Proposition 2.1].) A Lagrangian version of a similar optimal estimate was obtained by [Che].

We do not need this optimal estimate for the purpose of this note, but it becomes importantant to study spectral invariant.

We now define

$$\Phi_E : CF(X, H'; \Lambda_0) \to CF(X, H; \Lambda_0)$$

as follows. We consider $\mathfrak{x}$ and $\tilde{\gamma}$ so that:

(a) The (virtual) dimension of $\mathcal{M}(X, \mathcal{H}; \mathfrak{x}, \tilde{\gamma})$ is 0.
(b) $\mathcal{A}_H(\tilde{\gamma}) \le H'(\mathfrak{x}) + E$.

We then put

$$\langle \Phi_E(\mathfrak{x}), \tilde{\gamma}\rangle = T^{\mathcal{A}_H(\tilde{\gamma}) - H'(\mathfrak{x}) + \mathfrak{E}^-(\mathcal{H})} \#\mathcal{M}(X, \mathcal{H}; \mathfrak{x}, \tilde{\gamma})^{\mathfrak{s}}.$$

Here we take and fix a system of multisections $\mathfrak{s}$ of the moduli spaces $\mathcal{M}(X, \mathcal{H}; \mathfrak{x}, \tilde{\gamma})$ that is transversal to zero, compatible with the description of the boundary, and satsifies the inequality (b) above. We use it to define the right hand side.

We remark that the exponent of T in the right hand side is nonnegative because of Lemma 5.12.

Lemma 5.14.
$$\partial_E \circ \Phi_E - \Phi_E \circ \partial \equiv 0 \mod T^E.$$

Proof. We use the case of moduli space $\mathcal{M}(X, \mathcal{H}; \mathfrak{x}, \tilde{\gamma})$ satisfying (a) above and has virtual dimension 1. It boundary is described as in Theorem 5.11. The case (2) there, counted with sign, gives $\partial_E \circ \Phi_E$.

We consider the case (1) there. We need to consider the case when virtual dimension of $\mathcal{M}(X, H'; \mathfrak{x}, \mathfrak{x}', \alpha)$ is zero. Using S^1 equivariance of our Kuranishi structure and multisection, and Lemma 2.15, we find that such $\mathcal{M}(X, H'; \mathfrak{x}, \mathfrak{x}', \alpha)$ is an empty set after perturbation, unless $\alpha = 0$. In the case $\alpha = 0$ we can prove, in the same way as above, that
$$\mathcal{M}(X, H'; \mathfrak{x}, \mathfrak{x}', 0) = \mathcal{M}(X, H'; \mathfrak{x}, \mathfrak{x}', 0)^{S^1}$$
after perturbation. Therefore case (1) gives $\Phi_E \circ \partial$.

The proof of Lemma 5.14 is complete. □

We have thus defined a chain map
$$\Phi_E : CF(X, H'; \Lambda_0/T^E\Lambda_0) \to CF(X, H; \Lambda_0/T^E\Lambda_0). \tag{5.39}$$
We next put $\mathcal{H}'(\tau, t, x) = \mathcal{H}(-\tau, t, x)$. We use it in the same way to define the moduli space $\mathcal{M}(X, \mathcal{H}'; \tilde{\gamma}, \mathfrak{x})$. This moduli space has an oriented Kuranishi structure with corners and its boundary is described in a similar way as Theorem 5.11. (The proof of this fact is the same as the proof of Theorem 5.11.) If it is nonempty then we have
$$H'(\mathfrak{x}) \geq \mathcal{A}_H(\tilde{\gamma}) - \mathfrak{E}^+(\mathcal{H}'). \tag{5.40}$$

Remark 5.15. Here $\mathfrak{E}^+(\mathcal{H}')$ is certain constant depending only on $\mathcal{H}'$. The optimal value is $\mathcal{E}^+(H - H')$.

We put
$$\langle \Psi_E(\tilde{\gamma}), \mathfrak{x} \rangle = T^{H'(\mathfrak{x}) - \mathcal{A}_H(\tilde{\gamma}) + \mathfrak{E}^+(\mathcal{H}')} \# \mathcal{M}(X, \mathcal{H}; \mathfrak{x}, \tilde{\gamma})^{\mathfrak{s}}$$
and obtain a chain map
$$\Psi_E : CF(X, H; \Lambda_0/T^E\Lambda_0) \to CF(X, H'; \Lambda_0/T^E\Lambda_0). \tag{5.41}$$

Lemma 5.16. *We may choose $\mathfrak{E}^+(\mathcal{H}')$, $\mathfrak{E}^-(\mathcal{H})$ and such that the following holds for $\mathfrak{E} = \mathfrak{E}^+(\mathcal{H}') + \mathfrak{E}^-(\mathcal{H})$.*

(1) *$\Psi_E \circ \Phi_E$ is chain homotpic to $[\mathfrak{x}] \mapsto T^{\mathfrak{E}}[\mathfrak{x}]$.*
(2) *$\Phi_E \circ \Psi_E$ is chain homotpic to $[\tilde{\gamma}] \mapsto T^{\mathfrak{E}}[\tilde{\gamma}]$.*

Proof. For $S > 0$ we define $\rho_S : \mathbb{R} \to [0, 1]$ such that
$$\rho_S(\tau) = \begin{cases} 1 & \text{if } |\tau| < S - 1 \\ 0 & \text{if } |\tau| \geq S, \end{cases}$$
and put
$$\mathcal{H}^S(t, x) = \mathcal{H}(\rho_S(\tau, x)).$$
For $\mathfrak{x}_\pm \in \mathrm{Crit}(H')$ and $\alpha \in \Pi$, we use the perturbation of Cauchy-Riemann equation by the Hamilton vector field $H_{S,\tau,t}$ to obtain a moduli space $\mathcal{M}(X, \mathcal{H}^S; \mathfrak{x}_-, \mathfrak{x}_+; \alpha)$ in the same way. Its union for $S \in [0, S_0]$ also has a Kuranishi structure whose boundary is given as in Theorem 5.11 (1), (2) and $\mathcal{M}(X, \mathcal{H}^S; \mathfrak{x}_-, \mathfrak{x}_+; \alpha)$ with $S = 0, S_0$.

We consider the case $S = 0$. In this case the equation for $\mathcal{M}(X, \mathcal{H}^S; \mathfrak{x}_-, \mathfrak{x}_+; \alpha)$ is S^1 equivariant. Therefore it has an S^1 equivariant Kuranishi structure that is free for $\alpha \neq 0$. For $\alpha = 0$ we obtain an S^1 equivariant Kuranishi structure on $\mathcal{M}_0(X, \mathcal{H}^{S=0}; \mathfrak{x}_-, \mathfrak{x}_+; 0)$. Therefore, by counting the moduli space of virtual dimension 0 we have identity. (It becomes $[\mathfrak{x}] \mapsto T^{\mathfrak{E}}[\mathfrak{x}]$ because of the choice of the exponent in the definition.)

The case $S = S_0$ with S_0 huge gives the composition $\Psi_E \circ \Phi_E$.

(1) now follows from a cobordism argument.

The proof of (2) is similar. □

Using [FOOO1, Proposition 6.3.14] we have

$$HF(X, H'; \Lambda_0) \cong (\Lambda_0)^{\oplus b'} \oplus \bigoplus_{i=1}^{m'} \frac{\Lambda_0}{T^{\lambda'_i}\Lambda_0}$$

here λ'_i, $i = 1, \dots, m'$ are positive real number. It implies

$$HF(X, H'; \Lambda) \cong (\Lambda)^{\oplus b'}.$$

We remark $H(X, H'; \Lambda) \cong H(X; \Lambda)$ where the right hand side is the singular homology. (Note H' is time independent Hamiltonian that is a Morse function on X.)

Similarly we have

$$HF(X, H; \Lambda_0) \cong (\Lambda_0)^{\oplus b} \oplus \bigoplus_{i=1}^{m} \frac{\Lambda_0}{T^{\lambda_i}\Lambda_0}$$

and

$$HF(X, H; \Lambda) \cong (\Lambda)^{\oplus b}.$$

We take E sufficiently larger than $\mathfrak{E}$ and λ_i, λ'_i. Then we can use Lemma 5.16 to show $b = b'$. (See [FOOO6, Subsection 6.2] for more detailed proof of more precise results in a related situation.) The proof of Theorem 5.11 is now complete. □

Remark 5.17. (1) The argument of the last part of the proof of Theorem 5.11 shows that to prove the inequality (the homology version of Arnold's conjecture)

$$\#\mathfrak{P}(H) \geq \sum \mathrm{rank} H_k(X; Q) \tag{5.42}$$

for periodic Hamiltonian system with nondegenerate closed orbit, we do not need to use projective limit. We can use $HF(X, H; \Lambda_0/T^E\Lambda_0)$ for sufficiently large but fixed E. (Such E depends on H and H'.)

(2) During the above proof of the isomorphism $H(X, H; \Lambda) \cong H(X; \Lambda)$ we did not construct an isomorphism among them but showed only the coincidence of their ranks. Actually we can construct the following diagram

$$\begin{array}{ccc} CF(X, H'; \Lambda_0/T^{E'}\Lambda_0) & \xrightarrow{\Phi_{E'}} & CF(X, H; \Lambda_0/T^{E'}\Lambda_0) \\ \downarrow & & \downarrow \\ CF(X, H'; \Lambda_0/T^{E}\Lambda_0) & \xrightarrow{\Phi_{E'}} & CF(X, H; \Lambda_0/T^{E}\Lambda_0) \end{array} \tag{5.43}$$

for $E < E'$. Here the virtical arrows are composition of reduction modulo T^E and chain homotopy equivalence. (Note this map is not reduction modulo T^E. In fact the two chain complexes that are the target and the domain of the vertical arrows, are constructed by using different Kuranishi structures and multisections.) We can prove that Diagram 5.43 commutes up to chain homotopy. Then using a lemma similar to Lemma 5.8 we can extend Φ_E to a chain map $CF(X, H'; \Lambda_0) \to CF(X, H; \Lambda_0)$. We then can prove that it is a chain homotopy equivalence. This argument is a baby version of one developed in [FOOO1, Section 7.2].

Remark 5.18. As we mentioned in Introduction, there is an alternative (third) proof of (5.42) which does *not* use S^1 equivariant Kuranishi structure, for an arbitrary compact symplectic manifold X. Let H be a time dependent Hamiltonian whose 1 periodic orbit are all nondegenerate. Let $\varphi : X \to X$ be the symplectic diffeomorphism that is time one map of our time dependent Hamiltonian vector field. We consider a symplectic manifold $(X \times X, \omega \oplus -\omega)$. The graph

$$L(\varphi) = \{(x, \varphi(x)) \mid x \in X\}$$

is a Lagrangian submanifold in $X \times X$. Since the inclusion induces injective homomorphism $H(L(\varphi)) \cong H(X) \to H(X \times X)$ in the homology groups, [FOOO1, 3.8.41] implies that the Lagrangian Floer homology between $L(\varphi)$ with itself is defined, (after an appropriate bulk deformation). Again since $L(\varphi) \to X \times X$ induces injective homomorphism in the homology the spectral sequence in [FOOO1, Theorem 6.1.4] degenerates at E_2 stage and implies

$$HF((L(\varphi), b, \mathfrak{b}), (L(\varphi), b, \mathfrak{b}); \Lambda) \cong H(L(\varphi); \Lambda_0) = H(X; \Lambda_0).$$

(Here b is an appropriate bounding cochain and $\mathfrak{b}$ is an appropriate bulk class.) Since $L(\varphi)$ is Hamiltonian isotopic to the diagonal X, [FOOO1, Theorem 4.1.5] implies

$$HF((L(\varphi), b, \mathfrak{b}), (L(\varphi), b, \mathfrak{b}); \Lambda) \cong HF((L(\varphi), b, \mathfrak{b}), (X, b', \mathfrak{b}); \Lambda).$$

Note $L(\varphi) \cap X \cong \mathfrak{P}(H)$ and the intersection is transversal. (This is a consequence of the nondegeneracy of the periodic orbit.) Therefore the rank of the Floer cohomology $HF((L(\varphi), b, \mathfrak{b}), (X, b', \mathfrak{b}); \Lambda)$ is not greater than the order of $\mathfrak{P}(H)$. The formula (5.42) follows.

Note in the above proof we use injectivity of $H(L(\varphi)) \to H(X \times X)$ to show that $L(\varphi)$ (which is Hamiltonian isotopic to the diagonal) is unobstructed (after bulk deformation). Alternatively we can use the involution $X \times X \to X \times X$, $(x, y) \mapsto (y, x)$ to prove the unobstructedness of the diagonal. (See [FOOO7].)

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