

POTENTIALS OF HOMOTOPY CYCLIC A_∞ -ALGEBRAS

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ABSTRACT. For a cyclic A_∞ -algebra, a potential recording the structure constants can be defined. We define an analogous potential for a homotopy cyclic A_∞ -algebra and prove its properties. On the other hand, we find another different potential for a homotopy cyclic A_∞ -algebra, which is related to the algebraic analogue of generalized holonomy map of Abbaspour, Tradler and Zeinalian.

1. INTRODUCTION

We first recall the definition of cyclic inner products due to Kontsevich [Ko], which may be understood as constant invariant symplectic structures in the non-commutative geometry.

Definition 1.1. An A_∞ -algebra $(A, \{m_*\})$ is said to have a **cyclic inner product** if there exists a skew symmetric non-degenerate, bilinear map

$$\langle, \rangle : A \otimes A \rightarrow k$$

such that for all integer $k \geq 1$,

$$\langle m_k(x_1, \dots, x_k), x_{k+1} \rangle = (-1)^{K(\vec{x})} \langle m_k(x_2, \dots, x_{k+1}), x_1 \rangle. \quad (1.1)$$

Here, $(-1)^{K(\vec{x})}$ denotes the sign given by Koszul sign convention. Namely,

$$(-1)^{K(\vec{x})} = (-1)^{|x_1|'(|x_2|' + \dots + |x_{k+1}|')}, \quad (1.2)$$

where $|x|'$ is the shifted degree of x .

This notion for the A_∞ -algebras and A_∞ -categories is crucial in homological mirror symmetry, for example, as in the work of Kontsevich-Soibelman [KS] or Costello [Cos]. In particular, Costello has proved in [Cos] that the category of open topological conformal field theory is homotopy equivalent to the category of Calabi-Yau categories, where the Calabi-Yau category is a categorical generalization of a cyclic A_∞ -algebra.

The first application of this gadget is to define a potential for a cyclic A_∞ -algebra, which in physics, is called an action of a string field theory: Let (A, m_*^A) be a cyclic A_∞ -algebra. Let e_i be generators of A as a vector space, which is assumed to be finite dimensional. Define $\mathbf{x} = \sum_i e_i x_i$ where x_i are formal parameters with $\deg(x_i) = -\deg(e_i)$.

Definition 1.2. Define

$$\Phi^A(\mathbf{x}) = \sum_{k=1}^{\infty} \frac{1}{k+1} \langle m_k^A(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x}), \mathbf{x} \rangle \quad (1.3)$$

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This may be considered as a systematic way of gathering structure constants of a cyclic A_∞ -algebra. In the case of toric manifolds, this potential when restricted to the Maurer-Cartan elements, becomes the Landau-Ginzburg superpotential of the mirror B-model (see [CO],[FOOO1]).

The notion of cyclicity is not a homotopy invariant notion. For example, an A_∞ -algebra which is homotopy equivalent to a cyclic A_∞ -algebra may not be cyclic. Instead, it has a strong homotopy inner product, which was defined by the first author in [C]: for this, a cyclic inner product on A may be understood as a special kind of A_∞ -bimodule map $A \rightarrow A^*$ (Lemma 3.1, [C]). An A_∞ -bimodule quasi-isomorphism $A \rightarrow A^*$ is called as an infinity inner product by Tradler (see [T],[TZ] for example).

Definition 1.3 ([C], Definition 3.6). Let A be an A_∞ -algebra. We call an A_∞ -bimodule map $\phi : A \rightarrow A^*$ a **strong homotopy inner product** if there exists a cyclic A_∞ -algebra B with $\psi : B \rightarrow B^*$ and an A_∞ -quasi-isomorphism $f : A \rightarrow B$ such that the following diagram of A_∞ -bimodules over A commutes

$$\begin{array}{ccc} A & \xrightarrow{\quad g = \tilde{f} \quad} & B \\ \phi \downarrow & & \downarrow \psi \text{ cyc} \\ A^* & \xleftarrow{\quad g^* \quad} & B^* \end{array} \quad (1.4)$$

Here by $g : A \rightarrow B$, we denote the induced A_∞ -bimodule map $\tilde{f} = g$ where B is considered as an A_∞ -bimodule over A .

In this paper, we give a definition of the potential for strong homotopy inner products and prove its properties. It turns out that the definition of the potential 3.1 is very similar to that of (1.2), but the prove that they are indeed related is non-trivial and involves quite combinatorial arguments. Beyond the fact that it is quite natural to work with homotopy notions when dealing with homotopy algebras, sometimes it is necessary to work with homotopy notions directly. For example, in the work of Kontsevich and Soibelman [KS], they find a relation between cyclic cohomology of an A_∞ -algebra A and cyclic symmetry. Given a cyclic cohomology class, one obtains first a homotopy inner product on A , and then cyclic inner product in the minimal model. We refer readers to [CL] for the explicit formulas of this correspondence in terms of negative cyclic cohomology $HC^\bullet_-(A)$ and strong homotopy inner products.

Now, let us assume that the A_∞ -algebra is unital (see definition 4.1), and assume that the A_∞ -bimodule maps are also unital. Then, from the strong homotopy inner products $\{<, >_{p,q}\}$ we can define another potential as follows. (Here, $<, >_{p,q}$ is obtained from the (p, q) -component of the bimodule map ϕ . See (3.1))

Definition 1.4. Define

$$\Psi^A(\mathbf{x}) = \sum_{p,q \geq 0} \frac{1}{p+q+1} < \underbrace{\mathbf{x}, \dots, \mathbf{x}}_p, \underbrace{\mathbf{x}, \dots, \mathbf{x}}_q | I >_{p,q} \quad (1.5)$$

We prove that this potential is in fact *invariant under the gauge equivalence* for Maurer-Cartan elements. We also find its relation to the work of Abbaspour, Tradler and Zeinalian [ATZ], where this map corresponds to the algebraic analogue of the generalized holonomy map from the negative cyclic cohomology to the function ring of Maurer-Cartan elements.

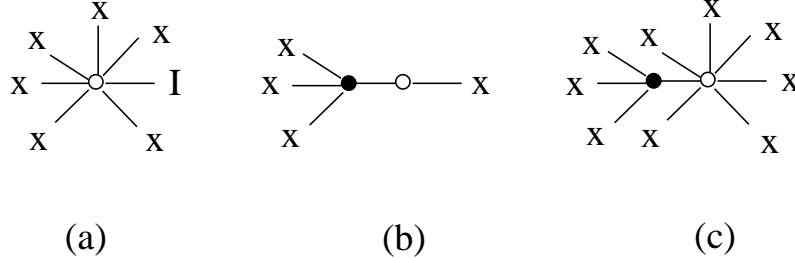


FIGURE 1. (a) Potential Ψ , (b) Cyclic Potential Φ , (c) Homotopy cyclic potential Φ

The following Figure 1 explains the differences of the expressions used in these potentials (without the coefficients). In the figure, the circle represent the strong homotopy inner product (following that of Tradler [T]) whose horizontal arrows are for the inputs from modules. The filled circle represent the A_∞ -operation m .

This paper may be considered as a continuation of the paper [C] to which we refer readers for the notations and further introductions, especially about the signs. Throughout the paper we assume that $H^\bullet(A)$ is finite dimensional. We thank H. Kajiura for sending us the unpublished manuscript (with Y. Terashima) on the decomposition theorem of A_∞ -algebras.

2. STRONG HOMOTOPY INNER PRODUCTS

We begin by proposing a modified definition of strong homotopy inner products, and discuss their equivalences and pull-backs.

We first make an observation that there exists certain subtlety in the direction of arrows in the diagram 1.4 in the definition of strong homotopy inner products. One could try to make the definition with the arrow $A \leftarrow B$ instead of $A \rightarrow B$, but the resulting diagram would become weaker as there may exists elements of A which is not covered by the image of the map $A \leftarrow B$ in general. The subtlety actually disappears if we have non-degeneracy in the chain level. The correct definition (which corresponds to exactly the non-commutative invariant symplectic two form) is rather in between these two definitions: to make the correct definition, we first recall the following characterization theorem of strong homotopy inner products from [C].

Theorem 2.1 ([C], Theorem 5.1). *An A_∞ -algebra A has a strong homotopy inner product if and only if there exists an A_∞ -bimodule map $\phi : A \rightarrow A^*$, satisfying the following three conditions.*

- (1) (*Skew symmetry*) For any $a_i, v, b_j, w \in A$,

$$\phi_{k,l}(\vec{a}, v, \vec{b})(w) = -(-1)^K \phi_{l,k}(\vec{b}, w, \vec{a})(v),$$

$$\text{with } |K| = (\sum_{i=1}^k |a_i|' + |v|')(\sum_{j=1}^l |b_j|' + |w|')$$

- (2) (*Closedness*) For any choice of a family (a_1, \dots, a_{l+1}) and any choice of indices $1 \leq i < j < k \leq l+1$, we have

$$(-1)^{K_i} \phi(\dots, \underline{a_i}, \dots)(a_j) + (-1)^{K_j} \phi(\dots, \underline{a_j}, \dots)(a_k) + (-1)^{K_k} \phi(\dots, \underline{a_k}, \dots)(a_i) = 0,$$

where the arguments inside ϕ are uniquely given by the cyclic order of the family (a_1, \dots, a_{l+1}) , and the signs K_* are given by the Koszul convention:

$$K_* = (|a_1|' + \dots + |a_*|')(|a_{*+1}|' + \dots + |a_k|'). \quad (2.1)$$

- (3) (*Homological non-degeneracy*) For any non-zero $[a] \in H^\bullet(A)$ with $a \in A$, there exists a $[b] \in H^\bullet(A)$ with $b \in A$, such that $\phi_{0,0}(a)(b) \neq 0$.

For non-degeneracy on the chain level, ϕ itself gives the strong homotopy inner product, otherwise the inner product obtained $\phi' : A \rightarrow A^*$ is only equivalent to ϕ .

The second condition is called *closed condition* since it is equivalent to the closed condition of the related non-commutative symplectic 2-form, and this plays a crucial role in proving the properties of the potential defined in this paper.

We also remark that in the proof of the Theorem 2.1, ϕ satisfying the three conditions, does not always become exactly a strong homotopy inner product (in the sense of definition 1.3), but only equivalent to a strong homotopy inner product (the equivalence is defined below).

Hence, we propose to define the strong homotopy inner products by the Theorem 2.1 because such a definition is equivalent to that of the non-commutative symplectic form as explained in [C].

Definition 2.1. Let A be an A_∞ -algebra. We call an A_∞ -bimodule map $\phi : A \rightarrow A^*$ a **strong homotopy inner product** if it is skew-symmetric, closed and homologically non-degenerate as in Theorem 2.1.

And A is called **homotopy cyclic A_∞ -algebra**, if there exists a strong homotopy inner product of A .

Then, the main result of [C] can be phrased as the following theorem.

Theorem 2.2. Let $\phi : A \rightarrow A^*$ be an A_∞ -bimodule map.

- (1) If ϕ is a strong homotopy inner product in the sense of definition 2.1, then there exists an A_∞ -algebra B with a cyclic inner product $\psi : B \rightarrow B^*$ and an A_∞ -quasi-isomorphism $\iota : B \rightarrow A$ satisfying the following commutative diagram of A_∞ -bimodule homomorphisms

$$\begin{array}{ccc} A & \xleftarrow{\tilde{\iota}} & B \\ \phi \downarrow & & \downarrow \psi \text{ cyc} \\ A^* & \xrightarrow{\tilde{\iota}^*} & B^* \end{array} \quad (2.2)$$

- (2) If there exists a cyclic A_∞ -algebra B with $\psi : B \rightarrow B^*$ and an A_∞ -quasi-isomorphism $f : A \rightarrow B$ such that the following diagram of A_∞ -bimodules over A commutes

$$\begin{array}{ccc} A & \xrightarrow{g=f} & B \\ \phi \downarrow & & \downarrow \psi \text{ cyc} \\ A^* & \xleftarrow{g^*} & B^* \end{array} \quad (2.3)$$

then, ϕ is a strong homotopy inner product in the sense of definition 2.1.

If $\phi_{0,0}$ is non-degenerate in the chain level, the old and the new definitions of a strong homotopy inner product are equivalent.

Remark 2.2. Hence the new definition of the strong homotopy inner product is a little stronger than the diagram using $A \leftarrow B$, a little weaker than the diagram using $A \rightarrow B$ and equivalent to the non-commutative symplectic two form.

Proof. In the non-degenerate case, from the proof of Theorem 2.1, one can find B with an A_∞ -isomorphism $f : A \rightarrow B$ with the commuting diagram (1.4). Hence one can find exact inverse of f to make the commuting diagram (2.2).

Also, the statement (2) can be checked without much difficulty from the commuting diagram, so we only consider the statement (1). We explain that the proof of the theorem 2.1 given in [C] is enough to prove the existence of the diagram (2.2): We recall from [C] that the first step of the construction of cyclic A_∞ -algebra B when A is only homologically non-degenerate was considering the minimal model $\iota : H^\bullet(A) \rightarrow A$ and consider the pull back $\iota^*\phi$.

$$\begin{array}{ccccc} A & \xleftarrow{\tilde{\iota}} & H^\bullet(A) & \xrightarrow{\tilde{f}} & H^\bullet(A) \\ \phi \downarrow & & \iota^*\phi \downarrow & & cyc \downarrow \\ A^* & \xrightarrow{\tilde{\iota}^*} & (H^\bullet(A))^* & \xleftarrow{\tilde{f}^*} & (H^\bullet(A))^* \end{array} \quad (2.4)$$

Then $\iota^*\phi$ is non-degenerate and skew symmetric and closed, and one proves the theorem for $\iota^*\phi$ to find $f : H^\bullet(A) \rightarrow H^\bullet(A)$ with the above commutative diagram. As the quasi-isomorphism f on $H^\bullet(A)$ is in fact an isomorphism, hence there exists explicit inverse f^{-1} and we obtain the diagram (2.2). \square

We can also prove the following corollary.

Corollary 2.3. *Let $\phi : A \rightarrow A^*$ be a strong homotopy inner product. Suppose we have an A_∞ -quasimorphism $f : A \rightarrow H^\bullet(A)$ with the commuting diagram*

$$\begin{array}{ccc} A & \xrightarrow{g=\tilde{f}} & H^\bullet(A) \\ \phi \downarrow & & \psi \downarrow cyc \\ A^* & \xleftarrow{\tilde{g}^*} & H^\bullet(A)^* \end{array} \quad (2.5)$$

then, there exists an A_∞ -quasimorphism $h : H^(A) \rightarrow A$ with the commuting diagram (with the same ψ as the above)*

$$\begin{array}{ccc} A & \xleftarrow{\tilde{h}} & H^\bullet(A) \\ \phi \downarrow & & \psi \downarrow cyc \\ A^* & \xrightarrow{\tilde{h}^*} & (H^\bullet(A))^* \end{array} \quad (2.6)$$

Proof. By the decomposition theorem of A_∞ -algebras, the map f has a right inverse A_∞ -quasi-homomorphism, say $h : H^*(A) \rightarrow A$ such that $f \circ h = id$. To see this, consider an A_∞ -isomorphism η

$$\eta : A \rightarrow A^{dc} := A^H \oplus A^{lc}$$

to the direct sum of the minimal A_∞ -algebra A^H and the linear contractible A^{lc} .

Let $\pi : A^{dc} \rightarrow A^H$ be the projection and $i : A^H \rightarrow A^{dc}$ be the inclusion where the both are A_∞ -quasimorphisms with $\pi \circ i = id$. As f is an A_∞ -quasimorphism, $f \circ \eta^{-1} \circ i : A^H \rightarrow H^\bullet(A)$ is an A_∞ -isomorphism, hence has an A_∞ -inverse say ξ .

Then, we define the right A_∞ inverse $h = \eta^{-1} \circ i \circ \xi$. The property $f \circ h = id$ can be checked immediately. The second diagram then follows from the first commuting diagram. \square

Now, we define equivalences between strong homotopy inner products.

Definition 2.3. Let $\phi : A \rightarrow A^*$ and $\psi : B \rightarrow B^*$ be strong homotopy inner products. They are called **equivalent** if there exists a cyclic symmetric A_∞ -algebra H with a commutative diagram:

$$\begin{array}{ccccc} A & \xleftarrow{qis} & H & \xrightarrow{qis} & B \\ \phi \downarrow & & \downarrow cyc & & \downarrow \psi \\ A^* & \longrightarrow & H^* & \longleftarrow & B^* \end{array}$$

One can actually choose H to be a minimal(or canonical) model.

Given a strong homotopy inner product $\phi : B \rightarrow B^*$, and an A_∞ -quasi-isomorphism $f : A \rightarrow B$, we may define a pullback $f^*\phi : A \rightarrow A^*$ as a composition : $f^*\phi = \tilde{f}^* \circ \hat{\phi} \circ \tilde{f}$

$$\begin{array}{ccc} A & \xrightarrow{\tilde{f}} & B \\ f^*\phi \downarrow & & \downarrow \phi \\ A^* & \xleftarrow{\tilde{f}^*} & B^* \end{array}$$

Proposition 2.4. $f^*\phi$ defines a strong homotopy inner product on A which is equivalent to ϕ .

Proof. Since $\phi : B \rightarrow B^*$ is skew-symmetric and closed, so is $f^*\phi$ by lemma 5.6 of [C]. It is not hard to check that $f^*\phi$ is also homologically non-degenerate as f is a quasi-isomorphism. Hence, $f^*\phi$, by the proposition 2.2 (1), is a strong homotopy inner product. Hence there exist an A_∞ -algebra C which is cyclic symmetric ($\psi : C \rightarrow C^*$), and A_∞ -quasi-homomorphism $h : C \rightarrow A$ with the following commutative diagrams.

$$\begin{array}{ccccc} C & \xrightarrow{\tilde{h}} & A & \xrightarrow{\tilde{f}} & B \\ \psi \downarrow & & f^*\phi \downarrow & & \downarrow \phi \\ C^* & \xleftarrow{\tilde{h}^*} & A^* & \xleftarrow{\tilde{f}^*} & (B)^* \end{array} . \quad (2.7)$$

From the diagram, it is easy to see that ϕ and $f^*\phi$ is equivalent in the sense of definition 2.3. \square

3. POTENTIALS

In this section we define a potential of a homotopy cyclic A_∞ -algebra and prove its properties. Let (A, m_*^A) be given a strong homotopy inner product $\phi : A \rightarrow A^*$. Recall that an A_∞ -bimodule map ϕ is given by a family of maps

$$\phi_{p,q} : A^{\otimes p} \otimes \underline{A} \otimes A^{\otimes q} \rightarrow A^*,$$

where the underlined A is to emphasize that it is an A -bimodule for reader's convenience. We denote by

$$\langle x_1, \dots, x_p, \underline{y_1}, \dots, y_q | w \rangle_{p,q} := \phi(x_1, \dots, x_p, \underline{y_1}, \dots, y_q)(w). \quad (3.1)$$

As in the cyclic case, let e_i be generators of A as a vector space, which is assumed to be finite dimensional. (One may use pull-back defined in the previous section using the inclusion $\iota : H^*(A) \rightarrow A$ in the case that $H^*(A)$ is finite dimensional). Define $\mathbf{x} = \sum_i e_i x_i$ where x_i are formal parameters with $\deg(x_i) = -\deg(e_i)$.

Now we give a definition a potential for the strong homotopy inner products.

Definition 3.1. The **potential** of an A_∞ -algebra $(A, m_*)^A$ with a strong homotopy inner product $\phi : A \rightarrow A^*$ is defined as

$$\begin{aligned} \Phi^A(\mathbf{x}) &= \sum_{N=1}^{\infty} \Phi_N^A(\mathbf{x}) \\ &:= \sum_{N=1}^{\infty} \sum_{p+q+k=N} \frac{1}{N+1} \langle \mathbf{x}, \mathbf{x}, \dots, \mathbf{x}, \underline{m_k^A(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})}, \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle_{p,q} \end{aligned} \quad (3.2)$$

The definition itself is somewhat similar to that of cyclic case(1.2). But in (1.2), the fraction $1/k$ was to cancel out repetitive contribution to the potential due to cyclic symmetry (1.1), whereas in the strong homotopy case, such cyclic symmetry of the rotation of arguments do not exist. Namely, in general

$$\langle e_1, \dots, \underline{m_i(e_j, \dots, e_{j+i-1})}, \dots, e_k | e_{k+1} \rangle \neq \langle e_2, \dots, \underline{m_i(e_{j+1}, \dots, e_{j+i})}, \dots, e_{k+1} | e_1 \rangle.$$

We later show that the combination of A_∞ -bimodule equation, skew-symmetry and closed condition will compensate the absence of the strict cyclic symmetry.

We explain how the potential behaves under pull-backs, and this will show the relation between the potentials of equivalent strong homotopy inner products. For A_∞ -quasi-isomorphism $h : B \rightarrow A$ the pull-back of a potential is defined as follows: We assume B is finite dimensional as a vector space, and denote by $\{f_*\}$ its basis, and introduce corresponding formal variables y_* as before. Suppose

$$h_k(f_{j_1}, \dots, f_{j_k}) = h_{j_1, \dots, j_k}^i e_i, \quad h_{j_1, \dots, j_k}^i \in \mathbf{k}.$$

Then, we set

$$x_i \mapsto h_{j_{11}}^i y_{j_{11}} + h_{j_{21}, j_{22}}^i y_{j_{21}} y_{j_{22}} + \dots + h_{j_{l1}, \dots, j_{lk}}^i y_{j_{l1}} \dots y_{j_{lk}} + \dots. \quad (3.3)$$

Then, one define the pull-back $h^* \Phi^A$ by using the above change of coordinate formula.

Theorem 3.1. *Let $\phi : A \rightarrow A^*$ be a strong homotopy inner products. Let B be a cyclic A_∞ -algebra with a quasi-isomorphism $h : B \rightarrow A$ providing the commutative diagram (2.2). Then, we have*

$$\Phi^B = h^* \Phi^A$$

Proof. The overall scheme of the proof, which is first to differentiate and then to compare, follows that of [C] (idea due to Kajiwara [Kaj] in the unfiltered case). The main difficulty, and the essential part of the proof is the first step where we take (formal) partial derivatives on each side. The following lemma shows that after partial differentiation, the fraction on each summand disappears.

Lemma 3.2.

$$\begin{aligned} \frac{\partial}{\partial x_i} \Phi_N^A(\mathbf{x}) &= \frac{\partial}{\partial x_i} \sum_{p+q+k=N}^{\infty} \frac{1}{N+1} \langle \mathbf{x}, \mathbf{x}, \dots, \mathbf{x}, \underline{m_k^A(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})}, \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle_{p,q} \\ &= \sum_{k=1}^{\infty} \langle \mathbf{x}, \mathbf{x}, \dots, \mathbf{x}, \underline{m_k^A(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})}, \mathbf{x}, \dots, \mathbf{x} | e_i \rangle_{p,q}. \end{aligned}$$

We assume the lemma for a moment and show the proof of the theorem using the lemma. Let $\{f_i\}$ be basis of $H^*(A)$, and let $\{y_i\}$ be corresponding formal variables for $\{f_i\}$, namely $\mathbf{y} := \sum_i y_i f_i$.

We let $h(\mathbf{y}) := \sum_{k \geq 1} h_k(\mathbf{y}^{\otimes k})$. Then

$$\frac{\partial}{\partial y_i} \Phi^{H^*(A)} = \sum_{k \geq 1} \langle m_k^{H^*(A)}(\mathbf{y}, \dots, \mathbf{y}), f_i \rangle$$

by cyclic symmetry, and

$$\begin{aligned} \frac{\partial}{\partial y_i} h^* \Phi^A &= \frac{\partial}{\partial y_i} \sum_{\substack{k \geq 1 \\ p+q+1=N}} \frac{1}{N+1} \langle h(\mathbf{y})^{\otimes p}, \underline{m_k^A(h(\mathbf{y}), \dots, h(\mathbf{y}))}, h(\mathbf{y})^{\otimes q} | h(\mathbf{y}) \rangle \\ &= \sum_{\substack{N \geq 1 \\ p+q+1=N}} \langle h(\mathbf{y})^{\otimes p}, \underline{m_k^A(h(\mathbf{y}), \dots, h(\mathbf{y}))}, h(\mathbf{y})^{\otimes q} | \frac{\partial}{\partial y_i} h(\mathbf{y}) \rangle \end{aligned}$$

by above lemma. From the diagram 2.2, we have $\psi = \tilde{h}^* \circ \hat{\phi} \circ \hat{h}$, where all maps are $H^*(A)$ -bimodule homomorphisms, consider following:

$$\begin{aligned} &\sum_{\substack{p,q \geq 0 \\ k \geq 1}} \psi(\mathbf{y}^{\otimes p}, \underline{m_k^{H^*(A)}(\vec{\mathbf{y}})}, \mathbf{y}^{\otimes q})(f_i) \tag{3.4} \\ &= \sum_{\substack{p,q \geq 0 \\ k \geq 1}} (\tilde{h}^* \circ \hat{\phi} \circ \hat{h})(\mathbf{y}^{\otimes p}, \underline{m_k^{H^*(A)}(\vec{\mathbf{y}})}, \mathbf{y}^{\otimes q})(f_i) \\ &= \sum_{\substack{p,q \geq 0 \\ k \geq 1}} \sum_{\substack{p_1+p_2+p_3=p \\ q_1+q_2+q_3=q}} \tilde{h}^*(\mathbf{y}^{\otimes p_3}, \phi(\hat{h}(\mathbf{y}^{\otimes p_2}), \underline{h(\mathbf{y}^{\otimes p_1}, \underline{m_k^{H^*(A)}(\vec{\mathbf{y}})}, \mathbf{y}^{\otimes q_1})}, \hat{h}(\mathbf{y}^{\otimes q_2})), \mathbf{y}^{\otimes q_3})(f_i) \\ &= \sum_{\substack{p,q \geq 0 \\ k \geq 1}} \sum_{\substack{p_1+p_2+p_3=p \\ q_1+q_2+q_3=q}} \phi(\hat{h}(\mathbf{y}^{\otimes p_2}), \underline{h(\mathbf{y}^{\otimes p_1}, \underline{m_k^{H^*(A)}(\vec{\mathbf{y}})}, \mathbf{y}^{\otimes q_1})}, \hat{h}(\mathbf{y}^{\otimes q_2}))(h(\mathbf{y}^{\otimes q_3}, \underline{f_i}, \mathbf{y}^{\otimes p_3})) \\ &= \sum_{\substack{p,q \geq 0 \\ k \geq 1}} \sum_{\substack{p_1+p_2+p_3=p \\ q_1+q_2+q_3=q}} \langle \hat{h}(\mathbf{y}^{\otimes p_2}), \underline{h(\mathbf{y}^{\otimes p_1}, \underline{m_k^{H^*(A)}(\vec{\mathbf{y}})}, \mathbf{y}^{\otimes q_1})}, \hat{h}(\mathbf{y}^{\otimes q_2}) | h(\mathbf{y}^{\otimes q_3}, \underline{f_i}, \mathbf{y}^{\otimes p_3}) \rangle \\ &= \sum_{\substack{N \geq 1 \\ p+q+1=N}} \langle h(\mathbf{y})^{\otimes p}, \underline{m_k^A(h(\mathbf{y}), \dots, h(\mathbf{y}))}, h(\mathbf{y})^{\otimes q} | \frac{\partial}{\partial y_i} h(\mathbf{y}) \rangle \\ &= \frac{\partial}{\partial y_i} h^* \Phi^A \end{aligned}$$

Here, we denote by $m_k(\vec{\mathbf{y}})$ the expression $m_k(\mathbf{y}, \dots, \mathbf{y})$ for simplicity. The last identity holds because the sum is over all $p_1 + p_2 + p_3 = p$ and $q_1 + q_2 + q_3 = q$

where p and q run over all nonnegative integers, and there is A_∞ -bimodule relation $\widehat{m^A} \circ \widehat{h} = \widehat{h} \circ \widehat{m^{H^*(A)}}$.

The summands of (3.4) are all zero except for $(p, q) = (0, 0)$ because ψ is a cyclic symmetric inner product. Hence,

$$\frac{\partial}{\partial y_i} h^* \Phi^A = \sum_{k \geq 1} \psi(m_k^{H^*(A)}(\vec{y}))(f_i) = \sum_{k \geq 1} \langle m_k^{H^*(A)}(\mathbf{y}, \dots, \mathbf{y}), f_i \rangle = \frac{\partial}{\partial y_i} \Phi^{H^*(A)}.$$

This proves the theorem. \square

Proof. We prove the lemma 3.2. Before we proceed, we give some remarks on the signs. The sign convention used in this paper and in [C] is the Koszul convention after the degree one shift. For simplicity, we omit the Koszul sign factor and the expressions will appear with $+$ if it agrees with the Koszul sign rule, $-$ if it is the negative of the Koszul sign. We illustrate this for two examples, from which the general convention can be easily understood. The first example is the A_∞ -equation with two inputs. We write

$$m_1 m_2(x_1, x_2) + m_2(m_1(x_1), x_2) + m_2(x_1, m_1(x_2)) = 0 \quad (3.5)$$

whereas the actual equation is

$$m_1 m_2(x_1, x_2) + m_2(m_1(x_1), x_2) + (-1)^{|x_1|'} m_2(x_1, m_1(x_2)) = 0.$$

The equation (3.5) will be also written as

$$m_1 m_2(x_1, x_2) = -m_2(m_1(x_1), x_2) - m_2(x_1, m_1(x_2)).$$

The second example is the equation for $\langle m_2(\underline{x}_1, x_2) | x_3 \rangle$. Note that ϕ being A_∞ -bimodule map $\phi : A \rightarrow A^*$ with the induced A_∞ -bimodule structure on A^* (see expression (3.3) [C] for the precise definition) implies the following actual equation.

$$\begin{aligned} & \langle m_2(\underline{x}_1, x_2) | x_3 \rangle + \langle m_1(\underline{x}_1), x_2 | x_3 \rangle + (-1)^{|x_1|'} \langle \underline{x}_1, m_1(x_2) | x_3 \rangle \\ & + (-1)^{|x_1|' + |x_2|'} \langle \underline{x}_1, x_2 | m_1(x_3) \rangle + (-1)^{|x_1|'} \langle \underline{x}_1 | m_2(x_2, x_3) \rangle = 0. \end{aligned}$$

In this paper, the above equation will be written simply as

$$\begin{aligned} & \langle m_2(\underline{x}_1, x_2) | x_3 \rangle + \langle m_1(\underline{x}_1), x_2 | x_3 \rangle + \langle \underline{x}_1, m_1(x_2) | x_3 \rangle \\ & + \langle \underline{x}_1, x_2 | m_1(x_3) \rangle + \langle \underline{x}_1 | m_2(x_2, x_3) \rangle = 0. \end{aligned}$$

Now, we begin the proof of the lemma. From now on, we replace m_k^A by m_k if there is no ambiguity. By taking a derivative, the expression becomes:

$$\frac{\partial}{\partial x_i} \sum_{p+q+k=N}^{\infty} \langle \mathbf{x}, \mathbf{x}, \dots, \mathbf{x}, \underline{m_k^A(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})}, \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle_{p,q} \quad (3.6)$$

$$= \sum_{\substack{p+q+k=N \\ r+s=k-1}} \langle \mathbf{x}, \dots, \mathbf{x}, \overbrace{m_k(\mathbf{x}, \dots, \mathbf{x}, \underline{e_i}, \mathbf{x}, \dots, \mathbf{x})}^{r,s}, \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle_{p,q} \quad (3.7)$$

$$+ \sum_{\substack{p+q+k=N \\ r+s=p-1}} \langle \underbrace{\mathbf{x}, \dots, \mathbf{x}}_r, \underline{e_i}, \underbrace{\mathbf{x}, \dots, \mathbf{x}}_s, \underline{m_k(\mathbf{x}, \dots, \mathbf{x})}, \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle_{p,q} \quad (3.8)$$

$$+ \sum_{\substack{p+q+k=N \\ r+s=q-1}} \langle \mathbf{x}, \dots, \mathbf{x}, \underline{m_k(\mathbf{x}, \dots, \mathbf{x})}, \underbrace{\mathbf{x}, \dots, \mathbf{x}}_r, \underline{e_i}, \underbrace{\mathbf{x}, \dots, \mathbf{x}}_s | \mathbf{x} \rangle_{p,q} \quad (3.9)$$

$$+ \sum_{p+q+k=N} \langle \mathbf{x}, \dots, \mathbf{x}, \underline{m_k(\mathbf{x}, \dots, \mathbf{x})}, \mathbf{x}, \dots, \mathbf{x} | e_i \rangle_{p,q}. \quad (3.10)$$

Now, the lemma can be proved by the following lemma. \square

Lemma 3.3. *The sum of the terms in (3.7), (3.8) and (3.9) equals to N times of the expression (3.10)*

Proof. To prove the lemma, we recall the notion of A_∞ -bimodule equation. The equation for A_∞ -bimodule homomorphism $A \rightarrow A^*$ is

$$\phi \circ \widehat{b_A} = b_{A^*} \circ \widehat{\phi} \quad (3.11)$$

with $b_A = m^A$ when A is considered to be an A_∞ -bimodule, and b_{A^*} is defined by canonical construction of the dual of the A_∞ -bimodule A . Here $\widehat{\phi}$ is the coalgebra homomorphism induced from ϕ (We refer readers to [C],[T] or [GJ] for details). Let us restrict the equation (3.11) to the case $(\mathbf{x}, \dots, \mathbf{x}, \underline{e_i}, \mathbf{x}, \dots, \mathbf{x}) \in A^{\otimes n} \otimes \underline{A} \otimes A^{\otimes m}$ where $n + m + 1 = N$. Then it becomes

$$\sum_{\substack{p+j_1=n \\ j_2+q=m}} \langle \mathbf{x}, \dots, \mathbf{x}, \underline{m_{j_1+j_2+1}(\overbrace{\mathbf{x}, \dots, \mathbf{x}}^{j_1}, \underline{e_i}, \overbrace{\mathbf{x}, \dots, \mathbf{x}}^{j_2})}, \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle_{p,q} \quad (3.12)$$

$$+ \sum_{\substack{k_1+k_2+j=n \\ p=k_1+k_2+1}} \langle \underbrace{\mathbf{x}, \dots, \mathbf{x}}_{k_1}, \underline{m_j(\mathbf{x}, \dots, \mathbf{x})}, \underbrace{\mathbf{x}, \dots, \mathbf{x}}_{k_2}, \underline{e_i}, \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle_{p,m}^{dum} \quad (3.13)$$

$$+ \sum_{\substack{l_1+l_2+h=m \\ q=l_1+l_2+1}} \langle \mathbf{x}, \dots, \mathbf{x}, \underline{e_i}, \underbrace{\mathbf{x}, \dots, \mathbf{x}}_{l_1}, \underline{m_h(\mathbf{x}, \dots, \mathbf{x})}, \underbrace{\mathbf{x}, \dots, \mathbf{x}}_{l_2} | \mathbf{x} \rangle_{n,q}^{dum} \quad (3.14)$$

$$= \sum_{\substack{p+k_1=m \\ k_2+q=n}} \langle \mathbf{x}, \dots, \mathbf{x}, \underline{m_{k_1+k_2+1}(\overbrace{\mathbf{x}, \dots, \mathbf{x}}^{k_1}, \underline{\mathbf{x}}, \overbrace{\mathbf{x}, \dots, \mathbf{x}}^{k_2})}, \mathbf{x}, \dots, \mathbf{x} | e_i \rangle_{p,q} \quad (3.15)$$

Remark 3.2. It is important to note that the expression in the summand (3.15) is obtained in $k := k_1 + k_2 + 1$ different ways according to the position of the (underlined) bimodule element $\underline{\mathbf{x}}$. Namely, different choices of a bimodule element still give rise to equivalent expressions.

Remark 3.3. Here the terms (3.12) and (3.15) in the above A_∞ -bimodule equation do appear in the process of derivation (3.6) but the terms (3.13) and (3.14) do not appear in (3.6). Hence we marked them as $<, >^{dum}$ for reader's convenience to indicate that they are dummy parts. We will show how all the dummy parts are canceled out or used in the subsequent process.

We say an expression such as in (3.12), \dots , (3.15) to be of (n, m) -type as it is obtained from the input $A^{\otimes n} \otimes \underline{A} \otimes A^{\otimes m}$. And for convenience, we will denote the summands as in (3.12), \dots , (3.15) to be $\sum_{(n, m)\text{-type}}$ instead of writing down specific conditions.

Note that the expression (3.12) equals (3.7) and (3.15) provides k times the expression (3.10) from the remark 3.3. Hence, we may use the above A_∞ -bimodule equation to turn (3.7) into k -times (3.10) together with dummy terms. Hence, to prove the Lemma 3.3, we need to find $N - k$ times the expression (3.10) from what are left out in (3.6) together with the new dummy terms.

Now, we explain the dummy expressions we add to the equation. We set $m_j(\vec{x}) := m_j(x, \dots, x)$ just for simplicity. The following are the dummy expressions to be added to the expression (3.6).

$$\begin{aligned}
(i) \quad & \sum_{p+j+k_1+k_2+1=N} < \mathbf{x}^{\otimes p}, m_j(\vec{x}), \mathbf{x}^{\otimes k_1}, \underline{e}_i, \mathbf{x}^{\otimes k_2} | \mathbf{x} >^{dum}, \\
(ii) \quad & \sum_{p+j+k_1+k_2+k_3+2=N} < \mathbf{x}^{\otimes p}, e_i, \mathbf{x}^{\otimes k_1}, m_j(\vec{x}), \mathbf{x}^{\otimes k_2}, \underline{x}, \mathbf{x}^{\otimes k_3} | \mathbf{x} >^{dum}, \\
(iii) \quad & \sum_{p+j+k_1+k_2+k_3+2=N} < \mathbf{x}^{\otimes p}, m_j(\vec{x}), \mathbf{x}^{\otimes k_1}, e_i, \mathbf{x}^{\otimes k_2}, \underline{x}, \mathbf{x}^{\otimes k_3} | \mathbf{x} >^{dum}, \\
(iv) \quad & \sum_{p+j+k_1+k_2+k_3+2=N} < \mathbf{x}^{\otimes p}, e_i, \mathbf{x}^{\otimes k_1}, \underline{x}, \mathbf{x}^{\otimes k_2}, m_j(\vec{x}), \mathbf{x}^{\otimes k_3} | \mathbf{x} >^{dum}, \\
(v) \quad & \sum_{p+k_1+k_2+l_1+l_2+2=N} < \mathbf{x}^{\otimes p}, m_j(\mathbf{x}^{\otimes l_1}, e_i, \mathbf{x}^{\otimes l_2}), \mathbf{x}^{\otimes k_1}, \underline{x}, \mathbf{x}^{\otimes k_2} | \mathbf{x} >^{dum}, \\
(vi) \quad & \sum_{p+j+k_1+k_2+1=N} < \mathbf{x}^{\otimes p}, m_j(\vec{x}), \mathbf{x}^{\otimes k_1}, \underline{x}, \mathbf{x}^{\otimes k_2} | e_i >^{dum}, \\
(vii) \quad & \sum_{p+j+k_1+k_2+1=N} < \mathbf{x}^{\otimes p}, \underline{e}_i, \mathbf{x}^{\otimes k_1}, m_j(\vec{x}), \mathbf{x}^{\otimes k_2} | \mathbf{x} >^{dum}, \\
(viii) \quad & \sum_{p+j+k_1+k_2+k_3+2=N} < \mathbf{x}^{\otimes p}, \underline{x}, \mathbf{x}^{\otimes k_1}, e_i, \mathbf{x}^{\otimes k_2}, m_j(\vec{x}), \mathbf{x}^{\otimes k_3} | \mathbf{x} >^{dum}, \\
(ix) \quad & \sum_{p+j+k_1+k_2+k_3+2=N} < \mathbf{x}^{\otimes p}, \underline{x}, \mathbf{x}^{\otimes k_1}, m_j(\vec{x}), \mathbf{x}^{\otimes k_2}, e_i, \mathbf{x}^{\otimes k_3} | \mathbf{x} >^{dum}, \\
(x) \quad & \sum_{p+j+k_1+k_2+k_3+2=N} < \mathbf{x}^{\otimes p}, m_j(\vec{x}), \mathbf{x}^{\otimes k_1}, \underline{x}, \mathbf{x}^{\otimes k_2}, e_i, \mathbf{x}^{\otimes k_3} | \mathbf{x} >^{dum}, \\
(xi) \quad & \sum_{p+k_1+k_2+l_1+l_2+2=N} < \mathbf{x}^{\otimes p}, \underline{x}, \mathbf{x}^{\otimes k_1}, m_j(\mathbf{x}^{\otimes l_1}, e_i, \mathbf{x}^{\otimes l_2}), \mathbf{x}^{\otimes k_2} | \mathbf{x} >^{dum}, \\
(xii) \quad & \sum_{p+j+k_1+k_2+1=N} < \mathbf{x}^{\otimes p}, \underline{x}, \mathbf{x}^{\otimes k_1}, m_j(\vec{x}), \mathbf{x}^{\otimes k_2} | e_i >^{dum},
\end{aligned}$$

The following is easy to check.

Lemma 3.4. *By applying skew symmetry condition, we have*

$$\begin{aligned}
(i) + (xii) &= 0, & (ii) + (ix) &= 0, & (iii) + (viii) &= 0, \\
(iv) + (x) &= 0, & (v) + (xi) &= 0, & (vi) + (vii) &= 0
\end{aligned}$$

Hence, the overall sum also vanishes:

$$(i) + (ii) + \dots + (xi) + (xii) = 0.$$

Hence we add all the dummy terms above to the expression (3.6) without changing the value. Notice that the expression (i) and (vii) already appeared in (3.13) and (3.14) and is used to turn (3.7) into k times (3.10).

In addition we need the following A_∞ -bimodule equation (3.11) which is obtained considering the case that the input e_i is *not* the bimodule element of the expression: Given $m, n \in \mathbb{N}$ with $n + m = N - 1$, we have

$$\begin{aligned}
& \sum_{(n,m)\text{-type}} (\langle \mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x}, m_j(\vec{\mathbf{x}}), \mathbf{x}, \dots, \mathbf{x}, \underline{\mathbf{x}}, \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle^{dum}) \quad (3.16) \\
& + \langle \mathbf{x}, \dots, \mathbf{x}, m_j(\vec{\mathbf{x}}), \mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x}, \underline{\mathbf{x}}, \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle^{dum} \quad (3.17) \\
& + \langle \mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x}, \underline{\mathbf{x}}, \mathbf{x}, \dots, \mathbf{x}, m_j(\vec{\mathbf{x}}), \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle^{dum} \quad (3.18) \\
& + \langle \mathbf{x}, \dots, \mathbf{x}, m_j(\mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x}), \mathbf{x}, \dots, \mathbf{x}, \underline{\mathbf{x}}, \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle^{dum} \quad (3.19) \\
& + \langle \mathbf{x}, \dots, \mathbf{x}, \underline{\mathbf{x}}, \mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x}, m_j(\vec{\mathbf{x}}), \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle^{dum} \quad (3.20) \\
& + \langle \mathbf{x}, \dots, \mathbf{x}, \underline{\mathbf{x}}, \mathbf{x}, \dots, \mathbf{x}, m_j(\vec{\mathbf{x}}), \mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle^{dum} \quad (3.21) \\
& + \langle \mathbf{x}, \dots, \mathbf{x}, m_j(\vec{\mathbf{x}}), \mathbf{x}, \dots, \mathbf{x}, \underline{\mathbf{x}}, \mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle^{dum} \quad (3.22) \\
& + \langle \mathbf{x}, \dots, \mathbf{x}, \underline{\mathbf{x}}, \mathbf{x}, \dots, \mathbf{x}, m_j(\mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x}), \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle^{dum} \quad (3.23)
\end{aligned}$$

$$\begin{aligned}
& = - \sum_{(n,m)\text{-type}} (\langle \mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \underline{\mathbf{x}}, \dots, \mathbf{x} | m_j(\mathbf{x}, \dots, \underline{\mathbf{x}}, \dots, \mathbf{x}) \rangle) \quad (3.24) \\
& + \langle \mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x}, \underline{m_j(\mathbf{x}, \dots, \underline{\mathbf{x}}, \dots, \mathbf{x})}, \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle \quad (3.25) \\
& + \langle \mathbf{x}, \dots, \mathbf{x}, \underline{m_j(\mathbf{x}, \dots, \underline{\mathbf{x}}, \dots, \mathbf{x})}, \mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle) \quad (3.26) \\
& + \langle \mathbf{x}, \dots, \underline{\mathbf{x}}, \mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x} | m_j(\mathbf{x}, \dots, \underline{\mathbf{x}}, \dots, \mathbf{x}) \rangle \quad (3.27) \\
& + \langle \mathbf{x}, \dots, \mathbf{x}, \underline{\mathbf{x}}, \mathbf{x}, \dots, \mathbf{x} | m_j(\mathbf{x}, \dots, \underline{\mathbf{x}}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x}) \rangle \quad (3.28) \\
& + \langle \mathbf{x}, \dots, \mathbf{x}, \underline{\mathbf{x}}, \mathbf{x}, \dots, \mathbf{x} | m_j(\mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \underline{\mathbf{x}}, \dots, \mathbf{x}) \rangle \quad (3.29) \\
& + \langle \mathbf{x}, \dots, \mathbf{x}, \underline{m_j(\mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \underline{\mathbf{x}}, \dots, \mathbf{x})}, \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle \quad (3.30) \\
& + \langle \mathbf{x}, \dots, \mathbf{x}, \underline{m_j(\mathbf{x}, \dots, \underline{\mathbf{x}}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x})}, \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle) \quad (3.31)
\end{aligned}$$

In the case that $n = m$, we consider only the above equation, but if $n \neq m$, then we also consider the similar equation for (m, n) -type also. Observe the following facts.

- For later purposes, we have separated certain dummy expressions in the LHS.
- Some of dummy expressions already appeared before.

$$(3.16) \sim (ii), (3.17) \sim (iii), (3.18) \sim (iv), (3.19) \sim (v),$$

$$(3.20) \sim (viii), (3.21) \sim (ix), (3.22) \sim (x), (3.23) \sim (xi).$$

- As (i) , (vii) has been used in (3.13), (3.14), the dummy expressions (among the list above the Lemma 3.4 which has not used elsewhere (yet) are (vi) and (xii)).
- Note the equivalence in (3.30) and (3.31) if the number of \mathbf{x} ahead of e_i in the following expression are the same:

$$\underline{m_j(\mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x}, \underline{\mathbf{x}}, \mathbf{x}, \dots, \mathbf{x})} = \underline{m_j(\mathbf{x}, \dots, \mathbf{x}, \underline{\mathbf{x}}, \mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x})}$$

Now, we show that in the right hand side of the equation, all the terms cancel out from skew symmetry. In the case that $n = m$, we have the cancellations (from skew symmetry)

$$(3.24) + (3.26) = 0, (3.27) + (3.25) = 0, (3.28) + (3.30) = 0, (3.29) + (3.31) = 0.$$

Hence we may consider the case that $n \neq m$. In this case, there are similar cancellations between (n, m) -type and (m, n) -type terms. Namely, if we use subscript (n, m) , (m, n) to denote (n, m) -type and (m, n) -type, we have

$$(3.24)_{(n, m)} + (3.26)_{(m, n)} = 0, \quad (3.24)_{(m, n)} + (3.26)_{(n, m)} = 0,$$

$$(3.27)_{(n, m)} + (3.25)_{(m, n)} = 0, \quad (3.27)_{(m, n)} + (3.25)_{(n, m)} = 0,$$

$$(3.28)_{(m, n)} + (3.30)_{(n, m)} = 0, \quad (3.28)_{(n, m)} + (3.30)_{(m, n)} = 0,$$

$$(3.29)_{(m, n)} + (3.31)_{(n, m)} = 0, \quad (3.29)_{(n, m)} + (3.31)_{(m, n)} = 0.$$

Hence, the right hand side always vanishes.) Consequently, if we collect all the remaining terms, there are (3.8) and (3.9) and (vi) and (xii). Now, we show that the addition of these expressions produce $N - k$ times (3.9), which proves the theorem. Let us list the remaining terms first.

$$(3.9) \quad \sum_{p+k+j_1+j_2+1=N} \langle \mathbf{x}^{\otimes p}, \underline{m_k(\vec{x})}, \mathbf{x}^{\otimes j_1}, e_i, \mathbf{x}^{\otimes j_2} | \mathbf{x} \rangle, \quad (3.32)$$

$$(3.8) \quad \sum_{p+k+j_1+j_2+1=N} \langle \mathbf{x}^{\otimes p}, e_i, \mathbf{x}^{\otimes j_1}, \underline{m_k(\vec{x})}, \mathbf{x}^{\otimes j_2} | \mathbf{x} \rangle, \quad (3.33)$$

$$(vi) \quad \sum_{p+k+j_1+j_2+1=N} \langle \mathbf{x}^{\otimes p}, m_k(\vec{x}), \mathbf{x}^{\otimes j_1}, \underline{\mathbf{x}}, \mathbf{x}^{\otimes j_2} | e_i \rangle, \quad (3.34)$$

$$(xii) \quad \sum_{p+k+j_1+j_2+1=N} \langle \mathbf{x}^{\otimes p}, \underline{\mathbf{x}}, \mathbf{x}^{\otimes j_1}, m_k(\vec{x}), \mathbf{x}^{\otimes j_2} | e_i \rangle. \quad (3.35)$$

Now we use closed condition with these terms to obtain (3.10).

- (1) By applying the closed condition in the theorem 2.1 to (3.9) and (xii), we obtain (here (a_i, a_j, a_k) corresponds to $(e_i, m_k(\vec{x}), \mathbf{x})$)

$$\begin{aligned} & \langle \underbrace{\mathbf{x}, \dots, \mathbf{x}, \underline{\mathbf{x}}, \mathbf{x}, \dots, \mathbf{x}}_s, m_k(\vec{x}), \mathbf{x}^{\otimes r} | e_i \rangle \\ & + \langle \mathbf{x}, \dots, \mathbf{x}, m_k(\vec{x}), \mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle \\ & + \langle \mathbf{x}^{\otimes r}, \underline{e_i}, \mathbf{x}^{\otimes s} | m_k(\vec{x}) \rangle = 0 \end{aligned}$$

In fact, we obtain s different such equations depending on the position of $\underline{\mathbf{x}}$ in the first line. Hence, the sum of expressions (3.9) and (xii) produces s times that of (3.10) as the last term equals the minus of (3.10):

$$\langle \mathbf{x}^{\otimes r}, \underline{e_i}, \mathbf{x}^{\otimes s} | m_k(\vec{x}) \rangle = - \langle \mathbf{x}^{\otimes s}, \underline{m_k(\vec{x})}, \mathbf{x}^{\otimes r} | \underline{e_i} \rangle$$

- (2) Similarly by applying the closed condition to (3.8) and (vi),

$$\begin{aligned} & \langle \mathbf{x}^{\otimes s}, m_k(\vec{x}), \underbrace{\mathbf{x}, \dots, \mathbf{x}, \underline{\mathbf{x}}, \mathbf{x}, \dots, \mathbf{x}}_r | e_i \rangle \\ & + \langle \mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x}, \underline{m_k(\vec{x})}, \mathbf{x}, \dots, \mathbf{x} | \mathbf{x} \rangle \\ & + \langle \mathbf{x}^{\otimes r}, \underline{e_i}, \mathbf{x}^{\otimes s} | m_k(\vec{x}) \rangle \\ & = 0. \end{aligned}$$

we obtain r different such equations depending on the position of $\underline{\mathbf{x}}$ in the first line.

Hence we obtain $r + s = N - k$ times the expression of (3.10), which proves the lemma 3.3. \square

4. POTENTIAL Ψ AND THE GENERALIZED HOLONOMY MAP

In this section, we consider another potential Ψ defined in the definition 1.4 for a unital homotopy cyclic A_∞ -algebra. We discuss its gauge invariance and its relationship with the algebraic analogue of generalized holonomy map in [ATZ].

Let us first recall the definition of a unit for A_∞ -algebra.

Definition 4.1. An element $I \in C^0 = C^{-1}[1]$ is called a unit if

$$\begin{cases} m_{k+1}(x_1, \dots, I, \dots, x_k) = 0 \text{ for } k \geq 2 \text{ or } k = 0 \\ m_2(I, x) = (-1)^{\deg x} m_2(x, I) = x. \end{cases} \quad (4.1)$$

We assume that the strong homotopy inner product $\phi : A \rightarrow A^*$ is an unital A_∞ -bimodule map, or $\phi_{k,l}(\vec{a}, v, \vec{b})(w)$ vanishes if one of a_i 's or b_i 's is a constant multiple of I .

We also recall the Maurer-Cartan elements and its gauge equivalences.

Definition 4.2. Let A be an A_∞ -algebra. An element $b \in A^1$ satisfying $m(e^b) = \sum_k m_k(a, \dots, a) = 0$ is called the Maurer-Cartan elements and we denote by $MC(A)$ the set of all Maurer-Cartan elements. Let $\mathcal{MC} := MC / \sim$ be the moduli space of Maurer-Cartan elements, whose gauge equivalence is defined as follows (definition 2.3 of [Fu]): b is *gauge equivalent* to \tilde{b} if there are one-parameter families $b(t) \in A^1[t], c(t) \in A^0[t]$ such that

- $b(0) = b, b(1) = \tilde{b}$, and
- $\frac{d}{dt}b(t) = \sum_{k \geq 1} m_k(b(t), \dots, b(t), c(t), b(t), \dots, b(t))$.

We remark that $b(t)$ is also a Maurer-Cartan element for any t (Lemma 4.3.7 of [FOOO]). Now, we prove the gauge invariance of the potential Ψ for Maurer-Cartan elements.

Proposition 4.1. *The potential $\Psi(x) = \sum_{p,q \geq 0} \frac{1}{p+q+1} \langle x^{\otimes p} \otimes \underline{x} \otimes x^{\otimes q} | I \rangle$ when restricted to the Maurer-Cartan elements MC is invariant under gauge equivalences. i.e. if $x(t)$ is a one-parameter family in the Maurer-Cartan solution space, then*

$$\frac{d}{dt} \Psi(x(t)) = 0.$$

Proof. We prove this proposition with the help of following lemmas.

Lemma 4.2. $\Psi(x)$ equals the following expression. $\Psi(x) = \sum_{k \geq 0} \langle \underline{x} \otimes x^{\otimes k} | I \rangle$.

Proof. By the closedness condition of ϕ , for any p and q we have

$$\begin{aligned} \langle x^{\otimes p} \otimes \underline{x} \otimes x^{\otimes q} | I \rangle &+ \langle x^{\otimes p+q} \otimes \underline{I} | x \rangle \\ &+ \langle x^{\otimes q} \otimes I \otimes \underline{x} \otimes x^{\otimes p-1} | x \rangle = 0. \end{aligned}$$

By definition of unital A_∞ -bimodule homomorphisms, we have

$$\langle x^{\otimes q} \otimes I \otimes \underline{x} \otimes x^{\otimes p-1} | x \rangle = 0,$$

and the above equation gives

$$\langle x^{\otimes p} \otimes \underline{x} \otimes x^{\otimes q} | I \rangle = - \langle x^{\otimes p+q} \otimes \underline{I} | x \rangle = \langle \underline{x} \otimes x^{\otimes p+q} | I \rangle,$$

where the last equality follows from the skew-symmetry of ϕ . This proves the lemma. \square

Lemma 4.3. $\sum_{\sigma \in \mathbb{Z}/n\mathbb{Z}} \langle \underline{a_{\sigma(1)}}, a_{\sigma(2)}, \dots, a_{\sigma(n-1)} | a_{\sigma(n)} \rangle = 0$.

Proof. Fix a_1, \dots, a_n and denote $[i, j] := \langle \dots, \underline{a_i}, \dots | a_j \rangle$. Then what we need to prove is

$$[1, n] + [2, 1] + \dots + [n, n-1] = 0.$$

The closedness condition of strong homotopy inner products gives

$$[i, j] + [j, k] = [i, k].$$

Hence, it follows that

$$[1, n] + [n, n-1] + \dots + [2, 1] = [1, n] + [n, 1] = 0.$$

□

Now we prove the above proposition. First, assume

$$\frac{d}{dt}x(t) = \sum_{i+j=k \geq 0} m_{k+1}(x(t)^{\otimes i} \otimes c(t) \otimes x(t)^{\otimes j}).$$

We denote x by $x(t)$ and c by $c(t)$, for it causes no problem in this proof.

Applying lemma 4.2, the fraction disappears and we get

$$\frac{d}{dt}\Psi(x) = \sum_{l \geq 0} \langle \sum_{i+j=k \geq 0} m_{k+1}(x^{\otimes i} \otimes c \otimes x^{\otimes j}) \otimes x^{\otimes l} | I \rangle \quad (4.2)$$

$$+ \sum_{l, m \geq 0} \langle \underline{x} \otimes x^{\otimes l} \otimes \sum_{i+j=k \geq 0} m_{k+1}(x^{\otimes i} \otimes c \otimes x^{\otimes j}) \otimes x^{\otimes m} | I \rangle \quad (4.3)$$

To prove that it is zero, we use the A_∞ -bimodule equation. Namely, we compute

$$(\phi \circ \widehat{m} - m^* \circ \widehat{\phi})(\sum_{l \geq 0} \underline{c} \otimes x^{\otimes l} + \sum_{l, m \geq 0} \underline{x} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m})(I),$$

which is a priori zero.

$$(\phi \circ \widehat{m})(\sum_{i \geq 0} \underline{c} \otimes x^{\otimes i})(I) = \sum_{l \geq 0} \langle \sum_{k \geq 0} m_{k+1}(c \otimes x^{\otimes k}) \otimes x^{\otimes l} | I \rangle \quad (4.4)$$

$$+ \sum_{l, m \geq 0} \langle \underline{c} \otimes x^{\otimes l} \otimes (\sum_{k \geq 1} m_k(x^{\otimes k})) \otimes x^{\otimes m} | I \rangle \quad (4.5)$$

and (4.5) is zero by Maurer-Cartan equation.

$$\begin{aligned} & (\phi \circ \widehat{m})(\sum_{i, j \geq 0} \underline{x} \otimes x^{\otimes i} \otimes c \otimes x^{\otimes j})(I) \\ &= \sum_{l, m \geq 0} \langle \sum_{k \geq 1} m_k(x^{\otimes k}) \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m} | I \rangle \end{aligned} \quad (4.6)$$

$$+ \sum_{l \geq 0} \langle \sum_{i \geq 1, j \geq 0} m_k(x^{\otimes i} \otimes c \otimes x^{\otimes j}) \otimes x^{\otimes l} | I \rangle \quad (4.7)$$

$$+ \sum_{l, m \geq 0} \langle \underline{x} \otimes x^{\otimes l} \otimes \sum_{i+j=k \geq 0} m_{k+1}(x^{\otimes i} \otimes c \otimes x^{\otimes j}) \otimes x^{\otimes m} | I \rangle \quad (4.8)$$

$$+ \sum_{l, m, n \geq 0} \langle \underline{x} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m} \otimes \sum_{k \geq 1} m_k(x^{\otimes k}) \otimes x^{\otimes n} | I \rangle. \quad (4.9)$$

Remark again, that (4.6) and (4.9) vanish by Maurer-Cartan equation. Observe also that

- (4.4)+(4.7)=(4.2),
- (4.8)=(4.3).

It remains to show that

$$(m^* \circ \hat{\phi})\left(\sum_{l \geq 0} \underline{c} \otimes x^{\otimes l} + \sum_{l, m \geq 0} \underline{x} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m}\right)(I) = 0.$$

Since I is the unit, we may easily verify that

$$(m^* \circ \phi)\left(\sum_{l \geq 0} \underline{c} \otimes x^{\otimes l}\right)(I) = \sum_{l \geq 0} \langle \underline{c} \otimes x^{\otimes l} | x \rangle, \quad (4.10)$$

$$(m^* \circ \phi)\left(\sum_{l \geq 0} \underline{x} \otimes x^{\otimes l} \otimes c\right)(I) = \sum_{l \geq 0} \langle \underline{x} \otimes x^{\otimes l} | c \rangle, \quad (4.11)$$

$$(m^* \circ \phi)\left(\sum_{l \geq 0, m \geq 1} \underline{x} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m}\right)(I) = \sum_{l, m \geq 0} \langle \underline{x} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m} | x \rangle.$$

In (4.10) and (4.11), for $l = 0$, we have

$$\langle c | x \rangle + \langle x | c \rangle = 0$$

by skew-symmetry. For remaining parts, we collect terms appropriately and use closedness condition to show that they all vanish. More precisely, for $k \geq 1$, we claim that

$$\langle \underline{c} \otimes x^{\otimes k} | x \rangle + \langle \underline{x} \otimes x^{\otimes k} | c \rangle + \sum_{l+m=k-1} \langle \underline{x} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m} | x \rangle = 0$$

But this follows from the previous lemma 4.3, by setting $a_1 = c, a_2 = \dots = a_{k+2} = x$. \square

Now, we discuss the potential Ψ and the algebraic generalized holonomy map. We refer readers to [ATZ] or [CL] for the relevant definitions of this construction.

First, recall from Proposition 6.1 of [CL] that given a negative cyclic cohomology class α of an A_∞ -algebra A , one obtains a bimodule map $\tilde{\alpha} : A \rightarrow A^*$. This provides a strong homotopy inner product, if α is in addition homologically non-degenerate. The definition 1.4 thus provides the potential Ψ^α using α . Combined with the above proposition, we prove

Theorem 4.4. *The potential Ψ provides a map $\Psi : HC_-^\bullet(A) \rightarrow \mathcal{O}(\mathcal{MC})$ defined by $\alpha \mapsto \Psi^\alpha|_{MC}$. Furthermore, this agrees with the algebraic analogue of generalized holonomy map of Abbaspour, Tradler and Zeinalian [ATZ].*

Proof. We only need to prove the relation with that of [ATZ] and we recall the construction of a map $\rho : HC_-^\bullet(A) \rightarrow \mathcal{O}(\mathcal{MC})$. Here we always work with reduced versions of negative cyclic or Hochschild (co)homologies.

Given a Maurer-Cartan element a of a unital A_∞ -algebra A , consider the expression (Definition 8 of [ATZ])

$$P(a) := \sum_{i \geq 0} I \otimes a^{\otimes i} = (I \otimes I) + (I \otimes a) + (I \otimes a \otimes a) + \dots$$

One can check that $P(a)$ is a Hochschild homology cycle from the unital property of I and the Maurer-Cartan equation. Note that Connes-Tsygan operator B of $P(a)$

vanishes on the reduced complex, due to the unit I . Hence, $P(a)$ can be considered as a negative cyclic homology cycle.

Hence, given a negative cyclic cohomology cycle $\alpha \in HC_-^\bullet(A)$, one can use the pairing $\langle, \rangle: HC_-^\bullet(A) \otimes HC_-^\bullet(A) \rightarrow k$ to define the map ρ as

$$\rho([a])([a]) := \langle \alpha, \sum_{i \geq 0} I \otimes a^{\otimes i} \rangle \quad (4.12)$$

Now, we compare the above expression with that of Lemma 4.2. We recall the following proposition from [CL].

Proposition 4.5 (Proposition 6.1 [CL]). *Let $\alpha \in C_{red}^\bullet(A, A^*)$ be a negative cyclic cocycle. We define*

$$\widetilde{\alpha}_0(\vec{a}, \underline{v}, \vec{b})(w) := \alpha_0(\vec{a}, v, \vec{b})(w) - \alpha_0(\vec{b}, w, \vec{a})(v).$$

Then $\widetilde{\alpha}_0$ is an A_∞ -bimodule map from A to A^ , satisfying the skew-symmetry and closedness condition.*

Here α_0 is the part of α which is dual to the inclusion of Hochschild homology cycles. Also, from the unital property, we have

$$\widetilde{\alpha}_0(\underline{a}, a, \dots, a)(I) = \alpha_0(a, \dots, a)(I) - \alpha_0(a, \dots, a, I)(a) = \alpha_0(a, \dots, a)(I)$$

Hence,

$$\langle \alpha, I \otimes a^{\otimes i} \rangle = \langle \alpha_0, I \otimes a^{\otimes i} \rangle = \alpha_0(a, \dots, a)(I) = \widetilde{\alpha}_0(\underline{a}, a, \dots, a)(I) = \langle \underline{a}, a, \dots, a | I \rangle$$

where the equality in the middle follows from the identification

$$\text{Hom}(A \otimes (A[1]/k \cdot 1)^{\otimes n}, k) \cong \text{Hom}((A[1]/k \cdot 1)^{\otimes n}, A^*).$$

Hence, each term of the function ρ of [ATZ] equals the potential Ψ in the paper given in the Lemma 4.2. This proves the theorem. \square

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