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Examples of Matrix Factorizations from SYZ*

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Abstract. We find matrix factorization corresponding to an anti-diagonal in $\mathbb{C}P^1 \times \mathbb{C}P^1$, and circle fibers in weighted projective lines using the idea of Chan and Leung of Strominger–Yau–Zaslow transformations. For the tear drop orbifolds, we apply this idea to find matrix factorizations for two types of potential, the usual Hori–Vafa potential or the bulk deformed (orbi)-potential. We also show that the direct sum of anti-diagonal with its shift, is equivalent to the direct sum of central torus fibers with holonomy $(1,-1)$ and $(-1,1)$ in the Fukaya category of $\mathbb{C}P^1 \times \mathbb{C}P^1$, which was predicted by Kapustin and Li from B-model calculations.

Key words: matrix factorization; Fukaya category; mirror symmetry; Lagrangian Floer theory

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1 Introduction

The Strominger–Yau–Zaslow (SYZ for short) [21] conjecture provides a geometric way to understand mirror symmetry phenomenons. Recently, Chan and Leung [4] have shown that in $\mathbb{C}P^1$ the Lagrangian Floer chain complex between the equator and a generic Lagrangian torus fiber corresponds by SYZ to the matrix factorization of the Landau–Ginzburg (LG for short) superpotential W. The general idea is as follows. To find a matrix factorization corresponding to a Lagrangian submanifold, say L_0, they consider a family of Floer chain complex of the pair (L_u, L_0) for all torus fibers L_u (with all possible holonomies), and use the information of holomorphic strips (which varies as L_u changes) and apply SYZ transformation to construct the matrix factorization for L_0 (see Section 2 for more details).

Their observation is very insightful to understand the homological mirror symmetry between Lagrangian Floer theory of toric manifolds and matrix factorization of LG superpotential W, but the procedure is known to work only for a $\mathbb{C}P^1$ (or a product of $\mathbb{C}P^1$ with Lagrangian submanifold given by product of equators). They also found the corresponding matrix factorization for $\mathbb{C}P^2$, but the description of Floer chain complex is not complete.

In this paper, we provide more evidence on this correspondence following their ideas. The first new example is the case of the anti-diagonal Lagrangian submanifold in the symplectic manifold $\mathbb{C}P^1 \times \mathbb{C}P^1$. In fact, Kapustin and Li already conjectured in [15] that the anti-diagonal should correspond to a specific matrix factorization of LG superpotential $W = x + \frac{q}{x} + y + \frac{q}{y}$, and we verify this conjecture using this procedure.

Proposition 1.1. *For $\mathbb{C}P^1 \times \mathbb{C}P^1$, the anti-diagonal Lagrangian submanifold corresponds to the following matrix factorization by SYZ transformation* (*in the sense of* [4])

$$(x+y)\left(1+\frac{q}{xy}\right) = x + \frac{q}{x} + y + \frac{q}{y}.$$

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For this, we deform generic Lagrangian torus fibers into specific forms via Hamiltonian isotopy, and analyze the Floer cohomology of the anti-diagonal, with "deformed" generic torus fibers, and apply Chan and Leung's SYZ transformation to find the corresponding matrix factorization.

From B-model calculations, Kapustin and Li further conjectured in [15] that in the Fukaya category, the direct sum of anti-diagonal A and its shift $A[1]$, is isomorphic to the direct sum of two fibers $T_{1,-1}$ and $T_{-1,1}$ of holonomy $(1,-1)$ and $(-1,1)$ respectively. We also verify this conjecture by computing the Floer cohomology and products between these objects, and finding a homomorphism which induces this isomorphism.

Theorem 1.2 (Theorem 5.11)**.** *In the derived Fukaya category of $\mathbb{C}P^1 \times \mathbb{C}P^1$, $A \oplus A[1]$ is equivalent to $T_{1,-1} \oplus T_{-1,1}$.*

Namely, $\mathbb{C}P^1 \times \mathbb{C}P^1$ has four Lagrangian torus fibers, whose Floer cohomology groups are non-vanishing. It is given by the central fiber T^2, with holonomy $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$. Central fiber with holonomy $(1,-1)$ (or $(-1,1)$), which we denote by $T_{1,-1}$ (or $T_{-1,1}$) has vanishing m_0, and hence is unobstructed. The anti-diagonal Lagrangian submanifold A, is monotone Lagrangian submanifold of minimal Maslov index 4, hence unobstructed. Hence the Lagrangian Floer cohomology among $\{T_{1,-1}, T_{-1,1}, A, A[1]\}$ can be defined, where $A[1]$ is regarded as an object of (derived) Fukaya category. In Section 5, we compute the Floer cohomology between these objects as well as several m_2 products between them to verify the conjecture.

Our second type of examples are weighted projective lines, which are toric orbifolds. There is an interesting new phenomenon due to *bulk deformation by twisted sectors of toric orbifolds*. First of all, these weighted $\mathbb{C}P^1$'s have Landau–Ginzburg mirror superpotential $W : \mathbb{C}^* \to \mathbb{C}$, and the first author and Poddar has recently developed a Lagrangian Floer cohomology theory for toric orbifolds, and superpotential W can be defined from the data of smooth holomorphic discs in toric orbifolds. We consider the Floer chain complex of a central fiber of the weighted $\mathbb{C}P^1$ and a generic torus fiber, and from this we can find the corresponding matrix factorization of W.

Proposition 1.3. *For a weighted $\mathbb{C}P^1$ with $\mathbb{Z}/m\mathbb{Z}$-singularity on the left and $\mathbb{Z}/n\mathbb{Z}$ on the right, the central fiber corresponds to the following matrix factorization by SYZ transformation:*

$$\left(1-\frac{z}{\alpha q}\right)\left(\sum_{k=0}^{n}\frac{q^{\frac{m}{n}k}}{\alpha^k}\left(\frac{q^{\frac{m+n}{n}}}{z}\right)^{n-k}-\sum_{k=1}^{m}\alpha^k q^k z^{m-k}\right)=z^m+\frac{q^{m+n}}{z^n}-\left(\alpha^m q^m+\frac{q^m}{\alpha^n}\right).$$

(See Sections 6 and 7.)

Then, we can turn on bulk deformation $\mathfrak{b}$ by twisted sectors to obtain a bulk deformed mirror superpotential $W^{\mathfrak{b}}$. This potential has additional terms from the data of orbifold holomorphic discs in toric orbifolds. Once bulk deformation $\mathfrak{b}$ is chosen (so that there is a torus fiber L whose Floer cohomology is non-vanishing), then we consider the Floer chain complex of L with a generic torus fiber to find a matrix factorization of bulk deformed LG superpotential $W^{\mathfrak{b}}$. In this case, we find the corresponding matrix factorization of $W^{\mathfrak{b}}$ by additionally considering orbifold holomorphic strips.

2 Preliminaries

We recall Strominger–Yau–Zaslow conjecture briefly. The classical form of mirror symmetry considers mirror pairs of Calabi–Yau 3-folds X and $\check{X}$, and the symplectic geometry (Gromov–Witten invariants) of X corresponds to complex geometry (periods) of $\check{X}$. The SYZ conjecture is, roughly speaking, a geometric tool to find the mirror manifold, as a dual torus fibration. We state the conjecture in the following form from [14] (see also [3]):

Conjecture 2.1 ([21]). *If two Calabi–Yau n-folds X and $\check{X}$ are mirror to each other, then there exist special Lagrangian fibrations $f : X \to B$ and $\check{f} : \check{X} \to B$, whose generic fibers are tori. Furthermore, these fibrations are dual, namely $X_b = H^1(\check{X}_b, \mathbb{R}/\mathbb{Z})$ and*

$$\check{X}_b = H^1(X_b, \mathbb{R}/\mathbb{Z}),$$

when X_b and $\check{X}_b$ are nonsingular torus fibers over $b \in B$.

Toric Fano manifolds X, which are torus fibrations over the moment polytopes, has a mirror given by a Landau–Ginzburg model $(\check{X}, W)$. Torus fibers become singular over the facets of the moment polytope, and the singularity of the fibration is measured by the Landau–Ginzburg superpotential W, which can be constructed from the Maslov index two holomorphic discs in X with boundary on torus fibers [9, 13]. The homological mirror symmetry due to Kontsevich (in this setting) asserts that the derived Fukaya category $\mathrm{DFuk}(X)$ of a toric Fano manifold X is equivalent, as a triangulated category, to the category of matrix factorizations $MF(\check{X}, W)$ of the mirror Landau–Ginzburg model $(\check{X}, W)$. The latter category is equivalent to the category of singularites $D_{\mathrm{Sg}}(\check{X}, W)$ (see Orlov [19]).

A matrix factorization of a Landau–Ginzburg model $(\check{X}, W)$ is a square matrix M of even dimensions with entries in the coordinate ring $\mathbb{C}[\check{X}]$ and of the form

$$M = \begin{pmatrix} 0 & F \\ G & 0 \end{pmatrix},$$

such that

$$M^2 = (W - \lambda)\mathrm{Id}$$

for some $\lambda \in \mathbb{C}$. It is well-known that M is a non-trivial element of $MF(\check{X}, W)$ only if λ is a critical value of W (see [19]).

The idea of Chan and Leung will be explained in more detail in the next section, but we first explain the Lagrangian Floer theory behind this correspondence. Let L_0, L_1 be a Lagrangian submanifold in a symplectic manifold (X, ω). Let J be a compatible almost complex structure. One considers J-holomorphic discs $u : (D^2, \partial D^2) \to (X, L)$ with Lagrangian boundary conditions, and denote by $\mathcal{M}_k(L, \beta)$ be the moduli space of such J-holomorphic discs of homotopy class $\beta \in \pi_2(X, L)$ with k boundary marked points. We denote by $\mu(\beta)$ the Maslov index of β. The dimension of the moduli space is given by $n + \mu(\beta) + k - 3$.

We further assume that L_0, L_1 are *positive* in the sense that any non-constant J-holomorphic discs have positive Maslov index. In particular, this implies that the (virtual) dimension of $\mathcal{M}_1(L, \beta)$ is always at least n if $\beta \neq 0$ and non-empty. And $\dim(\mathcal{M}_1(L, \beta)) = n$ exactly when $\mu(\beta) = 2$.

We define the Novikov ring

$$\Lambda = \left\{ \sum a_i T^{\lambda_i} \,\middle|\, a_i \in \mathbb{C},\ \lambda_i \in \mathbb{R},\ \lim_i \lambda_i = \infty \right\}.$$

The Lagrangian Floer chain complex $CF(L_0, L_1)$ is generated by intersection points $L_0 \cap L_1$ with coefficients Λ, and its differential is defined by

$$m_1(\langle p \rangle) = \sum_q n_\alpha(p, q) \langle q \rangle T^{\omega(\alpha)},$$

where the sum is over all $q \in L_0 \cap L_1$, and $n_\alpha(p, q)$ is the count of isolated J-holomorphic strips with boundary on L_0, L_1 (modulo translation action) of homotopy class α, and $\omega(\alpha)$ is the area

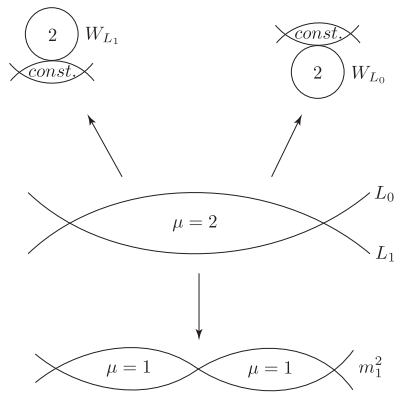


Figure 1. Degeneration of index 2 strips.

of such J-holomorphic strip. Such isolated strips have Maslov–Viterbo index one. We refer readers to [13, 17] for details.

In general, $m_1^2 \neq 0$ and hence the Lagrangian Floer cohomology cannot be defined in general. With the above positivity assumption, we have the following *Floer complex equation*

$$m_1^2 = (W_{L_1} - W_{L_0})\mathrm{Id},$$

where W_{L_i} is defined as follows: From the evaluation map $\mathrm{ev}_{0,\beta} : \mathcal{M}_1(L, \beta) \to L$ at the marked point, if $\mu(\beta) = 2$, the image of $\mathrm{ev}_{0,\beta}$ is a multiple of fundamental class $[L]$

$$\mathrm{ev}_{0,\beta}(\mathcal{M}_1(L, \beta)) = c_\beta [L]$$

as it is of dimension n, and β is of minimal Maslov index. Then we define

$$W_L := \sum_{\mu(\beta)=2} c_\beta T^{\omega(\beta)}. \tag{2.1}$$

The Floer complex equation $m_1^2 = (W_{L_1} - W_{L_0})\mathrm{Id}$, is obtained by analyzing the moduli space of holomorphic strips of Maslov–Viterbo index two (see Fig. 1). Some sequences of J-holomorphic strips of Maslov–Viterbo index two, can degenerate into broken J-holomorphic strips, each of which has index one, which contributes to m_1^2. Some sequence of J-holomorphic strips of index two can also degenerate into a constant strip together with a bubble holomorphic disc attached to either upper or lower boundary of the strip. Discs attached to upper (resp. lower) boundary contributes to W_{L_1} (resp. W_{L_0}) and it gives Id map since the J-holomorphic strip is constant.

In fact, one needs to considers Lagrangian submanifolds L_0, L_1, equipped with flat line bundles $\mathcal{L}_0 \to L_0$, $\mathcal{L}_1 \to L_1$, and the above setting can be extended to this setting. In such a case, we put an additional contribution of holonomy $\mathrm{hol}_{\mathcal{L}_i}(\partial\beta)$ for each β in (2.1).

Chan and Leung's idea is to compare the Floer complex equation and that of matrix factorization $M^2 = (W - \lambda)\mathrm{Id}$ (via their Fourier transform). For this, we take L_0 to be a fixed torus fiber (corresponding to the critical value λ) and vary L_1 as generic torus fibers with holonomy to obtain W as a function on the mirror manifold.

3 Chan–Leung's construction for $\mathbb{C}P^1$

We recall the result of Chan–Leung [4] in the case of $X = \mathbb{C}P^1$ for readers' convenience. Recall that $\check{X} = \mathbb{C}^*$, and the Landau–Ginzburg superpotential is $W = z + \frac{q}{z}$ where $q = T^t$ when $[0, t]$ is the moment polytope of X. (W can obtained from the disc potential $e^x T^u + e^{-x} T^{t-u}$ by substituting $z = e^x T^u$).

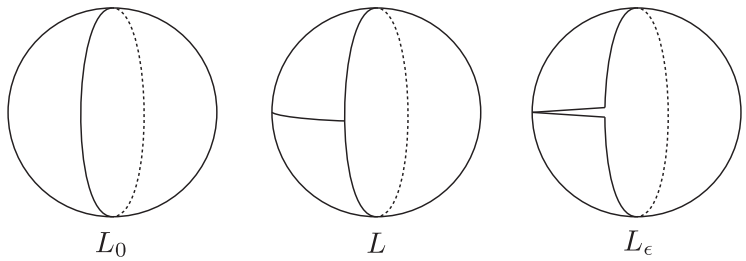


Figure 2. Hamiltonian defomation of L and the spike.

By removing north (N) and south (S) pole of X, we regard $X \setminus \{N, S\}$ as a circle fibration over $(0, t)$, and denote by u the coordinate in $(0, t)$, by y that of the fiber circle. Then the standard symplectic form ω equals $du \wedge dy$ on $X \setminus \{N, S\}$.

An equator with trivial holonomy (fiber at $t/2$) has non-trivial Floer cohomology, and it corresponds to the critical point $\sqrt{q}$ of W. By homological mirror symmetry, this should correspond to a skyscraper sheaf at the critical point, and by Orlov's result, we have a matrix factorization corresponding to it. The critical value of W is $2\sqrt{q}$ and the corresponding factorization of $W - 2\sqrt{q}$ is given in matrix form as

$$\begin{pmatrix} 0 & z - \sqrt{q} \\ 1 - \frac{\sqrt{q}}{z} & 0 \end{pmatrix}.$$

Chan–Leung's idea is to recover this matrix factorization from the geometry of torus(S^1) fibration. Let L_0 be the central fiber with trivial holonomy. We deform L_0 to $\tau : [0, 3] \to X$ as follows: (in (u, y) coordinate)

$$\tau(s) = \begin{cases} ((1-s)t/2, 0) & \text{if } 0 \le s \le 1, \\ ((s-1)t/2, 0) & \text{if } 1 \le s \le 2, \\ (t/2, 2\pi(s-2)) & \text{if } 2 \le s \le 3. \end{cases}$$

Namely, τ first goes along the zero section from center to the left pole, comes back to the center and at last turns around L_0. Let this deformation be denoted by L. Note that L still splits X into two equal halves. Since L is too singular, we slightly perturb L to L_ϵ so that it is still area bisecting, and hence Hamiltonian isotopic to central fiber. It is helpful to think of L as a limit of L_ϵ (see Fig. 2).

For each $u \in (0, t)$, we have a corresponding fiber L_u. Then L_u and L_ϵ meet at two points a and b, and there occur four holomorphic strips between them. Let $[w]$ be a homotopy class of a holomorphic strip w between a and b, namely $[w] \in \pi_2(X; L, L_u; a, b)$. Let $\partial_-[w]$ be the boundary of w on L_u. Taking the limit of L_ϵ, $\partial_-[w]$ is identified as an element of $\pi_1(L_u)$.

Now we define a function $\Psi_L^{a,b} : (0, t/2) \times \pi_1(L_u) \to \mathbb{R}$ as follows:

$$\Psi_L^{a,b}(u, [\gamma]) = \sum_{\substack{[w] \in \pi_2(X; L, L_u; a, b) \\ \partial_-(w) = [\gamma]}} \pm n([w]) \exp(\text{area}(w)) \text{hol}([\gamma]), \tag{3.1}$$

where the sign is due to the orientation of w, and $n([w])$ is the number of holomorphic discs representing $[w]$.

Note that the area of a Maslov index 2 holomorphic disc whose boundary is a toric fiber L_u is just given as u or $(t - u)$ (up to times 2π). If we identify $\mathbb{Z} \simeq \pi_1(L_u)$, then we have a complete

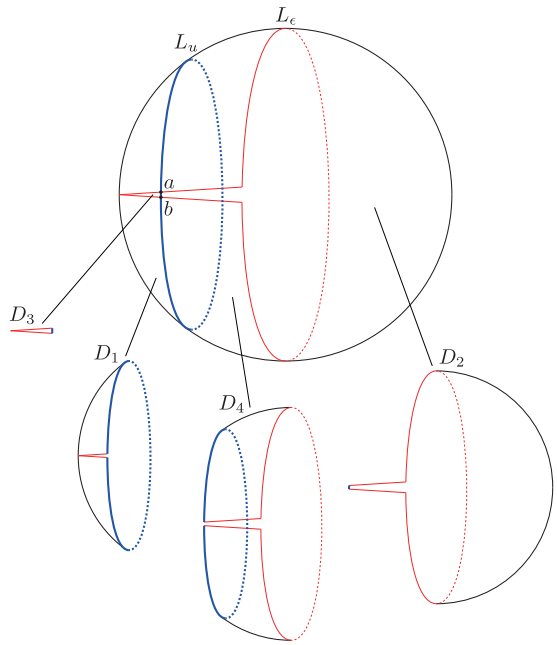


Figure 3. Disk splittings in $\mathbb{C}P^1$.

list of (3.1):

$$\Psi_L^{a,b}(u,v) = \begin{cases} \exp(u) & \text{if } v = 1, \\ -\exp(t/2) = -\sqrt{q} & \text{if } v = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\Psi_L^{b,a}(u,v) = \begin{cases} \exp(0) = 1 & \text{if } v = 0, \\ -\exp(t/2 - u) = -\dfrac{\sqrt{q}}{\exp(u)} & \text{if } v = -1, \\ 0 & \text{otherwise.} \end{cases}$$

The correspondence of the function values and holomorphic strips is given as follows. D_i are drawn in the above (Fig. 3)

$$\Psi_L^{a,b}(u,1) \longleftrightarrow D_1, \qquad \Psi_L^{a,b}(u,0) \longleftrightarrow D_2, \qquad \Psi_L^{b,a}(u,0) \longleftrightarrow D_3, \qquad \Psi_L^{b,a}(u,-1) \longleftrightarrow D_4.$$

Observe that the areas of discs are computed in the limit L.

With these functions we make a matrix-valued function Ψ_L by

$$\Psi_L(u,v) = \begin{pmatrix} 0 & \Psi_L^{a,b}(u,v) \\ \Psi_L^{b,a}(u,v) & 0 \end{pmatrix}. \tag{3.2}$$

Finally, Fourier transform of (3.2) following [5] can be obtained. Each entry of (3.2) is a function of the form $f = f_v \exp(\langle u, v\rangle)$, and for such a function f we define Fourier transform of f as

$$\hat{f} := \sum_{v\in\mathbb{Z}} f_v \exp(\langle u, v\rangle)\mathrm{hol}(v).$$

Since $\mathrm{hol}(v) = \exp(iyv)$, if we adopt a complex coordinate $z = \exp(u + iy)$, then

$$\hat{f} = \sum_{v\in\mathbb{Z}} f_v z^v.$$

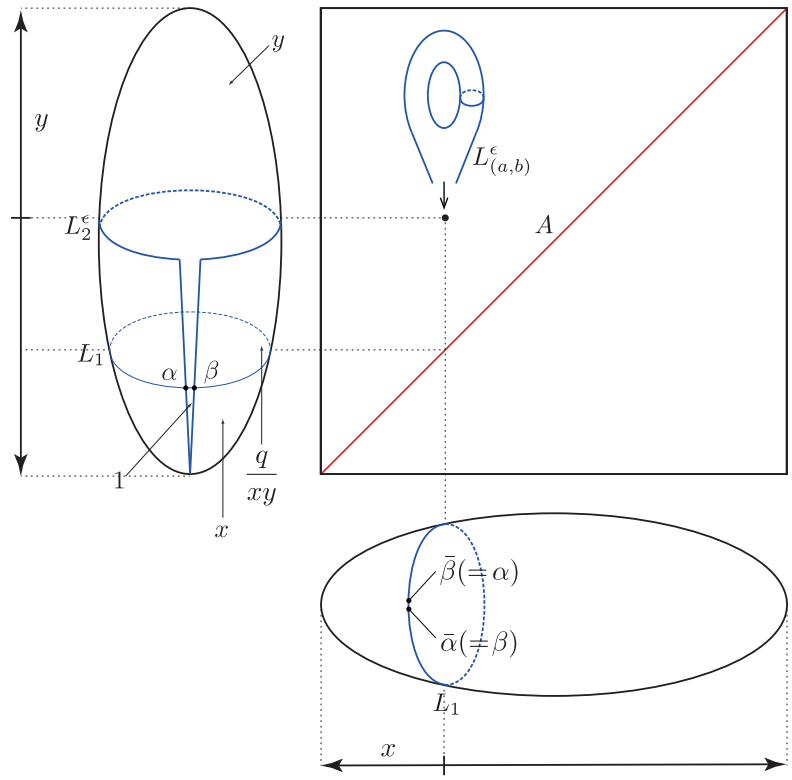


Figure 4. Intersection of antidiagonal and the deformation $L^\epsilon_{(a,b)}$ of $L_{(a,b)}$.

After the Fourier transform, we have

$$\Psi_L(z) = \begin{pmatrix} 0 & z-\sqrt{q} \\ 1-\frac{\sqrt{q}}{z} & 0 \end{pmatrix},$$

which is the desired factorization of $W - 2\sqrt{q}$.

4 Anti-diagonal A in $\mathbb{C}P^1 \times \mathbb{C}P^1$

Consider the anti-diagonal

$$A := \left\{([z:w],[\bar z:\bar w]) \,|\, [z:w] \in \mathbb{C}P^1\right\}$$

which is a Lagrangian submanifold of $\mathbb{C}P^1 \times \mathbb{C}P^1$, where both factors of $\mathbb{C}P^1$ have the same standard symplectic form. Let $\mu : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{R}^2$ be the moment map whose image is a square $P = [0,l]^2$ and $L_{(a,b)}$ be the moment fiber over $(a,b) \in P$. Then, $L_{(a,b)}$ is a torus isomorphic to $L_1 \times L_2$ where L_1 has radius a and L_2 has b. Note that if $a \neq b$, $L_{(a,b)}$ does not intersect A. If $a = b$, they intersect along a circle.

In the example of $\mathbb{C}P^1$, the central fiber was deformed, and its Floer chain complex with a generic torus fiber was considered, whereas for our case of anti-diagonal, we deform a generic torus fiber while keeping the anti-diagonal A fixed. Namely, for each torus fiber, we deform the second component L_2 using the same methods as we did for $\mathbb{C}P^1$ in the previous section and get L_2^ϵ as in Fig. 4. In fact, consider the "real" circle in $\mathbb{C}P^1$ corresponding to the fixed points of complex conjugation of $\mathbb{C}P^1$, and we may choose the deformation L_2^ϵ so that it is symmetric with respect to this real circle. (In the Fig. 4, the real circle is the vertical circle which bisects the spike of L_2^ϵ.)

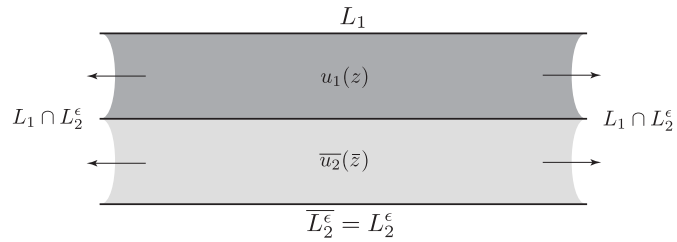


Figure 5. Gluing of strips.

Then, $L^\epsilon_{(a,b)} := L_1 \times L_2^\epsilon$ will meet the anti-diagonal A in at most two points. If $0 \le a \le b \le l$, they intersect precisely at two points and we can explicitly find out these two points. Let α and β be two intersection points of L_1 and L_2^ϵ as in the picture below. Then, it is easy to check that

$$A \cap L^\epsilon_{(a,b)} = \{(\alpha, \bar{\alpha}), (\beta, \bar{\beta})\}.$$

Note that L_2^ϵ will be preserved under the conjugation action on $\mathbb{C}P^1$ and (α, β) and (β, α) are two intersection points of L_1 and L_2^ϵ since $\bar{\alpha} = \beta$ in this case

Now, we have to find holomorphic strips from (α, β) to (β, α) and vice versa. The following proposition classifies all those strips in terms of holomorphic strips in the $\mathbb{C}P^1$ (which one might think of as the first or the second factor of $\mathbb{C}P^1 \times \mathbb{C}P^1$).

We introduce the following notation. We say that a holomorphic strip $u : \mathbb{R} \times [0, 1] \to M$ has a Lagrangian boundary condition (L_a, L_b) if the image of $(\mathbb{R}, 0)$ maps to L_a and that of $(\mathbb{R}, 1)$ maps to L_b.

Proposition 4.1. *There is a one to one correspondence between holomorphic strips with boundary $\left(A, L^\epsilon_{(a,b)}\right)$ in $\mathbb{C}P^1 \times \mathbb{C}P^1$ and holomorphic strips with boundary (L_2^ϵ, L_1) in $\mathbb{C}P^1$. Moreover, corresponding strips have the same symplectic area.*

The same holds for pairs $(L^\epsilon_{(a,b)}, A)$ and (L_1, L_2^ϵ).

Proof. Let $u = (u_1, u_2) : \mathbb{R} \times [0, 1] \to \mathbb{C}P^1 \times \mathbb{C}P^1$ be a holomorphic strip with boundary conditions

$$u(\cdot, 0) \in A, \qquad u(\cdot, 1) \in L_1 \times L_2^\epsilon,$$

and with asymptotic conditions

$$u(\infty, \cdot) = (\alpha, \beta), \qquad u(-\infty, \cdot) = (\beta, \alpha).$$

From the boundary conditions of u, we can conclude that u_1 and $\overline{u_2}$ agrees on one of boundary components, i.e. $u_1(s, 0) = \overline{u_2(s, 0)}$. Thus, if we define $u' : \mathbb{R} \times [-1, 1] \to \mathbb{C}P^1$ by (see Fig. 5)

$$u'(z) = \begin{cases} u_1(z) = u_1(s, t), & t \in [0, 1], \\ \overline{u_2}(\bar{z}) = \overline{u_2(s, -t)}, & t \in [-1, 0], \end{cases}$$

where we use the complex coordinate as $z = s + it$, then u' asymptotes to $\alpha(= \bar{\beta})$ and $\beta(= \bar{\alpha})$ at ∞ and $-\infty$ respectively as we take complex conjugate of u_2. Note that by the construction, L_2^ϵ is preserved by complex conjugation and hence, one of boundary components of the strip is still mapped to L_2^ϵ by $\overline{u_2}$.

Finally, since $\mathbb{C}P^1 \times \mathbb{C}P^1$ has the product symplectic structure induced from one on each factor, u' and u should have the same symplectic area. ■

Although the moduli spaces can be identified, the orientation of the moduli spaces work slightly differently (see [13] for details on the definition of canonical orientations). For example, for a holomorphic strip with boundary (L_2^ϵ, L_1), changing the orientation of L_2^ϵ and L_1 to the opposite orientation reverses the canonical orientation of the holomorphic strip. But for a holomorphic strip with boundary $\left(A, L^\epsilon_{(a,b)}\right)$, if we change the orientation of L_2^ϵ and L_1 at the same time, the orientation of the product $L^\epsilon_{(a,b)}$ remains the same, and so does the canonical orientation. Hence, even though the holomorphic strips for the calculation of the Floer cohomology $HF(L_2^\epsilon, L_1)$ in $\mathbb{C}P^1$ cancels in pairs (to produce a non-vanishing Floer cohomology of the equator), but the corresponding pairs of holomorphic strip with boundary $\left(A, L^\epsilon_{(a,b)}\right)$ do not cancel because they have the same sign from this consideration. This is why all the terms in the matrix factorization below (4.1) has positive signs.

By the above proposition, to find holomorphic strips for the anti-diagonal, and a deformed generic torus fiber, it suffices to find holomorphic strips bounding L_1 and L_2^ϵ which converge to α and β at $\pm\infty$. These are the same holomorphic strips, discussed in the previous section for $\mathbb{C}P^1$. Namely, the shape of the strip remain the same except that now we have deformed L_2 whose position is at b, not the central fiber of $\mathbb{C}P^1$.

Before we proceed, we recall the disc potential for $\mathbb{C}P^1 \times \mathbb{C}P^1$, whose terms correspond to holomorphic discs of index two, intersecting each toric divisor and having boundary lying in fibers of the moment map (see [11]):

$$e^\alpha T^{u_1} + e^{-\alpha} T^{l-u_1} + e^\beta T^{u_2} + e^{-\beta} T^{l-u_2},$$

where (α, β) represents a induced holonomy on the boundary of holomorphic discs, which may be identified with an element of H^1 of the torus. We denote $x = e^\alpha T^{u_1}$ and $y = e^{-\beta} T^{l-u_2}$ to obtain the potential (where $q = T^l$):

$$W = x + \frac{q}{x} + y + \frac{q}{y}.$$

Remark 4.2. We use the coordinate $y = e^{-\beta} T^{l-u_2}$ instead of $y = e^\beta T^{u_2}$ so that the upper-hemisphere of the second factor $\mathbb{C}P^1$ bounded by L_2 has the area y (Fig. 4). This is to get a symmetric form of factorization of W.

In Fig. 4, strips from α to β are strips of area 1 and $\frac{q}{xy}$, and those from β to α are strips of area x and y. Therefore, the resulting factorization of W is

$$(x+y)\left(1 + \frac{q}{xy}\right) = x + \frac{q}{x} + y + \frac{q}{y}. \tag{4.1}$$

These four strips contribute to m_1 with the same sign as we discussed above. This proves Proposition 1.1.

Remark 4.3. We can also compute the matrix factorization corresponding to the central moment fiber. It turns out to be an exterior tensor product of matrix factorization of the central fiber of each factor $\mathbb{C}P^1$ which is given in [4]. One can check that the following matrix factors $(W - \lambda)I_4$, where $\lambda = 4\sqrt{q}$:

$$\begin{pmatrix} 0 & 0 & z - \sqrt{q} & -1 + \frac{\sqrt{q}}{w} \\ 0 & 0 & w - \sqrt{q} & 1 - \frac{\sqrt{q}}{z} \\ 1 - \frac{\sqrt{q}}{z} & 1 - \frac{\sqrt{q}}{w} & 0 & 0 \\ -w + \sqrt{q} & z - \sqrt{q} & 0 & 0 \end{pmatrix}.$$

For the tensor product of matrix factorization, see [2].

5 Lagrangian Floer cohomology in $\mathbb{C}P^1 \times \mathbb{C}P^1$

In this section, we verify the conjecture that in the derived Fukaya category, the following two objects are the same:

$$A \oplus A[1], \qquad T_{(1,-1)} \oplus T_{(-1,1)}.$$

One is the direct sum of anti-diagonal A and its shift $A[1]$. The other is the direct sum of Lagrangian torus fiber at the center of the moment map image with holonomy $(1,-1)$, denoted as $T_{1,-1}$ or $(-1,1)$, denoted as $T_{-1,1}$. We denote by T_0 the central fiber of $\mathbb{C}P^1 \times \mathbb{C}P^1$ (without considering holonomies). We refer readers to Seidel's book [20] on the definition of derived Fukaya category. We just recall that in our case, we work with $\mathbb{Z}/2$-grading and by definition we have $HF^*(L[1], L') = HF^{*+1}(L, L')$.

5.1 Floer cohomology

First we compute $HF(T_{1,-1}, A)$ and $HF(T_{-1,1}, A)$. Note that $T_0 \cap A$ is a clean intersection, which is a circle S^1. Instead of working with the Bott–Morse version of the Floer cohomology, we move T_0 by Hamiltonian isotopy so that it intersects A transversely at two points. The Hamiltonian isotopy we choose are rotations in each factor of $\mathbb{C}P^1$ so that the equator of the circle is moved to the great circle passing through North and South pole. More precisely, if we identify $\mathbb{C}P^1$ as $\mathbb{C} \cup \{\infty\}$ and the equator with the unit circle in $\mathbb{C}$, then after isotopy, we obtain a Lagrangian submanifold L_0 obtained as a product of real line in the first component and imaginary line in the second component.

Locally on $\mathbb{C} \times \mathbb{C}$ we use (a, b) and (x, y) as coordinates of the first and the second factor, respectively. Let L_0 be the torus in $\mathbb{C}P^1 \times \mathbb{C}P^1$ given by the following equations

$$L_0 = \begin{cases} a = 0, \\ y = 0. \end{cases}$$

We denote L^R (resp. L^I) the great circle in $\mathbb{C}P^1$ corresponding to the real axis (resp. imaginary axis). We have $L_0 = L^R \times L^I$.

The anti-diagonal A (for which we will write L_1 from now on) can be expressed as

$$L_1 = \begin{cases} a - x = 0, \\ b + y = 0. \end{cases}$$

Let us calculate the Floer cohomology of the pair (L_0, L_1). They intersect at two points, $(0,0)$ and (∞, ∞). We denote $p = (0,0)$ and $q = (\infty, \infty)$. As explained in the previous section, given the holomorphic strips with boundary on (L_0, L_1), we can glue the first and the conjugate of second component of the strip to obtain a holomorphic strip with boundary on (L^I, L^R) (lower boundary on L^I, and upper boundary on L^R).

There are four such strips as seen in Fig. 6 (two strips from p to q, the other two from q to p) and these four strips have the same symplectic area.

As explained in the previous section, each of these strips are counted with the same sign in $\mathbb{C}P^1 \times \mathbb{C}P^1$ (different from the case of $\mathbb{C}P^1$). Hence two strips from p to q do not cancel out but adds up. In fact $m_1^2 \neq 0$ also, since

$$m_1^2 = W_{T_0} - W_A = W_{T_0}$$

and the potential W_{T_0} for the central fiber T_0 with a trivial holonomy $(1,1)$ is non-trivial, which is a sum of 4 terms corresponding to 4 holomorphic discs with boundary either on L^I or L^R in $\mathbb{C}P^1$.

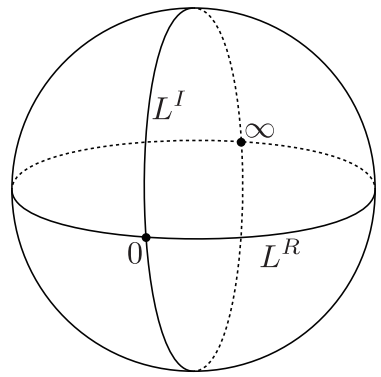


Figure 6. The first factor of $\mathbb{C}P^1 \times \mathbb{C}P^1$.

But for $T_{1,-1}$ or $T_{-1,1}$ (which induces the flat line bundles of the same holonomy on L_0), two strips from p to q cancel out due to holonomy contribution, and note that the corresponding potential $W_{T_{1,-1}} = W_{T_{-1,1}} = 0$. Thus from this cancellation of holomorphic strips we have $m_1(p) = m_1(q) = 0$.

Hence the Floer cohomology $HF(T_{1,-1}, A)$ (or $HF(T_{-1,1}, A)$) is generated by p, q and hence isomorphic to the homology of S^1 with Novikov ring coefficient. The similar argument works for $HF(A, T_{1,-1})$(or $HF(A, T_{-1,1})$), which is again generated by p, q.

The Floer cohomology $HF(T_{1,-1}, T_{1,-1})$ is a Bott–Morse version of the Floer cohomology (see [13]) and can be computed as in [7] or [9], and is isomorphic to the singular cohomology of the torus $H^*(T_0, \Lambda)$. The Floer cohomology $HF(A, A)$ is also isomorphic to the singular cohomology $H^*(A, \Lambda)$, as it is monotone and minimal Maslov index is 4 (see [17]).

5.2 Products

From now on, we don't distinguish L_0 and T_0 since they are clearly isomorphic in the Fukaya category. We assume that L_0 is the central torus fiber in $\mathbb{C}P^1 \times \mathbb{C}P^1$ which is equipped with a flat complex line bundle of holonomy $(1, -1)$ (or $(-1, 1)$), but we will omit it from the notation for simplicity. And by L_1, we denote the anti-diagonal Lagrangian submanifold (for which we used the notation A before).

Lemma 5.1. *Consider the product*

$$m_2 : \ HF(L_1, L_0) \times HF(L_0, L_1) \to HF(L_1, L_1)$$

we have that $m_2(p, p) = [p] \pm [L_1]T^{l/2}$. *Here* $T^{l/2}$ *is an area of the upper* (*or lower*) *hemisphere of each factor* $\mathbb{C}P^1$.

Lemma 5.2.

$$m_2 : \ HF(L_1, L_0) \times HF(L_0, L_1) \to HF(L_1, L_1)$$

we have that $m_2(q, q) = [q] \mp [L_1]T^{l/2}$.

Lemma 5.3.

$$m_2 : \ HF(L_0, L_1) \times HF(L_1, L_0) \to HF(L_0, L_0)$$

we have that $m_2(p, p) = [p] \pm [L_0]T^{l/2}$.

Lemma 5.4.

$$m_2 : \ HF(L_0, L_1) \times HF(L_1, L_0) \to HF(L_0, L_0)$$

we have that $m_2(q, q) = [q] \mp [L_0]T^{l/2}$.

Lemma 5.5. *For the product*

$$m_2 : \ HF(L_1, L_0) \times HF(L_0, L_1) \to HF(L_1, L_1),$$

we have $m_2(p, q) = m_2(q, p) = 0$.

Remark 5.6. It turns out that the products $m_2(p, q)$, $m_2(q, p)$ for

$$m_2 : \ HF(L_0, L_1) \times HF(L_1, L_0) \to HF(L_0, L_0)$$

do not vanish. But this won't be needed in our arguments of equivalence later

Proof of Lemma 5.1. The proof breaks into two parts, (i) one for the actual counting of strips and (ii) the other for the Fredholm regularity of these strips.

(i) The holomorphic triangle contributing to m_2 in this case can be considered as a holomorphic strip $u : \mathbb{R} \times [0, 1] \to \mathbb{C}P^1 \times \mathbb{C}P^1$ with

$$u(\cdot, 0) \subset L_0, \qquad u(\cdot, 1) \subset L_1$$

and a marked point $z_0 = [t_0, 1]$ on the upper boundary of the strip, which is used as an evaluation to L_1. Hence the first and second factor of holomorphic triangle can be again glued as in the previous section to give a holomorphic strip in $\mathbb{C}P^1$ with boundary on (L^R, L^I) in $\mathbb{C}P^1$, but both ends of the holomorphic strips converge to 0 (the first component of p). For convenience, we also call 0 as p.

Note that both L^R and L^I are preserved by the complex conjugation so that we can freely use this kind of process. Note also that after gluing, the marked point for evaluation lies in the interior of the strip.

From [16] such holomorphic strips can be decomposed into simple ones, and in this case, homotopy class of any holomorphic strip is given by the union of strips (in fact an even number of unions to come back to p). Since we are interested in the case that the dimension of the evaluation image is either 0 or two, the number of strips must be less than or equal to two. Since it starts and ends at p, the number is either 0 or 2.

First we consider the case of 0, or equivalently a constant triangle. In this case, we can use the following theorem of the first author in preparation

Theorem 5.7 ([6])**.** *Let L_a, L_b, L_c be Lagrangian submanifolds in a $2n$-dimensional symplectic manifold M, such that all possible intersections among them are clean. If $L_a \cap L_b \cap L_c = \{p\}$, p contributes to energy zero part of the product $([L_a \cap L_b]) \times ([L_b \cap L_c])$ in $HF(L_a, L_b) \times HF(L_b, L_c)$ non-trivially as $p = L_a \cap L_b \cap L_c$, if and only if*

$$\dim_{\mathbb{R}}(L_a \cap L_b) + \dim_{\mathbb{R}}(L_b \cap L_c) + \dim_{\mathbb{R}}(L_c \cap L_a) + \angle L_a L_b + \angle L_b L_c + \angle L_c L_a = 2n.$$

The notion of an angle is defined in [1]. In our case, $L_a = L_c = L_0$ and $L_b = L_1$ and hence $\angle L_c L_a = 0$ and also it is not hard to see from the definition of an angle that if L_a and L_b intersect transversely,

$$\angle L_a L_b + \angle L_b L_a = n.$$

Thus, $n + \angle L_a L_b + \angle L_b L_c + \angle L_c L_a$ is equal to $2n$. Hence, we have p as an energy-zero component of $m_2(p, p)$.

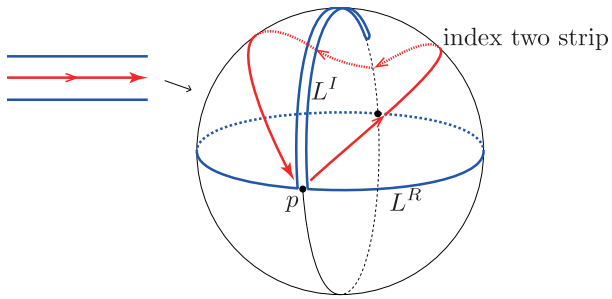


Figure 7. Index two strips bounding L^I and L^R.

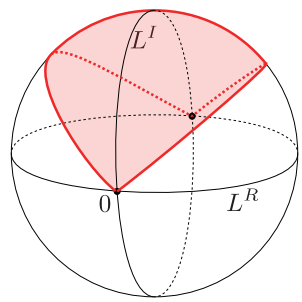


Figure 8. The shape of a lune.

Now, we consider the case that the holomorphic strip covers half of $\mathbb{C}P^1$. It is in fact easy to find such holomorphic strips, covering half of $\mathbb{C}P^1$, starting from p ending at p. The image of the strip is a disc with boundary either on real or imaginary circle in $\mathbb{C}P^1$, and one of the lower or upper boundary covers the circle once, and the other boundary covers part of the segment and comes back to p. (This strip of index two usually appears to explain the bubbling off in $\mathbb{C}$ with Lagrangian submanifolds $\mathbb{R}$ and unit circle S^1.)

These holomorphic strips of Maslov–Viterbo index two, lies inside a holomorphic disc in $\mathbb{C}P^1$ (of Maslov index two) with boundary on L^I or L^R, and there are 4 such discs (see Fig. 7). Thus, there are 4 homotopy classes of Maslov–Viterbo index two holomorphic strips from p to p and we denote them as $\beta_1, \ldots, \beta_4$.

Consider $\mathcal{M}_1(L_0, L_1, \beta_i, p, p)$ the moduli space of holomorphic strips of (Maslov–Viterbo) index two as described above starting and ending at p, with one marked point in the upper boundary of the strip for $i = 1, \ldots, 4$. The boundary $\partial\mathcal{M}_1(L_0, L_1, \beta_i, p, p)$ is well understood, and exactly has two possible components, one is from the broken strip of from p to q and to p, and the other is the bubbling off of a Maslov index two disc attached to a constant strip at p. In the former case, the marked point is located in either component of the broken holomorphic strips, and in the latter case, one of the coordinate of the marked point is free to move along the bubbled disc.

We can compare the orientations of the bubbled discs for each β_i's and they correspond to the potential W of L_0, and with the holonomy $(1, -1)$ or $(-1, 1)$, all these terms cancel out. Similarly, the evaluation images of the first type of boundary from the broken strips also are mapped exactly twice, since given an index one strip, there are two adjacent strips to it. And as the signs cancelled in W, the signs of the images for the first type of boundary should be opposite too. Thus, this shows that actually the boundaries of $\mathcal{M}_1(L_0, L_1, \beta_i, p, p)$ for $i = 1, \ldots, 4$ matches with opposite signs and the union gives a cycle in $L_1 \cong \mathbb{C}P^1$.

Hence this shows that $m_2(p, p)$ is a constant multiple of $[L_1]$. And it is enough to find the constant. Given such an index two strip, we consider the glued strip in $\mathbb{C}P^1$, and we evaluate at the marked point which is in the middle line of the glued strip. By varying the strip, it is

not hard to see that the image of evaluation map covers "half" of the disc, or a spherical lune (Fig. 8), connecting p and q once. But there are 4 discs and these 4 lunes together cover the whole $\mathbb{C}P^1$. This shows that the constant is one, and we have $m_2(p,p) = \pm[L_1]T^{l/2}$.

(ii) One can show that these strips are Fredholm regular from the following explicit formulation. First we identify the holomorphic strip with the upper half-disc $D_+ = \{z \in \mathbb{C} \mid |z| \leq 1, \operatorname{Im}(z) \geq 0\}$ with punctures at $-1, +1 \in D^2$, which are identified with $-\infty, \infty$ of the strip. Then, consider a holomorphic map $u : D_+ \to \mathbb{C}$ with semi-circle of ∂D_+ mapping to the unit circle of $\mathbb{C}$, and real line segment of ∂D_+ mapping to real line of $\mathbb{C}$. All such maps of degree two (whose images covers D^2 once) are given by

$$z \in D_+ \mapsto \frac{(z-a)(z-b)}{(1-az)(1-bz)} \tag{5.1}$$

for a real number $a, b \in (-1, 1)$, or

$$z \in D_+ \mapsto \frac{(z-\alpha)(z-\overline{\alpha})}{(1-\alpha z)(1-\overline{\alpha} z)} \tag{5.2}$$

for some $\alpha \in D^2$.

(To see this one starts with the generic form of a product of two Blaschke factors, and define an involution $u(z) \to \overline{u(\overline{z})}$ and find its fixed elements.) Since the holomorphic discs in $\mathbb{C}$ with boundary on S^1 are always Fredholm regular, the fixed elements by involution are again Fredholm regular. ■

Proof of Lemma 5.2. All the arguments are the same as the proof of Lemma 5.1 except on the sign in front of $[L_1]$. Hence, we only need to compare orientations. Note that the moduli space of holomorphic strips from p to p of index two with boundary on (L^R, L^I) in $\mathbb{C}P^1$, gives rise to the moduli space of holomorphic strips from q to q by rotating 180 with boundary on (L^R, L^I) in $\mathbb{C}P^1$. If L^R lies at the center of the disc which contains the strip, then this this process reverses the orientation of L^R, but not the orientation of L^I. (If L^I lies at the center, orientation of L^I is reversed, but that of L^R is fixed.)

As the rest of the ingredients for the orientation of the moduli space and evaluation map to the anti-diagonal remain the same, the resulting evaluation image has the opposite sign. ■

Proof of Lemma 5.3. The proof is somewhat similar to that of Lemma 5.1.

A holomorphic triangle contributing to m_2 in this case can be considered as a holomorphic strip $u : \mathbb{R} \times [0,1] \to \mathbb{C}P^1 \times \mathbb{C}P^1$ with

$$u(\cdot, 0) \subset L_1, \qquad u(\cdot, 1) \subset L_0$$

and a marked point $z_0 = [t_0, 1]$ on the upper boundary of the strip, which is used as an evaluation to L_0.

Hence the first and second factor of holomorphic triangle can be again glued as in the previous section to give a holomorphic strip in $\mathbb{C}P^1$ with boundary on (L^I, L^R) of $\mathbb{C}P^1$, but with both ends converging to p.

Again, the same argument as in the previous lemma shows that constant strip do contribute to m_2 in this case, and also the Maslov–Viterbo index two strips are to be considered. The relevant moduli space of holomorphic strips have 4 connected components, and their boundaries cancel out. Hence the evaluation image defines a 2-dimensional cycle in L_0, or a constant multiple of unit $[L_0]$.

Thus it is enough to find the constant. For this we use the explicit form of the holomorphic strip (5.1), (5.2). We may find a holomorphic strip of index two, sending z_0 to $(t_1, t_2) \in L_0$.

After gluing of first and second component of the strip, we may find a holomorphic map from D_+ sending 0 to $t_1 \in \mathbb{R} \subset \mathbb{C}$ and i to $\overline{t_2} \in S^1 \subset \mathbb{C}$ (up to automorphism of a strip, we may assume that $(0, i)$ corresponds to z_0).

By inserting these numbers to (5.1), we obtain

$$a + b = \frac{(t_1 - 1)(\overline{t_2} + 1)}{i(1 - \overline{t_2})}, \qquad ab = t_1.$$

Or from (5.2), we obtain

$$\alpha + \overline{\alpha} = \frac{(t_1 - 1)(\overline{t_2} + 1)}{i(1 - \overline{t_2})}, \qquad \alpha\overline{\alpha} = t_1.$$

Thus a, b or α, $\overline{\alpha}$ are (real or conjugate) pair of solutions of the quadratic equation

$$x^2 - \frac{(t_1 - 1)(\overline{t_2} + 1)}{i(1 - \overline{t_2})} x + t_1 = 0.$$

(one can check that the coefficient of x is real). If we choose $t_1 < 1$ to be almost as big as 1, and choose t_2 to be close to -1, then the coefficient of x is very close to 0 whereas t_1 is almost equal to 1. Thus the quadratic equation has a unique conjugate pair of complex solutions, both of which lies in the unit disc (since $|\alpha|^2 < 1$). Thus, this shows that the constant c of $m_2(p, p) = c[L_0]T^{l/2}$ equals ± 1. Hence, this proves the lemma. ■

The proof of Lemma 5.4 is exactly the same as that of Lemma 5.2 and omitted.

Proof of Lemma 5.5. We begin the proof of Lemma 5.5. The products

$$HF(L_1, L_0) \times HF(L_0, L_1) \to HF(L_1, L_1),$$

given by $m_2(p, q)$ or $m_2(q, p)$ are zero since $HF(L_1, L_1) \cong H^*(\mathbb{C}P^1, \Lambda)$ has no degree one classes. (This is because holomorphic strips connecting p and q have odd Maslov–Viterbo index, which is the dimension of the moduli space of holomorphic strips). ■

5.3 Floer cohomology between torus with different holonomies

First we consider the case of a cotangent bundle of a torus. Let L be a Lagrangian torus $T^n \subset T^*T^n$ Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two different flat line bundles on L. We prove that the Floer cohomology of the pairs $HF\big((L, \mathcal{L}_1), (L, \mathcal{L}_2)\big)$ vanishes if $\mathcal{L}_1 \neq \mathcal{L}_2$.

Proposition 5.8. *The Floer cohomology $HF((L, \mathcal{L}_1), (\phi(L), \phi_*(\mathcal{L}_2))$ vanishes if $\mathcal{L}_1 \neq \mathcal{L}_2$.*

Proof. Since L is a torus, we identify L as $\mathbb{R}^n/\mathbb{Z}^n$, and define a Morse function $f : T^n \to \mathbb{R}$ by

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \cos(2\pi x_i). \tag{5.3}$$

It is immediate to check that the critical points set is

$$\{(a_1, a_2, \ldots, a_n) |\ a_i = 0 \text{ or } 1/2 \text{ for } i = 1, \ldots, n\}.$$

Denote the holonomy of $\mathcal{L}^0$ (or $\mathcal{L}^0$) along the i-th generator of T^n by h_i^0 (or h_i^1). Let $I = (1/6, 5/6), J = (4/6, 8/6)$. Define

$$\mathcal{S} = \{L_1 \times \cdots \times L_n \subset \mathbb{R}^n/\mathbb{Z}^n \mid L_i = I \text{ or } J \text{ for } i = 1, \ldots, n\}.$$

$\mathcal{S}$ defines an open cover of T^n. The line bundle $\mathcal{L}^0$ (and $\mathcal{L}^1$) may be described by local charts on $\mathcal{S}$ as follows. We explain how to glue trivial lines bundles on the open sets of $\mathcal{S}$

$$\phi: \ L_1 \times \cdots \times L_n \times \mathbb{C} \mapsto L_1' \times \cdots \times L_n' \times \mathbb{C}$$

sends $(x_1, \ldots, x_n, l) \to (x_1, \ldots, x_n, l')$ where $l' = b_1 b_2 \cdots b_n l$ with

$$b_i = \begin{cases} 1 & \text{if } L_i = L_i', \\ 1 & \text{if } x_i \in (1/6, 2/6), \\ h_i^0 & \text{if } x_i \in (4/6, 5/6) \text{ and } L_i = I, \quad L_i' = J, \\ 1/h_i^0 & \text{if } x_i \in (4/6, 5/6) \text{ and } L_i = J, \quad L_i' = I. \end{cases}$$

It is easy to check that this defines the flat line bundle $\mathcal{L}^0$.

Now we compute the boundary map in the Floer complex. First, we fix some sign convention about Morse complex. Recall the following rules, for a submanifold $P \subset L$ and $x \in P$,

$$N_x P \oplus T_x P = T_x L.$$

Also

$$N_x P_1 \oplus N_x P_2 \oplus T_x(P_1 \cap P_2) = T_x L$$

determines the orientation of $P_1 \cap P_2$ at x. Now, we denote $W^u(x)$, $W^s(x)$ to be the unstable and stable manifold of x for the given Morse function f on L. Then, we set

$$TW^s(x) \oplus TW^u(x) = T_x L. \tag{5.4}$$

Finally, we set the orientation of the moduli space $\mathcal{M}(x, y)$ of the trajectory moduli space as

$$W^s(y) \cap W^u(x) = \mathcal{M}(x, y).$$

Now, we consider the function f given by (5.3). Unstable manifolds of f can be written as products of intervals $[0, 1/2)$ or $(1/2, 0]$, and intervals are canonically oriented. Hence we assign the product orientations on the unstable manifolds.

Lemma 5.9. *Let $x = [a_1, a_2, \ldots, a_n]$, $y = [b_1, \ldots, b_n]$ where for a fixed i, $a_i = 0$, $b_i = 1/2$ and $b_j = a_j$ for $j \neq i$. Then, the trajectory space $\mathcal{M}(x, y)$ has the canonical orientation $(-1)^A \partial_i$ where A is the number of $j < i$ with $a_j = 0$. Here ∂_i is the ith standard basis vector of $\mathbb{R}^n$.*

Proof. First, from the orientation convention, we can identify $NW^u = TW^s$. Hence,

$$NW^s(y) \oplus NW^u(x) \oplus T\mathcal{M}(x, y) = NW^s(y) \oplus TW^s(x) \oplus T\mathcal{M}(x, y) = TL. \tag{5.5}$$

It is easy to check that

$$(-1)^A \partial_i \oplus TW^u(y) = TW^u(x),$$

where A is the number of $j < i$ with $a_j = 0$, by comparing two unstable manifolds. Hence, from (5.4), we have

$$TW^s(x) \oplus (-1)^A \partial_i = TW^s(y).$$

Hence, combining with (5.5) and denoting $T\mathcal{M}(x, y) = (-1)^B \partial_i$, we have

$$NW^s(y) \oplus TW^s(y) \cdot (-1)^A (-1)^B = TL.$$

Hence, we have $TL \cdot (-1)^{A+B} = TL$, which proves the lemma. ■

The lemma implies that actual Morse boundary map is given as follows by comparing the coherent orientation with the flow orientation.

$$\partial_{\text{Morse}} x = (-1)^A (1-1) y = 0.$$

Now, in the case of the Floer complex twisted by flat bundles, we have

$$\partial\big((a_1, \dots, a_n)\big) = \sum_{\text{for each } a_i = 0} (-1)^{A_i} = \left(1 - \frac{h_i^0}{h_i^1}\right)(a_1, \dots, a_{i-1}, a_i + 1/2, a_{i+1}, \dots, a_n).$$

If $\mathcal{L}^0 = \mathcal{L}^1$, we have $h_i^0/h_i^1 = 1$, hence all boundary maps vanish and we obtain the singular cohomology of the torus T^n. If $\mathcal{L}^0 \neq \mathcal{L}^1$, we first assume that $h_i^0 \neq h_i^1$ for all i, and show that the complex has vanishing homology.

In fact, the above complex, with an assumption $h_i^0 \neq h_i^1$ for all i, is chain isomorphic to the same complex with the following new differential

$$\widetilde{\partial}\big((a_1, \dots, a_n)\big) = \sum_{\text{for each } a_i = 0} (-1)^{A_i} (1)(a_1, \dots, a_{i-1}, a_i + 1/2, a_{i+1}, \dots, a_n).$$

Here chain isomorphism can be defined as

$$\Psi([a_1, \dots, a_n]) = \left(\prod_{i \text{ with } a_i = 0} (1 - h_i^0/h_i^1)\right)[a_1, \dots, a_n].$$

It is easy to check that $\Psi\partial = \widetilde{\partial}\Psi$, and there is an obvious inverse map.

The new complex with $\widetilde{\partial}$ may be considered as the reduced homology complex of the standard simplex Δ^{n-1}, hence has a vanishing homology. The face corresponding to $[a_1, \dots, a_n]$ contains i-th vertex if and only if $a_i = 0$.

Now, consider the general case that $h_i^0 \neq h_i^1$ if and only if $i \in \{i_1, \dots, i_k\}$ where $k \geq 1$. The chain complex we obtain has non-trivial differential only for the terms containing $(1 - h_i^0/h_i^1)$ for $i \in \{i_1, \dots, i_k\}$ and hence the chain complex decomposes into several chain sub-complexes with only non-trivial differentials within. And by using the result in the first case, we obtain the proposition. ■

So far, we have discussed the case in the cotangent bundle of the torus. For our case, it follows from the spectral sequence of [18].

Lemma 5.10.

$$HF(T_{1,-1}, T_{-1,1}) \cong 0.$$

Proof. T_0 (and hence $T_{1,-1}$ and $T_{-1,1}$) is a monotone Lagrangian submanifold, and hence by [18], there is filtration of the Floer differential

$$m_1 = m_{1,0} + m_{1,N} + m_{1,2N} + \cdots,$$

where N is the minimal Maslov number of T, which can be easily modified to the case of flat complex line bundles. By usual spectral sequence argument, we obtain the vanishing of the homology of m_1 differential, since the homology of $m_{1,0}$ vanishes in our case. ■

5.4 Equivalence

To show that $A \oplus A[1]$ is equivalent to $T_{1,-1} \oplus T_{-1,1}$. We find

$$\Phi_1 \in \mathrm{Hom}_{\mathrm{DFuk}}(A \oplus A[1], T_{1,-1} \oplus T_{-1,1}), \qquad \Phi_2 \in \mathrm{Hom}_{\mathrm{DFuk}}(T_{1,-1} \oplus T_{-1,1}, A \oplus A[1]),$$

such that $\Phi_1 \circ \Phi_2 = \mathrm{Id}$, and $\Phi_2 \circ \Phi_1 = \mathrm{Id}$ in the derived Fukaya category of $\mathbb{C}P^1 \times \mathbb{C}P^1$. We write

$$\Phi_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}.$$

Namely,

$$\begin{aligned}
&\alpha_1 \in \mathrm{Hom}(A, T_{1,-1}), \beta_1 \in \mathrm{Hom}(A, T_{-1,1}),\\
&\gamma_1 \in \mathrm{Hom}(A[1], T_{1,-1}), \delta_1 \in \mathrm{Hom}(A[1], T_{-1,1}),\\
&\alpha_2 \in \mathrm{Hom}(T_{1,-1}, A), \beta_2 \in \mathrm{Hom}(T_{-1,1}, A[1]),\\
&\gamma_2 \in \mathrm{Hom}(T_{-1,1}, A), \delta_2 \in \mathrm{Hom}(T_{-1,1}, A[1]).
\end{aligned}$$

We choose

$$\Phi_1 = \begin{pmatrix} p & q \\ q & p \end{pmatrix}, \qquad \Phi_2 = \frac{1}{2T^{l/2}} \begin{pmatrix} p & -q \\ -q & p \end{pmatrix}.$$

Theorem 5.11. *We have*

$$\begin{aligned}
&\Phi_1 \circ \Phi_2 = \pm\mathrm{Id} \in \mathrm{Hom}(A \oplus A[1], A \oplus A[1]),\\
&\Phi_2 \circ \Phi_1 = \pm\mathrm{Id} \in \mathrm{Hom}(T_{1,-1} \oplus T_{-1,1}, T_{1,-1} \oplus T_{-1,1}).
\end{aligned}$$

Therefore in the derived Fukaya category of $\mathbb{C}P^1 \times \mathbb{C}P^1$, $A \oplus A[1]$ *is equivalent to* $T_{1,-1} \oplus T_{-1,1}$.

Proof. This follows from Lemmas 5.1, 5.2, 5.3, 5.4 and 5.5 and the fact that $[p] = [q]$ in the Bott–Morse Floer cohomology of A or T. We note that for $\Phi_2 \circ \Phi_1$, the products of the following type,

$$HF(T_{1,-1}, A) \times HF(A, T_{-1,1}) \to HF(T_{1,-1}, T_{-1,1}),$$

are automatically zero, due to Lemma 5.10.

Note also that $[L_0]$ and $[L_1]$ play a role of units in $HF(L_0, L_0)$ and $HF(L_1, L_1)$, respectively. ■

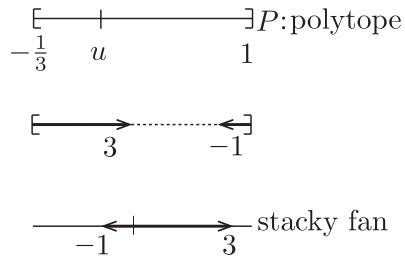
6 Teardrop orbifold

We show that the correspondence between the Floer complex equation and the matrix factorization continues to hold for a teardrop orbifold. Such correspondence can be divided into two levels. The first level is regarding smooth discs. Namely, we consider the Floer complex equation, only involving smooth holomorphic strips (and discs). Then, we obtain a smooth potential or the Hori–Vafa Landau–Ginzburg potential and the correspondence holds on this level. Here, by smooth holomorphic strips or discs, we mean a holomorphic maps from a smooth domain Riemann surface with boundary, and by definition of holomorphicity, they locally lift to uniformizing chart of the target orbifold point, and hence when their images contain an orbifold point, it meets the point with multiplicity (see [10]).

For the second level we consider bulk deformations by twisted sectors, and hence obtain the corresponding bulk potential or bulk Landau–Ginzburg potential which has additional terms

corresponding to orbi-discs. We consider the Floer complex equation, involving smooth and orbifold holomorphic strips (and discs), which are maps from orbifold Riemann surfaces with boundary. Then the correspondence between the Floer complex equation and the matrix factorization continues to hold for bulk deformed cases.

Let X be the orbifold obtained from the following stacky fan: Then, X is an orbifold with



one singular point with $(\mathbb{Z}/3\mathbb{Z})$-singularity.

6.1 The case of Hori–Vafa potential

The Hori–Vafa Landau–Ginzburg potential can be constructed (see [10]) in this case by considering smooth holomorphic discs of Maslov index two:

$$W(z) = z^3 + \frac{q^4}{z}, \tag{6.1}$$

where z^3 is due to the fact that smooth holomorphic discs around the orbifold point has to wrap around it 3 times (see [11] for a general procedure of boundary deformation to construct such a potential from the moduli of holomorphic discs).

We briefly review how to obtain the above expression of the potential (6.1) as above. Since index 2 discs correspond to the vectors in the stacky fan [10], we have the following description of index 2 holomorphic discs,

$$e^{3x}T^{3\left(u-\left(-\frac{1}{3}\right)\right)} + e^{-x}T^{1-u} = e^{3x}T^{1+3u} + e^{-x}T^{1-u},$$

where the power of e represents the holonomy factors, and that of T represents the area of discs (see (3.1)). In particular, u is a position in the interior of the moment polytope. Note that we multiply 3 to $(u + 1/3)$ to obtain the area of the *smooth discs*.

One get the expression of W as in (6.1), by substituting $z = e^x T^{1+3u}$ and $q = T^{1/3}$. Then, the total area of the teardrop orbifold will give the term

$$T^{1-(-1/3)} = T^{4/3} = q^4.$$

Denote L_u by the torus fiber over $u \in [-1/3, 1]$ where we identify P with the interval $[-1/3, 1] \subset \mathbb{R}$. Let L be the balanced fiber L_0 (i.e. the moment fiber over $u = 0$).

Let α be a holonomy around L which is one of solutions of

$$3z^2 - \frac{1}{z^2} = 0$$

or equivalently, $3\alpha^4 = 1$. Here, the holonomy α is not unitary but, the first author proved in [8] that one can define a Floer cohomology with non-unitary line bundles. As in the picture, we can list up all strips which bound L_u and L. Then, the similar technique to the one given in Section 3 will give the corresponding holomorphic functions. Note below that there are two more discs D_5 and D_6 which are not easily visible in Fig. 9. We will be able to find these in the development picture (see Fig. 10).

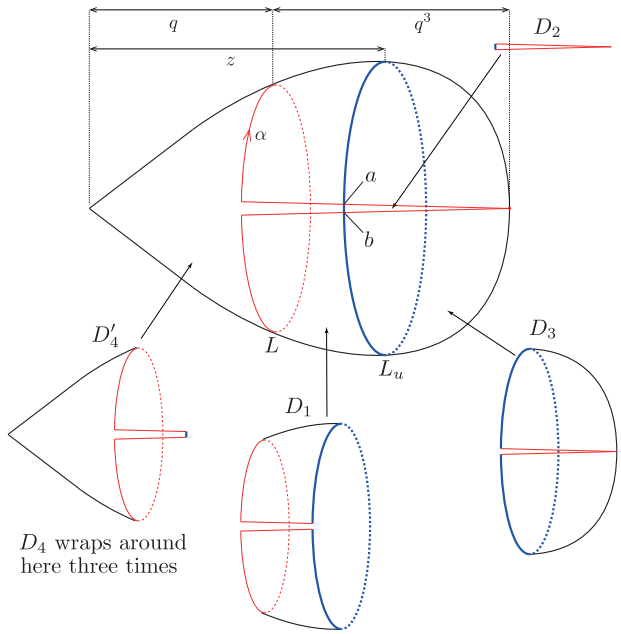


Figure 9. Disk splitting in the teardrop orbifold.

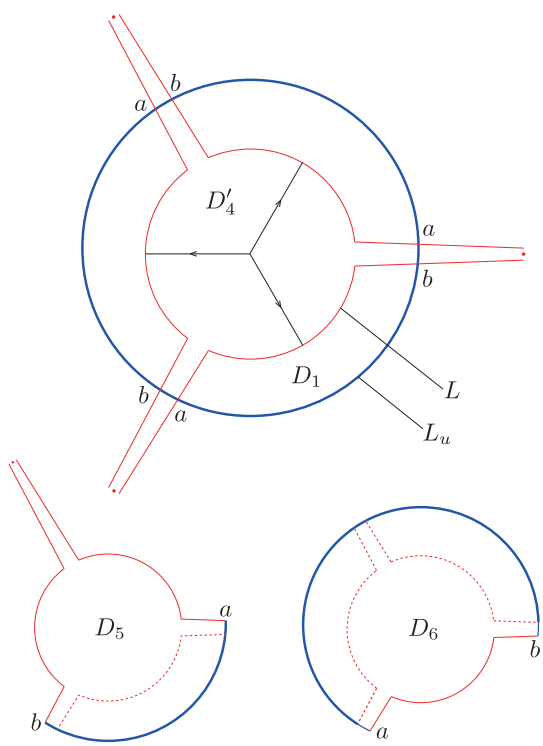
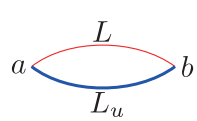


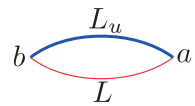
Figure 10. Development figure.

(1) strips from a to b:



(i) The disc D_1 in the pictures leads to the term $-\frac{z}{\alpha q}$.
(ii) In the limit, D_2 degenerates so that D_2 corresponds to 1.

(2) strips from b to a



(*iii*) D_3 gives rise to the term $\frac{q^4}{z}$.

(*iv*) Since we only consider smooth disc in this subsection, D_4 in the picture indeed wraps around the singular point three times to give $-q^3\alpha^3$. (We will consider nontrivial orbi-discs in the next subsection.)

(*v*) D_5 can be seen in the development figure below which is clearly smooth since it covers the cone at the singular point three times. It leads to $-zq^2\alpha^2$.

(vi) Likewise, D_6 corresponds to the term $-z^2q\alpha$.

In conclusion,

$$\left(1-\frac{z}{\alpha q}\right)\left(\frac{q^4}{z}-q^3\alpha^3-zq^2\alpha^2-z^2q\alpha\right)=z^3+\frac{q^4}{z}-\left(q^3\alpha^3+\frac{q^3}{\alpha}\right). \tag{6.2}$$

By definition of α, $q^3\alpha^3+\frac{q^3}{\alpha}$ is a critical value of $z^3+\frac{q^4}{z}$.

Proposition 6.1. *The Lagrangian torus fiber for $u=0$, corresponds to the matrix factorization (6.2) of $W=z^3+\frac{q^4}{z}$ with critical value $\lambda=q^3\alpha^3+\frac{q^3}{\alpha}$.*

6.2 The case of bulk deformed orbi-potential

Now, we turn on bulk deformation by twisted sectors. Namely, for $\nu=[1]\in\mathbb{Z}/3$, we can consider X_ν which is an isolated point, whose fundamental class is $1_\nu\in H^0(X_\nu)$. Then we take $\mathfrak{b}=c1_\nu$, with $c\in\Lambda_+$. Here,

$$\Lambda_+=\left\{\sum a_iT^{\lambda_i}\in\Lambda\,\middle|\,\lambda_i>0\right\}.$$

Since there is an insertion from twisted sectors, now we also include orbifold holomorphic discs (orbi-discs for short). (We refer readers to [10] for details of the following constructions. See [12] for bulk deformations in the case of toric manifolds.)

We first find the bulk-deformed mirror. As we have chosen $\mathfrak{b}=c1_\nu$, we need to consider orbifold holomorphic discs with several orbifold interior marked points with $\mathbb{Z}/3$ singularity, mapping to X_ν where each generator of the local group at orbifold marked point is mapped to ν. By simple degree consideration, the orbi-disc with only one orbifold interior marked point contributes to the potential, and such holomorphic orbi-disc is classified in [10]. In this case, there is a unique holomorphic orbi-disc D_4', which covers the cone once. The additional information from the orbi-disc(which has area $u+1/3$) can be described as follows.

$$e^{3x}T^{3u+1}+e^{-x}T^{1-u}+ce^xT^{u+1/3}$$

or we can write $z=e^xT^{u+1/3}$ and $T=q$, which gives

$$W^{\mathfrak{b}}=z^3+\frac{q^{4/3}}{z}+cz.$$

We remark that the bulk deformation which makes the fiber L_u for any $u<1/3$ to have a non-trivial Floer cohomology is $c=T^{2/3-2u}-3T^{2u+2/3}$ or $c=q^{2/3-2u}-3q^{2u+2/3}$. (If $u\geq 1/3$, then $c\notin\Lambda_+$. In fact, fibers for $1>u>1/3$, can be displaced from itself by using the open set obtained by removing the cone point.)

The critical point equation for $W^{\mathfrak{b}}$ is

$$3z^2 - \frac{q^{4/3}}{z^2} + c = 0,$$

whose solution is denoted as $q^{1-u}\alpha$. Then the critical value of the bulk deformed potential is

$$q^{3-3u}\alpha^3 + q^{1/3+u}/\alpha + cq^{1-u}\alpha.$$

Now, we look at the corresponding matrix factorization.

We repeat (6.2), with the additional orbifold holomorphic strip contribution (underlined term below), which is D_4' in the development picture (Fig. 10), to obtain the following matrix factorization:

$$\begin{aligned}\left(1 - \frac{z}{q^{1-u}\alpha}\right)&\left(\frac{q^{4/3}}{z} - q^{3-3u}\alpha^3 - zq^{2-2u}\alpha^2 - z^2q^{1-u}\alpha - \underline{cq^{1-u}\alpha}\right)\\ &= z^3 + \frac{q^{4/3}}{z} + cz - \left(q^{3-3u}\alpha^3 + q^{1/3+u}/\alpha + cq^{1-u}\alpha\right).\end{aligned} \tag{6.3}$$

Proposition 6.2. *For the teardrop orbifold X, with the bulk deformation $\mathfrak{b} = c1_\nu \in H^0(X_\nu)$, the Lagrangian fiber L_u for $u < 1/3$ corresponds to the matrix factorization (6.3) of the bulk deformed potential $W^{\mathfrak{b}}$.*

7 Weighted projective lines

Finally, we study the case of weighted $\mathbb{C}P^1$s with general weights at ends. Let X be a weighted projective line with $\mathbb{Z}/m\mathbb{Z}$-singularity on the left and $\mathbb{Z}/n\mathbb{Z}$ on the right, i.e. X is obtained by dividing $\mathbb{C}^2 \setminus \{(0,0)\}$ by the following action of $\mathbb{C}^*$ (m, n are assumed to be relatively prime):

$$\rho \in \mathbb{C}^* : \ (z, w) \mapsto (\rho^n z, \rho^m w).$$

Recall that $\left[-\frac{1}{m}, \frac{1}{n}\right]$ is the moment polytope of X. (Similarly, one can also work on the case of toric orbifold of dimension one, which corresponds to the interval as a polytope with integer labels m, n at each end point. In this setting, m, n do not need to be relatively prime. But we leave the details of this general case to the interested reader.)

Thus, in this case, smooth holomorphic discs of Maslov index 2 can be described as follows:

$$e^{mx}T^{1+mu} + e^{-nx}T^{1-nu}.$$

As before, we make a substitution for this equation by $z = e^x T^{\frac{1}{m}+u}$ and $q = T^{1/m}$, which is coherent to the computation for the teardrop. Therefore, the LG potential W is given as follows:

$$W = z^m + \frac{q^{m+n}}{z^n}.$$

In terms of the *smooth* Floer theory, the fiber at $u = 0$ has a non-vanishing Floer homology. This is because two smooth holomorphic discs (left, and right) has the same area at $u = 0$.

As before, we consider the deformation L of the central fiber L_0 and consider intersection with a general fiber L_u at $u \in \left(0, \frac{1}{n}\right) \subset \left[-\frac{1}{m}, \frac{1}{n}\right]$. The total area of X corresponds to

$$T^{\frac{1}{n}-\left(-\frac{1}{m}\right)} = q^{\frac{m+n}{n}},$$

and L splits the total area into q and $q^{m/n}$ (see Fig. 11).

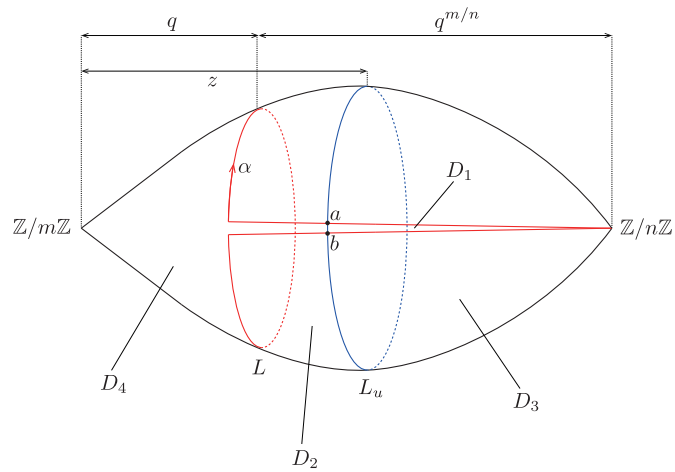


Figure 11. Disk splittings in weighted projective lines.

Let α be a (non-unitary) holonomy of L, which is given by one of the solutions of the equation

$$mz^{m-1} - \frac{n}{z^{n+1}} = 0. \tag{7.1}$$

Then we count index 2 holomorphic strips as we did above, counting visible ones from Fig. 11 and those from development figures which are obtained by letting singularities at both ends being ∞. (Strips can cover the region of D_3 and D_4 several times.) We remark that only D_1 and D_2 are strips from a to b, and any other strips such as D_3, D_4 and those given by development figures are strips from b to a.

After counting all such strips, we have the factorization as

$$\begin{aligned}\left(1-\frac{z}{\alpha q}\right)&\left(\sum_{k=0}^{n}\frac{q^{\frac{m}{n}k}}{\alpha^k}\left(\frac{q^{\frac{m+n}{n}}}{z}\right)^{n-k}-\sum_{k=1}^{m}\alpha^k q^k z^{m-k}\right)\\ &=z^m+\frac{q^{m+n}}{z^n}-\left(\alpha^m q^m+\frac{q^m}{\alpha^n}\right).\end{aligned} \tag{7.2}$$

(The first factor in the left-hand side of (7.2) counts the strips from a to b.) One can easily see that $\alpha^m q^m + \frac{q^m}{\alpha^n}$ is a critical value of $W = z^m + \frac{q^{(m+n)}}{z^n}$, comparing with (7.1). Note that if $m = 3$ and $n = 1$, then the result coincides with the one we have obtained in the previous section.

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References

[1] Alston G., Floer cohomology of real Lagrangians in the Fermat quintic threefold, arXiv:1010.4073.

[2] Ashok S.K., Dell'Aquila E., Diaconescu D.E., Fractional branes in Landau–Ginzburg orbifolds, *Adv. Theor. Math. Phys.* **8** (2004), 461–513, hep-th/0401135.

[3] Auroux D., Mirror symmetry and T-duality in the complement of an anticanonical divisor, *J. Gökova Geom. Topol. GGT* **1** (2007), 51–91, arXiv:0706.3207.

[4] Chan K., Leung N.C., Matrix factorizations from SYZ transformations, in Advances in Geometric Analysis, *Adv. Lect. Math.*, Vol. 21, International Press, Somerville, MA, 2011, 203–224, arXiv:1006.3832.

[5] Chan K., Leung N.C., Mirror symmetry for toric Fano manifolds via SYZ transformations, *Adv. Math.* **223** (2010), 797–839, arXiv:0801.2830.

[6] Cho C.-H., Constant triangles in Fukaya category, in preparation.

[7] Cho C.-H., Holomorphic discs, spin structures, and Floer cohomology of the Clifford torus, *Int. Math. Res. Not.* **2004** (2004), no. 35, 1803–1843, math.SG/0308224.

[8] Cho C.-H., Non-displaceable Lagrangian submanifolds and Floer cohomology with non-unitary line bundle, *J. Geom. Phys.* **58** (2008), 1465–1476, arXiv:0710.5454.

[9] Cho C.-H., Oh Y.-G., Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds, *Asian J. Math.* **10** (2006), 773–814, math.SG/0308225.

[10] Cho C.-H., Poddar M., Holomorphic orbi-discs and Lagrangian Floer cohomology for toric orbifolds, arXiv:1206.3994.

[11] Fukaya K., Oh Y.-G., Ohta H., Ono K., Lagrangian Floer theory on compact toric manifolds. I, *Duke Math. J.* **151** (2010), 23–174, arXiv:0802.1703.

[12] Fukaya K., Oh Y.-G., Ohta H., Ono K., Lagrangian Floer theory on compact toric manifolds. II. Bulk deformations, *Selecta Math. (N.S.)* **17** (2011), 609–711, arXiv:0810.5654.

[13] Fukaya K., Oh Y.-G., Ohta H., Ono K., Lagrangian intersection Floer theory: anomaly and obstruction, *AMS/IP Studies in Advanced Mathematics*, Vol. 46, Amer. Math. Soc., Providence, RI, 2009.

[14] Gross M., The Strominger–Yau–Zaslow conjecture: from torus fibrations to degenerations, in Algebraic Geometry – Seattle 2005, *Proc. Sympos. Pure Math.*, Vol. 80, Amer. Math. Soc., Providence, RI, 2009, Part 1, 149–192.

[15] Kapustin A., Li Y., D-branes in Landau–Ginzburg models and algebraic geometry, *J. High Energy Phys.* **2003** (2003), no. 12, 005, 44 pages, hep-th/0210296.

[16] Kwon D., Oh Y.-G., Structure of the image of (pseudo)-holomorphic discs with totally real boundary condition, *Comm. Anal. Geom.* **8** (2000), 31–82.

[17] Oh Y.-G., Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. I, *Comm. Pure Appl. Math.* **46** (1993), 949–993.

[18] Oh Y.-G., Floer cohomology, spectral sequences, and the Maslov class of Lagrangian embeddings, *Int. Math. Res. Not.* **1996** (1996), no. 7, 305–346.

[19] Orlov D.O., Triangulated categories of singularities and D-branes in Landau–Ginzburg models, *Tr. Mat. Inst. Steklova* **246** (2004), 240–262.

[20] Seidel P., Fukaya categories and Picard–Lefschetz theory, *Zürich Lectures in Advanced Mathematics*, European Mathematical Society (EMS), Zürich, 2008.

[21] Strominger A., Yau S.T., Zaslow E., Mirror symmetry is T-duality, in Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, 1999), *AMS/IP Studies in Advanced Mathematics*, Vol. 23, Editors C. Vafa, S.T. Yau, Amer. Math. Soc., Providence, RI, 2001, 275–295.