THE REPRESENTABILITY CRITERION FOR GEOMETRIC DERIVED STACKS

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The aim of these notes is to study the proof of Lurie's representability criterion in the case of derived stacks. The reference for the proof is [HAG2].

Let k be a field of characteristic zero.

☛ **Question.**— *Let* X *be a derived stack, how can I check in practice that* X *is a geometric derived stack ?*

☛ **Theorem.**— *Let* X *be a derived stack. The following conditions are equivalent.*

1) X *is an* n*-geometric derived stack;*

2) X *satisfies the following three conditions,*

   a) *The truncation* $t_0(X)$ *is an* n*-geometric stack.*

   b) X *has an obstruction theory.*

   c) *(nilcompleteness) For any* $A \in \mathcal{A}lg_k^{cdg_{\leq 0}}$*, the natural morphism*

$$X(A) \longrightarrow \varprojlim_k X(A_{\leq k})$$

   is a weak equivalence of simplicial sets.

Before giving the proof of the theorem, let us recall all the definitions one needs to have in mind.

DEFINITIONS

Derived stack

Let $\mathcal{A}lg_k^{cdg_{\leq 0}}$ be the big ∞-category of negatively graded commutative differential graded k-algebras, endowed with the Grothendieck topology generated by *étale* morphisms. A *derived stack* X is then an ∞-sheaf of spaces on that big site,

$$X \in \mathcal{S}h(\mathcal{A}lg_k^{cdg_{\leq 0}}, \text{ét})$$

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We are going to give the definition of an n-geometric stack by induction on n,

- A derived stack is (-1)-geometric if it is representable.
- A map of derived stacks $f : X \to Y$ is (-1)-representable if for any map from an affine stack $\mathrm{Spec}(A) \to Y$, the pullback $X \times_Y \mathrm{Spec}(A)$ is representable.

  Remark.— A map $f : X \to Y$ is (-1)-representable if its fibres are (-1)-geometric.
- A map of derived stacks $f : X \to Y$ is (-1)-smooth if it is (-1)-representable and for any map $\mathrm{Spec}(A) \to Y$ the pullback

$$X \times_Y \mathrm{Spec}(A) \to \mathrm{Spec}(A)$$

  is a smooth map between affine stacks.

$\boxed{-1 \longrightarrow 0}$

- Let X be a derived stack, a 0-atlas of X is a small family of (-1)-smooth morphisms $\{\mathrm{Spec}(A_i) \to X\}$ such that the morphism

$$\coprod_i \mathrm{Spec}(A_i) \longrightarrow X$$

  is an epimorphism.
- A derived stack X is 0-geometric if,
  a) the derived stack X admits a 0-atlas;
  b) the diagonal morphism $X \to X \times X$ is (-1)-representable.

  Remark.— The diagonal condition guaranties compatibility on intersections.
- A morphism of derived stacks $f : X \to Y$ is 0-representable if for any $\mathrm{Spec}(A) \to Y$, the pullback $X \times_Y \mathrm{Spec}(A)$ is 0-geometric.
- A morphism of derived stacks $f : X \to Y$ is 0-smooth if it is 0-representable and for any $\mathrm{Spec}(A) \to Y$, there exists a 0-atlas $\{U_i\}$ of $X \times_Y \mathrm{Spec}(A)$, such that each composite morphism $U_i \to \mathrm{Spec}(A)$ is smooth.

  $\ldots \to n$-atlas $\to n$-geometric derived stack $\to n$-representable morphism $\to n$-smooth morphism $\to \ldots$

Definition.— *Following the same ideas, one defines the notion of n-geometric stacks.*

Obstruction theory

A derived stack X has an *obstruction theory* if it satisfies certain conditions, that in this context are equivalent to:

$$\begin{cases} \text{X has a cotangent complex;} \\ \text{X is infinitesimally cartesian.} \end{cases}$$

Definition.— *A derived stack* X *is infinitesimally cartesian if, for every* $A \in \mathcal{A}lg_k^{cdg_{\leq 0}}$, $M \in \mathcal{M}od_A$ *and* $d \in H_0(\mathrm{Der}(A, M))$, *the following diagram is cartesian*

$$\begin{array}{ccc} F(A \oplus_d \Omega M) & \longrightarrow & F(A) \\ \downarrow & \lrcorner & \downarrow \\ F(A) & \longrightarrow & F(A \oplus M) \end{array}$$

where $A \oplus_d \Omega M$ *is defined as the the fibre product in the* ∞*-category of square zero extensions of* A,

$$\begin{array}{ccc} A \oplus_d \Omega M & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow d \\ A & \xrightarrow{\text{triv}} & A \oplus M \end{array}$$

THE PROOF

We are only going to prove the 'if' part of the theorem, which is the useful part in practice.

☛ **Idea.—** Prove the criterion by induction on n and use the fact that the 'derived' part of a derived stack is made of square zero extensions, which are controlled by the cotangent complex.

Initialisation at (−1)

Let X be a derived stack such that $t_0(X)$ is an affine scheme and such that X has an obstruction theory and is nilcomplete. We wish to prove that X is an affine derived scheme.

The very first thing to do is to find a suitable atlas for X. To do so, we are going to take an atlas of $t_0(X)$ — itself — and lift it to X. This is the goal of the following lemma.

Lemma.— *For any étale map* $\mathrm{Spec}(A_0) \to t_0(X)$, *there exists an étale map* $\mathrm{Spec}(A) \to X$ *such that the following square is cartesian,*

$$\begin{array}{ccc} \mathrm{Spec}(A_0) & \longrightarrow & t_0(X) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & X \end{array}$$

Remark.— Let us recall that being étale is the same as being formally unramified and of finite presentation: $f : X \to Y$ is étale iff $\mathbb{L}_{X/Y} \simeq 0$ and $f : t_0(X) \to t_0(Y)$ is finitely presented.

Proof.— For this we will build inductively a family of morphisms $f_i : \mathrm{Spec}(A_i) \to X$ that will be closer and closer to étaleness. Precisely, we ask that

$$H_j(\mathbb{L}_{\mathrm{Spec}(A_i)/X}) = 0 \text{ for } j \leq i.$$

We also ask that A_i be i-truncated and that for all i, we have a morphism $A_{i+1} \to A_i$ that induces isomorphisms on H_j for $j \leq i+1$.

Given such a sequence of approximations, we can take the limit of the tower and set

$$A = \varprojlim_i A_i$$

Because X is Postnikov continuous, we are then supplied with a morphism $f : \mathrm{Spec}(A) \to X$ such that

$$\begin{array}{ccc} \mathrm{Spec}(A_0) & \longrightarrow & t_0(X) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & X \end{array}$$

is cartesian. So we only need to show that f is étale.

Let M be an A-module, then by definition

$$\mathrm{Der}_X(\mathrm{Spec}(A), M) = \mathrm{Map}_{X/\mathcal{A}\mathrm{ff}}(\mathrm{Spec}(A \oplus M), X)$$

Choose a Postnikov tower for M

$$M \simeq \varprojlim_i M_{\leq i}$$

Then we have a Postnikov tower for the algebra $A \oplus M$,

$$A \oplus M \simeq \varprojlim_i (A_i \oplus M_{\leq i})$$

By this we deduce that

$$\mathrm{Der}_X(\mathrm{Spec}(A), M) \simeq \varprojlim_i \mathrm{Der}_X(\mathrm{Spec}(A_i), M_{\leq i})$$

Then by construction, for every i,

$$\mathrm{Der}_X(\mathrm{Spec}(A_i), M_{\leq i}) \simeq \mathrm{Map}_{\mathrm{Spec}(A_i)}(\mathbb{L}_{\mathrm{Spec}(A_i)/X}, M_{\leq i}) \simeq 0$$

Then

$$\mathbb{L}_{\mathrm{Spec}(A)/X} \simeq 0.$$

And $t_0(\mathrm{Spec}(A)) = \mathrm{Spec}(A_0) \to t_0(X)$ is finitely presented by assumption. So $\mathrm{Spec}(A) \to X$ is étale.

Now we only need to build such a family of approximations

$$U_0 \to U_1 \to \ldots \to X$$

We do it inductively.

Initialisation at 0.

From the adjunction

$$\mathcal{S}\mathrm{t} \underset{t_0}{\overset{i}{\rightleftarrows}} \infty\text{-}\mathcal{S}\mathrm{t}$$

we get a morphism

$$\mathrm{Spec}(A_0) \xrightarrow{u} i t_0(X) \to X$$

Looking at the cofibre sequence, with $U_0 = \mathrm{Spec}(A_0)$,

$$u^*\mathbb{L}_{it_0(X)/X} \to \mathbb{L}_{U_0/X} \to \mathbb{L}_{U_0/it_0(X)}$$

But because $U_0 \to t_0(X)$ is étale, then $\mathbb{L}_{U_0/it_0(X)} \simeq 0$ and we get a quasi-isomorphism

$$u^*\mathbb{L}_{it_0(X)/X} \simeq \mathbb{L}_{U_0/X}$$

Finally, from what we know about Postnikov towers, we have that $\mathbb{L}_{it_0(X)/X}$ is 1-connected and so is $\mathbb{L}_{U_0/X}$.

Hence we have built the first approximation $U_0 \to X$.

Induction $n \to n+1$.

Suppose we have built $U_n \to X$. We have the morphisms

$$\mathbb{L}_{U_n} \to \mathbb{L}_{U_n/X} \to (\mathbb{L}_{U_n/X})_{\leq n+2} \simeq H_{n+2}(\mathbb{L}_{U_n/X})[n+2]$$

The composition defines a square zero extension of $A_{n+1} \to A_n$ of A_n by $H_{n+2}(\mathbb{L}_{U_n/X})[n+2]$.

Thanks to the obstruction theory of X, we are then supplied with a new map

$$U_n \to U_{n+1} \to X$$

satisfying the required assumption. □

☛ **Back to the proof.**— Thanks to the lemma, we have an étale morphism $U \to X$ with U affine and $t_0(U) \simeq t_0(X)$. This means in particular that for every 0-truncated cdga A, we have

$$U(A) \simeq X(A)$$

Induction $n \to n+1$

Suppose that for any n-truncated $A \in \mathcal{A}lg_k^{cdg_{\leq 0}}$, we have $U(A) \simeq X(A)$. And let $A \in \mathcal{A}lg_k^{cdg_{\leq 0}}$ be $n+1$-truncated. Then A is a square zero extension of $A_{\leq n}$. Then because both U and X have obstruction theories, we deduce that $U(A) \simeq X(A)$.

Finally because both U and X are nilcomplete, we deduce that for any $A \in \mathcal{A}lg_k^{cdg_{\leq 0}}$,

$$U(A) \simeq X(A) \Longrightarrow U \simeq X$$

Induction $n \to n+1$

Suppose that the criterion is proved for n-geometric derived stacks and let X be a derived stack which is nilcomplete, has an obstruction theory and such that $t_0(X)$ is an n-geometric stack.

To begin, we show that $X \to X \times X$ is n-representable. Let U be an affine derived stack, then $Y = U \times_{X\times X} X$ satisfies the criterion of representability: by stability under pullbacks, Y is nilcomplete and has an obstruction theory. Furthermore, because $t_0(X)$ is an $(n+1)$-geometric stack, $t_0(Y)$ is an n-geometric stack.

We now have to build an $(n+1)$-atlas for X. Let $Y_0 \to t_0(X)$ be an $(n+1)$-atlas. Then it is possible to lift it to an $(n+1)$-atlas of X thanks to the following lemma.

Lemma.— *Let* $U_0 \to t_0(X)$ *be a smooth morphism with* U_0 *an affine stack, then there exists an affine derived stack* U *and a smooth morphism* $U \to X$ *such that the following square is cartesian,*

$$\begin{array}{ccc} U_0 & \longrightarrow & t_0(X) \\ \big\downarrow & \lrcorner & \big\downarrow \\ U & \longrightarrow & X \end{array}$$

Which can be proved exactly as the previous lemma. Thanks to this lemma, we know we can lift 0-altases and by induction, we can lift n-atlases.

REFERENCES

[HAG2] Toën, B., & Vezzosi, G. (2004). *Homotopical Algebraic Geometry II: geometric stacks and applications.* arXiv preprint math/0404373.