CONFIGURATION SPACES AND BRAIDS

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Today we'll see configuration spaces and braids. Before doing this, let me recall some definitions.

Let X be a topological space (here it means at least locally connecte, Hausdorff, second countable etc). Practically it is going to be a CW-complex. For $k \geq 0$, $\mathrm{Conf}_k(X)$ is the set of ordered configurations in X. The group Σ_k acts on the right on $\mathrm{Conf}_k(X)$. The space $B_k(X) := (\mathrm{Conf}_k(X))_{\Sigma k}$ is the space of unordered configurations in X. Last week we have computed configuration spaces of the empty set, of the point, the interval, etc.

1 SECTION 2.2

Assume we have a continuous function $f : X \to Y$ that is injective. Then for every k, we have an injective function $f^k : X^k \to Y^k$. Then it is very easy to see that f induces a map $\mathrm{Conf}_k(f) : \mathrm{Conf}_k(X) \to \mathrm{Conf}_k(Y)$. In addition, it is compatible with the symmetric group action. That means we also get a map $B_k(f) : B_k(X) \to B_k(Y)$.

What we have is actually two functors

$$\mathrm{Conf}_k : \mathsf{Top}^{\mathsf{inj}} \longrightarrow \mathsf{Top}^{\Sigma_k}$$

and

$$B_k : \mathsf{Top}^{\mathsf{inj}} \longrightarrow \mathsf{Top}$$

Proposition 1.1. — *If $f : X \to Y$ is an open embedding. Then $\mathrm{Conf}_k(f)$ and $B_k(f)$ are also open embeddings.*

Proposition 1.2. — *The map*

$$\coprod_{i+j=k} \mathrm{Conf}_i(X) \times \mathrm{Conf}_j(Y) \times_{\Sigma_i \times \Sigma_j} \Sigma_k \longrightarrow \mathrm{Conf}_k(X \amalg Y)$$

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is an homeomorphism. Then

$$B_i(X) \times B_j(Y) \longrightarrow B_k(X \amalg Y)$$

is an homeomorphism.

Definition 1.3. — *Define*

$$B(X) := \coprod_{k \geq 0} B_k(X)$$

Proposition 1.4. — *One has*

$$B(X \amalg Y) = B(X) \times B(Y).$$

Assume $\mathcal{B}$ is a basis for X. Consider

$$\mathcal{B}_k = \{U \subset X | U \simeq \amalg_{i=1}^k U_i, \quad U_i \in \mathcal{B}\}$$

Then one can equip a partial order: $U \leq V$ if and only if $U \subset V$ and $\pi_0(U) \to \pi_0(V)$ is surjective.

It can then be extended to

$$\mathcal{B}_k^\Sigma := \{(U, \sigma) | U \in \mathcal{B}_k, \sigma : \{1, \ldots, k\} \to \pi_0(U)\}$$

Then the partial order is given by $(U, \sigma) \leq (V, \tau)$ if $U \leq V$ and $\tau(i) = \pi_0(U \to V) \circ \sigma(i)$.

If $U \in \mathcal{B}_k$, then

$$B_k^0(U) := \{\{x_1, \ldots, x_k\} | U_i \cap \{x_1, \ldots, x_k\} \neq \emptyset\} \simeq \prod_{i=1}^k U_i$$

is a subset of $B_k(X)$.

If $(U, \sigma) \in \mathcal{B}_k^\Sigma$, then one has

$$\mathrm{Conf}_k^0(U, \sigma) \subset \mathrm{Conf}_k(X) \simeq \prod_{i=1}^k U_{\sigma(i)}.$$

Proposition 1.5. — *The collection of* $B_k^0(U)$ *and the collection of* $\mathrm{Conf}_k^0(U, \sigma)$ *are basis of* $B_k(X)$ *and* $\mathrm{Conf}_k(X)$.

The proof is very easy. $\mathrm{Conf}_k(X)$ is a subpace of X^k with subspace topology, so a basis element in X^k is going to be a product of elements in a basis of X. In order to find a basis of $\mathrm{Conf}_k(X)$, one only need to prove that for any point x in a a product of opens in X^k is contained in $\mathrm{Conf}_k^0(X, \sigma)$. This is obvious. The only problem can come frome the fact that opens in the product could intersect. This is why we need Hausdorff property.

Proposition 1.6. — *The map* $\mathrm{Conf}_k(X) \to B_k(X)$ *is a covering space.*

When M^n *is a manifold, then both* $\mathrm{Conf}_k(M)$ *and* $B_k(M)$ *are also nk-manifolds. So we get functors*

$$(\mathsf{Man}^{\mathsf{inj}}, \amalg) \longrightarrow (\mathsf{Man}, \times)$$

2 SECTION 2.3

☞ THEOREM 2.1 (Fadell-Neuwirth). — *For* M *a manifold and* $k > l$*, we have a (homotopy) fibre sequence*

$$\mathrm{Conf}_{k-l}(\mathrm{M} - \{x_1, \ldots, x_l\}) \to \mathrm{Conf}_k(\mathrm{M}) \to \mathrm{Conf}_l(\mathrm{M})$$

The result does not depend on the choice of the point x_1 *to* x_l*. The projection map is actually a fibre bundle.*

COROLLARY 2.2. — *As a consequence, there is a long exact sequence of homotopy groups*

$$\cdots \to \pi_i(\mathrm{Conf}_{k-l}(\mathrm{M}-\{\ldots\})) \to \pi_i(\mathrm{Conf}_k(\mathrm{M})) \to \pi_i(\mathrm{Conf}_l(\mathrm{M})) \to \ldots$$

and for M *simply connected of dimension* $n \geq 3$*, then* $\mathrm{Conf}_k(\mathrm{M})$ *is also simply connected, so* $\mathrm{B}_k(\mathrm{M})$

$$\pi_1(\mathrm{B}_k(\mathrm{M})) \simeq \Sigma_k$$

Moreover if M^2 *is not the sphere or* $\mathbf{RP}^2$*, then* $\mathrm{Conf}_k(\mathrm{M})$ *is aspherical.*

The proof is very easy and uses the long exact sequence. The proof is trivial for $k = 1$ and the rest follows by induction.

PROPOSITION 2.3. — *If* M *has dimension* $n \geq 3$*, then*

$$\pi_1(\mathrm{Conf}_k(\mathrm{M})) \simeq \prod^k \pi_1(\mathrm{M})$$

and

$$\pi_1(\mathrm{B}_k(\mathrm{M})) \simeq \prod^k \pi_1(\mathrm{M}) \rtimes \Sigma_k$$

PROPOSITION 2.4. — *If* M *has a non-empty boundary* N*, then* $\mathrm{Conf}_l(\mathrm{M}) \to \mathrm{Conf}_k(\mathrm{M})$ *admits a homotopy section for* $l < k$*.*

COROLLARY 2.5. — *For* $n \geq 3$ *and* $k \geq 0$ *and* $i \geq 0$*,*

$$\pi_i(\mathrm{Conf}_k(\mathbf{R}^n)) \cong \prod_{j=1}^{k=1} \pi_i(\vee_j \mathrm{S}^{n-1})$$

The proof follows again using the long exact sequence and the fact that $\pi_i(\mathrm{Conf}_k(\mathbf{R}^n)) \to \pi_i(\mathrm{Conf}_{k-1}(\mathbf{R^n}))$ has a section.

3 SECTION 2.4

Let's now talk of braid groups. They were introduced by Artin. A braid is a loop in $\mathrm{Conf}_k(\mathbf{R}^2)$. So the braid group is the fundamental group of Conf_k or B_k.

Define

$$\mathbf{B}_k := < \sigma_1, \ldots, \sigma_{k-1} | \sigma_i\sigma_j = \sigma_j\sigma_i \quad |i-j| > 1, \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \quad |i-j| = 1 >$$

Then one can build a map $\mathbf{B}_k$ to $\pi_1(\mathrm{B}_k(\mathbf{R}^2))$. This map sends σ_i to a braid switching the paths i and $i+1$. It is easy to see that this is a group homomorphism.

There is an obvious map $\mathbf{B}_k \to \Sigma_k$ sending σ_i to $\tau_i = (i, i+1)$. Similarly, there is also a homomorphism $\pi_1(\mathrm{B}_k(\mathbf{R}^2)) \to \Sigma_k$ since $\mathrm{Conf}_k(\mathbf{R}^2) \to \mathrm{B}_k(\mathbf{R}^2)$ is a Σ_k-cover.

It is then possible to reduced to the case of the pure braid group $\mathbf{P}_k$ (the kernel of the map $\mathbf{B}_k \to \Sigma_k$) and $\pi_1(\mathrm{Conf}_k(\mathbf{R}^2))$.

Then one can use the long exact sequence again to prove the claim by induction, using that $\pi_1(\mathbf{R}^2 - \{x_1, \ldots, x_{k-1}\})$ is a free group and the following lemma.

Lemma 3.1. — *One has*

$$\mathbf{P}_k \cong \mathrm{U} \rtimes \mathbf{P}_{k-1}$$

where U *is the kernel of* $\mathbf{P}_k \to \mathbf{P}_{k-1}$.

This lemma is a bit hard to prove because we don't have a presentation for $\mathbf{P}_k$. So we use an alternative lemma

Lemma 3.2. — *If* D_k *is the inverse image of* Σ_{k-1} *in* $\mathbf{B}_k$. *Then*

$$\mathrm{D}_k \simeq \mathrm{U} \rtimes \mathbf{B}_{k-1}$$

Then follows a discussion about Schrieir sets and fundamental groups of graphs.