INTRODUCTION TO CONFIGURATION SPACES I

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We are following that one paper from Knudsen called *Configuration spaces in algebraic topology*.

Let me write down an outline:

- Definitions and goals;
- Motivation;
- Configuration spaces of $\mathbf{R}^n$ and $\mathbf{S}^1$;
- First properties

Definition 0.1. — *For a topological space* X, *the* configuration space *of ordered points in* X *is the subspace of* X^k*:*

$$\mathrm{Conf}_k(X) := \{(x_1, \ldots, x_k) / \forall i \neq j x_i \neq x_j\}$$

with the subspace topology.

We can observe that there is an evident action of Σ_k on $\mathrm{Conf}_k(X)$ by permuting the indices. Explicitely, I mean

$$(\sigma, (x_1, \ldots, x_k)) \longmapsto (x_{\sigma(1)}, \ldots, x_{\sigma(k)})$$

Definition 0.2. — *The* unordered configuration space *of* X *is the quotient*

$$B_k(X) := (\mathrm{Conf}_k(X))_{\Sigma_k}.$$

The goal of Ben's paper is to understand the topology of these configuration spaces through the understanding of their homotopy groups and (co)homology groups. In particular in the case where X is a manifold.

Derived Seminar, November 2019, Pohang, Korea.

1.1 *Invariants*

Most of this paper just uses pretty elementary algebraic topology. And I think there is some categorical flavour.

Basically some people study configuration spaces really I think because they give good homeomorphism invariants of spaces: for example in knot theory. Let me give you some of this motivation. And I should follow his numbering, I am going to try to follow it. Sometimes I am going to quote him because he has some slogans.

Here is a slogan: 'The homotopy type of a fixed configuration space is a homeomorphism invariant of the background manifold'. Ok, let us see some example.

Examples:

- $B_2(\mathbf{R}^m) \simeq B_2(\mathbf{R}^n)$ if and only if $m = n$.
- $B_2(\mathbf{T}^2 - *) \not\simeq B_2(\mathbf{R}^2 - \mathbf{S}^0)$. But $\mathbf{T}^2 - * \simeq \mathbf{S}^1 \vee \mathbf{S}^1 \simeq \mathbf{R}^2 - \mathbf{S}^0$.

This indicates that configuration spaces are sensitive to proper homotopy types.

Here is the best example, I think it is a really nice example.

Lens spaces: for p, q relatively prime

$$\mathrm{L}(p,q) := \mathbf{D}^2 \times \mathbf{S}^1 \amalg_{\mathbf{T}^2} \mathbf{D}^2 \times \mathbf{S}^1$$

where the torus is embedding either via the identity or along a (p,q)-knot.

☞ Theorem 1.1 (Reidemeister). — *One has*

- $\mathrm{L}(p,q_1) \simeq \mathrm{L}(p,q_2)$ *if and only if* $q_1 q_2 = \pm n^2 (\mathrm{mod} p)$;
- $\mathrm{L}(p,q_1) \cong \mathrm{L}(p,q_2)$ *if and only if* $q_1 = \pm q_2^{\pm 1} (\mathrm{mod} p)$.

☞ Theorem 1.2 (Longoni-Salvatore). — *They show that*

$$\mathrm{Conf}_2(\mathrm{L}(7,1)) \not\simeq \mathrm{Conf}_2(\mathrm{L}(7,2))$$

but $\mathrm{L}(7,1) \simeq \mathrm{L}(7,2)$ *and* $\mathrm{L}(7,1) \not\cong \mathrm{L}(7,2)$.

1.2 *Braids*

There is a very nice presentation of the group

$$\pi_1(\mathrm{B}_k(\mathbf{R}^2))$$

as braids.

1.3 *Embeddings*

A) The space of framed (= rectilinear) embeddings from $\coprod_k \mathbf{R}^n$ to $\mathbf{R}^n$ is homotopy equivalent to $\mathrm{Conf}_k(\mathbf{R}^n)$. The collection of these spaces forms a topological operad called E_n;

B) There is this thing called factorisation homology: take the structure of an E_n-algebra locolly on a manifold and paste it globally. I probably won't say very much. Factorisation homology probes a space by sending points to this manifold;

C) One more thing: embedding calculus. The idea is you have $\mathrm{Emb}(\mathrm{M}, \mathrm{N})$ and you have an approximation tower $\mathrm{T}_1\mathrm{Emb}(\mathrm{M}, \mathrm{N}), \leftarrow \mathrm{T}_2\mathrm{Emb}(\mathrm{M}, \mathrm{N}) \leftarrow \dots$ etc. Each time you get closer and closer to $\mathrm{Emb}(\mathrm{M}, \mathrm{N})$.

2 CONFIGURATION SPACES OF EXAMPLES

We have

I) $\mathrm{Conf}_k(\emptyset) := \emptyset$ if $k > 0$ and $*$ otherwise;

II) $\mathrm{Conf}_0(\mathrm{X}) := *$;

III) $\mathrm{Conf}_k(*) := \emptyset$ if $k > 1$ and $*$ otherwise;

IV) $\mathrm{Conf}_k(\mathbf{R}) \simeq \coprod_{\Sigma_n} *$ and $\mathrm{B}_k(\mathbf{R}) \simeq *$. Precisely, one can homeomorphically map $\mathrm{B}_k((0,1))$ to the space Δ^k of tuples $(t_0, \dots, t_k)$ in $(0,1)$ with $\sum t_i = 1$. This map is the 'gap' map sending $(x_1 < \dots < x_k)$ to $(x_1, x_2 - x_1, \dots, x_k - x_{k-1}, 1 - x_k)$.

V) In general $\mathrm{Conf}_k(\mathbf{R}^n)$ is not known. But $\mathrm{Conf}_2(\mathbf{R}^n) \simeq \mathbf{S}^{n-1}$ and $\mathrm{B}_2(\mathbf{R}^n) \simeq \mathbf{RP}^{n-1}$. To see this, map homoemorphically $\mathrm{Conf}_2(\mathbf{R}^n)$ to $\mathbf{S}^{n-1} \times \mathbf{R}_{>0} \times \mathbf{R}^n$ by sending

$$(x_1, x_2) \longmapsto \left(\frac{x_2 - x_1}{\|x_2 - x_1\|}, \|x_2 - x_1\|, \frac{x_1 + x_2}{2} \right)$$

VI) $\mathrm{Conf}_k(\mathbf{S}^1)$ is known. It is homotopy equivalent to $\coprod_{\Sigma_{k-1}} \mathbf{S}^1$. And $\mathrm{B}_k(\mathbf{S}^1) \simeq \mathbf{S}^1$.

I suggest you give your students the exercice of computing $\mathrm{Conf}_2(\mathbf{S}^1)$ because it's all about cutting and pasting: $\mathrm{Conf}_2(\mathbf{S}^1) = \mathbf{T}^2 - \mathrm{diag} \simeq \mathbf{S}^1$. And $\mathrm{B}_2(\mathbf{S}^1)$ is homotopy equivalent to a Möbius Band.