CGP DERIVED SEMINAR

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1. July 15: Kyeong-Seog Lee: Motivic integrations

Let me start with some motivation. Let X and Y be smooth projective Calabi–Yau varieties over $\mathbb{C}$, birationally equivalent, meaning that I have some open set U_X in X which is isomorphic to another open set U_Y in Y.

Then there is a theorem by Batyrev who says that the Betti numbers of X and Y are the same. Later Kontsevich made the stronger statement, that for every p and q the Hodge numbers $h^{p,q}(X) = h^{p,q}(Y)$ for all p and q.

Let me sketch how they approach this problem, with the idea of (half of) the proof.

Let X be an algebraic variety over $\mathbb{C}$. By Deligne there is a so-called mixed Hodge structure on $H^k(X,\mathbb{C})$, which means there is a natural increasing weight filtration

$$0 = W_{-1} \subset W_0 \subset \cdots \subset W_{2k} = H^k(X,\mathbb{Q}),$$

and a decreasing Hodge filtration

$$H^k(X,\mathbb{C}) = F^0 \supset F^1 \supset \cdots \supset F^k \supset F^{k+1} = 0.$$

such that $F^\bullet$ satisfies $Gr_\ell^W H^k(X,\mathbb{Q}) = W_\ell/W_{\ell-1}$ is a pure Hodge structure of weight ℓ.

$$h^{p,q}(H^k(X,\mathbb{C})) = \dim_{\mathbb{C}}(F^p Gr^W_{p+q} H^k(X) \cap \overline{F^q Gr^W_{p+q} H^k(X)})$$

When I have an arbitrary algebraic variety, I can define this kind of Hodge Deligne-number and Danilov–Khovanski tells me there's a mixed Hodge structure on $H^k_c(X,\mathbb{C})$. For X a complex algebraic variety, I define

$$E(X) = \sum_{0\le p,q\le \dim X}\ \sum_{0\le k\le 2\dim X} (-1)^k h^{p,q}(H^k_c(X,\mathbb{C}))u^p v^q.$$

The theorem is that if X is a union of locally closed pieces then $E(X) = \sum E(X_i)$ where X_i are these pieces. Moreover, $E(X\times Y) = E(X)E(Y)$, and finally if $Y \xrightarrow{F} X$ is a locally trivial fibration with respect to the Zariski topology then $E(Y) = E(X)E(F)$.

Let $\operatorname{Var}/\mathbb{C}$ be the category of complex alebraic varieties.

The Grothendieck ring of $\operatorname{Var}/\mathbb{C}$ is the free Abelian group generated by isomorphism classes of varieties, modulo the relation $[x]-[y]-[X\smallsetminus Y]$ where Y is closed in X.

This has a natural ring structure via the product.

You can localize $K_0(\operatorname{Var}/\mathbb{C})$ by inverting the affine line, and then define $\widehat{M_{\mathbb{C}}}$, and get a sequence of embeddings of $\mathbb{Z}[u,v]$ into the completion of $\mathbb{Z}[u,v,\frac{1}{uv}]$ so if $X\mapsto E(X)$ become the same in $\mathbb{Z}[u,v,\frac{1}{uv}]$ completed then they were already the same.

There are lots of definitions we have to discuss to make this precise.

Let me call an element M in $K_0(\mathrm{Var}/\mathbb{C})$ *d-dimensional* if there is an expression

$$M = \sum m_i [X_i]$$

in $K_0(\mathrm{Var}/\mathbb{C})$ with $m_i \in \mathbb{Z}$ and X_i an algebraic variety of dimension at most d, and there exists no such expression like this with all dimensions at most $d-1$.

Then this gives a map $K_0(\mathrm{Var}/\mathbb{C}) \to \mathbb{Z} \cup \{-\infty\}$ and this factors through $K_0(\mathrm{Var}/\mathbb{C})$ localized at $\mathbb{L}^1$, and I can define

$$F_d(K_0(\mathrm{Var}/\mathbb{C})[\mathbb{L}^{-1}])$$

to be

$$\{M \in K_0(\mathrm{Var}/\mathbb{C})[\mathbb{L}^{-1}] : \dim M \le d\}$$

and $F^d = F_{-d}$, and we let $\widehat{M_{\mathcal{C}}}$ be the inverse limit of $M_{\mathbb{C}}/F^d M_{\mathbb{C}}$. So you want $\mathrm{Var}/\mathbb{C} \to K_0(\mathrm{Var}/\mathbb{C}) \to M_{\mathbb{C}} \to \widehat{M_{\mathcal{C}}}$.

This will be accomplished by *motivic integration.*

The final goal is to have $[X] = \int_{J_\infty(X)} \mathbf{1} d\mu_X$, so I should define J_∞ and $\mathbf{1}$ and $d\mu$.

Let X be a scheme of finite type over k. An m-jet of X is a morphism $\operatorname{Spec} k[t]/(t^{m+1}) \to X$. Then $J_m(X)$ is the scheme of all m-jets into X. Then because I have a natural projection $J_m(X) \to J_{m-1}(X)$, I can take the inverse limit to define $J_\infty(X)$. We call this the *arc space* of X. This is represented by $\operatorname{Spec} k[[t]]$.

Let me give two examples. Let X be affine. If X is $\mathbb{A}^n$ then $J_m(X)$ is maps from $k[x_1, \ldots, x_n] \to k[t]/(t^{m+1})$. So $x_i \mapsto x_i^{(0)} + \cdots + x_i^{(m)} t^m$, and this is free so $J_m(X)$ is $\mathbb{A}^{n(m+1)}$.

Now let Y be a hypersurface in $\mathbb{A}^n$ defined by $f = 0$.

Then $J_m(Y)$ is maps $\operatorname{Spec} k[t]/(t^{m+1}) \to Y$, which is maps $k[x_1, \ldots, x_n]/(f) \to k[t]/(t^{m+1})$, and f can be written as $f^{(0)} + \cdots + f^{(m)} t^m$ and so you get the zeros of $f^{(0)}, \ldots, f^{(m)}$.

Finally, let me write a proposition. When X is a smooth scheme of dimension n over k then $J_m(X)$ is locally an $\mathbb{A}^{nm}$-bundle over X. In particular $J_{m+1}(X)$ is locally an $\mathbb{A}^n$-bundle over $J_m(X)$.

I have my domain $J_\infty(X)$ and I want to define $d\mu$.

A subset A in $J_\infty(X)$ is *stable* if for every large enough integer m, let me define A_m to be $\pi_m(A)$ is a constructible subset of $J_m(X)$ (here π_m is the projection from the inverse limit to J_m), A is $\pi_m^{-1}(A_m)$, and $A_{m+1} \to A_m$ is a locally trivial $\mathbb{A}^n$-bundle.

Then I can define a measure $\mu_X(A)$ as follows. We say that it's $[A_m]\mathbb{L}^{-nm} \in M_k$. For large enough m, we have $[A_{m+1}] = [A_m]\mathbb{L}^n$, so $\mu_X(A)$ is well-defined.

A measurable set is the same philosophy as constructible sets, and then define the measure here.

When you remember real analysis very well, you can define a measurable function, and define a function F from $J_\infty(X) \to \mathbb{N}_{\ge 0} \cup \infty$ is measurable if for every s in $\mathbb{N}_{\ge 0}$, $\mathbb{F}^{-1}(s)$ is measurable and $F^{-1}(\infty)$ is measure 0.

The typical measurable function in this case, for Y a subscheme, is $\mathrm{ord}_Y : J_\infty(X) \to \mathbb{N} \cup \{\infty\}$, where, $\theta : \operatorname{Spec} k[[t]] \to X$, so it's a map $\mathcal{O}_X \to k[[t]]$, and this is, $\mathrm{ord}_Y(\theta)$ is the supremum of e such that $\theta(I_Y) \subset (t^e)$.

Let X be a smooth k-scheme and Y a smooth subscheme of X. Then the motivic integration of $\mathbb{L}^{-\operatorname{ord}_Y}$ on X is defined to be

$$\int_{J_\infty(X)} \mathbb{L}^{-\operatorname{ord}_Y} d\mu_X = \sum_{s=0}^{\infty} \mu(\operatorname{ord}_Y^{-1}(s))\mathcal{L}^{-s}.$$

When Y is empty, then the order of Y is uniformly 0 then

$$\int_{J\infty(X)} \mathbb{L}^{-\operatorname{ord}_Y} d\mu_X = \mu(\operatorname{ord}_Y^{-1}(0)),$$

and this is some kind of $\mathbb{A}^\infty$ bundle over X, and so $J_m(X)$ is just an $\mathbb{A}^{nm}$-bundle over X. I defined my integral by $\mu_X(A) = [A_m]\mathbb{L}^{-nm}$, and so the class in the Grothendieck ring is just $[X][\mathbb{A}^{nm}][\mathbb{L}^{-nm}]$ and this is $[X]$ in M_k, and then I just complete.

This trivial integral is nothing but $[X]$ itself.

The second remark is that when Y^{n-1} is a smooth divisor in X^n, my $J_s(Y)$ is locally $\mathbb{A}^{(n-1)s}$-bundle over Y, and I can do this type of integral again, and $\operatorname{ord}_Y^{-1}(s)$, let me not explain this calculation, is $\pi_{s-1}^{-1} J_{s-1}(Y) - \pi_s^{-1} J_s(Y)$.

Then because this class is nothing but $[Y]\mathbb{L}^{(n-1)s}$ and I can define my motivic integration

$$\int_{J_\infty(X)} \mathbb{L}^{-\operatorname{ord}_Y} d\mu_X = [X-Y] + \sum_{s=1}^{\infty} [Y](\mathbb{L}-1)\mathbb{L}^{-s}\mathbb{L}^{-s}$$

and the final result is that this is

$$[X-Y] + \frac{[Y]}{[\mathbb{P}^1]}.$$

Finally we can prove Kontsevich's theorem. First let me say, let $Z \xrightarrow{f} X$ be a proper birational morphism of smooth k-schemes and D an effective divisor on X. Then

$$\int_{J_\infty(X)} \mathbb{L}^{-\operatorname{ord}_D} d\mu_X = \int_{J_\infty(Z)} \mathbb{L}^{-\operatorname{ord}_{f^{-1}D + K_{Z/X}}} d\mu_Z.$$

So you can compute this integral after pulling back if you modify it in this way.

You know that when you have a function $g : g^{-1}A \to A$. Then

$$\int_A h(f) d\mu = \int_{g^{-1}A} h(f) \operatorname{Jac}(g) d\mu$$

and this Jacobian looks something like this $K_{Z/X}$ term.

Let me give two examples.

First, let Z be a blowup of X along Y of codimension c, and $D = \varnothing$. Then the canonical bundle formula says that $K_{Z/X}$ is $(c-1)E$, and then

$$[X] = \int_{J_\infty(X)} \mathbb{L}^{-\operatorname{ord}\varnothing} d\mu_X = \int_{J_\infty(Z)} \mathbb{L}^{-\operatorname{ord}_{(c-1)E}} d\mu_Z = [Z-E] + \frac{[E]}{[\mathbb{P}^c]},$$

and in the blowup case, this is a $\mathbb{P}^c$ bundle over Y, so this is

$$[X-Y] + [Y] = [X].$$

Theorem 1.1 (Kontsevich)**.** *Let X and Y be smooth projective Calabi–Yaus and birational. Then $h^{p,q}(X) = h^{p,q}(Y)$.*

Proof. I can find a dominant smooth projective variety Z above X and Y via proper birational maps.

Since these are Calabi–Yau, $K_{Z/X} = K_Z - f^*K_X = K_Z$, and the same for $K_{Z/Y}$.

Then by my formula

$$\begin{aligned}[X] &= \int_{J_\infty(Z)} \mathbb{L}^{-\operatorname{ord}_{K_{Z/X}}} \\ &= \int_{J_\infty(Z)} \mathbb{L}^{-\operatorname{ord}_{K_{Z/Y}}} \\ &= [Y]\end{aligned}$$

in $\widehat{M_k}$. □

So you have this map

$$\operatorname{Var}/k \to K_0(\operatorname{Var}/k) \to K_0(\operatorname{Var}/k)[\mathbb{L}^{-1}] \to \widehat{M_k}$$

and we only just found out that the last map here is injective. The composite certainly isn't. So we wonder what you can say if $[X] = [Y]$ under this map.

Zargar changed the framework to $DM_{gm}^{eff}(k)$, an ∞-category related to Voevodsky motives, and this goes to $M(k,R)$ in K_0 of stable ∞-categories and you get an injection at the level of K_0.

We say that X and Y are K-equivalent if there is a span $X \leftarrow Z \to Y$ with $f^*K_X = g^*K_Y$. Then we can ask about $M(X)$ and $M(Y)$ and the theorem is that we have

$$M_{num}(X)_{\mathbb{Q}} \cong M_{num}(Y)_{\mathbb{Q}}.$$

I think this is a quite powerful tool and it has many applications. I proved a result using it, and it might have more.