CGP DERIVED SEMINAR

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1. March 25: Kyoung-Seog Lee: Introduction to Voevodsky motives

Because there are few, sorry for the two audience members, but since there are only two overlaps, let me briefly recall what I did last time and then go to Voevodsky motives.

Last time I explained why Grothendieck introduced Chow motives. I could only give the definition and construction and couldn't go further than that. Recall the construction of Chow motives, I'll spend twenty minutes or so, Grothendieck's original motivation was to construct, his idea was that there should be a universal cohomology theory taking values in a $\mathbb{Q}$-linear category of motives $M(k)$, so that every smooth projective variety over k and every Weil cohomology to graded K-algebras, he wants to find some kind of category of motives $M(k)$ and a functor h from varieties to motives so that for every such cohomology there is a unique functor from $M(k)$ making the triangle commute.

Let me briefly recall how he constructed that kind of category. I start from the category of smooth projective varieties over k and then I define the so-called category of correspondences, the objects is the same as the smooth projective varieties. The morphisms $\mathrm{Mor}(X,Y)$ is the Chow group $\mathrm{CH}^{\dim X}_{\mathbb{Q}}(X\times Y)$. That's, $Z^r(S)$ for S a scheme or variety is the free Abelian group on the codimension r cycles on S modulo rational equivalence (please do not ask what this is, I explained this for 20 minutes last time). If I had another equivalence, numerical or homological equivalence, I could have another group. So we call this Chow motives because we're using rational equivalence. If we chose numerical equivalence we'd call this maybe numerical motives.

If we choose two varieties, you can't add morphisms generically. You can take a graph of f and graph of g in the correspondences and can add these in this cycle group. Then I choose a pseudo-Abelian closure $\widetilde{CV_k}$, inputting a kernel and cokernel for each projector. Call this M_k^+. Then I localize this and obtain $M_k = M_k^+[\mathbb{L}^{-1}]$, inverting the Lefschetz motive, and this is my Chow motives. This is a pseudo-Abelian category (it has kernels and cokernels of idempotents).

I will also tell you why I, Grothendieck's original idea was to prove the Riemann hyothesis for this thing. Let me recall Grothendieck's standard conjecture C. This says that for X a smooth projective variety over k, he thinks that $h(X)$ should be $h^0(X)\oplus\cdots\oplus h^{2\dim X}(X)$ and then by a realization functor we have $H^*(X) = H^0(X)\oplus\cdots\oplus H^{2\dim X}(X)$ and from this we have the Riemann hypothesis for finite fields.

I can give you two applications that I think are important. Last time we had a very hard time to construct this one, so I want to tell you why this thing is maybe useful. I want to explain what kind of thing we get from this, some kind of possible applications.

Let X be a smooth projective variety over k, then H^* a Weil cohomology functor. By the axioms I have a cycle map $\gamma_i : Z^i(X) \to H^{2i}(X)$, and let me call $A^i(X)$ the image of γ_i.

For example for $k = \mathbb{C}$ and H^* usual singular cohomology, then

$$A^i(X) \subset H^{2i}(X,\mathbb{Q}) \cap H^{i,i}(X,\mathbb{C}).$$

Or another example for char $k = p$, and H^* the ℓ-adic cohomology, then

$$\mathbb{Q}_\ell A^i(X) \subset (H_\ell^{2i}(X)(1)(i))^{G_k}.$$

The Hodge conjecture in the first case and Tate conjecture 1 in the second case says that these are the same. So now I can make these kinds of conjectures for motives. I can say M is an *effective motive* meaning it belongs to M_k^+ and $A^i(M)$ is $\mathrm{im}\gamma_i$. Then the Hodge conjecture for M is that $A^i(M) = H^{2i}(M,\mathbb{Q}) \cap H^{i,i}(M,\mathbb{C})$ while the Tate conjecture 1 for M is that if k is finitely generated over its prime field then $\mathbb{Q}_\ell A^i(M) = H_\ell^{2i}(M)^{G_k}$.

Proposition 1.1. *For M and N in M_k, the Hodge conjecture for $M \oplus N$ implies the Hodge conjecture for M and N, and the same is true for the Tate conjecture 1.*

Theorem 1.1. *When X is $\mathcal{SU}_C(r,\mathcal{L})$ vector bundles of rank r with fixed degree with* $\deg \mathcal{L}$ *coprime to r for C a curve then I can give a nice such decomposition into the curve and its Jacobian, which lets me prove the Hodge conjecture is true.*

This kind of application is possible.

One more application of Manin, who learned from Grothendieck.

Theorem 1.2 (Manin)**.** *Let X be a smooth projective* 3*-fold, unirational over k a finite field. Then X satisfies the Riemann hypothesis.*

Maybe you think this was known by Deligne, but this is prior to Deligne. This is one approach. The idea of the proof, by unirational, there's a birational map $\mathbb{P}^3 \dashrightarrow X$.

This is characteristic p so I have Abhyankar's result, and I can get $\mathbb{P}^3 \leftarrow \tilde{X} \to X$, with a blowup of a smooth center in $\mathbb{P}^3$, so this gives $h(\tilde{X}) \cong h(\mathbb{P}^3) \oplus \bigoplus h(\mathrm{pt})^{\otimes} \oplus h(C_i)$ and these satisfy the Riemann hypothesis by previous results. Then $h(X)$ fits in $h(\tilde{X})$ so X satisfies the Riemann hypothesis.

Let me take a five minute break and then go to triangulated categories of motives.

I realized that this is maybe the right title. In the 1980s, there was a conjecture of Beilinson and Deligne independently.

Conjecture 1.1. When k is a field then there exists an Abelian tensor category MM_k of mixed motives containing Grothendieck's category of pure (homological) motives as the semi-simple object and some conditions or properties.

There should be some category containing this? Why? Deligne constructed mixed Hodge structures. When X is a smooth variety, we have

$$H^{2i}(X) = \bigoplus_{p+q=2i} H^{p,q}(X)$$

and if I have a Frobenius action on X then I have $H^{2i}(X,\mathbb{Q}_\ell)$ with eigenvalue q^i.

When X is non-projective or singular then you have no such thing but you have a weight filtration $W_j H^i(X)$ and the associated graded is pure. They think that, the motive is defined only for smooth projective varieties but even for singular varities you should have this one and the weight filtration should give the Hodge filtration.

People tried really hard to construct such categories. I don't know, maybe not really hard, but some people tried. They did not succeed. If you have time and energy, maybe you can try.

Remark 1.1. (1) This is still open and thought to be hard.

(2) (suggestion of Deligne) It's hard to construct this but constructing the bounded derived category of these motives might be easier. Maybe one reason is that the construction of mixed Hodge structures used this so much.
(3) There has been lots of progress. People really constructed tensor triangulated categories $DM(k)$ that have many expected properties of $D^b(MM_k)$. People really succeeded in this, but what one really wants is to construct a t-structure in $DM(k)$ to give MM_k.

There's something called non-commutative motives, constructed by Tabuada and Kontsevich. Today I only consider the original construction of Voevodsky.

Let me just say a few words about triangulated categories of motives. There are several constructions of this triangulated category of motives by Hanamura, Levine, and Voevodsky, who independently constructed tensor triangulated categories. They are expected to be equivalent. Indeed the Levine construction is equivalent to Voevodsky. People also expect Hanamura's to be equivalent. All three give the same $\mathbb{Q}$-motivic cohomology theory. Today I'll follow Voevodsky.

Indeed the construction is quite similar to the construction of Chow motives. Let k be a field and Sm/k is the category of smooth schemes over k, so one thing is good, because I erased the proper projective condition, admitting open varieties. It's not good that I'm in the smooth case but that will be removed. I'll construct a functor from Sm/k to $DM(k,\mathbb{Q})$ and later extend it to a functor from general schemes. Okay so let X and Y be smooth schemes over k. Then $c(X,Y)$ is the free Abelian group generated by integral closed subschemes W in $X\times Y$ such that $W\to X$ is finite and surjective over a connected component of X.

Then this is a really amazing fact, imposing this simple condition I can define composition.

Lemma 1.1. *If X_1, X_2, and X_3 are smooth schemes over k and $\phi\in c(X_1,X_2)$ and $\psi\in c(X_2,X_3)$, then you can really easily check that $\psi\circ\phi$, defined as $p_{13*}(p_{12}^*\phi\cap p_{23}^*\psi)$.*

Then I can define a new category of smooth correspondences over k with the same objects and morphisms $\mathrm{Hom}(X,Y)=c(X,Y)$ which is an additive category. Then I can define the so-called homotopy category $K^b(\mathrm{Sm\,Cor}/k)$ of $\mathrm{Sm\,Cor}/k$. I want to impose two conditions.

Let T be the class of complexes of the following two forms:

(1) For $X\in\mathrm{Sm}/k$, I have $[X\times\mathbb{A}^1]\to[X]$.
(2) For X in Sm/k and every $U\cup V=X$ open covering, I have

$$[U\cap V]\to[U]\oplus[V]\to[U]\cup[V]$$

and if these are in T then I define $\bar{T}$ as the minimal thick subcategory of K^b containing T.

Then whenever I have a thick subcategory of a triangulated subcategory, I can define the quotient by $\bar{T}$, the localization. Some people take this as the definition of the triangulated category of motives, but Voevodsky goes further and takes the pseudoAbelian closure. This is $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$. It has a very nice property. From the beginning, because I've modded out by T, it means that I have a Mayer–Vietoris sequence and as Philsang said, one might ask that this be a tensor triangulated category, and indeed it is.

Also when k admits resolution of singularity, some people say this can be removed these days, but anyway, then this contains effective Chow motives fully faithfully.

Then these motives have nice properties, all the properties you might expect. Let me give a few remarks and finish. Voevodsky said it's easy to construct but hard to deal with these. So he gave a sheaf theoretic definition of these motives. I don't have time to say this. In here the ingredients are Nisnevich topology, which I never saw before Voevodsky, and algebraic K-theory, and also homotopy invariant sheaves.

Let me tell you properties. This $DM : \mathrm{Sm}/k \to DM(k)$ can be extended to schemes of finite type over k. Then I can say that

1. Some Künneth formula holds: $M_{\mathrm{gm}}(X \times Y) \cong M_{\mathrm{gm}}(X) \otimes M_{\mathrm{gm}}(Y)$,
2. homotopy invariance, (i.e. $M_{\mathrm{gm}}(X \times \mathbb{A}^1) \cong M_{\mathrm{gm}}(X)$ for all X)
3. a Mayer–Vietoris sequence,
4. a blowup formula
5. a projective bundle formula
6. compactly supported cohomology, which extends beyond the smooth projective case.

If you want to see another definition of $DM(k)$ maybe I can personally talk to you.