CGP DERIVED SEMINAR

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1. Rune Haugseng: February 26: Yet another introduction to algebraic K-theory

I'm supposed to tell you something about algebraic K-theory. We heard a lot about motives and algebraic K-theory last time so I thought I would focus on the part I actually understand which is how you define K-theory.

Let me start with the warm-up, which is the Grothendieck group. Suppose $\mathcal{C}$ is a category with some notion of weak equivalences and a 0 object and some notion of direct sum (coproducts), and some sort of notion of a short exact sequence.

If you have this data (I'll be precise later) then you can define an Abelian group $K^0(\mathcal{C})$ generated by the objects of the category with the following relations:

(1) If c and c' are weakly equivalent then $[c] = [c']$, i.e., $c \sim c'$ in the equivalence relation generated by weak equivalences. In the simplest examples this will be isomorphisms.
(2) We want the direct sum to agree with addition in the group, $[c \oplus c'] = [c] + [c']$
(3) Given a short exact sequence

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d] \\ 0 \arrow[r] & c \end{tikzcd}$$

you want $[b] = [a] + [c]$.

Example 1.1.

- (Grothendieck, late fifties) Let $\mathcal{C}$ be the category of algebraic vector bundles on a nice variety X. Here isomorphisms are the weak equivalences. This gives $K_0(X)$.
- As a special case, think of vector bundles on the affine scheme of R, and these are finitely generated projective modules over R. This is usually $K_0(R)$. Since we restrict to projective modules, all short exact sequences split and we can forget the third relation. So for R a field, every projective module is free, and so you get $\mathbb{Z}$, one for each integer and then you get the negatives.
- If K is a number field you get the ideal class group of K and a copy of $\mathbb{Z}$, which I guess is just a consequence of how projective modules over $\mathcal{O}(K)$ look.
- If X is a topological space, you can look at $K_0(\mathbb{Z}[\pi_1 X])$ and this connects to geometric topology, containing the Wall finiteness obstruction, which measures whether you're a CW complex.

The number field gives something that number theorists are interested in, this gives something interesting to geometric topologists, if you plug in vector bundles on a scheme it might be interesting to algebraic geometers.

- You could plug in a small triangulated category with the short exact sequences the distinguished triangles. This is a non-example because this contains enough information to define K_0 but not the higher K-groups we'll come to later.
- So far all the examples have the weak equivalences be the isomorphisms. Let me give an example where they're not. If you want to define $K_0(X)$ for an arbitrary scheme X, then you need nice chain complexes of sheaves of $\mathcal{O}_X$-modules, the "perfect complexes with globally finite Tor-amplitude". The point is that in this case you use quasi-isomorphism.
- If you consider the category of pointed finite sets, and short exact sequences are pushouts of injective maps, and you get that K_0 of finite sets is $\mathbb{Z}$.

You can do more fancy things, cell complexes over and under a fixed space X, which gives Waldhausen's version and so on.

In the early 60s you get $K_1(R)$ and $K_2(R)$ explicitly with what look like parts of long exact sequences. Bass also constructed negative groups.

Quillen defined (in the 70s) higher K groups $K_n(\mathcal{C}) = \pi_n K(\mathcal{C})$ for a space $K(\mathcal{C})$ and eventually for some of the other examples you need a construction written up by Waldhausen in the early 80s, the $S.$ construction, although the story is that this was due to Graeme Segal but never appeared in print.

What I'll try to do in this talk is try to explain the Waldhausen $S.$ construction of K-theory. We can define K_0 as taking place via three steps.

(1) First, if we have some notion of weak equivalence, we can identify the weakly equivalent objects in the monoid of objects of $\mathcal{C}$ under direct sum
(2) You add negatives for the direct sum, something you can do for any monoid, called the group completion, in this case a commutative monoid
(3) Add the relations that split short exact sequences.

The goal is to explain how the $S.$ construction is a kind of homotopical analogue of these three steps. I want to explain in turn a homotopical version of each one of them.

So we want some way of inverting morphisms. So for this I have to talk about simplicial objects. We write $\mathbf{\Delta}$ for the category whose objects are finite non-empty ordered sets and $[n]$ is the set $\{0 < \cdots < n\}$ and the morphisms are order-preserving maps between these.

For example, we can define a map d_i from $[n-1]$ to $[n]$ which is the inclusion where you skip i (coface) or in the other direction we can do s_i by repeating i once (codegeneracy). Every morphism in the category can be written as composites of face and degeneracy maps, but not uniquely, there are some relations. I won't tell you the relations because I think they're quite useless. The point is that you can think of the simplicial as looking like this:

$$\longrightarrow$$

$$[0] \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; [1] \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; [2] \qquad\qquad \cdots$$

$$\longrightarrow$$

now a simplicial set is a functor $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X} \mathrm{Set}$ so $\mathrm{Set}_{\Delta} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set})$ so X is a collection of sets X_0, X_1, and so on, along with face maps $X_n \to X_{n-1}$ and degeneracies $X_n \to X_{n+1}$.

I'll tell you a way to get a space out of a simplicial set and then how to get a simplicial set out of a category so putting them together we'll get a topological space out of a category.

So let me define geometric simplices. There's a functor from Δ to topological spaces which takes $[n]$ to a geometric n-simplex

$$|\Delta^n| = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : 0 \le x_i; \sum x_i = 1\}.$$

You can define maps between these, if you have a map $\varphi : [n] \to [m]$ a map of ordered sets, you can define a map $|\Delta^n| \xrightarrow{\varphi} |\Delta^m|$ where

$$(x_0, \dots, x_n) \mapsto (y_0, \dots, y_m)$$
$$y_i = \sum_{j \in \varphi^{-1}(i)} x_j$$

and this gives the face and degeneracy maps. The relations I'm not telling you give the identities on faces of faces, and the degeneracies are some kind of projections.

So now having this geometric simplex functor, I formally get from that an adjunction relating simplicial sets to topological spaces, with left adjoint geometric realization and right adjoint the *singular simplicial set*. I start with a space and take $(\mathrm{Sing}\, T)_n$ to be $\mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, T)$.

The left adjoint geometric realization is given by a coend

$$\int_{\mathbf{\Delta}^{\mathrm{op}}} X \times |\Delta^*|$$

or a coequalizer of

$$\coprod_{\varphi:[m]\to[n]} X_n \times |\Delta^m| \rightrightarrows \coprod_n X_n \times |\Delta^n|.$$

You should think of this as giving you a topological space given by building up using X as a blueprint, by taking a copy of a geometric n-simplex for every $\sigma \in X_n$ and glue them in along their faces, according to the face maps and collapse degenerate faces.

Here's something else I can do, I can define a functor from $\mathbf{\Delta}$ into categories, by taking $[n]$ to the category $0 \to 1 \to \cdots \to n$, also called $[n]$. Again formally this gives rise to an adjunction between simplicial sets and categories, where the right adjoint is N, the nerve, and the left adjoint h, kind of the "homotopy category" described by a coend but colimits in categories are not very nice.

This nerve has a very explicit description. The n-simplices of the nerve are $\mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C})$, which is the sequences of n composable morphisms in $\mathcal{C}$. In this explicit description, the face maps correspond to composing, the face maps are composition at x_i if i is not 0 or n, and by forgetting the first or last morphism at the ends. The degeneracies are given by inserting identity maps.

Let me tell you some facts about this functor N. In fact it's fully faithful, a morphism of simplicial sets between nerves is exactly the same thing as a functor between the categories. Then for a general simplicial set I can't say much about h, but if I apply h to $\mathrm{Sing}\, T$ then I get something equivalent to the fundamental groupoid $\pi_{\leq 1} X$, which is given by taking objects the points and the morphisms from p to q the homotopy classes of paths from p to q. It tells you all the fundamental groups of X at the same time.

If we start from $\mathcal{C}$ we can build a topological space by first taking the nerve and then the geometric realization, $\|\mathcal{C}\| \coloneqq |N\mathcal{C}|$, often called the *classifying space of* $\mathcal{C}$, and formally there is a map from the nerve of $\mathcal{C}$ to $\mathrm{Sing}\,\|\mathcal{C}\|$.

I claim that $\|\mathcal{C}\|$ is a homotopical upgrade of inverting the morphisms in $\mathcal{C}$—if you start by inverting morphisms of $\mathcal{C}$ you get a groupoid.

I want to say something about what this looks like, $\|\mathcal{C}\|$ has a point for every object of $\mathcal{C}$ and then an edge relating those two points for each morphism $p \to q$, and then you add in two-cell for all composable pairs of morphisms, and keep going.

In particular, you're making the morphisms invertible because edges have inverses.

The set of components $\pi_0\|C\|$ is the quotient of the objects of $\mathcal{C}$ by the equivalence relation generated by $c \sim d$ if there exists a morphism from c to d.

You can also show that $\pi_{\leq 1}\|\mathcal{C}\| \cong \mathcal{C}[\mathcal{C}^{-1}]$.

I can view this construction taking $\mathcal{C}$ ot $\mathcal{C}[\mathcal{C}^{-1}]$ as left adjoint to the inclusion of groupoids into categories, and this construction $|-| : \mathrm{Set}_\Delta \to \mathrm{Top}$, you can view as modelling the ∞-version of this, left adjoint to the inclusion of ∞-groupoids into ∞-categories. We're regarding a category as a kind of ∞-category, and then doing this left adjoint there.

I guess, well, this was mainly a remark to those who know something about ∞-categories already, so not very useful maybe.

Let me tell you about this, if I start with a monoid M in Set, then I can define a category $\mathbb{B}M$ which has one object $*$ and $\mathrm{Hom}_{\mathbb{B}M}(*, *)$ is M with composition multiplication in M.

I can apply this construction to $\mathbb{B}M$, and get $\|\mathbb{B}M\|$ which is BM, which is called the classifying space of M, and so if G is a group, then $\|\mathbb{B}G\|$ is exactly the usual space $BG \cong K(G, 1)$, the universal space where π_1 is G.

Let me say one more sentence to finish the first section. For K-theory we had a category with weak equivalences, and we wanted to upgrade modding out equivalent objects, so we take $\|w\mathcal{C}\|$ for some subcategory $w\mathcal{C}$ spanned by weak equivalences.

2. Group completion

Right, so before the break we were talking about modding out by weak equvilances homotopically, now this is a homotopical version of group completion. If I had a monoid, I can define $BM = \|\mathbb{B}M\| = \|N\mathbb{B}M\|$ which is obtained by inverting morphisms in $\mathbb{B}M$, which is the same as adding negatives or inverses to M. Indeed,

BM is the same as BM_{gp} and since $M_{\mathrm{gp}} \cong \Omega BM$. So then ΩBM implements group completion of monoids in sets.

We want to do something similar for "monoids" in topological spaces. We have to be, you have to think what is the right notion of a monoid. For K-theory, we'd like $\|w\mathcal{C}\|$ to be a "monoid" via direct sum or coproduct in $\mathcal{C}$. This might not be literally a monoid in the strict sense. But this is likely not strictly associative, that $a \oplus (b \oplus c)$ and $(a \oplus b) \oplus c$, they're probably not strictly equal but canonically isomorphic.

The classical way to solve this is to replace the category $\mathcal{C}$ with a different one where the sum is associative. The modern thing to do, and you kind of have to do it in a sense, is a notion of monoid that's homotopy coherent. Let me start by saying this in the classical sense. If I have M a category with products (such as sets) then I can define an associative monoid as the data of a functor X from $\mathbf{\Delta}^{\mathrm{op}}$ to M such that $X_n \to X_1^{\times n}$ (coming from the inclusions of $[1]$ into $[n]$) is an isomorphism.

Let me try and justify that, X_0 says that this is isomorphic to a point, and so we have the degeneracy map $* \cong X_0 \to X_1$, which is a point, which tells you the unit of the monoid. If I look at X_2, the condition tells me that it's isomorphic to $X_1 \times X_1$, and the face map tells me that I have a binary product, which tells me the multiplication. Then if I look in X_3, there's an isomorphism to $X_1^{\times 3}$ and looking at the face maps I see that this is associative.

$$\begin{tikzcd}
X_1^{\times 3} \arrow[r, "(m,\mathrm{id})"] \arrow[d, "(\mathrm{id},m)"] & X_1^{\times 2} \\
X_1^{\times 2} & X_2
\end{tikzcd}$$

and you can show that the other data tells you nothing else.

For a monoid M in Set the corresponding map $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ is exactly $N\mathbb{B}M$.

Definition 2.1. An A_∞-*monoid* in Top (for the purpose of this talk) is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Top}$ such that $X_n \to X_1^{\times n}$ is a weak homotopy equivalence.

A monoidal category corresponds to functors from $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}$ so that $X_n \to X_1^{\times n}$ is an equivalence of categories.

In particular, explicitly in the case of $\mathcal{C}$ a category with coproducts, we can define $\mathcal{C}^{\otimes} : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}$ by taking $\mathcal{C}^{\otimes}_n$ to be the category of diagrams

$$\begin{tikzcd}
 & 0 \arrow[dl] \arrow[dr] & & 0 \arrow[dl] \arrow[dr] & & 0 \arrow[dl] \arrow[dr] & \\
X_1 \arrow[dr] & & X_2 \arrow[dl] & & \cdots & & X_n \\
 & X_{12} \arrow[dr] & & & & & \\
 & & \ddots \arrow[dr] & & {} \arrow[dl] & & \\
 & & & X_{1n}. & & &
\end{tikzcd}$$

If I remember right Milnor showed that $|-|:\mathrm{Set}_\Delta^{\mathrm{op}}\to\mathrm{Top}$ preserves homotopy equivalence. Using that you can show that if $\mathcal{C}$ is a monoidal category viewed as $\mathcal{C}^\otimes:\mathbf{\Delta}\to\mathrm{Cat}$ then $\|\mathcal{C}\|$ is an A_∞ monoid. For $X:\mathbf{\Delta}^{\mathrm{op}}\to\mathrm{Top}$ we can define $|X|$ by some coequalizer formula, and then for X an A_∞-monoid in Top we have $BX_1\coloneqq|X|$.

There is a canonical map $X_1\to\Omega BX_1$ which comes about by adjoint to $\Sigma X_1\to BX_1$.

Then this gives a morphism of A_∞ monoids, the loop space is always an A_∞ monoid, by concatenation of loops. This is even an A_∞ group, which is an A_∞ monoid such that $\pi_0(X_1)$, that's a monoid, and if that's actually a group, we say that this is an A_∞ group.

The fact is that ΩBX_1 is the universal A_∞ group with a map from X. In particular, if you take π_0 you get the group completion, $\pi_0\Omega BX_1\cong(\pi_0X_1)_{\mathrm{gp}}$. You can think of this as the ∞ analog of what I started with in Set. You can think of this as the ∞-category with one object, and then we invert morphisms to have inverses, invert morphisms to get an ∞ groupoid, and then we recover a monoid as the endomorphisms of the base point.

How do we apply this to K-theory? We start with $\mathcal{C}$ and take $\|w\mathcal{C}\|$ and that's an A_∞-monoid, and we can form its group completion in this sense as $\Omega B\|w\mathcal{C}_1\|$, and if $\mathcal{C}$ was one of these examples where all short exact sequences split, then this is already the K-theory space of $\mathcal{C}$. So $K_n(\mathcal{C})$ is $\pi_n(\mathcal{C})$.

If I want to split short exact sequences then I need something more complicated.

I won't make precise the notion of a Waldhausen category, but I can build a new category called $S_n\mathcal{C}$ which consists of diagrams

$$\begin{tikzcd}
0 \arrow[r] & A_01 \arrow[r]\arrow[d] & A_{02} \arrow[r]\arrow[d] & \cdots \arrow[r] & A_{0n}\arrow[d] \\
 & 0 \arrow[r] & A_{12} \arrow[r]\arrow[d] & \cdots \arrow[r] & A_{1n}\arrow[d] \\
 & & 0 & & \vdots \arrow[d] \\
 & & & \ddots & \vdots \arrow[d] \\
 & & & & 0.
\end{tikzcd}$$

I should say a Waldhausen category is a category $\mathcal{C}$ of cofibrant objects with cofibrations, weak equivalences, 0 and $\coprod$ and pushouts of cofibrations exist and are cofibrations. So these diagrams, the morphisms that are horizontal are cofibrations and all the squares are pushouts.

These categories have a natural simplicial structure $S_\cdot\mathcal{C}:\mathbf{\Delta}^{\mathrm{op}}\to\mathrm{Cat}$ where $wS_n\mathcal{C}$ is the subcategory where the morphisms are objectwise weak equivalences, I have this simplicial diagram of categories, and I can take $\|wS_\cdot\mathcal{C}\|$, then the loop space of that.

This is the definition of $K(\mathcal{C})$, so for $\mathcal{C}^{\coprod}$ the diagram I had before, then I can get a map $w\mathcal{C}^{\coprod}$ to $wS_\cdot\mathcal{C}$ by taking split exact sequences. If I look in level two, the data corresponds to a pair, [unintelligible], and so you get a map from $\Omega B\|w\mathcal{C}\| \to \Omega\Big|\|wS_\cdot\mathcal{C}\|\Big|$, and you can think of this as quotienting out by splitting exact sequences.

So $S_0\mathcal{C}$ is a point, $S_1\mathcal{C}$ is $0 \to X \to 0$ which is just $\mathcal{C}$ itself. Then an object in $S_2\mathcal{C}$ is a short exact sequence, so this is the category of short exact sequences, so I add in a two simplex when I take geometric realization which has as its boundaries the entries, so that the middle entry (as a loop) is the sum of the other two entries.

I'll say one more word. Each category is a Waldhausen category, and $K(S_2\mathcal{C})$ has maps (two of them) to the K-theory of $\mathcal{C}$, and the additivity says that $K(S_2\mathcal{C})$ is equivalent to $K(\mathcal{C})\times K(\mathcal{C})$ and similarly for higher n, which shows that $K(S_\cdot\mathcal{C})$ is an A_∞-monoid (in fact an A_∞-group) and I can deloop it, and iterate that, applying the $S_\cdot$ construction as many times as I want, and doing that I get an Ω-spectrum. So it's not just a space, it's a spectrum.

I should definitely stop.