CGP DERIVED SEMINAR

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1. Feb 12: Taesu Kim: t-structures on triangulated categories

So actually I have to say, I changed my original plan, apologies for that. I'm going to use a little bit different lanugage for talking about triangulated categories and T-structures. Suppose we have a category $\mathcal{C}$ with a 0 object, meaning initial and terminal, and suppose we have homotopy pushouts. Let me not elaborate on the precise definition of this, but you can think of it as a usual pushout, where the diagram only commutes up to some homotopy, and the uniqueness for the universal property is replaced by uniqueness up to some contractible choice in some sense, and that's the kind of rough definition of a homotopy pushout. We'll consider a category which admits all possible homotopy pushouts, and the diagram we'll think of will be of this type:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

and we'll call this a distinguished triangle. I'll denoe this by Δ from now on. If we look at this diagram, for instance,

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

this is a homotopy pushout so this is always a distinguished triangle. Another example is the suspension

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X[1] \end{tikzcd}$$

and then we see a rotation:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] \arrow[d] & X[1] \arrow[d] \\ & 0 \arrow[r] & Y[1] \end{tikzcd}$$

and all of these are distinguished.

So another is

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[rr, bend left] & Y \arrow[d] \arrow[rr, bend left] & X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ 0 \arrow[r] & Z \arrow[rr, bend right] & 0 \arrow[r] & Z' \end{tikzcd}$$

but this doesn't have to be unique.

One last example, suppose we are given

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] & X \arrow[r] \arrow[d] & Z \\ 0 \arrow[r] & Z' & 0 \arrow[r] & X' & 0 \arrow[r] & Y' \end{tikzcd}$$

then we get

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\ 0 \arrow[r] & Z' \arrow[r] \arrow[d] & Y' \arrow[d] \\ & 0 \arrow[r] & X' \end{tikzcd}$$

and we get that

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & Y' \arrow[d] \\ 0 \arrow[r] & X' \end{tikzcd}$$

is a homotopy pushout.

Last time Yong-Geun talked about $K(A)$, where you had complexes and chain maps. Today I'll talk about the derived category $D(A)$, where this is given by definition as follows, we take $C(A)$ and localize by the quasi-isomorphisms, and we have the following universal property, if $C(A)$ to D' takes quasi-isomorphisms to isomorphisms then there is a unique functor from $D(A) = C(A)[W^{-1}]$ to D' making the triangle commute.

That's an example of a triangulated category, and we have to specify what the triangles are, and the answer is

$$\begin{tikzcd} X_* \arrow[r, "f"] \arrow[d] & Y_* \arrow[d] \\ 0 \arrow[r] & C(f) \end{tikzcd}$$

and the suspension is degree shift by 1. The homology functor from $D(A)$ to A is the cohomology, and this is well-defined because quasi-isomorphism does not affect homology.

Now I consider $\tau_0 D(A)$ consisting of X the objects in $D(A)$ so that $H^i(X_*) = 0$ if $i \neq 0$. We consider all those objects, and these objects, we denote $\tau_0 D(A)$. We can show that this is equivalent to A.

We get an Abelian category inside the derived category, and the motivation for t-structures is to invert this. So we let D be an arbitrary triangulated category.

We are going to identify some Abelian category inside of this, which will contain some important information. We consider two full subcategories $D_{\geq 0}$ and $D_{\leq 0}$, this is a *t-structure* if it satisfies the following axioms. Let me make the notation $D_{\geq n} = D_{\geq 0}[n]$ and similarly for $D_{\leq n}$

(1) $D_{\geq 1} \subset D_{\geq 0}$ and $D_{\leq 1} \supset D_{\leq 0}$
(2) $\mathrm{Hom}_D(X,Y) = 0$ if $X \in D_{\leq 0}$ and $Y \in D_{\geq 1}$
(3) For any $X \in D$ there exists a distinguished triangle

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & X'' \end{tikzcd}$$

with $X' \in D_{\leq 0}$ and $X'' \in D_{\geq 1}$.

The *core* or *heart* of D is defined to be $D_{\geq 0} \cap D_{\leq 0}$, and it turns out that this is an Abelian category.

Here is a digression, not for the t-structure story, suppose we have a distinguished triangle $X \to Y \to Z$, and we assume that $\mathcal{C}$ is additive, and we consider the following sequence of Abelian groups,

$$\to \mathrm{Hom}_D(-,X) \to \mathrm{Hom}_D(-,Y) \to \mathrm{Hom}_D(-,Z) \to$$

and

$$\to \mathrm{Hom}_D(Z,-) \to \mathrm{Hom}_D(Y,-) \to \mathrm{Hom}_D(X,-) \to$$

are long exact, this is a fact that will be useful soon.

I'll define $\tau_{\geq 0} : D \to D_{\geq 0}$ and $\tau_{\leq 0} D \to D_{\leq 0}$.

Suppose X is an object, and we have $X' \to X \to X''$ a triangle, with X' in $D_{\leq 0}$ and X'' in $D_{\geq 1}$. We define $\tau_{\geq 0}(X)$ to be X'. For another object Y, we choose a pushout diagram $Y' \to Y \to Y''$ a triangle, and if we have $u : X \to Y$, then take the composition $X' \to X \to Y$, and taking $W = X'$ we get

$$\mathrm{Hom}_D(X',Y') \to \mathrm{Hom}_D(X',Y) \to \mathrm{Hom}_D(X',Y'')$$

and by our axioms, $\mathrm{Hom}_D(X',Y'')$ vanishes, so $\mathrm{Hom}_D(X',Y') \to \mathrm{Hom}_D(X',Y)$ is an isomorphism, so there is a lift to give a value for $\tau_{\geq 0}(u)$.

So we get a functor $\tau_{\leq 0}$ and similarly $\tau_{\geq 1}$ and we can see that $\tau_{\leq 0}$ is right adjoint to inclusion $D_{\leq 0} \to D$ and similarly $\tau_{\geq 1}$ is left adjoint to $D_{\geq 1} \to D$.

By using this functor we get $\tau_{\leq n}$ and $\tau_{\geq n}$.

So now we have the two truncation functors $\tau_{\leq m}$ and $\tau_{\geq n}$ which commute in the following sense, one can have $\tau_{\leq n} X \to X \to \tau_{\geq n+1} X$ as a distinguished triangle, and then we have $\tau_{\leq m} X \to X \to \tau_{\geq n} X$ and we have a unique isomorphism making the following diagram commute

$$\begin{tikzcd} \tau_{\leq m} X \arrow[r] \arrow[d] & X \arrow[r] & \tau_{\geq n} X \arrow[d] \\ \tau_{\geq n}\tau_{\leq m} X \arrow[rr, dashed] & & \tau_{\leq m}\tau_{\geq n} X \end{tikzcd}$$

By the way, all proofs are in the next talk or omitted. Then $\tau_{[n,m]} = \tau_{\geq n}\tau_{\leq m}$ and for $m = n$ we define $\tau_{[n,n]}$, which is $H^n D$.

In particular, when n is zero, H^0 goes from D to the core of D.

Theorem 1.1. *The core of D is Abelian and H^0 is cohomological (i.e., a distinguished triangle maps to a long exact sequence).*

Suppose $F : D \to D'$ respects the t-structure in the sense that $F(D_{\geq 0}) \subset D'_{\geq 0}$ and $F(D_{\leq 0}) \subset D'_{\leq 0}$ then we call this t-exact. So F should also respect the triangulated structure.

Then F restricts to the core, $F|_{\mathrm{Core}(D)} : \mathrm{Core}(D) \to \mathrm{Core}(D)$.

In the last part of my talk I'm going to talk about examples. In the derived category of an Abelian category, let me give an example of a t-structure. So $D_{\geq 0}$ consists of objects with no cohomology for $i < 0$ and $D_{\leq 0}$ consists of objects with no cohomology for $i > 0$. So then the core consists of X such that $H^i(X) = 0$ for $i \neq 0$.

The functor here from $D(A)$ to A sends X to $H^n(X)$, usual cohomology.

The next example is perverse sheaves, let me not give full details, all these things are from the original, the famous work by these Beilinson–Bernstein–Deligne, in their famous paper "Faisceaux Pervers", which contains t-structures and t-exact functors.

The next example is, since Damien introduced spectra, I felt an obligation to mention, the homotopy category of spectra. Recall that a spectrum is a sequence of pointed topological spaces $\{X_n\}$ equipped with $\Sigma X_n \to X_{n+1}$. For Ω-spectra, we have a map $X_n \xrightarrow{\cong} \Omega X_{n+1}$, and a map of spectra is f_n between X_n and X_{n+1}, and we require a commuting condition.

The stable homotopy groups are

$$\pi_k^s(X) = \lim_{n \to \infty} \pi_{k+n}(X_n)$$

and this stabilizes so that this is a well-defined notion. We consider a map between spectra, it's a π_*^s-isomorphism if $f_* : \pi_*^s(X) \to \pi_*^s(Y)$ is an isomorphism for each index.

Then we localize spectra with respect to these stable isomorphisms, and get the homotopy category of spectra.

Here there is a triangulated category structure, we have homotopy pushouts, the cone of f, and then you have Σ as the shift. The t-structure you give to this category is something like this: $h\,\mathrm{Sp}_{\geq 0}$ are X such that $\pi_i^s(X) = 0$ if $i < 0$ and similarly for ≤ 0. For this t-structure the core can be shown to be equivalent to the category of Abelian groups.

I'll continue next time.