CGP DERIVED SEMINAR

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1. January 29: Yong-geun Oh: reminder on triangulated categories

This talk will be very elementary and may bore some of you, but going slowly may be helpful for some of the audience.

Let me start with category theory, with Abelian categories, what this is about. Let me first introduce additive categories, which have some kind of linear structure. We exploit the knowledge of modules where we know what linearity is. Many definitions are using some kind of functor which maps something in the given category to a category of modules. Then many objects in this abstract category are defined by representatives of functors satisfying certain properties.

Definition 1.1. An *additive category* $\mathcal{C}$ is a category such that

(1) for any pair (X, Y) of objects, the space of morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has the structure of an additive group, an Abelian group, or $\mathbb{Z}$ modules, and the composition map is linear, so given two such groups, $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$, this map is bilinear,
(2) there is a zero object 0, such that the space of morphisms $\mathrm{Hom}_{\mathcal{C}}(0, 0) = 0$,
(3) for any pair (X, Y) in the objects of the category, the functor $\mathcal{C} \to \mathrm{Mod}(\mathbb{Z})$
$$(-) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, -) \times \mathrm{Hom}_{\mathcal{C}}(Y, -)$$
is representable, meaning that there exists $X \coprod Y$ such that for any W, we have
$$\mathrm{Hom}_{\mathcal{C}}(X \coprod Y, W) \cong \mathrm{Hom}_{\mathcal{C}}(X, W) \times \mathrm{Hom}_{\mathcal{C}}(Y, W),$$
(now this definition works for any category),
(4) a similar statement for the other way, $X \prod Y$ represents
$$(-) \mapsto \mathrm{Hom}_{\mathcal{C}}(-, X) \times \mathrm{Hom}_{\mathcal{C}}(-, Y)$$
as well—and in fact $X \coprod Y$ and $X \prod Y$ are isomorphic and so can be denoted $X \oplus Y$ which has natural maps $i_1 : X \to X \oplus Y$ and $i_2 : Y \to X \oplus Y$.

This is the object level, given maps from $X \to W$ and $Y \to W$ then there is a unique map making the diagram commute:

$$\begin{tikzcd}
X \arrow[dr, "i_1"'] \arrow[drr, "f"] & & \\
& X \oplus Y \arrow[r, dotted, "h"] & W \\
Y \arrow[ur, "i_2"] \arrow[urr, "g"'] & &
\end{tikzcd}$$

Now the next thing we are going to do, we're given an additive category, but working with a module category, w can always define kernels and cokernels, and we want the same thing in this additive category. We'll state the universal property, defining kernel and cokernel as representatives of a certain functor.

So say $\mathcal{C}$ is an additive category, and Z an object of it. Then we are going to, for a given morphism $f : X \to Y$, we'll associate a similar morphism in the category of modules, we associate $\mathrm{Hom}_{\mathcal{C}}(-, f)$, this is again a functor, so we have

$$\mathrm{Hom}_{\mathcal{C}}(Z, f) : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$$

So we'll consider this $\mathrm{Hom}_{\mathcal{C}}(-, f)$ as a functor from $\mathcal{C}$ to $\mathrm{Mod}(\mathbb{Z})$. So we can talk about kernels. Then we have a functor which I denote

$$(-) \mapsto \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(-, f)).$$

More explicitly, for each object W we look at the kernel of $\mathrm{Hom}_{\mathcal{C}}(W, f)$, which is a map from $\mathrm{Hom}_{\mathcal{C}}(W, X)$ to $\mathrm{Hom}_{\mathcal{C}}(W, Y)$.

This is a subset of the domain, $\phi \in \mathrm{Hom}_{\mathcal{C}}(W, X)$ such that $f \circ \phi = 0$.

We want this to be representable, so when this functor is representable, its representing object we call the *kernel* of f and denote it $\ker\ f$. Unlike the kernel of linear homomorphisms. It's not a subset of X necessarily. It's another object in the abstract category, and at the moment $\ker f \subset X$ does not make sense.

This is the object level, I want to state the characterizing property. By definition, we have the following isomorphisms:

$$\mathrm{Hom}_{\mathcal{C}}(W, \ker f) \cong \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(W, f)).$$

Similarly, the cokernel is defined by considering another functor

$$(-) \mapsto \mathrm{Coker}(\mathrm{Hom}_{\mathcal{C}}(f, -)).$$

Let me state several properties of kernels. Suppose $f : X \to Y$, in an additive category the kernel may or may not exist. Suppose $f : X \to Y$ has a kernel. Then we have a natural morphism (natural transformation) which I'll denote $\beta : \mathrm{Hom}_{\mathcal{C}}(-, \ker f) \to \mathrm{Hom}_{\mathcal{C}}(-, X)$. How is it defined? Let Z be an object of $\mathcal{C}$ and consider $\beta_Z : \mathrm{Hom}_{\mathcal{C}}(Z, \ker f) \to \mathrm{Hom}_{\mathcal{C}}(Z, X)$. Let

$$\alpha \in \mathrm{Hom}_{\mathcal{C}}(Z, \ker f) \cong \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(Z, f)),$$

which is a subset of the domain $\mathrm{Hom}_{\mathcal{C}}(Z, X)$. Therefore we have this map β_Z. The property of morphisms you can check. In particular, let's apply this to $Z = \ker f$, then we have the identity element of $\mathrm{Hom}_{\mathcal{C}}(\ker f, \ker f)$, then β of this identity element defines a map $\ker\ f \to X$, which we denote α. In this way, we cannot think of it as a subset but at least there's a natural map, and the universal property of kernels, the kernel of f has the universal property that for any morphism $e : W \to X$, whenever you look at $f : X \to Y$, whenever $f \circ e = 0$, there is a lift

$$\begin{tikzcd} W \arrow[r, "e"] \arrow[dr, dashed] & X \arrow[r, "f"] & Y. \\ & \ker\ f \arrow[u, "\alpha"] & \end{tikzcd}$$

Now I want to denote images or coimages.

Definition 1.2. Assume $\alpha : \ker f \to X$ has cokernel, then we denote $\mathrm{coker}(\alpha)$ as the coimage of f. Similarly, if the associated canonical map for the cokernel $\gamma : Y \to \mathrm{coker}\, f$ has a kernel, then we denote it by the image of f.

One important lemma is the following.

Lemma 1.1. *Assuming all these exist, there is a natural map from the coimage to the image.*

I'll give the proof of this.

Proof. We have

$$\ker f \xrightarrow{\alpha} X \xrightarrow{f} Y.$$

We have a natural map from X to the cokernel $\operatorname{coker}(\alpha : \ker\ f \to X)$ By definition, $f \circ \alpha = 0$ so there is a map δ from this cokernel to Y. Then we consider a similar diagram, we have

$$\operatorname{coker}(\alpha) \xrightarrow{\delta} Y \xrightarrow{\gamma} \operatorname{coker} f$$

and we have an inclusion of $\ker \gamma$ into Y. We check that $\gamma \circ \delta = 0$, so by universal properties there is a map from $\operatorname{coker}(\alpha)$ to $\ker(\gamma)$. □

In general this map may not be an isomorphism. Here comes the definition of an Abelian category.

Definition 1.3. An additive category $\mathcal{C}$ is called an *Abelian* category if

(1) any morphism has a kernel and cokernel, and
(2) the natural map from the coimage of f to the image of f is an isomorphism for any f.

You may wonder what is going on. Here are some examples of additive categories which are not Abelian.

(1) First you have topological categories. If you have Banach spaces, sometimes you need to do things that aren't algebraic, like closure. In Banach spaces (normed complete vector spaces with morphisms the bounded linear maps), the kernel of f is canonically defined, but the cokernel is not. For linear things, it's usually the quotient of the target by the image. So there is no canonical way of defining a Banach space on the quotient. The cokernel of f is $Y/\overline{\operatorname{im} f}$. You can check that this satisfies the defining property of a cokernel. But the problem is, there exists a homomorphism f such that both $\ker f = 0$ and $\operatorname{coker} f = 0$ but f is not an isomorphism. If you densely define an operator.
(2) this is more algebraic, this is the category of filtered modules over a filtered ring, maybe this is more relevant to the Fukaya category.

Let me take a five minute break.

Now I want to talk about triangulated categories, and the motivation will be from example, with the homotopy category of complexes, once we talk about this, the homotopy category of complexes over an additive category, this will eventually give us triangulated categories.

So let's talk about the homotopy category of complexes. Let's say $\mathcal{C}$ is an additive category, then a complex in $\mathcal{C}$ is a sequence $X = (X^n, d_X^n)$ for $n \in \mathbb{Z}$ where X^n is an object and d_X^n is a map from X^n to X^{n+1} satisfying that consecutive compositions are zero in $\operatorname{Hom}_{\mathcal{C}}(X^n, X^{n+2})$.

The morphisms between two complexes X and Y are the $\{f^n\}$, a sequence of maps $f^n : X^n \to Y^n$ that makes, that commute, so that the diagram commutes

$$\begin{tikzcd} X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"] & X^{n+1} \arrow[d, "f^{n+1}"] \\ Y^n \arrow[r, "d_Y^n"] & Y^{n+1} \end{tikzcd}$$

Let $C(\mathcal{C})$ be the category of complexes. This is again an additive category, let me not talk about this but you do most things termwise.

Definition 1.4. The *shifted complex*, for k an integer and X a complex in $C(\mathcal{C})$, a new complex $X[k]$, is defined by

$$X[k]^n = X^{n+k}$$
$$d^n_{X[k]} = (-1)^k d_X^{n+k}.$$

For a given morphism $f : X \to Y$, we denote by $f[k] : X[k] \to Y[k]$ the map $f[k]^n = f^{n+k}$.

Given a category, you can quotient by a collection of morphism, we'll define the homotopy category as the quotient by some collections.

Definition 1.5. A morphism $f : X \to Y$ in $C(\mathcal{C})$ is called *homotopic to zero* if there exist morphisms $s^n : X^n \to Y^{n-1}$ such that

$$f^n = s^{n+1} d_X^n + d_Y^{n-1} s_n.$$

Lemma 1.2. *The composition, we want to quotient this category by a collection of morphisms, we want this to be multiplicative, so the composition*

$$\mathrm{Hom}_{C(\mathcal{C})}(X,Y) \times \mathrm{Hom}_{C(\mathcal{C})}(Y,Z) \to \mathrm{Hom}_{C(\mathcal{C})}(X,Z)$$

maps

$$\mathrm{Ht}(X,Y) \times \mathrm{Hom}_{C(\mathcal{C})}(Y,Z) \sqcup \mathrm{Hom}_{C(\mathcal{C})}(X,Y) \times \mathrm{Ht}(Y,Z)$$

to $\mathrm{Ht}(X,Z)$*. So this allows us to define*

Definition 1.6. The category $K(\mathcal{C})$ is defined by, at the object level it's the category of complexes $C(\mathcal{C})$, and the morphisms from X to Y are the maps in $C(\mathcal{C})$ from X to Y quotiented by $\mathrm{Ht}(X,Y)$, i.e., the set of homotopy classes of chain maps. There are many ways of changing the morphisms, you can try to define these as some kind of homology.

The category of complexes will be an Abelian category of $\mathcal{C}$ is Abelian, but $K(\mathcal{C})$ may not be Abelian even if $\mathcal{C}$ is Abelian. So you need to go beyond in general.

Definition 1.7. Assume that $\mathcal{C}$ is Abelian. We denote by $Z^k(X)$ the kernel of d_X^k and by $B^k(X)$ the image of d_X^{k-1} and $H^k(X)$ the cokernel of the canonical map $B^k(X) \to Z^k(X)$, in other words $Z^k(X)/B^k(X)$, the kth cohomology group of X.

So H^k is an additive functor from $C(\mathcal{C})$ to $\mathcal{C}$, and $H^k(X)$ satisfies $H^0(X[k])$. We have canonical short exact sequences that hold in cohomology groups in the category of complexes in Abelian groups. For example, let me state $X^{k-1} \to Z^x(X) \to H^k(X) \to 0$. You have to check things because you don't have subgroups and so on. I want to state one important proposition.

Proposition 1.1. *You can define the notion of exact sequence, and for a short exact sequence* $0 \to X \to Y \to Z \to 0$ *in* $C(\mathcal{C})$ *you have a long exact sequence*

$$\cdots \to H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(X) \to \cdots$$

The only thing you might want to check is the existence of the connecting morphism, but I want to skip it. So many of the basic homological algebra tools work in Abelian categories. Mitchell's theorem says that any Abelian category is embedded into a category of modules, so Abelian categories are in many senses equivalent to the category of modules.

So now I want to talk about mapping cones. This works for any additive category. Suppose $f : X \to Y$ are morphisms in the category of complexes $C(\mathcal{C})$.

Definition 1.8. The mapping cone $M(f)$ of $f : X \to Y$ is the object in $C(\mathcal{C})$ defined by, $M(f)^n = X^{n+1} \oplus Y^n$ with differential the lower triangular matrix

$$\begin{pmatrix} d^n_{X[1]} & 0 \\ f^n & d_Y \end{pmatrix}$$

There is a natural map $\alpha(f) : Y \to M(f)$ and $\beta(f) \to X[1]$ defined by $\alpha(f)^n = \begin{pmatrix} 0 \\ \mathrm{id}_{Y^n} \end{pmatrix}$ and $\beta(f)^n = \begin{pmatrix} \mathrm{id}_{X^{n+1}} & 0 \end{pmatrix}$.

Here is an important lemma satisfied by the mapping cone. For any morphism $f : X \to Y$, there is a map $\phi : X[1] \to M(\alpha(f))$ such that

(1) ϕ is an isomorphism in $K(\mathcal{C})$ the homotopy category and
(2) the diagram

$$\begin{tikzcd} Y \arrow[r, "\alpha(f)"] \arrow[d] & M(f) \arrow[r, "\beta(f)"] \arrow[d] & X[1] \arrow[r, "{[1]}"] \arrow[d] & Y[1] \arrow[d] \\ F \arrow[r] & M(f) \arrow[r] & M(\alpha(f)) \arrow[r, "\beta(\alpha(f))"] & Y[1] \end{tikzcd}$$

commutes.

Maybe I will write down what this ϕ is, and its homotopy inverse.

Proof.

$$M(\alpha(f))^n = Y^{n+1} \oplus M(f)^n = Y^{n+1} \oplus X^{n+1} \oplus Y^n$$

and we define $\phi^n : X[1]^n \to M(\alpha(f))^n$ as

$$\begin{pmatrix} -f^{n+1} \\ \mathrm{id}_{X^{n+1}} \\ 0 \end{pmatrix}$$

and $\psi^n = (0, \mathrm{id}_{X^{n+1}}, 0)$ is a homotopy inverse of ϕ. You can check that $\psi \circ \phi = \mathrm{id}_{X[1]}$ and the other direction is homotopic to the identity. Also $\psi \circ \alpha(\alpha(f)) = \beta(f)$ and $\beta(\alpha(f)) \circ \phi \cong -f[1]$. $\square$

Definition 1.9. A *triangle* $X \to Y \to Z \to X[1]$ in $K(\mathcal{C})$ is *distinguished* if it is isomorphic to the basic triangle $X \to Y \to M(f) \to X[1]$ for some f.

So a triangle is called a distinguished triangle if the triangle is isomorphic to the basic triangle. Now let me state the basic properties will be the same as the definition of a triangulated category if you replace the homotopy category by any general additive category.

Theorem 1.1. *The collection of distinguished triangles above satisfy the following.*

TR0 *Any triangle isomorphic to a distinguished triangle is distinguished.*

TR1 *(existence of enough distinguished triangles) if X is in $\mathrm{Ob}(K(\mathcal{C}))$, then $X \to X \to 0 \to X[1]$ is distinguished.*

TR2 *Any $f : X \to Y$ in $K(\mathcal{C})$ can be embedded into a distinguished triangle, meaning that $X \xrightarrow{f} Y \to Z \to X[1]$ is for some Z and maps.*

TR3 *(rotation) If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$, then the rotation*

$$Y \to Z \to X[1] \to Y[1]$$

is distinguished.

TR4 *Given two distinguished triangles $X \to Y \to Z \to X[1]$, then $X' \to Y' \to Z' \to X'[1]$ and maps $u : X \to X'$ and $v : Y \to Y'$ making the diagram commute, this can be extended to a morphism of triangles.*

TR5 *(the most mysterious axiom, the octahedron axiom)—then I'll stop. Suppose given two triangles $X \xrightarrow{f} Y \to Z' \to X[1]$ and $Y \xrightarrow{g} Z \to X' \to Y[1]$, and there is a distinguished triangle $X \xrightarrow{g \circ f} Z \to Y' \to X$, then there exist triangle $Z' \to Y' \to X' \to Z'[1]$:*

$$\begin{tikzcd}
X \arrow[rr] \arrow[rrrr, bend left] & & Y \arrow[rr] \arrow[dl] & & Z \arrow[dl] \arrow[ddll, bend left] \\
& Z' \arrow[ul] \arrow[dr] & & X' \arrow[ll] \arrow[ul] & \\
& & Y' \arrow[ur] \arrow[uull, bend left] & &
\end{tikzcd}$$

and various things commute and are distinguished.

For the homotopy category all these are satisfied.

Now here is the definition, you replace the homotopy category with an additive category which satisfies these five axioms, and the prototype is the homtopy category of an Abelian category.