CGP DERIVED SEMINAR

GABRIEL C. DRUMMOND-COLE

1. January 15: Damien Lejay: Introduction to spectra

This is going to be informal, so this has two meanings. First, stop me whenever you want, ask a question. On my part, it means that I can lie and not give you the full details. All right? So last time we saw an algebraic cohomology in the sense of Weyl. I'll remind you of the definition and then we'll see what the definition tells us.

A *cohomology theory* is a functor H^* from smooth projective varieties, contravariant, it should go to graded Abelian groups, so they'll be graded commutative algebras with a non-degenerate trace,

$$H^* : SPV^{\mathrm{op}} \to K - \mathrm{alg}^{\mathrm{Tr}}.$$

So K here is characteristic zero field. All my varieties are over $\mathbb{C}$, an algebraically closed field. In the domain, you have the symmetric monoidal structure of taking the product, and on the right you have the tensor product, and you require this to be symmetric monoidal, taking the product to the tensor product. You should have some shifts, Tate twists, and blah blah blah, because it's complicated.

If you only satisfy this, you know nothing from this, where is the algebraic geometry. What you always require on top of that is a transformation $H^*_{\mathrm{cl}}(-,k) \to H^*$. Then every time you give yourself a variety, you should have something in H^*, and this is compatible with pullback, and non-degenerate, the trace of a point is 1, so you can recover something non-trivial.

You can think of this as a kind of sheaf of rings, and you have a morphism of rings from the classical object to any of your homology theories. Anything is below this initial object, the classical cohomology. It's like the way $\mathbb{Z}$ controls a lot about rings, this classical cohomology gives this kind of control.

One of the examples of such a thing is the topological cohomology, the cohomology with values in any field that you like. I was thinking that this is very related to this, but maybe this is not a cohomology theory. It's a bit as for quantum field theories, maybe it's too hard and you look at those that are topologically invariant. So we know that $H^*(-,K)$ is topologically invariant. People have asked for a long time, how do I make this representable, say that this is the same as the maps into some object, $H^*(-,K) \cong \mathrm{Hom}(-,?)$.

In algebraic topology the answer is via spectra.

You have what is called a generalized cohomology theory and what is called a generalized multiplicative theory. You could ask to have cohomology groups, and how they behave linearly, and this is spectra. If you add a multiplicative structure, you get something called ring spectra. Then later we can add a multiplicative structure.

There's a very famous paper in 1945 of Eilenberg and Steenrod that I invite you to read, it's only 6 pages and absolutely marvelous. I'll write the axioms that they give for a homology theory. Then we'll simplify the axioms and change the category a bit. A (co)homology theory is a functor that takes pairs (X, A) (where X is a space and A a closed subspace) to $H_n(X, A)$ (I put no coefficients), which should be functorial, it means that for $f : (X, A) \to (Y, B)$ (that is, a map $f : X \to Y$ so that $f(A) \subset B$) I get maps

$$H_n(X, A) \xrightarrow{f_*} H_n(Y, B)$$

and a boundary map

$$H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) = H_{n-1}(A, \varnothing),$$

and I want this data to be functorial too.

I also want to posit long exact sequences

$$\cdots \to H_q(X) \to H_q(X, A) \to H_{q-1}(A) \to \cdots$$

and I ask that this be a (long) exact sequence.

The final axiom is excision, whenever you have $U \subset A \subset X$ with U open and its closure contained in an open set V inside A, you have

$$H_q(X \smallsetminus U, A \smallsetminus U) \cong H_q(X, A)$$

which I don't understand but let me say something I do understand that's equivalent

$$H_q(X, A) \cong H_q(X/A, A/A)$$

in good cases, and the final axiom that is no longer used is

$$H_q(*) = 0$$

if $q \neq 0$.

One thing you can see with some computation, you need $H_*(\varnothing) = 0$, so you can factor this through the category of pointed spaces. So you can go to pointed topological spaces, where every space has a specified point. The point is initial and is still the terminal object, and $\mathrm{Ab}^{\mathbb{Z}}$ is pointed too, by zero, so $\mathrm{Top}_* \to \mathrm{Ab}^{\mathbb{Z}}$ is the *reduced homology.* The axiom becomes a bit more manageable if we write them for reduced homology theories. This you can think of as a baby example of a factorization, we go toward more representability by doing this. Then H_* we say is a functor and when we take a map of pointed spaces, you can take the quotient which should give you exact sequences in good cases $\widetilde{H}_q(X) \to \widetilde{H}_q(Y) \to \widetilde{H}_q(Y/X) \to \cdots$

One axiom I forgot, not always included, is that homology should commute with disjoint unions. Another property that you can derive from this is, there is something called a suspension, and one property of a suspension is that $\widetilde{H}_q(\Sigma X) \cong \widetilde{H}_{q-1}(X)$, and this is something that fills this square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

and we do this homotopy invariantly, so this is a cone with its end glued to a point.

So instead of the boundary, you can ask $\widetilde{H}_q(\Sigma X) \cong \widetilde{H}_{q-1}(X)$. I forgot to say that if $X \sim Y$ is a homotopy equivalence then you get an induced equivalence.

So the things you get are homotopy invariant, shift along the suspension functor, and give you exact sequences. If you want you also have $\widetilde{H}_n(\bigvee X_i) = \bigoplus H_n(X_i)$.

Nom I want to talk about representability. If A is nice and you have a functor $A^{\mathrm{op}} \xrightarrow{F} \mathrm{Set}$ which commutes with all small limits. Then F gets a (left) adjoint G. Because the category of sets is generated by a point, you have a single object of A^{op} which is the value at a point. Then $F(X) = \mathrm{Hom}(X, G(*))$.

The exact sequence for quotients and for disjoint unions is about commutation with limits, but the homotopy axiom and the suspension axiom are not.

Let's say you have a map of groups between a group G and an Abelian group A. Because A is Abelian, you can always factor through the quotient $G/[G,G]$. We have functors that respect the homotopy thing and suspension thing. So we should take topological spaces and go to something where we have inverted or quotiented out by homotopies, where the objects are homotopy types of topological spaces.

One way to build this is to keep the same objects and add inverses of homotopy equivalences. If you do this in a universal way, you get this homotopy category. Now suppose you have a functor F from Top to $\mathcal{C}$ so that when you take a homotopy equivalence it goes to an isomorphism, then you can always factor F through the localization to this homotopy category.

What this means is that our homology or cohomology theories are homotopy invariant and should factor through this homotopy category.

The only thing is that by doing this you destroy all your understanding of the category. So what people do to understand this, they put a model structure on Top, that's a lot of extra data to add but then we can have a fine-grained understanding when we invert all the things.

Because you're invariant by the shift functor, you have to localize again to say that the shift is invertible. So now $\mathrm{Top}[he^{-1}][\Sigma^{-1}]$ is the category of spectra.

There are many models for these spectra, but I'll give one model. I'll go back to generalized cohomology theory and do that step by step in a kind of stupid way.

So when I write $H^n(X)$ I imagine I have a special space so that this is $[X, E_n]$, equivalence classes of continuous maps up to homotopy. Let's say for every n I can represent that functor. Then since

$$\widetilde{H}^n(\Sigma X) cong \widetilde{H}^{n-1}(X).$$

So I get $[\Sigma X, E_n] \cong [X, E_{n-1}]$, and $[\Sigma X, E_n]$ is isomorphic to $[X, \Omega E_n]$. So I can ask that $E_0 \cong \Omega E_1$ et cetera, so that $E_{n-1} \cong \Omega E_n$.

This is called an Ω-spectrum, a bunch of pointed topological spaces and homotopy equivalences like this. Then we'd say $E^n(X) \cong [X, E_n]$. In the literature when people start speaking of spectra, they use E^* for the cohomology theory.

You can put a model structure on spectra (sort of) and the best example is the Eilenberg–MacLane spectrum, there's a space $K(n,\mathbb{Z})$, where e.g., $K(1,\mathbb{Z}) \cong S^1$, and $K(n-1,\mathbb{Z})$ is equivalent to $\Omega K(n,\mathbb{Z})$. You could do the same thing with A.

Now I'll mention one of the coolest things about Ω-spectra. I should tell you about homotopy groups. I give myself an Ω-spectrum E_*, with $E_0 \cong \Omega E_1$ etc.

I know that $\pi_i(E_0) \cong \pi_i(\Omega E_1) = \pi_{i+1}(E_1)$, and so this is also the same as $\pi_{i+2}(E_2)$, and is also $\pi_{i+n}(E_n)$. I can then define $\pi_0(E)$ to be $\pi_0(E_0)$. For $\pi_1(E)$ I can define $\pi_1(E)$ to be either $\pi_1(E_0)$ or $\pi_1(E_1)$.

Now there's something I can do, I can define $\pi_{-1}(E)$ to be $\pi_0(E_1)$. So now I have the negative groups that I can define only starting at a certain point, $\pi_{-n}(E) =$

$\pi_0(E_n) \cong \pi_1(E_{n+1}) \cong \cdots$, so now this has π_i in each direction. Now a homotopy equivalence is a map of pointed spectra that induces an equivalence of all π_i.

This is the same as Ω-spectra where I've inverted homotopy equivalences. So now I want to be able to invert homotopy equivalences. I have a concrete category, and once I invert these maps, I get the category of spectra that I want, the localization of Ω-spectra.

Now I have the Brown representability theorem, which says that generalized cohomology theories, up to homotopy, are the same as Ω-spectra, up to homotopy. You can also represent maps between cohomology theories as maps between spectra.

So say you define a reduced cohomology theory $\mathrm{Top}_*^{\mathrm{op}} \xrightarrow{\widetilde{H}^*} \mathrm{Ab}^{\mathbb{Z}}$, and there is a functor from $\mathrm{Top}_*^{\mathrm{op}}$ to $\mathrm{Sp}^{\mathrm{op}}$, called Σ^∞, and

$$\widetilde{H}^*(X) \cong \mathrm{Hom}_{\mathrm{Sp}}(\Sigma^\infty X, E)$$

Well, this needs shifts so let me say instead that $\widetilde{H}^n(X) \cong [X, E_n]$, and that is the solution to the problem of motives in this case.

This is a linear version, we didn't have the monoidal structure. There is a monoidal structure on spectra, topological spaces have the product structure. For pointed topological spaces, then the product becomes something called the smash product $\wedge$. One way is that you take two pointed topological spaces, which is the quotient of the product $X \times Y$ by the (pointed) sum $X \wedge Y$. This has the property that you want which is that $\mathrm{Hom}_*(X \wedge Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$, and this smash product can be defined on the category of spectra. If you take Ω-spectra it's not easy to define.

I'll tell you about another model, which is called just a pre-spectrum. A pre-spectrum is a bunch of pointed spaces and instead of having homotopy equivalences you just have maps $\Sigma E_n \to E_{n+1}$, what is absolutely clear is that an Ω-spectrum gives you a spectrum, because you can use the adjunction to get from $E_n \cong \Omega E_{n+1}$ to $\Sigma E_n \to E_{n+1}$, but we don't ask about anything being a homotopy equivalence. With prespectra you can easily define the smash product. Then I can tell you the suspension spectrum. Take X a pointed space and then suspend it n times. This is a prespectrum. This is a priori not an Ω spectrum. The nice way to go from pointed topological spaces to spectra goes to prespectra, which might not land in Ω-spectra, to do that you want to localize at homotopy equivalences. This is what people call Σ^∞. If you suspend S^0 many times you get the sphere spectrum $\mathbb{S}$, probably the most important spectrum, so this one represents the stable homotopy groups.

I have other things to say, but anyway that's another model. People also use other models. I wanted to introduce those to give the smash product of prespectra.

If you have E and F you want to define $(E \wedge F)_{2n}$, which will be $E_n \wedge F_n$ and I'll define $(E \wedge F)_{2n+1}$ as $E_n \wedge F_n \wedge S^1$, so this is the suspension $\Sigma(E_n \wedge F_n)$. With a lot of work you can see that this is compatible with homotopies. So you can say what is a monoid object in this homotopy category. I want a monoid in the category of spectra, this is a *ring spectrum*, so you have $E \wedge E \to E$ with associativity, unit, so on, up to homotopy. Then the cohomology theory becomes multiplicative.

So this gives an equivalence between multiplicative cohomology theories and ring spectra. Up to now we were talking about non-multiplicative theories. These are central to people doing stable homotopy theory. The sphere spectrum is the unit for the smash product, so it has the same universal property that $\mathbb{Z}$ has for rings, it's hard to understand, we don't know its homotopy groups. But it acts everywhere.

With the Eilenberg–MacLane spectrum, you can turn a ring into HR, the ring spectrum, and you have a morphism of ring spectra $\mathbb{S} \to HR$, including $H\mathbb{Z}$, so you can go below $\mathbb{Z}$ to $\mathbb{S}$, and this is what people want to do to go to *spectral* algebraic geometry. This is stuff, K-theory, cobordism, between $\mathbb{S}$ and $H\mathbb{Z}$.

I could speak about topological K-theory, maybe, with my remaining time. So K-theory has things in all directions, not just in positive degrees.

So I'll define K-theory as a spectrum, I'll define an Ω-spectrum. Because of my equivalence that's the same as defining the cohomology theory. If X is compact, then I'll define, and then I'll extend by universal constructions to other spaces.

So $K^0(X)$ is the set of equivalence classes of finite dimensional vector bundles on X, complex vector bundles and because I can always take the sum of vector bundles, I get the direct sum, and I want an Abelian group, and there's a universal construction to give a group, with pairs of vector bundles (V_1, V_2) with the relation $(V_1, V_2) \sim (V_1 \oplus Z, V_2 \oplus Z)$ where Z is anything.

Since X is compact, any vector bundle is a factor of a free bundle. I can always find a Z so that $V_1 \oplus Z \cong \mathbb{C}^n$. So my Abelian group of $(\mathrm{Vect}(X), \oplus)$ as (V, n) with $n \in \mathbb{Z}$, I think of this as $V - \mathbb{C}^n$. This is representable already by a topological space, there is a candidate for that. People in algebraic geometry can say that this is the same as the homotopy classes of maps from X to $BU \times \mathbb{Z}$, this is BU the classifying space for linear bundles of varying dimension and n is the virtual class.

I said that I want a spectrum, so I need to define the other guys. I could say that $K^{-1}(X)$, this has to be $[X, \Omega(BU \times \mathbb{Z})]$ and K^{-2} is $[X, \Omega^2(BU \times \mathbb{Z})]$. Then Bott periodicity says that $K^{-2}(X) \cong K^0(X)$. So this has non-trivial groups in both directions.

There are variants, like with real bundles, where you have BO, and then you have eight-fold periodicity. That's an example of a spectrum, in fact a ring spectrum, which comes from the external tensor product.

The literature is enormous on that.