# CGP DERIVED SEMINAR

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## 1. January 8,2019: Jun Yong Park: Notion of motives and Grothendieck's proposal

Sorry I'm late. Thank you all for coming, and thank you to the organizers for the kind invitation to speak.

Today I want to talk a little bit about motives, I can't do too much in one hour. Let me tell you something we already know very well. Let me talk about Weil cohomology. When you have a variety, say, a smooth projective variety $X$, you want to study its cohomology with some coefficients, $H^i(X,k)$. There are several variants. Maybe the most obvious is singular, but what we're really doing is Betti cohomology $H^i_{\mathrm{cl}}(X/\mathbb{C};\mathbb{C})$. Then you could do (algebraic) de Rham cohomology $H^i_{\mathrm{dR}}(X/\bar{\mathbb{Q}};\mathbb{Q})$. Then there are $\ell$-adic versions, so you could do étale, $H^i_{\text{ét}}(X/\bar{\mathbb{F}}_{q=p^r},\mathbb{Q}_\ell)$, or the crystalline, $H^i_{\mathrm{cris}}(X/\bar{\mathbb{F}}_q,\mathbb{Q}_\ell)$, where $\mathbb{Q}_\ell$ is the $\ell$-adic rationals, the field of fractions of the $\ell$-adic integers.

One can compute these things for $X$ and the punchline is that these all coincide when you formulate and compute them.

So these are vector spaces and the Betti numbers are all the same. What one wants to know is what is going on, why are they all coinciding? To show that these coincide, you need to show that the various things know how to talk to each other.

Besides cup product there is extra structure. So if you compare singular and de Rham cohomology, you get the Hodge structure, the Hodge filtration, if you consider the projection $\pi:\mathcal{H}^1(E,\mathbb{C})\cong\mathbb{C}^2$, then you can project this to $H^{0,1}(E;\mathbb{C})\cong\mathbb{C}$. I pass from de Rham to Hodge and look at the singular class. We know that the integer lattice lies inside the $\mathbb{C}^2$ plane, and that tells you the isomorphism classes of elliptic curves via the so-called period mapping.

Why do you want to compute this cohomology, because you want to know about the variety. The extra structure is obtained via this "comparison" which lets you distinguish different elliptic curves via the period mapping.

So that's interesting, extra structure, the Hodge filtration, that's interesting stuff. If you compare the singular with the $\ell$-adic, screw the crystalline, we're going to be human beings for today, so if you compare étale to singular, what do you get? The Galois action, the absolute Galois group, if $X$ is over a finite field $\mathbb{F}_p$, then I take the closure, so $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ is what we're looking at, so this is $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_q)$. Understanding this is easy because this is finite, the real deal is doing this over $\mathbb{Q}$. This acts on the étale cohomology as long as $\ell$ is not $p$.

Now if you have a group action on a vector space you get a representation, and what do you want to do? you want to count the rational points of $X(\mathbb{F}_q)\to\mathbb{F}_q$. If I consider $X(\bar{\mathbb{F}}_p)$, I have a natural "geometric Frobenius", and this $\mathrm{Frob}_q$ raises the coordinate $X$ to $X^q$, and the number of fixed points, which is a finite set, coincides with the number of points this variety has over this finite field. Why? Why?

Because this makes you do the American college student's dream: $(x+y)^n = x^n+y^n$, you don't need transversality, this is a bad explanation for why these two things coincide but it does. The number of fixed points can be captured by the trace formula, so what is the trace formula I want to talk about. You all learned, say, the Lefschetz fixed point formula when you were a child, which counts the fixed point of a continuous map. So I'll talk about the Grothendieck–Lefschetz–Hopf trace formula, which tells me that

$$|X(\mathbb{F}_q)| = \sum_{i=0}^{2\dim\ X} (-1)^i \operatorname{tr}(\mathrm{Frob}_q^* : \mathcal{H}^i_{\text{ét}}(X/\bar{\mathbb{F}}_q; \mathbb{Q}_\ell) \to \mathcal{H}^i_{\text{ét}}(X/\bar{\mathbb{F}}_q; \mathbb{Q}_\ell))$$

and this is the number of fixed points, that's what the trace formula does. That coincides with $\mathbb{F}_q$-rational points for finite fields. What's lying behind this is that the absolute Galois group is procyclic, profinite, and this is generated topologically by the Frobenius map. There's nothing else to look at because that's the generator.

So if I want to count the number of points of $\mathbb{A}^1$, it's smooth but not projective. You need to modify, so count for $\mathbb{P}^1$. How many points does it have over $\mathbb{F}_q$? It's $q+1$. According to your little formula it should be

$$\sum_{i=0}^{2} \operatorname{tr}(\mathrm{Frob}_q^* : H^i \to H^i)$$

and so you know that this has $\mathbb{Q}_\ell$ in $i = 0$ and $i = 2$. You know the cohomology because you're a human being, and then you become more of a human being with the Artin comparison. So how does the induced operator act? It's $q^0$ in degree 0 and $q^1$ in degree 2. These are one dimensional vector spaces, so the trace is the sum of the eigenvalues. So $\lambda_0$ is 1 because Frobenius acts trivially on the point. So then $\lambda_2$ has to be $q$. There is a guy named Deligne, I think it's Pierre Deligne, and one thing he proved is that the étale cohomology is of Tate type and étale pure. So this gives an independent proof. You get weight $k$ Hodge structures, what Tate type means, it's semi-simple, the representation is a direct sum of $\mathbb{Q}_\ell$. If it's étale pure, then the weight is what Kim said, its weight should be $i/2$.

So how are you going to write this up with a non-projective or singular variety.

What I'm saying is, back in the 60s, Alexander Grothendieck saw this. At the end of the day all of the cohomology theories have the same Betti numbers, and their extra structures talk to each other.

So what are the axioms of Weil cohomology?

(1) It should be a contravariant functor from smooth projective connected varieties over some field to graded vector spaces (or maybe $k$-algebras) (over some other field).
(2) $H^i(X,k) = 0$ when $i < 0$ or $i > 2\dim X$, and
(3) $H^{2\dim(X)}(X) \cong k$,
(4) Poincaré duality: there is a perfect pairing $H^i(X) \times H^{2\dim X - i} \to k$,
(5) The Künneth formula holds, $H^n(X \times Y) \cong \bigoplus_{i+j} H^i(X) \otimes_k H^j(X)$.
(6) The existence of a cycle map $\eta^i$, an Abelian group of algebraic cycles, this is a smooth connected subvariety of codimension $i$ equipped with a map $\eta^i(X) \to H^{2i}(X)$.
(7) The weak Lefschetz isomorphism (I won't discuss this)
(8) The hard Lefschetz isomorphism (I won't discuss this)

Once you have a Weil cohomology, when they satisfy these conditions, they should coincide as vector spaces in terms of ranks, and even then each of the structures have their own stories, period isomorphisms, point counts, and so on.

So what Grothendieck asked was what is *behind* cohomology, he said there should be an object, a $\mathbb{Q}$-linear Abelian category in which all the Weil cohomology would factor through, $\mathcal{M}_k$, the category of motives over $k$.