CGP DERIVED SEMINAR

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1. Calin Lazariou: A_∞ structures on categories of matrix factorizations

Everything in the mathematics literature here is both trivial and trivially wrong. Not so much is known about this either in mathematics or in physics.

Why is the obvious idea trivial? Let A be a dg category. For any two objects, the space of morphisms $\operatorname{Hom}_A(a,b)$ is R-module. Then R is a unital commutative ring. This is already an A_∞ category of a very particular type. There's nothing to do.

So what would you do? You'd consider a minimal model. So the first (and failed) attempt. Any A_∞ algebra has an anti-canonical (dg) and canonical (minimal) model, which is finite dimensional if the homology is finite dimensional. So assume that A is (cohomologically) hom-finite, compact, or proper (these are all the same thing, please ask Kontsevich why he changed the terminology three times in the past ten years). I will taken $\operatorname{Hom}_A(a,b)$ to be $\mathbb{Z}/2\mathbb{Z}$-graded, and I'll denote this $\operatorname{Hom}^k(a,b)$. That is, $\bigoplus H_d^\alpha \operatorname{Hom}^k(a,b)$ is finite dimensional over $\mathbf{k}$. Here $\mathbf{k}$ is a field in R and R is a $\mathbf{k}$-algebra, and $\mathbf{k}\cdot 1 = \mathbf{k}$.

If you have this, then you have a minimal model, which is realized on the total cohomology category $H(A)$. It's the category which has the same objects as A and the homs are the graded R-modules of cohomology. This is completely trivial by the minimal model theorem.

In the case of a proper dg category, this has the pleasing property that it's a finite dimensional model.

The anti-minimal model has the unpleasing property that the underlying space is infinite dimensional.

This is trivial! All you have done is the Kadeishvili minimal model theorem with more than one object. It's also *wrong*, misdirected, wrong in the philosophical sense of Kant. It's the wrong question to ask, it's the wrong way to think about this.

The question is not to find the minimal model. You haven't done anything. There is a traditional line Laudal, various people, Manetti, that says there are these Massey products controlling the deformation theory and the nicest way to arrange these is with a minimal model. Say I want the moduli stack of an object a, you can build it by representing a deformation functor $\operatorname{Def}_A(a)$. You can represent local Artinian rings, and this can be written as the deformation functor of the commutator L_∞ algebra induced on $\operatorname{End}_A(a) = \operatorname{Hom}_A(a,a)$ by the minimal model of $\operatorname{End}_A(a,a)$.

If you're interested in deformations you can do this, build a moduli stack $\mathcal{M}_a$, an ∞-stack in general. This is again trivial in the sense that it was well-known before, you just put objects in what was known before, and again, not so interesting, because what you really want is to understand the *structure* of $\mathcal{M}_a$. There are physical reasons to expect this to be a non-commutative Calabi–Yau scheme.

So you can find the literature on this but this is the wrong problem. So what's the right way to think about it? The right way to think about the problem is via string field theory. This really works correctly if you have some sort of "Calabi–Yau"-ness. Let me explain what I mean. Your dg category, as I said, I pick some base field and can consider it as a dg category over $\mathbf{k}$, it's $\mathbb{Z}/2\mathbb{Z}$-graded, and I'm fixing μ in $\mathbb{Z}/2\mathbb{Z}$. I say A is *μ-Calabi–Yau* if there exist cyclic *homologically non-degenerate* linear maps

$$\mathrm{tr}_a : \mathrm{End}_A(a) \to \mathbf{k}$$

of degree μ for every a in A. By non-degenerate I mean that the bilinear pairing defined by taking $\mathrm{Hom}_A(a,b) \times \mathrm{Hom}_A(b,a) \to \mathbf{k} \to \mathbf{k}[\mu]$, this is a dg map with zero in the target, cyclic so that $(u,v) = \mathrm{tr}_a(v \otimes u) = \mathrm{tr}_b(u \otimes v)$ and this defines a non-degenerate bilinear form on the cohomology. I required my space to be hom-finite; otherwise I'd need to topologize and require perfectness. You can never have a non-degenerate bilinear form on two vector spaces of infinite dimension. You want this to be invariant up to sign up to the obvious permutation. You want it to be compatible with the differentials here, so that the trace of a boundary is zero, and it should induce a nondegenerate pairing on the cohomology.

I can write down the other properties explicitly:

$$\begin{aligned}
tr_a(v \circ u) &= (-1)^{|u||v|}\,\mathrm{tr}_b(u \circ v) && \\
\mathrm{tr}_a((dv) \circ u + (-1)^{|v|} v \circ (du)) &= 0 && \\
\mathrm{tr}_a(v \circ u) &= 0 && \text{unless } |v| + |u| = \mu \\
\overline{\mathrm{tr}_a} : H(\mathrm{End}_A(a)) &\to \mathbf{k}[\mu] && \text{is nondegerate.}
\end{aligned}$$

A $\mathbb{Z}/2\mathbb{Z}$-graded category with these maps, such a category, with degree μ non-degenerate traces, is usually called a *Calabi–Yau* category, and this is the extension to the dg world, except that you only require the non-degeneracy at the homological level.

I will tell you the interesting problem. What does this have to do with matrix factorizations.

Theorem 1.1. *Let X be a smooth Stein manifold which is holomorphically Calabi–Yau in the sense that its canonical line bundle is trivial. Let W be a holomorphic function on X such that the critical set is compact (in this case finite). Then the $\mathbb{Z}/2\mathbb{Z}$-graded dg category of matrix factorizations $PF(X,W)$, of projective analytic factorizations of W is proper and μ-Calabi–Yau with $\mu \equiv d \pmod 2$ where d is the dimension of X as a complex manifold.*

One of the nicest types of Landau–Ginzburg pairs is (X,W) where X is Stein and W is holomorphic. I insist on this compactness to get a proper category.

What is K_X? It's the top wedge product $\wedge^d T^*X$ (the holomorphic cotangent bundle) is trivial, isomorphic as a holomorphic line bundle to $\mathcal{O}_X$.

There is a particular example of Gromov's principle that says that the topological and holomorphic classifications coincide in this setting (Stein) so topologically trivial (first Chern class vanishes) implies holomorphically trivial.

If you try to do a non-Calabi–Yau version, then you get an anomaly in the $U(1)_X$ symmetry. So twisting with K_X like Pantov, Katzarkov, Pomerleano, Orlov, et cetera, have done, is physically wrong. Then you have to do something very weird on the other side to the Fukaya category. What is behind is that the correct data, you have to build an open-closed field theory.

Why did I mention compact? There's a version of this category, the so-called correct version, which doesn't require Stein, which is *not* the version they have proposed. There's something called DF which is again triangulated and $\mathbb{Z}/2\mathbb{Z}$-graded and makes sense for any X complex non-compact, and any W holomorphic with compact critical locus. There's a hypercohomology description, but this is a 2-periodic thing.

Of course any affine variety is a Stein analytic space, and in that case you can do an algebraic version of this category, but this is a much nicer statement, I think.

So what is $PF(X,W)$? They are pairs (P,D) where P is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathcal{O}(X)$-module, degreewise projective and finitely generated. And D is an endomorphism of this module such that $D^2 = W$. The morphisms are the obvious ones, if I give you $a_1 = (P_1, D_1)$ and $a_2 = (P_2, D_2)$, then the hom space in PF is $\underline{\mathrm{Hom}}_{\mathcal{O}(X)}(P_1, P_2)$, with the defect differential

$$d_{a_1,a_2}(f) = D_2 \circ f - (-1)^{|f|} f \circ D_1.$$

There's a Serre–Swan theorem for Stein manifolds. Their condition is satisfied by the sheaf of holomorphic functions on a Stein manifold. [Something about Cartan Theorem B.] This says that finitely generated projective $\mathcal{O}(X)$-modules are equivalent to holomorphic vector bundles.

There's a general result that if X is a non-compact complex manifold with $K_X \cong \mathcal{O}(X)$. Let W have compact critical locus. Then $DF(X,W)$ is proper and d (mod 2)-Calabi–Yau. This is the twisted Dolbeault category of the holomorphic factorizations. The objects are (E,D) where E is a $\mathbb{Z}/2\mathbb{Z}$-graded holomorphic vector bundle and $D \in \Gamma(X, \mathrm{End}(E))$ such that $D^2 = W\mathrm{id}$.

The morphisms between two is $\mathcal{A}^{0,*}(X, \mathrm{Hom}(E_1, E_2))$ equipped with the differential $\bar{\partial} + \partial_{a_1,a_2}$, where this latter on ω is $D_2 \circ \omega - (-1)^{|\omega|} \omega \circ D_1$.

The way I prove this with Dmitry is by combining Serre's original result with sophisticated spectral sequence arguments. This is a very general example.

I didn't introduce this notion of Calabi–Yau category, of course.

This is still not what you need. I will tell you in a moment how this is induced by a holomorphic volume form. But you need more, you need this notion of a Calabi–Yau structure, which is more than these traces.

Definition 1.1. Let A be proper, $\mathbf{k}$-linear (I'll assume $\mathbf{k}$ of characteristic zero, my interest is in $\mathbb{C}$) dg category. A cochain level Calabi–Yau structure (of degree μ) on A is a linear map from the cyclic complex $\theta : CC_*(A) \to \mathbf{k}[\mu]$, so

(1) $\theta \circ \delta = 0$
(2) $\theta_* : HC(A) \to \mathbf{k}[\mu]$ induces nondegenerate traces on $H(A)$ via precomposition with the natural map q from $H(A)$ to the Hochschild complex, and then this gives a natural map to to the cyclic homology. So this restriction is a homologically non-degenerate trace.

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So that's the Calabi–Yau structure. They only cared about the cohomology class, but this is a trivial extension, this was basically introduced by Kontsevich–Soibelman. To be precise, string field action is a *strict* cyclic structure, where the traces induecd by θ are nondegenerate at the cochain level.

So either this cyclic structure is established at the level of the minimal model or you topologize and require a perfect pairing. Everything you see here is defined for any A_∞ category. I can consider a minimal A_∞ category which is proper, and there require nondegeneracy off-shell.

The punchline, the point, there's a theorem, the particular case was proved by Sklyarov, that says the cohomologically non-degenerate traces of $DF(X,W)$ have a natural extension to a chain level μ-Calabi–Yau structure which is induced by a cubic open string field theory (in the sense of Witten). Cubic means that you have only, you have a dg model, but the trace is non-degenerate off-shell. This is something with compact supports. The trace is induced by the volume form. You do a gauge-fixing procedure, trying to find a quasi-isomorphic model by projecting on a small tubular neighborhood of your critical locus.

A *minimal Calabi–Yau* structure or strictly cyclic minimal A_∞-category is a minimal A_∞-category which is proper, the spaces are finite dimensional, and the traces are strictly cyclic with respect to the A_∞ structure, so

$$\langle f_0, m_n(f_1,\ldots,f_n)\rangle = (-1)^{\text{whatever}}\langle f_1, m_n(f_2,\ldots,f_n,f_0)\rangle.$$

In practice this was hard to construct, and Sklyarov gives you such a theory. You replace DF with a compactly supported version DF_c, which naturally includes in DF, so the objects are the same, but the morphisms are *compactly supported* forms of type $0,*$ as before. If Ω is a volume form, a holomorphic section of $K_X\backslash\{0\}$, then you have for $\omega \in \mathrm{End}_{DF_c}(E,D)$ the following:

$$\mathrm{tr}_c(\omega) = \int \Omega\wedge \mathrm{str}(\omega)$$

and you have $\delta_W = \delta + \partial_{a_1+a_2}$, and this is a *perfect trace* if $DF_c(X,W)$ is topologized using the Fréchet topology. Then the cubic string field action is the functional S with

$$S(\phi) = \int_X \Omega\wedge[\mathrm{str}(\phi\delta_W\phi) + \frac{2}{3}\mathrm{str}(\phi^3)]$$

a ($\mathbb{Z}/2\mathbb{Z}$-graded twisted-by-W, categorified: $\phi\in\mathrm{End}(A) = \bigoplus_{a,b}\mathrm{Hom}(a,b)$) Chern–Simons type action).

But this only makes sense on the compactly supported one, and it uses smooth things, none of the algebraic geometers and few of the complex geometers would touch this.

Then what you do, the idea is the following, how does that object transfer into something defined on the other category. These are dg categories. You can prove that the map induced on cohomology is an isomorphism, so that HDF_c is a quasi-equivalence. If we know anything about quasi-equivalences, there should be a (non-unique) quasi-equivalence. This is not just an ordinary map, it's an A_∞ quasi-isomorphism. I'm sure you've seen this at least for algebras. It inverts i. It's an ordinary thing that commutes with differentials, but it has an inverse with many pieces.

You want to make a choice, getting rid of anything smooth, Fréchet, et cetera. You choose some tubular neighborhood of the (compact) critical locus and try to construct π_1 as a projector, I won't give the formula, and then π_n are given by some universal formula using π_1 and some property. This depends on the choice of infinitesimal neighborhood. You take some sort of inductive limit in which this neighborhood shrinks to Z_W and in that limit you use a residue theorem of [unintelligible]–Andersson (not Grothendieck, you need to upgrade this, a representation of Bochner–Martinelli type) so when you do this you find that all the θ_n of the corresponding Calabi–Yau structure have an expression in terms of these W–A residues.