CGP DERIVED SEMINAR

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1. March 6: Taesu Kim: Introduction to the Fukaya category IV (or III)?

Let (M,ω) be a symplectic manifold and let L an oriented spin closed Lagrangian submanifold. We put some conditions on these geometric objects, for instance

- that the first Chern class of M is 2-torsion,
- that μ_L which lives in $H^1(L,\mathbb{Z})$, called the Maslov class of L, vanishes, and
- that $[\omega]\pi_2(M,L) = 0$.

Call these conditions (*). The first two of these are to give us a $\mathbb{Z}$-grading on Floer cochain complexes. The final condition is to prevent so-called "disk bubbling." Let me explain what this means later, so that we have the $\partial^2 = 0$ condition.

The spin condition is needed to put an orientation on the moduli space of pseudoholomorphic disks. We need this to appropriately count the number of rigid elements so that we can define the differential of the chain complex.

This is our geometric setting. Here are, what we called the Fukaya category, this is an A_∞-category $F(M,\omega)$. Its objects are Lagrangian submanifolds satisfying (*). The Floer chain complex between L_1 and L_2 is the direct sum over intersection points of Λp, where Λ is the Novikov ring. We assume $L_1 \pitchfork L_2$ for this definition. Later we'll modify this in some way. We should consider the Hamiltonian isotopy $\phi^t_{H_{L_1,L_2}}$ associated to $H_{L_1,L_2} \in C^\infty([0,1]\times M;\mathbb{R})$ so that $L_1 \pitchfork \phi^1_{H_{L_1,L_2}}(L_2)$ and then define $CF(L_1,L_2)$ as $CF(L_1,\phi^1_{H_{L_1,L_2}})$ which a priori depends on H_{L_1,L_2} and do this in a way that makes these transversal. This is a Λ-module with $\mathbb{Z}$-grading.

This construction anyway includes the case $CF(L,L)$. What about composition rules. We consider a map u from the disk with fixed holomorphic structure to M with fixed compatible almost complex structure J. We mark points z_0 to z_d on the boundary of the disk. We are given $L_1,\ldots,L_d,L_0$, objects in $F(M,\omega)$ (i.e. Lagrangians satisfying (*) which transversally intersect).

[pictures]

The conditions are that $u(z_i) = p_i$, and u is J-holomorphic in that $\bar\partial_J(u) = 0$. The image of the arc between z_i and z_{i+1} should lie in L_i and $[u] = \beta$. Then

$$\widetilde{\mathcal{M}}(p_1,\ldots,p_d,p_0,\beta)$$

is the set of such maps. In nice cases this has a manifold structure and the expected dimension (assuming generic J) this is $\operatorname{ind}(D\bar\partial_J(u)) = d+1+\operatorname{ind}(U)$, where $\operatorname{ind}(U)$ is the Maslov index of the disk, and so this is $d+1-\sum_{i=1}^d |p_i| + |p_0|$. Then we can reduce by an equivalence relation and get

$$\mathcal{M}(p_1,\ldots,p_d,p_0,\beta) = \widetilde{\mathcal{M}}(\vec{p},\beta).$$

The dimension of $PSL(2,\mathbb{R})$ is three, so the expected dimension is $d-2-\sum_1^d |p_i| + |p_0|$.

Then $\overline{\mathcal{M}}(\vec{p},\beta)$ can be defined, a compactification so that its boundary consists of maps with image nodally glued disks. Since $[\omega]\pi_2(M,L_j)=0$, the area of this disk should be zero and so this cannot happen.

So now we can define the composition between these morphisms

$$\mu^d : CF(L_{d-1},L_d)\otimes\cdots\otimes CF(L_0,L_1)\to CF(L_0,L_d)$$

by

$$\mu^d(p_d,\dots,p_1)=\sum_{q\in L_d\cap L_0}\#\mathcal{M}((\vec{p},q),\beta)T^{\omega(beta)}q$$

where the sum is over $q\in L_d\cap L_0$, where $\sum|p_i|-|p_0|=d-2$, and β.

We have to check that the operators μ^d satisfy the A_∞ relations. For $d-1=\sum|p_i|-|p_0|$, then it's a compact 1-manifold, so a disjoint union of intervals and circles. So the signed count of a boundary of this 1-dimensional moduli space is zero.

This says that the signed count of nodal configurations of the appropriate dimension with one node are zero. Then this can be parameterized by gluing parameters somewhere, and near the limit there is a one-to-one correspondence between the nodal and glued smoothed configurations. The glued configurations, counting them is about the composition of two copies of μ^j for some smaller j. Then the sum being zero says that this sum of compositions is zero. Let me put the sign as a $\pm$ and that's how we get the A_∞ relation. Hence the Fukaya category is an A_∞ category.

One important point is that it's cohomologically unital, so that $HF(M,\omega)$ is unital.