CGP DERIVED SEMINAR

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1. February 20: Kyoung-Seog Lee: Hochschild homology of DG categories

Today I will talk about both homology and cohomology of something. First let me discuss Hochschild homology and cohomology of algebras.

Let $\mathbf{k}$ be a commutative ring and R be a $\mathbf{k}$-algebra and M be an R-R-bimodule. Here R can be a non-commutative $\mathbf{k}$-algebra.

In this setting I can associate a simplicial $\mathbf{k}$-module $M\otimes R^{\otimes *}$ with

$$[n]\mapsto M\otimes R^{\otimes n}$$

and for concreteness, $M\otimes R^{\otimes 0}=M$.

I will make a complex

$$0\leftarrow M\xleftarrow{\delta_0-\delta_1} M\otimes R\xleftarrow{d} M\otimes R\otimes R$$

with $d=\sum(-1)^i\partial_i$.

Here $\delta_i(m\otimes r_1\otimes\cdots\otimes r_n)$ is

$$\begin{cases} mr_1\otimes r_2\otimes\cdots\otimes r_n & i=0\\ m\otimes r_1\otimes\cdots\otimes r_ir_{i+1}\otimes\cdots\otimes r_n & 0<i<n\\ r_nm\otimes r_1\otimes\cdots\otimes r_{n-1} & i=n.\end{cases}$$

And $\sigma_i(m\otimes r_1\otimes r_n)=m\otimes\cdots\otimes r_i\otimes 1\otimes r_{i+1}\otimes\cdots\otimes r_n$.

So call $C(M\otimes R^{\otimes *})$ the chain complex above, and then

Definition 1.1. The Hochschild homology $H_n(R,M)$ is the homology $H_nC(M\otimes R^{\otimes *})$.

When we look at $M\otimes R\to M$, the differential takes $m\otimes r$ to $mr-rm$, so $H_0(R,M)\cong M/[M,R]$. In the same setting I can define a cosimplicial $\mathbf{k}$-module, where n goes to $\operatorname{Hom}_{\mathbf{k}}(R^{\otimes n},M)$, this is $\mathbf{k}$-linear maps from $R^{\otimes n}$ to M. We can again define a cochain complex

$$0\to M\to \operatorname{Hom}_{\mathbf{k}}(R,M)\to\operatorname{Hom}_{\mathbf{k}}(R\otimes R,M)\to\cdots$$

and let me call this $C\operatorname{Hom}_{\mathbf{k}}(R^{\otimes *},M)$ and d is defined the same way, $d=\sum(-1)^i\partial^i$ and let me define ∂^i as follows. This is a $\mathbf{k}$-module of functions, so $(\partial^i f)(r_0,\ldots,r_n)$ is

$$\begin{cases} r_0f(r_1,\ldots,r_n) & i=0\\ f(r_0,\ldots,r_ir_{i+1},\ldots,r_n) & 0<i<n\\ f(r_0,\ldots,r_{n-1})r_n & i=n.\end{cases}$$

We can define $\sigma^if(r_1,\ldots,r_n)=f(r_1 ldots,r_i,1,r_{i+1},\ldots,r_n)$.

Definition 1.2. The Hochschild cohomology $H^*(R,M)$ is the $\mathbf{k}$-module which is the cohomology of the cochain complex $H^n(C\operatorname{Hom}_{\mathbf{k}}(R^{\otimes *},M))$.

You have $0 \to M \to \operatorname{Hom}(R,M)$. If I have m, this goes to $\partial^0(m) - \partial^1(m)$, this is a function, which when you apply it to r, by definition, this is $rm - mr$. So $H^0(R,M)$ consists of the m in M such that $rm = mr$.

Let us compute $H^1(R,M)$. This is very closely related to derivations. I'll write

$$0 \to M \xrightarrow{d} \operatorname{Hom}(R,M) \xrightarrow{d} \operatorname{Hom}(R\otimes R, M)$$

and if I take f in $\operatorname{Hom}(R,M)$ it goes to $\partial^0 f - \partial^1 f + \partial^2 f$ and

$$(\partial^0 f - \partial^1 f + \partial^2 f)(r_0 \otimes r_1) = r_0 f(r_1) - f(r_0 r_1) + f(r_0) r_1$$

which means that $f(r_0r_1) = r_0 f(r_1) + f(r_0)r_1$.

So the kernel of d is nothing but the set of $\mathbf{k}$-linear maps $f : R \to M$ satisfying this condition, which we call the $\mathbf{k}$-*derivation* condition. So $\operatorname{Der}_{\mathbf{k}}(R,M)$.

I should mod this out by the image of M, so M goes to $\operatorname{Hom}(R,M)$, so m goes to f_m which is $r \mapsto rm - mr$, and you can check that $f_m(r_0r_1)$ is a derivation:

$$\begin{aligned} f_m(r_0r_1) &= r_0r_1m - mr_0r_1 \\ &= r_0(r_1m - mr_1) + (r_0m - mr_0)r_1 \\ &= r_0 f_m(r_1) + f_m(r_0)r_1. \end{aligned}$$

So we call the principal derivations

$$\operatorname{PDer}_{\mathbf{k}}(R,M) = \langle f_m \rangle.$$

So

$$H^1(R,M) \cong \operatorname{Der}_{\mathbf{k}}(R,M)/\operatorname{PDer}_{\mathbf{k}}(R,M).$$

Definition 1.3. Let R be a commutative $\mathbf{k}$-algebra. We can define the *Kähler differential* of R over $\mathbf{k}$ is

$$\Omega_{R/\mathbf{k}} = R\langle dr | d\alpha = 0 : \alpha \in \mathbf{k}\rangle.$$

So if $R = \mathbb{C}[x_1,\ldots,x_n]$ then $\Omega_{R/\mathbf{k}} = R\langle dx_1,\ldots,dx_n\rangle$.

This is an example,

Proposition 1.1. *Let R be a commutative $\mathbf{k}$-algebra and M be an R-R-bimodule, $rm = mr$. Then $H_0(R,M) \cong M$ and $H_1(R,M) \cong M \otimes_R \Omega_{R/\mathbf{k}}$.*

This is dual to derivation, this is dual to 1-forms.

Hochschild cohomology is related to derivations; homology is related to 1-forms in R.

When R is a polynomial ring, then $H_1(R,R) \cong \Omega^1_{R/\mathbf{k}}$ and $H^1(R,R) \cong T^1_{R/\mathbf{k}}$.

Let $R = \mathbb{C}[x]$, and $\mathbf{k} = \mathbb{C}$, and let us compute $\operatorname{Der}_{\mathbf{k}}(R,R)$. This is, by definition, $\mathbf{k}$-linear homomorphisms $R \to R$ such that $f(r_0r_1) = r_0 f(r_1) + f(r_0)r_1$. In this case, this is a function, a $\mathbf{k}$-linear map. $f(x) = 1f(x) + f(1)x$. This implies that $f(1) = 0$. Then $f(x^2) = 2xf(x)$.

I want to claim that $\operatorname{Der}_{\mathbf{k}}(R,R) \cong R\langle \frac{\partial}{\partial x}$.

So then for $\mathbb{C}[x]$ the principal derivations are 0 so $H^1(R,R) \neq 0$.

Exercise 1.1. Let $R = \mathbf{k}[x]/(x^{n+1} = 0)$. Then if $\frac{1}{n+1} \in R$, we hav $H_i(R,R) \cong H^i(R,R) \cong R/(x^nR)$ for all $i \geq 1$.

When R is $\mathbb{C}[x_1, \dots, x_n]$, $M = R$, and $\mathbf{k} = \mathbb{C}$, then $H_0(R,R) \cong R$ and $H_1(R,R) \cong Rdx_1 \oplus \cdots \oplus Rdx_n \neq 0$ and $H^1(R) \cong R\frac{\partial}{\partial x_1} \oplus \cdots \oplus R\frac{\partial}{\partial x_n}$ then this is nonzero too.

On the other hand for $R = \mathbb{C}$, you get $H_i(R,R) \cong H^i(R,R) \cong 0$.

Let me show you one more example. This first homology is related to Kähler differentials. Let me give you one more, related to H^2. As I told you, H^2 is related to deformation. Let me show you. So a *square zero extension* of R by M is a $\mathbf{k}$-algebra E with $E \xrightarrow{\epsilon} R$ a projection such that $\ker \epsilon$ is an ideal of square zero and $M \cong \ker \epsilon$ as R-modules. So $0 \to M \to E \to R \to 0$ is short exact. This is called a Hochschild extension if $0 \to M \to E \to R \to 0$ is $\mathbf{k}$-split. This is an algebra, so as a $\mathbf{k}$-module, it's isomorphic to $R \oplus M$. As an algebra, I have a multiplication, I have $(r_1, m_1)(r_2 m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2 + f(r_1, r_2))$.

So $f : R \otimes R \to M$, and because this is an associative algebra, we should have $(r_1, 0)(r_2, 0)(r_3, 0)$ gives some condition. So you get $(r_1 r_2, f(r_1, r_2))(r_3, 0) = (r_1 r_2 r_3, f(r_1, r_2) r_3 + f(r_1 r_2, r_3))$ and I can do the other side and add up, and you eventually get a condition that f is a *cycle*, that

$$r_0 f(r_1, r_2) - f(r_0 r_1, r_2) + f(r_0, r_1 r_2) - f(r_0, r_1) r_2 = 0$$

and this is nothing but $df(r_0, r_1, r_2)$, which is just $\partial^0 - \partial^1 + \partial^2 - \partial^3$. And from this associative rule, this says that $f \in Z^2(\ )$ of our cochain complex. If I choose another section, I had to choose a section, and if I choose another σ' I get another f' and we can check that the difference is in $B^2(\ )$ of our cochain complex. I want to say that this kind of extension, the equivalence class of Hochschild extensions is in one to one correspondence with $H^2(R,M)$. If M and R are commutative, then I have some commutative version which corresponds to another version of Hochschild cohomology.

Why is this kind of thing interesting? When M is R, then this kind of diagram is something like this. If I have $\operatorname{Spec} \mathbf{k}[\epsilon] \to \operatorname{Spec} \mathbf{k}$ and have $\operatorname{Spec} R \to \operatorname{Spec} \mathbf{k}$, then this diagram, this algebra.

So what this means, if you look at $R = \mathbf{k}[x]/(x^2)$, you have this kind of sequence:

$$0 \to (x) \to R \to \mathbf{k} \to 0$$

and this is exactly that situation. As Damien said, when I have this kind of $0 \to R \to E \to R \to 0$, then it means that I have some kind of, you have the deformation space of Spec R, and here you have some kind of choice of direction, to deform the algebra. This has this kind of feeling.

[Is it true that $H^2(R,R)$ is the same as equivalence classes of flat algebras so that when I point at $\mathbf{k}$, it reduces to R?]

Yes. So H^2 measures deformations of a certain kind of structure. Here it's deformations of algebra. This is some feeling I have.

I believe you have some feeling of this now.

Let me just state some general feeling. Let me write some general theorems that I think are quite important.

Let me give another definition of Hochschild homology. Let me define R^e to be $R \otimes_{\mathbf{k}} R^{\mathrm{op}}$. This op means it's the $\mathbf{k}$-algebra with $r\dot{s} = sr \in R$. Then this is a $\mathbf{k}$-module, and then a right R-module M is the same as a left R^{op}-module. Then an R-R-bimodule is a left R^e-module, $(r \otimes s)m = rms$. In the same way, it's also a right R^e-module.

You can check that, using the bar resolution, if R is flat over $\mathbf{k}$ then $H_*(R,M) \cong \operatorname{Tor}_*^{R^e}(M,R)$. If R is projective over $\mathbf{k}$ then $H^*(R,M) \cong \operatorname{Ext}^*_{R^e}(R,M)$. Here R is an R-R-bimodule, and M is one, so you can make everything a left R^e-module.

Let X be a smooth (projective?) variety over $\mathbf{k}$, let $\mathbf{k} = \bar{\mathbf{k}}$ of characteristic zero. Then

$$HH_*(X) \coloneqq H^*(X \times X, \Delta_*\mathcal{O}_X \otimes^L \Delta_*\mathcal{O}_X))$$

and

$$HH^*(X) \coloneqq \operatorname{Hom}^*_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X).$$

Theorem 1.1 (Hochschild–Kostant–Rosenberg)**.** *X is a smooth projective variety of dimension n, then*

$$HH_i(X) \cong \bigoplus_{p=0}^{n} H^{i+p}(X, \Omega^p_X)$$

and

$$HH^i(X) \cong \bigoplus_{p=0}^{n} H^{i-p}(X, \wedge^p T_X).$$

Finally let me discuss the Hochschild homology and cohomology of a dg algebra or category. Let C^* be a dg algebra. Then C^* is a dg bimodule over C^*, and $HH_*(C) = C \otimes^{\mathbb{L}}_{C\otimes C^{\mathrm{op}}} C$ and $HH^*(C) = \mathbb{R}\operatorname{Hom}_{C\otimes C^{\mathrm{op}}}(C,C)$.

Whenever I have X a smooth projective variety, I can consider the derived category $D(X)$, and it is known that this has a so-called *strong generator*. Let $\mathcal{E}$ be a strong generator of $D(X)$. I can consider $C = \mathbb{R}\operatorname{Hom}(\mathcal{E},\mathcal{E})$. I mean I have a category, and a generator, and I have the endomorphism algebra of the generator. Then I can consider the Hochschild homology and cohomology of this algebra to be the Hochschild homology and cohomology of this category.

For example, $D(\mathbb{P}^1)$, this is generated by $\langle \mathcal{O}, \mathcal{O}(1)\rangle$, the strong generator is $\mathcal{O} \oplus \mathcal{O}(1)$, and C is the Kroenecker quiver on $\bullet \rightrightarrows \bullet$. You can compute that $HH^2(C) = 0$ because $\mathbb{P}^1$ is a rigid variety. So the right hand side of HKR is easy to compute. Sometimes we can compute it.