CGP DERIVED SEMINAR

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1. Feb 6: Christophe Wacheux: A_∞-structure on Fukaya categories II

I have way too much stuff. Last time I showed the formula of the derivative that we will, the differential of the Floer complex.

$$\partial(p) = \sum_{\substack{q \in L_0 \pitchfork L_1 \\ [u]:\mu([u])=1}} \#\mathcal{M}(p,q;[u],J)T^{\omega[u]}q.$$

In order to make it so that this is a zero dimensional manifold, I want to count only Maslov index 1 holomorphic disks. Because I have an orientation, in the good case, a spin structure on the Lagrangian, I can do this and get an orientation on the moduli space and get signs so that I can now count with signs, and that's the count I put here. This $\omega[u]$ is $\int_D u^*\omega$. There is lots of reason for this not to be well-defined. This is a compactness issue which is taken care of using the Gromov compactness theorem.

This, as we saw earlier, belongs to Λ^R, the Novikov ring.

The plan today is to try and define some of the things, I want you all to see the actual formula for the k-ary operation, and then after that I discuss as many details as possible.

Just to mention, the theorem was to show that

Theorem 1.1 (Floer)**.** *If $k = \mathbb{Z}_2$ and $[\omega].\pi_2(M, L_i) = 0$, then*

(1) *∂ is well-defined,*
(2) *$\partial^2 = 0$,*
(3) *$HF(L,L) \cong H_*(L,\mathbb{Z}_2)$, and*
(4) *$HF(L_0,L_1)$ doesn't depend on the choice of J, of isotopy class of L_i*

This result helps prove the Arnold conjecture, at least in this case. Then it was extended to another very nice setting. It was extended to the case which is called monotone, also a very important case, Yong-Geun did it, that $\int_D u^*\omega = \lambda\mu([u])$ for u representing a class in $\pi_2(M,L)$.

Now I'll make a huge jump to define, to give any sense to this formula, this is assuming everything works fine. I should write "AEWF," and I'll try to give sense to what this acronym means. Now the product that we will call m_2, Now you have a pair of pants [sic]. It's the disk with three marked points, each of which is sent to one intersection (between L_0, L_1, and L_2 pairwise). So z_0 goes to q in the intersection of L_0 and L_2 and z_i to p_i in the intersection of L_{i-1} and L_i. Then we have

$$m_2 : CF(L_1,L_2) \otimes CF(L_0,L_1) \to CF(L_0,L_2)$$

given by

$$m_2(p_2 \otimes p_1) = \sum_{\substack{q \in L_1 \pitchfork L_2 \\ [u]|\mu[u]=0}} \#\mathcal{M}(p_1,p_2,q;[u],J)T^{\omega[u]}q.$$

This is the formula. And I guess now we start to understand that we have some pattern going on. The whole general operation $m_k : CF(L_{k-1}, L_k) \otimes \cdots \otimes CF(L_0, L_1) \to CF(L_0, L_k)$ is given by

$$m_k(p_k \otimes \cdots \otimes p_1) = \sum_{\substack{q \in L_0 \pitchfork L_k \\ [u] | \mu([u]) = 2-k}} \mathcal{M}(p_1, \ldots, p_k, q; [u], J) T^{\omega[u]} q.$$

You can guess, these verify A_∞ relations. Maybe this is not worth taking time to verify.

I really want to address a bunch of problems. If I want an honest A_∞ structure, then I need to define a grading. So now I want to talk about Maslov index and grading of CF. First we need an orientation on the Lagrangian if you want to work over something other than $\mathbb{Z}_2$, then you will need to be able to define a consistent grading. The way to do this is with the Maslov index. If you have your Lagrangian manifold living in some big ambient symplectic manifold, if you look at the tangent space, then T_pL is a Lagrangian subspace of T_pM. What are the Lagrangian subspaces of M? If I give you a distribution of Lagrangian subspaces, does it integrate to a Lagrangian submanifold? So if I say now, I look at, just, on T_pM it's just the same as $\mathbb{R}^{2n}$, I can locally trivialize, and I define $LG(n)$ as the set of Lagrangian vector spaces of $\mathbb{R}^{2n}$, the "Lagrangian Grassmanian." There is a result which tells you that this is isomorphic to $U(n)/O(n)$. You can look at $\det^2 : U(n)/O(n) \to S^1$, and this goes, for $\pi_1[\det^2] : \pi_1(U(n)/O(n)) \to \mathbb{Z}$, and that's an isomorphism.

Essentially this is going to define μ. I'll define two things, I'm going to define the Maslov index of a holomorphic strip, the Maslov class of the Lagrangian, and the degree of a point. Let me write, now given u a J-holomorphic strip, let me remind you how this looks [picture].

If I look at $u^*_{\mathbb{R}\times\{i\}} TL_i : [0,1] \to LG(n)$ and call that ℓ_i, then my path of Lagrangian subspaces, since L_0 and L_1 intersect transversally, then

$$\ell_0(0) \pitchfork \ell_1(0) \text{ and } \ell_0(1) \pitchfork \ell_1(1).$$

As you said, Gabriel, I want to identify $(\mathbb{R}^{2n}, \omega_0) \cong (\mathbb{C}, \omega)$ and I want to say that $\ell_0 \in LG(n)$, there exists an $A_0 \in GL_n(\mathbb{C})$ such that $A_0(\ell_0(0)) = \mathbb{R}^n$ and $A_0(\ell_1(0)) = i\mathbb{R}^n$.

Now I call this $\lambda(t) \coloneqq A_0^{-1}(e^{i\frac{\pi}{2}t}\mathbb{R}^n)$. [pictures].

Now I identify what is the path going from the tangent to the tangent, between $\ell_0(0)$ and $\ell_1(0)$. I can do the same stuff with a different identification for $\lambda_1(t)$, between $\ell_0(1)$ and $\ell_1(1)$. Now I will define the Maslov index of a J-holomorphic strip.

Definition 1.1. Define $\gamma : [0,1] \to LG(n)$ by $\gamma = \ell_0 \bullet \lambda_1 \bullet \ell_1^{-1} \bullet \lambda_0^{-1}$ and then

$$\mu([u]) \coloneqq \pi_1[\overset{2}{\det}][\gamma]$$

and that's the Maslov index of a strip.

We've defined the Maslov index of a strip. In order to start doing all this business I need a spin structure, which is a choice of a section in the double cover of $U(n)$, or $O(n)$.

Now to define a $\mathbb{Z}$-grading, I will need exactly to make sure that $\mu[u]$ depends only on $|p| - |q|$ but not on $[u]$ even if I didn't define it yet. To make it happen, one thing is to ask for $2c_1(TM) = 0$, which, now you get a hint for why this works in $\mathbb{Z}_2$. Why do you need this? I'll define something more elaborate than a Lagrangian, something we called a [unintelligible]Lagrangian submanifold, taking a universal cover of the Lagrangian Grassmannian.

So $c_1(TM)$ tells me, take $\Theta \in \wedge^n T^* M \otimes \mathbb{C}$, then $\varphi(D) = \arg(\Theta|_D) \in S^1$. Now you define $\tilde{\varphi}(D)$, a choice of smooth lift of $\varphi(D)$, and since $\pi_1(LG(n))$ is $\mathbb{Z}$, you can think of the universal cover,