CGP DERIVED SEMINAR

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1. January 16, 2018: Byunghee An, bar and cobar

Today I am going to talk about bar and cobar constructions again, between categories of algebras and coalgebras.

I think that one of the goals is to explain this diagram. You have a category Alg of algebras and a category Alg_∞ of ∞ algebras, and a category Cog of coalgebras, and a full subcategory of fibrant and cofibrant objects. Among these four categories we can think of algebras as a non-full subcategory of A_∞ algebras. We can think of coalgebras which are fibrant and cofibrant as a full subcategory of coalgebras. We want to define functors between these:

$$\begin{tikzcd} \mathrm{Alg} \arrow[r] \arrow[d, shift left, "B"] & \mathrm{Alg}_\infty \arrow[d, "B_\infty"] \\ \mathrm{Cog} \arrow[u, shift left, "\Omega"] & \mathrm{Cog}_{cf} \arrow[l] \end{tikzcd}$$

and B_∞ will be an equivalence and all of these will induce equivalences on the homotopy categories.

I will define everything but this is my goal.

Let's start with algebras Alg. This is the category of unital augmented dg algebras over $\mathbf{k}$ a field, objects are (A,ϵ) where A is a unital algebra and ϵ is an algebra map $A\to\mathbf{k}$, and this has a model category structure where the weak equivalences are the quasi-isomorphisms, the fibrations are the degreewise surjections, and the cofibrations are the maps with the left-lifting property against trivial fibrations.

It is known

Theorem 1.1. *This data defines a model structure on* Alg.

Now I want to define the category of coalgebras, so let me denote Cog' the category of coaugmented dg coalgebras, an object consists of a complex C with differential d, a coproduct Δ, a counit η and a coaugmentation ϵ. We require that d is a coderivation against Δ, so that $(d\otimes 1+1\otimes d)\Delta = \Delta d$, that $\eta\epsilon = 1_{\mathbf{k}}$.

For example, let (V,d) be a complex. Then $T^c(V)$, the "tensor coalgebra" on V, is $\bigoplus_{n\geq 0} V^{\otimes n}$, and the coproduct is defined by the sum of all possible separations.

$$\Delta(v_1\otimes\cdots\otimes v_n) = \sum (v_1\otimes\cdots\otimes v_i)\boxtimes(v_{i+1}\otimes\cdots\otimes v_n).$$

So for example $\Delta(v) = 1\boxtimes v + v\boxtimes 1$.

There is a canonical projection $T^c(V)\to V$ taking the V summand. But $T^c(V)$ is not cofree on V. This does not have a universal property, that coalgebra maps to $T^c(V)$ are the same as maps to V. Suppose we have C a coalgebra and take a chain map $C\to V$. Then there need not be a lift to $T^c(V)$. The answer is no.

So I want a smaller (full) subcategory Cog whose objects are *cocomplete*, meaning that $C = \sqcup \ker(C\to C^{\otimes n}\to (C/\mathbf{k})^{\otimes n})$ where $C\to C^{\otimes n}$ is the iteration $\Delta^{(n)}$ of Δ. If you take the coproduct enough times it has the base field element in at least one factor. It cannot be decomposed in a trivial way any more.

We can easily see that the tensor coalgebra is cocomplete, any element is in the form, a finite sum of $v_1 \otimes \cdots v_n$, and if you take the coproduct to $n+1$ copies, there will be some 1 somewhere. Actually it is not only cocomplete, it is cofree in the category of cocomplete coalgebras. For any C in cocomplete coalgebras and any chain map from $\bar{C}$ to V we can always find a map from C to $T^c(V)$. These should be counital maps and send $\mathbf{k}$ to 0. So $T^c(V)$ is cofree on V in Cog.

So let's let C be a cocomplete coalgebra and A a unital augmented algebra. Then we want to consider $\mathrm{Hom}_{\mathbf{k}}(C, A)$, well, really, we want chain maps that are compatible with augmentations. We want a differential and algebra structure, and I'll put this in Alg, and the differential will be D and the multiplication $*$. The differential is $D(f) = d_A f - (-1)^{|f|} f d_c$. The product is $\mu_A(f \otimes g)\Delta_c$, the unit is $\eta_A \circ \eta_C$.

The nontrivial thing to check is that d is a derivation with respect to the product.

$$D(f * g) = Df * g + (-1)^{|f|} f * Dg.$$

The left hand side is

$$d_A \mu_A (f \otimes g)\Delta_C - (-1)^{|f|+|g|} \mu_A (f \otimes g)\Delta_C d_C,$$

and the Δ_C and d_C have compatibility and can be interchanged, and likewise d_A and μ_A, so we get

$$\begin{aligned} &\mu_A(d_A \otimes 1 + 1 \otimes d_A)(f \otimes g)\Delta_C - (-1)^{|f|+|g|}\mu_A(f \otimes g)(d_C \otimes 1 + 1 \otimes d_C)\Delta_C \\ &\quad = \mu_A(d_A f \otimes g + (-1)^{|f|} f \otimes d_A g) - (-1)^{|f|+|g|}\mu_A((-1)^{|g|} f d_C \otimes g + f \otimes g d_C) \\ &\qquad\qquad = \mu_A(Df \otimes g + (-1)^{|f|} f \otimes Dg)\Delta_C \end{aligned}$$

but this is the multiplication in A and the coproduct in C so this is

$$Df * g + (-1)^{|f|} f * Dg,$$

which is the right-hand side.

We call $\tau \in \mathrm{Hom}^1_{\mathbf{k}}(C, A)$ a *twisting cochain* if $D\tau + \tau * \tau = 0$ and $\epsilon\tau\epsilon = 0$. We define a set $\mathrm{Tw}(C, A)$ as the set of all twisting cochains.

For a given A you get a contravariant functor $C \mapsto \mathrm{Tw}(C, A)$. We need to check functoriality, that if you have a map $C' \to C$ that you get a map $Tw(C, A) \to Tw(C', A)$, by postcomposing.

We need to check that this is a twisting morphism. If $\tau \in \mathrm{Tw}(C, A)$ and $f : C' \to C$, we need to check that $D(\tau \circ f) + (\tau \circ f) * (\tau \circ f) = 0$.

But this is

$$\begin{aligned} &d_A(\tau f) - (-1)(\tau f)d_{C'} + \mu_A(\tau f \otimes \tau f)\Delta_{C'} \\ &\quad = (d_A \tau - (-1)\tau d_C)f) + \mu_A(tau \otimes \tau)(f \otimes f)\Delta_{C'} \\ &\qquad = D(\tau)f + \mu_A(tau \otimes \tau)\Delta_C f \\ &\qquad\qquad = (D(\tau) + \tau * \tau)f = 0. \end{aligned}$$

This functor is nice. It's representable, and I want to give an explicit representation, which is the bar construction. We define BA as the tensor coalgebra $T^c(S\bar{A})$, where SA is the shift of the algebra A. Then the differential is $\sum 1^{\otimes -} d_A \otimes 1^{\otimes -}$ plus another term using the (shifted) algebra b_2 (which is $s^{-1}\mu s \otimes s$) which is $\sum 1^{\otimes -} \otimes b_2 \otimes 1^{\otimes -}$.

Then $BA \in \mathrm{Cog}$, and the canonical projection is to $S\bar{A}$, and by postcomposition you have a morphism to $\bar{A}$, which we denote τ_0. I want to show that τ_0 is a twisting cochain. The projection map is degree 0 and the other map is degree 1, so I want to check that $D(\tau_0) + \tau_0 * \tau_0 = 0$.

But this is, well, $\tau_0 = S^{-1}\pi$. Then it's the same as saying that

$$d_A(S^{-1}\pi) - (-1)(S^{-1}\pi)d_{BA} + \mu_A(S^{-1}\pi \otimes S^{-1}\pi)\Delta_{BA} = 0.$$

If we put $v_1 \otimes \cdots \otimes v_n$ where $n \geq 3$ then the projection right away gives 0. We need to check or $v_1 \otimes v_2$. If you take the latter, then you get, by taking the definition, you get $S^{-1}b_2(v_1, v_2)$, and the differential d_{BA} is some d_A terms and a $b_2(v_1, v_2)$ term. One of these vanishes because of the projection; the other gives the A_∞ relation. The case with just v is easier.

So what I proved here is that τ_0 is a twisting cochain, it's contained in the set $\mathrm{Tw}(BA, A)$. Now I want to prove that $\mathrm{Tw}(C, A)$ is bijective with $\mathrm{Hom}_{\mathrm{Cog}}(C, BA)$. I wnat to show this representation statement. So a map $\tau : C \to A$ gives a map $C \to BA$ by the universal property. To prove bijectivity, we need to check that $\tilde{\tau} \circ \tau_0 \in \mathrm{Tw}(C, A)$.

But this is not hard. The equation for $D(\tau_0 \circ \tilde{\tau}) + (\tau_0 \circ \tilde{\tau}) * (\tau_0 + \tilde{\tau})$ can be rewritten (as previously $(D\tau_0 +$
$tau_0 * \tau_0) \circ \tilde{\tau} = 0$.

Dually, this construction, we started with a fixed A and get a contravariant functorp Dually if we fix a coalgebra then we get a covariant functor by assigning the same set. It's corepresentable, and the elements represented by it are "cobar." Let's have a break.

Damien asked why we consider twisting cochains. I said I don't know why.

[Christophe: They are the first nontrivial examples of Maurer–Cartan elements. These are very simple elements on which we can express the calculus on A_∞ categories. Knowing these is enough to reconstruct your A_∞ category. You can reduce to calculating these. This corresponds in the Fukaya category, say, to a very precise calculus.]

You can use a twisting cochain to deform the A_∞ structure. So I want to define a functor $\Omega : \mathrm{Cog} \to \mathrm{Alg}$ and will show that Ω and B are adjoint to each other.

Now I fix a coalgebra C, and whenever we have an algebra A we can define a set of twisting cochains $\mathrm{Tw}(C, A)$, and this is functorial, $A \mapsto \mathrm{Tw}(C, A)$, covariantly. So we should prove that a morphism $A \to A'$, by postcomposing you get a map $\mathrm{Tw}(C, A) \to \mathrm{Tw}(C, A')$. We should check that $D(f\tau) + (f\tau) * (f\tau) = 0$, and this is the same as $f(D\tau + \tau * \tau) = 0$, so that this functor is well-defined. Moreover it is actually representable by an element "Cobar," ΩC, which is nothing but the tensor algebra $T(S^{-1}\bar{C})$, this is the tensor algebra. So we want to regard this as an algebra, so we need a differential, and $d = \sum 1^{\otimes -} \otimes d_C \otimes 1^{\otimes -} + \sum 1^{\otimes -} \otimes S^{-1}\Delta \otimes 1^{\otimes -}$. Here $S^{-1}\Delta$ is something like $(S^{-1} \otimes S^{-1})\Delta S$. We need to check that d is actually a derivation of the tensor product. I don't want to check the details.

There's a canonical map, something like $C \to S^{-1}C \to \Omega C$, this is a degree 1 map, and we can denote this by, well, I want to show that this is in $\mathrm{Tw}(C, \Omega C)$. So we need to check that $D(iS^{-1}) + (iS^{-1}) * (iS^{-1}) = 0$ but this is

$$d_{\Omega C}(iS^{-1}) - (-1)iS^{-1}d_C + \mu_{\Omega C}(iS^{-1} \otimes iS^{-1})\Delta_C$$

but this is just

$$d_{\Omega C} is^{-1} - (-1)id_{S^{-1}C}S^{-1} + \mu_{\Omega C}S^{-1}\Delta.$$

but this is the definition of $d_{\Omega C}$, and so this is very complicated but conceptually this is nothing but the definition of the differential on ΩC. So this $\tilde{\iota}$ is a twisting cochain, and the functor from algebras to sets $A \mapsto \mathrm{Tw}(C, A)$ is represented by ΩA, it's $\mathrm{Hom}_{\mathrm{Alg}}(\Omega C, A)$. To see this we have the kind of situation $C \to A$, and the canonical inclusion $C \to \Omega(C)$, but this tensor algebra is a free object in the category of algebras, so we can always find a morphism from $\Omega(C)$ to A so we only need to prove the other way, that the composition of $\tilde{\tau} \circ \tilde{i}$ is a twisting morphism. But this is the same as like $\tilde{\tau}(D\tilde{\iota} + \tilde{\iota} \star \tilde{\iota})$, and we showed that $\tilde{\iota}$ is a twisting cochain, so this is zero.

So what we've shown is that there are two bijections

$$\mathrm{Hom}_{\mathrm{Cog}}(C, BA) \cong \mathrm{Tw}(C, A) \cong \mathrm{Hom}_{\mathrm{Alg}}(\Omega C, A),$$

there are such bijections, and we have these two functors, and these are adjoint to each other. So Ω is left adjoint. This is bar and cobar.

I wnat to mention a model structure on the category of coalgebras. A map is a weak equivalence in Cog if and only if $\Omega(f)$ is a weak equivalence in algebras. The cofibrations in Cog are the degreewise injective morphisms. The fibrations are the morphisms which have the right lifting property against trivial cofibrations.

Theorem 1.2. *(Lefèvre–Hasegawa)*

- *This data gives a model structure on* Cog *and* Ω *preserves cofibrations and trivial cofibrations and* B *preserves fibrations and trivial fibrations, so* (Ω, B) *are a Quillen adjunction. Actually* Ω *and* B *are Quillen equivalences. In other words they induce an equivalence of homotopy categories.*
- *All objects in* Alg *are fibrant; all objects in* Cog *are cofibrant. An algebra* A *is cofibrant if and only if it is a retract of* ΩC *for some* C *in* Cog *and a coalgebra* C *is cofibrant if and only if it is isomorphic as an underlying graded coalgebra to* $T^c V$ *for some* V*.*
- *If* A *and* A' *are fibrant and cofibrant in* Alg*, then* $f \sim g$ *as maps* $A \to A'$ *if and only if there exists* $h : A \to A'$ *of degree* -1 *with* $h\mu_A = \mu_{A'}(f \otimes h + h \otimes g)$ *and* $f - g = d_{A'}h + hd_A$*. There is a dual statement for coalgebras.*

So we have

$$\begin{tikzcd} \mathrm{Alg} \arrow[d, shift left, "B"] & \\ \mathrm{Cog} \arrow[u, shift left, "\Omega"] & \mathrm{Cog}_{cf} \arrow[l] \end{tikzcd}$$

and to complete the corner, we should pass to Alg_∞ the category of augmented strongly unital A_∞ algebras, which is equivalent to considering non-unital A_∞ algebras. This has this sequence of maps (A, b_n), where $b_n : A^{\otimes n} \to A$, with all maps of degree 1. The unital means there is a unit element and it should be zero unless $n = 2$.

As before we want to define something like $\mathrm{Hom}^\bullet_{\mathbf{k}}(C, A)$ for $C \in \mathrm{Cog}$ and $A \in \mathrm{Alg}_\infty$. We want to equip this with, this has an A_∞ structure and there is a unit and stuff like that.

If f is a map, and we want to define $b_1(f) = b_1^A f - (-1)^{|f|} f d_C$. For $n \geq 2$, we have

$$b_n(f_1, \ldots, f_n) = b_n^A(f_1 \otimes \cdots \otimes f_n)\Delta^{(n)},$$

and we need to check that $\mathrm{Hom}(C,A)$ with these multiplications is an A_∞ algebra. It's not hard to check this but I won't prove it all. I'll show one simple thing. If $n=2$, then $b_2(1\otimes b_1)+b_2(b_1\otimes 1)+b_1b_2$ should be zero, and this is

$$(-1)^{|f_1|}b_2^A(f_1\otimes(b_1^Af_2-(-1)^{|f_2|}f_2d_C))\Delta+b_2^A((b_1^Af_1-(-1)^{|f_1|}f_1d_C)\otimes f_2)\Delta \\ +b_1^A(b_2^A(f_1\otimes f_2)\Delta)-(-1)^{|f_1|+|f_2|+1}b_2^A(f_1\otimes f_2)\Delta d_C$$

and [some cancellation]. So one can check A_∞ relations, so this defines an A_∞ algebra structure on the Hom set.

The equation is something like

$$\sum_{n\geq 1} b_n(\tau\otimes\cdots\otimes\tau)=0,$$

and now $\tau\in\mathrm{Hom}^0_{\mathbf{k}}(C,A)$.

There is a canonical way to consider an algebra by setting all higher multiplications to be zero, so this equation is the same as $b_1(\tau)+b_2(\tau\otimes\tau)=0$. So then in this case by carefully considering the sign, this is nothing but the equation $D\tau+\tau*\tau=0$.

So we want to define the set $\mathrm{Tw}_\infty(C,A)$ as the solutions to this equation.

Then this comment $\mathrm{Alg}\subset\mathrm{Alg}_\infty$, if A is an algebra, then this set is the same as $\mathrm{Tw}(C,A)$.

As before one can regard this as a functor $\mathrm{Cog}\to\mathrm{Set}$ which sends the coalgebra C to $\mathrm{Tw}_\infty(C,A)$. I'll skip the proof of functoriality (this is actually very easy, you just pull $f:C'\to C$ out to the left, this is a standard argument we've used many times).

This functor is representable. You define $B_\infty A$ for an A_∞ algebra as $T^c(A)$, the reduced version, then we want to define a differential, the differential is the sum of $1^{\otimes -}\otimes b_i\otimes 1^{\otimes -}$, and one can check that d is a coderivation with respect to the coproduct. But I want to skip. So it's almost the end. So I want to show that representability $\mathrm{Tw}(C,A)\cong\mathrm{Hom}_{\mathrm{Cog}}(C,B_\infty(A))$, I want to prove this, and before I prove that, let's consider the canonical projection $B_\infty A\to A$, this is degree 0, and I want to show that this is a twisting cochain in $\mathrm{Tw}_\infty(B_\infty,A)$, and then by universal properties, for any twisting cochain $\tau:C\to A$ this lifts to a coalgebra map $C\to B_\infty A$ which pulls the canonical twisting cochain on $B_\infty A$ to the given one on C.

Let me mention some facts. If V is a graded vector space then the set of A_∞-structures on V is in one to one correspondence with coalgebra differentials on TSV. If A and A' are A_∞ algebras, then $\mathrm{Hom}_{\mathrm{Alg}_\infty}(A,A')$ is in one to one correspondence with $\mathrm{Hom}_{\mathrm{Cog}}(B_\infty A,B_\infty A')$. This means that B_∞ is a functor from A_∞ algebras to Cog, it's actually fully faithful.

Theorem 1.3. *If C is a coalgebra, then C is fibrant cofibrant if and only if $C\cong B_\infty A$ for some A_∞ algebra.*

This implies that the functor is essentially (quasi-)surjective. Then this is very close to an equivalence of categories, it's a (quasi-)equivalence. Moreover, C in Cog has a *minimal model* where $I\in\mathrm{Cog}_{cf}$ and $I\xrightarrow{\sim}C$ and there exists $f^{-1}\in\mathrm{Cog}$ if and only if there is an inverse in $\mathrm{Ho}(\mathrm{Cog})$. The theorem is that any cocomplete coalgebra has a minimal model, and on the other hand, if $A_{\min}$ is a minimal model for A, then this minimal model, this is a kind of A_∞ algebra with $b_1=0$. Of course, this is quasi-isomorphic to A.

Then the functor B_∞ makes a bridge between the two minimal models. B_∞ of a minimal model of A. is isomorphic to the minimal model for $B_\infty(A)$.

Now I can draw my diagram.

$$\begin{tikzcd}
\mathrm{Alg} \arrow[r] \arrow[d, shift left, "B"] & \mathrm{Alg}_\infty \arrow[d, "B_\infty"] \\
\mathrm{Cog} \arrow[u, shift left, "\Omega"] & \mathrm{Cog}_{cf} \arrow[l]
\end{tikzcd}.$$

And if you take homotopy categories everything is an equivalence of categories. So in some sense we have four different descriptions of one algebra, but homotopically they are all the same, there are no new homotopic descriptions. In particular, the A_∞ description is homotopically not new but gives several types, that's the story I wanted to tell. I will stop here.