# CGP DERIVED SEMINAR

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Okay, my title is the bar functor. I want to define functors between the categories of augmented algebras and the coaugmented coalgebras. I'll also use $A_\infty$ algebras. I have a non-full subcategory inclusion of algebras and $A_\infty$ algebras; I also have a full subcategory inclusion of cofibrant fibrant coalgebras and coalgebras. I have bar and cobar between algebras and coalgebras. I want to define a bar functor $B_\infty$ from $A_\infty$ algebras to the cofibrant fibrant coalgebras which is an equivalence of categories. I want to show that all of these functors induce equivalences on the homotopy categories.

Actually, using this, one can think, well,

**Proposition 1.1.** *Let $A$ be an $A_\infty$ algebra over $\mathbf{k}$. Then there exists an algebra $U(A)$ such that $A \to U(A)$ is a quasi-isomorphism, this is just the composition of the three functors, $\Omega B_\infty A$.*

Let's start with algebras. $\mathbf{k}$ is a field and algebras are always augmented and unital. Unital means it has an element 1. Augmented means there is an map $\epsilon : A \to \mathbf{k}$ which sends 1 to 1. This has a model structure with weak equivalences the quasi-isomorphisms and fibrations the surjections. The cofibrations have some lifting property.

There is a theorem, that these three define a model structure on differential graded (augmented unital) algebras.

Now I want to define coalgebras and the bar and cobar construction. Here I want augmented, dg coalgebras, it's basically a chain complex, it has a differential and a grading, and it has another operation, called a coproduct, $(C, \Delta, d, \epsilon, \eta)$. This is $\Delta : C \to C \otimes C$ satisfying $\Delta \circ d = (1 \otimes d + d \otimes 1) \circ \Delta$. Our $\epsilon$ is a coaugmentation $\mathbf{k} \to C$ and $\eta$ is a counit $C \to \mathbf{k}$. This satisfies $\eta \circ \epsilon = 1_{\mathbf{k}}$.

Now let $V$ be a complex. Then $TV = \bigoplus V^{\otimes n}$. One can assign a coproduct here like this: $\Delta : TV \to TV \boxtimes TV$ (introducing $\boxtimes$ to separate the tensors) then

$$(v_1 \otimes \cdots \otimes v_n) \mapsto \sum (v_1 \otimes \cdots \otimes v_i) \boxtimes (v_{i+1} \otimes \cdots \otimes v_n).$$

This sum ranges from 0 to $n$, and one can prove that this defines a coproduct.

I want to define a coaugmentation and unit map. You have an inclusion $k \to V^{\otimes 0}$ and the projection is the counit.

So to show that this has a coalgebra structure I write $T^c V$, regarding this as a coalgebra.

So if you have a complex $V$ then we can make $TV$ an algebra, and this construction is kind of a free functor, from complexes to algebras, and this satisfies some universal property. We have a canonical map $V \to TV$, and given a map from $V \to C$, a dg map to an algebra map $C$.

As a coalgebra $TV$ is not cofree. If you have a map $T^cV \to V$, even if you have a map, a $\mathbf{k}$-linear map $C \to V$, we might not be able to fill with an arrow $C \to T^cV$.

So maybe our, there is another construction to make something slightly bigger, a bigger coalgebra which is cofree, but in this talk I want to shrink our category of coalgebras. Instead of thinking of all coalgebras, I want to consider some "cocomplete" coalgebras. We say $C$ a coalgebra is cocomplete if $C = \bigcup \ker(C \to C^{\otimes n} \to (C/\mathbf{k})^{\otimes n})$. If you apply the coproduct $(n-1)$ times, then mod out by the scalar part at each factor, the kernel here means that if we take some iterated coproduct, then there is eventually some scalar in each factor. Then Cog is our category of cocomplete coalgebras.

One can prove that $T^cV \to V$ is now a cofree cocomplete coaugmented coalgebra. Any element is a sum up to a finite length of tensors. If you take the coproduct more than $n$ times you will have at least one scalar factor. So it's cocomplete (conilpotent). Now it's cofree in this category.

Now let $A$ be a dg algebra and consider a chain complex of $\mathbf{k}$-linear maps $\mathrm{Hom}_{\mathbf{k}}(C, A)$, and these are both graded, and we want a differential on here and a multiplication. So $d(f) = d \circ f - (-1)^{|f|} f \circ d$. Then the product $f * g$ (for $\mu$ the product on $A$) is $\mu \circ (f \otimes g) \circ \Delta$. We need to check that this differential is a derivation with respect to this product, but having done so, then $\mathrm{Hom}_{\mathbf{k}}(C, A)$ is a dg algebra.

Here's a definition.

**Definition 1.1.** A *twisting cochain* $\tau$ in $\mathrm{Hom}^1(C, A)$ is a element that satisfies $d\tau + \tau * \tau = 0$

So then $Tw(C, A)$ will be the set of twisting cochains, and this gives for fixed $A$ a functor from Cog to sets. This is a subset of $\mathrm{Hom}(C, A)$. If it defines a functor it should be contravariant.

So given $f : C \to D$, we can ask if $\tau \circ f$ is a twisting cochain for $\tau$ twisting in $D$. Then we see $d(\tau \circ f) + (\tau \circ f) * (\tau \circ f) = d \circ \tau \circ f + \tau \circ f \circ d + \mu \circ (\tau \circ f) \otimes (\tau \circ f) \circ \Delta$ and since $f$ commutes with coproducts and differentials this is $(d\tau + \tau * \tau) \circ f$ which is zero.

So this is really a functor. This functor is (co)representable by $BA$, so $Tw(C, A) \cong \mathrm{Hom}(C, BA)$, and so replacing $C$ with $BA$ we get a special (universal) twisting cochain $\tau_0$.

So $BA = T^c(SA)$ along with a differential $D$ which has two components, one which looks like $1^{\otimes a} \otimes d_{sA} \otimes 1^{\otimes b}$ and the other given by $1^{\otimes a} \otimes b_2 \otimes 1^{\otimes (b-1)}$. You have another functor, fixing $C$ instead of $A$, defining a set $Tw(C, A)$, and one can prove that this is a functor. The representation is denoted by $\Omega C$, so that Hom (in algebra) between $\Omega C$ and $A$ is in bijection with $Tw(C, A)$. Indeed $\Omega C$ is the tnsor algebra of $T(s^{-1}C)$ along with a differential.

So $\Omega$ and $B$ are adjoint to each other.

**Theorem 1.1.** *(Lefévre–Hasegawa) these form a Quillen equivalence. This means that they preserve the model category structure and moreover induce an equivalence of categories on the homotopy level.*

*Secondly, all algebras are fibrant and all coalgebras are cofibrant. So $A$ is cofibrant if and only if it is a retract of a cobar of something; $C$ is fibrant if and only if it is quasi-free.*

*The homotopy relation between two maps, $f$ and $g$ are morphisms between $A$ and $A'$ fibrant and cofibrant objects. Then $f \sim g$ if and only if there exists a homotopy $h : A \to A'$ of degree $-1$ such that (some augmentation condition is satisfied) and $h \circ \mu_A = \mu_B \circ (f \otimes h + h \otimes g)$ and $f - g = dh + hd$.*

I didn't say anything about the model category structure in coalgebras. In coalgebras, the weak equivalences are those whose image $\Omega(f)$ is a weak equivalence in algebras. The cofibrations are injections. The fibrations have the lifting property.

Let's see how this result is related to $A_\infty$-algebras and minimal models. An $A_\infty$ algebra (maybe with augmented strict unit), as before, $C$ is a coalgebra, cocomplete (or conilpotent) and we want to consider all the $\mathbf{k}$-linear maps from $C$ to $A$. By using the $A_\infty$ algebra structure, we can define an $A_\infty$ structure on $\mathrm{Hom}(C, A)$ as $b_n(f_1 \otimes \cdots \otimes f_n) = b_n^A \otimes (f_1 \otimes \cdots \otimes f_n) \circ \Delta^{(n)}$.

As before I want to define twisting cochains, as the set of morphisms $\tau$ satisfying the Maurer–Cartan equation,

$$\sum b_n(\tau, \dots, \tau) = 0$$

Now fix $A$. Whenever we choose a coalgebra $C$, we can assign a set $\mathrm{Tw}(C, A)$, and one can prove that this is functorial and moreaever representable, by $B_\infty A$. Then $\mathrm{Tw}(C, A) \cong \mathrm{Hom}_{\mathrm{Cog}}(C, B_\infty A)$. Practically, $B_\infty A$ is $T^c(sA)$ with some differential. Then actually, in fact, if you use $V$ a graded vector space then there is a bijection of sets between $A_\infty$ structures on $V$ and differentials on the coalgebra structures on $T(sV)$. Another fact, the hom sets, $\mathrm{Hom}_{A_\infty}(A, A') \cong Hom_{\mathrm{Cog}}(B_\infty A, B_\infty A')$, so $B_\infty$ as a functor from $A_\infty$ algebras to coalgebras is fully faithful.

Moreover, there's a theorem

**Theorem 1.2.** *(Lefèvre–Hasegawa) This functor $B_\infty$ is essentially surjective to cofibrant-fibrant objects.*

*So any cofree coalgebra has some $A_\infty$ algebra with equivalent $B_\infty$.*

*Secondly, for each $C$ there exists a minimal model and if $A$ has $A_{\min}$, then $B_\infty(A_{\min})$ is the minimal model of $B_\infty(A)$.*

So we have a diagram that I drew at the beginning.

$$\begin{tikzcd}
\mathrm{Alg} \arrow[r] \arrow[d, shift left, "B"] & \mathrm{Alg}_\infty \arrow[d, "B_\infty"] \\
\mathrm{Cog} \arrow[u, shift left, "\Omega"] & \mathrm{Cog}_{cf} \arrow[l, "\text{fully faithful}"']
\end{tikzcd}$$

I was supposed to say more but let me just give one more example. I mentioned about the classification of fibrant and cofibrant objects. Actually in Alg, when $A$ is cofibrant then it must be free as a graded algebra but the converse is not true. So say $|1| = 1$, then $\mathbf{k} \oplus \mathbf{k}1 + \cdots$ and $d(1) = 1 \otimes 1$. Then by using this you can show that this has homology that vanishes except at the bottom, and so you have a trivial "cofibration" from the trivial algebra. Then it's clear that $\mathbf{k}$ is fibrant cofibrant, so that we should be able to invert the map up to homotopy. But we can't.