CGP DERIVED SEMINAR

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1. November 7: Christophe Wacheux: Homotopy Algebras

(My understanding of) A_∞ algebras and A_∞ categories. I'll define what is an A_∞ algebra and I don't know if I'll define the categories, we'll see along the way. For reference, I'm following the work of Keller, if you take A_∞ algebra the first work you thought of is Keller, his student K. Lefévre-Hasegawa, but not Kontsevich, which apparently adopts a very different approach. I'm convinced it has its merits and all, but it was inaccessible and doesn't give a good introduction.

Okay, now I'm going to set $\mathbf{k}$ a field, V a graded $\mathbf{k}$-vector space, so maybe sometimes I'll just say GVS, so I mean I have $V = \bigoplus_{p\in\mathbb{Z}} V^p$ and I define $V[q]$ by $(V[q])^p := V^{p+q}$ and you will see that I'll make an important use of this shift. If $v \in V^p$, then it is said to be homogeneous of degree p and we say $|V| = p$.

The category G, I don't know if there is conventional notation GrV, has objects graded $\mathbf{k}$-vector spaces and morphisms, let M and L be two graded vector spaces, then $\mathrm{Hom}_{\mathrm{GrV}}(M,L)$ is a graded vector space, in category theory it means something I guess when the homs are again an object, with component

$$\mathrm{Hom}_{\mathrm{GrV}}(M,L)^r := \prod_{p\in\mathbb{Z}} \mathrm{Hom}_{Vec}(M^p, L[r]^p).$$

So f is said to be of degree r.

So of course you have to pay attention to how you define your morphisms even though they might look the same. If you shift then the degree will change.

Now I will define what is called the monoidal structure. First I'll define $M\otimes L$ to be a graded vector space with

$$(M\otimes L)^n = \bigoplus_{p+q=n} M^p \otimes_{\mathbf{k}} L^q$$

where here we have the tensor product of vector spaces, the usual tensor product.

Now if $f: M\to M'$ and $g: L\to L'$, I also need to define what is $f\otimes g$, this will go from $M\otimes L \to M'\otimes L'$, and I define this so that $|f\otimes g| = |f| + |g|$, which is a consequence of my definition, I can define it by saying that for v and w homogeneous

$$(f\otimes g)(v\otimes w) = (-1)^{|g||v|} f(v)\otimes g(w).$$

This is some trick because to get the symbols from the one order to the other order you should permute the g and the v. This amounts to a choice of a map $M\otimes L \to L\otimes M$, $x\otimes y \mapsto (-1)^{|x||y|} y\otimes x$.

The neutral element for $\otimes$ is $e := \begin{cases} e^0 = k \\ e^n = \{0\},\ n\neq 0.\end{cases}$ So now GrV is a symmetric monoidal category. An interesting point here is that the morphisms of graded vector spaces are again graded vector spaces. Now if I have (M, d_M) a cochain complex, meaning that $d_M \in \mathrm{Hom}_{\mathrm{GrV}}(M,M)^1$ satisfying $d_M^2 = 0$, and for (M, d_M)

and (L, d_L) two complexes, we equip $\mathrm{Hom}_{\mathrm{GrV}}(M, L)$ with differential δ where $\delta^r : \mathrm{Hom}_{\mathrm{GrV}}(M, L)^r \to \mathrm{Hom}_{\mathrm{GrV}}(M, L)^{r+1}$, with $\delta^r(f) = d_L \circ f - (-1)^{|f|} f \circ d_M$ of degree f.

I guess, then, f and f', morphisms of graded vector spaces (or maybe I'd better reduce to maps of complexes), are homotopic if $f - f' = \delta(h)$ for some $h \in \mathrm{Hom}_{\mathrm{GrV}}(M, L)^{|f|-1}$.

A note is that h and h' homotopic induce the same maps on cohomology. Normally I'm also supposed to set, if I shift, I set $d_{V[1]} = -d_V$.

Now we are ready for A_∞ algebras.

Definition 1.1. An A_∞ algebra is a graded vector space A with maps $b_n : (A[1])^{\otimes n} \to A[1]$ such that the degree of b_n is 1, for $n \geq 1$.

Here I should stop and make a big comment. Sometimes you want maps $A^{\otimes n}$ to A of degree $2 - n$. Understanding the difference of signs is sometimes an annoying thing.

So I just wanted to say that the link between m_n and b_n, if what I read in Lefèvre-Hasegawa, there is a formula linking the b_n and the formula linking the m_n, there are no pluses or minuses linking the b_n, but for m_n there are signs, and he said that, yes, there is no precise, no canonical choice of signs between the m_n, and, which, I think this is, uh, [some discussion]

Let's write the formula b_n satisfies.

$$\sum_{i+j+\ell=n} b_{i+1+\ell} \circ (\mathbf{1}^{\otimes i} \otimes b_j \otimes \mathbf{1}^{\otimes \ell}) = 0$$

for all $n \geq 1$.

Several comments. The advantage of defining b_n like this, now I have maps that are all of the same degree, and also, because I take this as a convention, with this I don't have sign troubles, but I'll have sign issues.

I never apply it to an element, I've never come across it. Okay, so now a representation, if I take, or realization, [pictures].

In this case he has a sign of $(-1)^{ij+\ell}$, but he says there's no canonical choice, so.

Okay so what do we have? For $n = 1$ you have $b_1 \circ b_1 = 0$ so $(A[1], b_1)$ is a complex.

For $n = 2$, I can have $b_2(\mathbf{1} \otimes b_1) + b_2(b_1 \otimes \mathbf{1}) + b_1(b_2) = 0$. If you remember the formula I erased, we know that b_2 goes from $A[1] \otimes A[1] \to A[1]$. Then $A[1] \otimes A[1]$ is a complex with differential $d_{A[1]} \otimes \mathbf{1} + \mathbf{1} \otimes d_{A[1]}$. Now if I write $\delta(b_2)$ I get that it is $d_{A[1]} b_2 - (-1)^{|b_2|} b_2 \circ (d_{A[1]} \otimes \mathbf{1} + \mathbf{1} \otimes d_{A[1]})$, and we know that this is equal to zero by the A_∞ equation (A_2). This means that b_2 is a morphism of complexes.

Now this is where it gets funny. This is also supposed to be like the graded Leibniz rule, because b_2 is actually the multiplication but here b_2 is defined on $A[1]$ so you have to get back, that's the discussion we had with you, so actually $m_2(x, y) = (-1)^{|x|} s^{-1} b_2(sx, sy)$. Normally if we check the formula, we should find out that, I'm going to switch it, I'm going to change in the formula, so I have

$$d_A \circ m_2(x, y) = m_2(d_A(x) \otimes y) + (-1)^x m_2(x, d_A(y))$$

which is graded Leibniz. Next, (A_3) implies that

$$b_2 \circ (b_2 \otimes \mathbf{1} + \mathbf{1} \otimes b_2) + b_1 \circ b_3 + b_3 \circ (b_1 \otimes \mathbf{1} \otimes \mathbf{1}) + b_3 \circ (\mathbf{1} \otimes b_1 \otimes \mathbf{1}) + b_3 \circ (\mathbf{1} \otimes \mathbf{1} \otimes b_1) = 0$$

and when you switch to m_2 you get associativity of m_2 up to a homotopy which is more or less m_3.

For $n > 3$ you have a quadratic equality up to higher homotopy. Also, a consequence of what I said, if $b_n = 0$ for all $n \geq 3$, then we have a dg algebra and vice versa a dg algebra gives you an A_∞ algebra with $b_n = 0$ for $n \geq 3$.

What about $n = 0$? If I try to adapt the formula, allowing $n = 0$? Then applying bluntly what happens, in that case, $b_0 : \mathbf{k} \to A[1]$, you get that this would modify all the equations, and (A_0) now says that $b_1 \circ b_0 = 0$ but (A_1) tells youo that $b_1^2 = -b_2(\mathbf{1} \otimes b_0 + b_0 \otimes \mathbf{1})$ so this is what is called, this is not zero, this is what is called weak A_∞-algebra or curved A_∞ algebra. In Keller and Lefévre-Hasegawa, they say little is known. We'll speak about the rest of this next week.