EMILY RIEHL: ∞-CATEGORIES FROM SCRATCH (THE BASIC TWO-CATEGORY THEORY)

GABRIEL C. DRUMMOND-COLE

I'm trying to be really gentle on the category theory, within reason. So in particular, I'm expecting a lot of people won't have seen a two-category at all. You guys may know too much to be an ideal practice audience, but if you see anything.

The goal of the talks this week will to be give a precise account of the development of ∞ dimensional categories. We'll talk about limits and colimits, adjunctions, fibrations, modules (profunctors), the Yoneda lemma, and Kan extensions. That will be the scope of the basic theory.

There will be a number of instantiations of the categories I'll consider, but this will be model-independent, it will apply in a number of contexts but will be agnostic to which context.

I'll call these *∞-categories*, this is a technical term that I'll explain, they live in an *∞-cosmos*. This won't be an axiomitization of the ∞-categories, but of the things we need to prove theorems with them.

This will be optimized for $(\infty-1)$-categories, which are roughly ∞-dimensional categories such that morphisms above dimension 1 are invertible. There are many models of this. Some models include quasicategories (Joyal), complete Segal spaces (Rezk), Segal categories (Simpson), marked simplicial sets (Verity). We'll see that the development of the category theory is preserved and reflected by nice functors that change models. You won't need to know what any of these things mean.

The way this works, ∞-categories and the functors between them live in something called the *homotopy 2-category*. This will be a strict two-category comprised of objects called ∞-categories, which I'll denote A, B, C. A morphism will be an *(∞-)functor* which I'll write $A \xrightarrow{f} B$. The two-morphisms or 2-cells will be *natural transformations*.

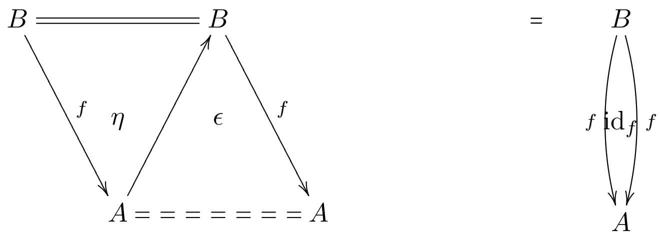
One example to keep in mind is categories, functors, and natural transformations.

So what can we do in a 2-category? It has an underlying ordinary 1-category. We also have composition of 2-cells, which compose in two different directions. If I'm given a 2-cell $\alpha : f \to g$ and $\beta : g \to h$, then I get $\beta \cdot \alpha$ from $f \to h$. So you have identity two-cells so that the morphisms are the objects of a category themselves. This is *vertical* composition. I also have horizontal composition, if I have $\alpha : f \to g$ and $\gamma : h \to k$ with the targets of the first equal to the sources of the second to get $\gamma\alpha$ from hf to kg.

This is pasting, I can compose vertically and then horizontally or horizontally and then vertically, then the compositions agree. More generally, if I have a diagram with two-cells, all consistently oriented, I get a unique composite.

I claim that this is a good framework to develop our basic category theory.

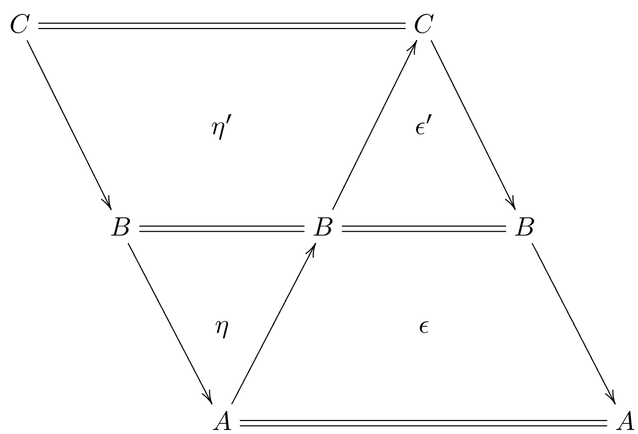
Definition 0.1. An *adjunction* between ∞-categories consists of a pair of 1-categories A, B, a pair of functors $u : A \to B$ and $f : B \to A$, and a pair of natural transformations $\eta : \mathrm{id}_B \to uf$ and $\epsilon : fu \to id_A$ which satisfy



and similarly for the other composition.

This can be defined internally internally for any kind of ∞-category. So let me show some of what you can do.

Proposition 0.1. *Adjunctions compose. If I have $f' : C \to B$ and $f : B \to A$ then we can define a 2-cell for $f'f$ and we can paste together. I get the composite of the identities from drawing a diagram like this:*



You can also define an equivalence

Definition 0.2. An *equivalence* consists of a pair of ∞-categories A and B, a pair of functors u and f, and a pair of natural isomorphisms $id_B \to uf$ and $fu \to id_A$.

Exercise 0.1. *Show that any equivalence can be promoted to an* adjoint equivalence *by changing one of the* 2*-cells.*

Corollary 0.1. *Any equivalence is both a left and right adjoint.*

Proposition 0.2. *Given an adjunction between A and B and equivalences $A \cong A'$ and $B \cong B'$, then there is an adjunction between A' and B'.*

The proof is that we can make A' and A adjoint in either direction and likewise for B.

Exercise 0.2.

- *If f is left adjoint to u and $f \cong f'$ then f' is left adjoint to u.*
- *If f and f' are adjoint to u then $f \cong f'$, and finally*
- *$f \cong f'$ and f is an equivalence then so is f'.*

The homotopy 2-category in which we've been working is the quotient of an ∞-cosmos. Let me motivate this a bit. If you want to do the homotopy theory of something, you might hope that it lives in a simplicial model category. You can say that two homotopy theories are the same if their model categories are equivalent in some sense.

I'll show that a good framework to do our kind of theory is for something enriched over quasicategories, not Kan complexes. We actually won't need those details.

Definition 0.3. An ∞-*cosmos* is a simplicially enriched category, it's something that has objects (an object is called an ∞-category), homs between objects which are quasicategories, a particular sort of simplicial set, and a specified class of maps called *isofibrations*. This should satisfy axioms. Let me pause and say that there are canonical equivalences, that $A \to B$ is an equivalence if and only if $map(X, A) \to map(X, B)$ is an equivalence of quasicategories. Trivial

- completeness: K has (simplicially enriched) terminal object 1, pullbacks of isofibrations, cotensors by simplicial sets, meaning that with U a simplicial set and A an ∞-category then A^U is my cotensored object.
- isofibrations: are closed under composition, contain isomorphisms, and are stable under pullback. Further, if $U \xrightarrow{i} V$ is an inclusion of simplicial sets and $E \xrightarrow{p} B$ is an isofibration then $E^V \to E^U \times_{B^U} B^V$ is an isofibration. We call an isofibration that is a weak equivalence a trivial fibration.
- cofibrations: there exists a lift

$$\begin{array}{ccc} & & E \\ & \nearrow & \downarrow \\ A & \longrightarrow & B. \end{array}$$

There are some consequences

- trivial fibrations: It follows from the definitions that trivial fibrations are closed under composition, contain isomorphisms, are stable under pullbacks, and $E^V \to E^U \times_{B^U} B^V$ is a trivial fibration if either p is a trivial fibration or i is a trivial cofibration in the Joyal model structure.
- factorization: Another consequence, we can factor any $A \to X$ as a section of a trivial fibration followed by an isofibration. This is what goes into the proof of Ken Brown's lemma.
- representable isofibrations: If p is an isofibration then for all X, $map(X, E)$ is a fibration onto $map(X, B)$.

We can have the optional axiom of Cartesian closure, that $map(A \times B, C) \cong map(A, C^B) \cong map(B, C^A)$.

Ret me remind you that a *quasi-category* is a simplicial set A with fillers for inner horns

$$\begin{array}{ccc} \Lambda^{n,k} & \longrightarrow & A \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

There is a model category structure on simplicial sets for quasi-categories by Joyal such that the cofibrations are monomorphisms, fibrant objects are quasi-categories, fibrations between quasi-categories are isofibrations (so I have lifting problem solutions as follows)

$$\begin{array}{ccc} \Lambda^{n,k} & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & B \end{array} \qquad \begin{array}{ccc} \Delta^0 & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ \mathbb{I} & \longrightarrow & B \end{array}$$

where $\mathbb{I}$ is the pushout of

$$\begin{array}{ccc} \Delta^1 \cup \Delta^1 & \xrightarrow{[02]\cup[13]} & \Delta^3 \\ \downarrow & & \\ * & & \end{array}$$

Equivalences are simplicial homotopy equivalences using $\mathbb{I}$, it's given by a pair of maps f and g so that there exists

$$\begin{array}{ccc} & & A \\ & \overset{\text{id}}{\nearrow} & \uparrow p_0 \\ A & \longrightarrow & A^{\mathbb{I}} \\ & \underset{gf}{\searrow} & \downarrow p_1 \\ & & A \end{array}$$

and likewise for B.

How about some examples. We have these for categories, quasicategories, for complete Segal spaces, Segal categories, naturally marked simplicial sets, each of which other than the first is a model of $(\infty,1)$-categories. Each has a model category structure, all of which are equivalent. Taking isofibrations as fibrations, then weak equivalences are the equivalences in my cosmos.

There are a few other examples. Certain models of $(\infty - n)$-categories are objects in an ∞-cosmos; similarly higher complete Segal spaces. Finally, if B is in an ∞-cosmos then K/B is a slice cosmos whose objects are isofibrations with codomain B.

This axiomatizes what is needed for their basic homotopy theory.

Now we can forget most of this. I want to now tell you about the quotient.

Definition 0.4. The homotopy 2-category K_2 of an ∞-cosmos K has as its objects the ∞-categories. The morphisms will be vertices in the mapping quasicategory, so that the 2-category and the ∞-cosmos have the same underlying category.

A 2-cell between $f, g : A \to B$ is represented by a 1-simplex in the quasicategory. We say that two parallel 1-simplices represent the same 2-cell if and only if they are connected by a 2-cell whose third face is degenerate.

There's a slicker way at this definition. Equivalently, there's a functor from quasicategories to categories which takes a quasicategory to its homotopy category. Then hom are $ho(map(A,B))$. That's the sense in which this is a quotient of the ∞-cosmos. Let me now tell you the most important result of today. We have a priori two distinct notions of equivalence.

Proposition 0.3. *If I have a functor $f : A \to B$ in an ∞-cosmos K, then the following are equivalent:*

(1) *f is an equivalence in the ∞-cosmos K*
(2) *f is an equivalence in K_2.*

So we can work 2-categorically and capture the right thing. For 2 to imply 1, we need that if f is an equivalence in K and $f \cong f'$ in K_2 then f' is an equivalence. The proof uses that $A^{\mathbb{I}}$ is a path object for A. Now if f has an equivalnec inverse g, then $fg \cong id_B, gf \cong id_A$, so then by 2 out of three we know that by 2 out of 3, f is an equivalence in K.

If i apply the homotopy category functor to the equivalence $map(X, A) \xrightarrow{\sim} map(X, B)$, I get an equivalence $hom(X, A) \xrightarrow{\sim} hom(X, B)$. Then by the Yoneda lemma, a representable equivalence is an equivalence in K_2.

That's a good place to stop.