SIMONS CENTER WORKSHOP ON HOMOLOGICAL METHODS IN QUANTUM FIELD THEORY

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1. September 29: Owen Gwilliam, Introduction to the BV formalism

I'll start with an introduction to core ideas and then give one way of making that precise. I'll talk about the way that Kevin Costello organized it in his approach to perturbative field theories.

Let's talk about the motivating picture. Okay. So I guess the question you might ask is what you want from a quantum field theory. You have $\mathcal{F}$, a space of fields, sections of some bundle on a manifold. What we actually work with is $\mathcal{O}(\mathcal{F})$, they are functions on $\mathcal{F}$, these are observables or measurements you might make. What we really want is an expected value map

$$\langle \quad \rangle \mathcal{O}(\mathcal{F}) \to \mathbb{R}$$

This is a probabilistic thing and it's what you really want to know. When you're working perturbatively, maybe you want to expand in $\hbar$ instead of working in $\mathbb{R}$.

So a quick slogan, Batalin-Vilkovisky is a homological approach to making $\langle \quad \rangle$.

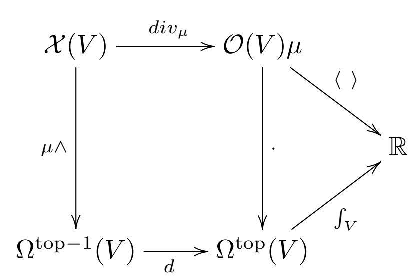
I want to explain what to do in a finite dimensional toy model where you already know how integration works.

Let's imagine I have a finite dimensional manifold. For me, it'll always be a vector space, V, and μ will be some probability measure on V. I'm going to be very, I apologize about this, my functions $\mathcal{O}(V)$, they have an expected value map to $\mathbb{R}$. One way to think about that is I go to some class of top forms, $f \mapsto f\mu \to \int f\mu$. So the expected value map factors through the top forms.

If I'm thinking homologically, maybe I want to understand the kernel of the map. to $\mathbb{R}$.

Question 1.1. *Which functions have zero expectation value?*

There is something we know is killed off. Anything that's the boundary of a top $-$ 1-form is zero, we know $\int_V d\lambda = 0$.



We know how to keep going on the bottom row and we get the de Rham complex. If I do the upper row, I get the polyvector fields, with div_μ instead of μ directly.

Then you can write down properties that div_μ has, so maybe if you can axiomatize the top row maybe it will scale to infinite dimensions.

I'd like to write down some formulas that will show up in a lot of talks. If $\mu = e^S dvol$, then $div_\mu = \iota_{dS} + div_{Leb}$. This Lesbegue divergence is $\frac{partial^2}{\partial x \partial \xi} = \Delta_{BV}$.

Some features to note, if you take the divergence of a wedge product, it satisfies this funny formula

$$\div(a \cdot b) = div(a) \cdot b + (-1)^a a \cdot div(b) + (-1)^a \{a, b\}$$

and $\{\ ,\ \}$ is a degree one Poisson bracket on polyvector fields (in this case the Schouten bracket).

This says that polyvector fields become a shifted Poisson algebra via $\{\ ,\ \}$, and arise from the "shifted symplectic vector space" $V \oplus V^*[1]$.

One of the kinds of data I'm going to be looking for is a shifted Posison structure on the observables. I can rewrite contraction with dS, up to a sign, $\iota_{dS} = \{S, \quad\}$ and this lets me get rid of exterior differentiation.

The most important thing to note, perhaps, is if you lok at the divergence operator, there's a piece that you can try to handle separately. There's a piece depending on S and a piece that's from the construction of the vector space. So if we drop the Lesbegue divergence, we have $\mathcal{X}(V) \to \mathcal{O}(V)$ via $\{S, \quad\}$ with $X \mapsto X(S)$, and the cokernel is functions on the critical points of S, the Jacobian quotient, and these are the "observables for the classical field theory." This suggests the following idea, which is to change the differential to $\{S, \quad\} + \hbar \div_{Leb}$. What does this do? At $\hbar = 0$, I get a resolution of $\mathcal{O}(Crit\ S)$, which is the classical theory. At $\hbar = 1$, I get the quantum observables.

This idea of somehow, a version of deformation quantization, where you try to find a way to add a differential weighted by $\hbar$ which lets you capture something like polyvector fields with divergence.

You can get Wick's lemma in the finite dimensional case with a Poisson distribution, and you can do this in a way that gives you the combinatorics of Feynman diagrams as well, so this has some power even in the toy model.

If I say $\mu = e^{S/\hbar}$, then I get $\frac{\iota_{dS}}{\hbar} + \div_{Leb}$ and multiplying through by $\hbar$ I get what I wrote before.

That was part one, the motivating idea for BV. Now I want to talk about how to set that up for field theory. I'll start with the classical version and deform from classical to quantum.

I'll give the definition that Kevin works with. I'll start with a $\mathbb{Z}$-graded vector bundle, with total finite rank, let me say $E = \bigoplus_n E^n$, andthe space of fields will be smooth sections, $\mathcal{E} = C^\infty(M, E)$. These are the "fields." I had a shifted symplectic vector space giving us the bracket. So E will be a shifted symplectic space from the beginning. So I'll have a vector bundle map $\langle\ ,\ \rangle : E \otimes E \to Dens_M$ which is fiberwise non-degenerate, cohomological degree one, and skew-symmetric in an appropriate sense. I'll use this to build a pairing on compactly supported sections of E,

$$\langle\ ,\ \rangle \mathcal{E}_c \times \mathcal{E}_c \to \mathbb{R}$$

via

$$(\phi, \psi) \mapsto \int_M \langle \phi, \psi \rangle_{loc}$$

which makes $\mathcal{E}_c$ into a "shifted symplectic space." This is the kinematics and now I need to give you an action functional, so I'll give you S as part of my data, $S \in \mathcal{O}(\mathcal{E}) := \widehat{Sym}(\mathcal{E}^*) = \prod Hom_{cont}(\mathcal{E}^{\otimes n}, \mathbb{R})_{\mathbb{S}_n}$. The action functional should satisfy constraints. I'll write $S = S_2 + S_3 + \cdots$, where $S_k \in Sym^k$. So we should have that

$$S_k(\phi_1, \ldots \phi_k) = \sum_j \int_M (D_{j1}\phi_1) \cdots (D_{jk}\phi_k)\mu_j$$

so this is Lagrangian density.

The goal is to have a chain complex $(\mathcal{O}(\mathcal{E}), \{S, \quad\})$ (observables for the classical theory). For $\{S, \quad\}$ to form a differential. I keep writing this bracket as if it makes sense. There's problems just taking the formulas for the Poisson bracket. If you use those formulas it just doesn't work. You end up trying to multiply distributions. I don't want to talk about that, but I claim that local functionals have a bracket and say that $\{S, S\} = 0$. This is a giant caveat that you need to set up this theory. The bracket is not defined on all of $\mathcal{O}(\mathcal{E})$. So this condition ensures that you have a chain complex, where the shifted Poisson structure is not defined on the whole space.

Let me put an example up here, the example of perturbative Chern-Simons. I'll start with an oriented three-manifold, and I'll use the Lie algebra $\mathfrak{g} = su(n)$. My space of fields will be sections of a graded vector bundle, it'll be $\Omega^i \otimes \mathfrak{g}$ in degree $i-1$. If I have two fields, I'll define a pairing

$$\langle \alpha, \beta \rangle = \pm \int_M Tr(\alpha \wedge \beta)$$

and an action functional which will have only S_2 and S_3. So it'll be

$$S(\alpha) = \int_M Tr(\alpha \wedge d\alpha) + \frac{2}{3} \int_M Tr(\alpha \wedge \alpha \wedge \alpha).$$

So here's an example. There's a natural dg Lie algebra associated with every such field theory, there's some moduli space and we're describing a formal neighborhood of this theory. So another thing to say is, what are the first order deformations or symmetries of this theory. It's really what you would expect, I hope.

There's derivations, local ones, which preserve the symplectic pairing, that preserve the bracket on $\mathcal{E}$. This is the Lie algebra I'm interested in, $SympDer_{loc}$, and there's a more succinct description. Since this has a symplectic structure sitting around, I can send $f \in \mathcal{O}_{loc}(\mathcal{E})$ to $\{f, \}$ and because everything is a vector space, this is surjective and the kernel is the constants. That's a different way of describing the symplectic derivations. Maybe the important thing, this map is a degree $+1$ map. So what does this this thing $\mathcal{O}_{loc}/\mathbb{R}$ describe? It encodes first order deformations of S as a classical BV theory, that's H^0, and H^{-1} encodes first order symplectic automorphisms, local symmetries of the theory.

Why did I introduce this thing at all? That's also where obstructions to quantizations live? It's really computable in a lot of cases, and you discover obstructions and anomalies that are familiar, and this tells you some terms to add to your action functional that are interesting.

I gave you a definition of a classical BV theory, there was no volume form in sight, I had a differential on functions that behaves like an action functional from physics. Now I want to do the quantum version. I want to take functions on $\mathcal{E}$ and deform the differential. The pairing on $\mathcal{E}$ produces an operator Δ_{BV} that is only defined on some sub-part of the functions on $\mathcal{E}$. This was the Lesbegue divergence earlier.

You need to find some way to get around that problem. It ends up making you want to multiply distributions. It doesn't work on local things even.

So what's the trick? Mollify everything in sight. Smooth things out. The definition will rely on a technical analytic choice. I'll hone in on a particular class of elliptic complexes. Recall the action functional had a quadratic bit, it's like $S_2(\phi) = \langle \phi, Q\phi \rangle$. I'll require two things about Q to make mollification tractable.

One thing is that Q is an operator on $\mathcal{E}$ of degree one, it makes this a chain complex, I want this to be an elliptic complex. If you want this on Lorentzian spaces, you have to do something different. I'll also require a degree -1 operator Q^* so that $[Q, Q^*]$ is a generalized Laplacian. A generalized Laplacian, the principal symbol should be a metric. For Chern-Simons theory, $Q = d$ and $Q^* = d^*$.

So what's the point, I get this Laplacian D, and now what, given a parametrix Φ (an inverse up to a smoothing operator), then I get an operator Δ_Φ, which is ι_Φ, satisfying the formula from earlier,

$$\Delta_\Phi(a \cdot b) - \Delta_\Phi(a) \cdot b - (-1)^a a \Delta_\Phi(b) + (-1)^a \{a, b\}_\Phi.$$

Maybe this isn't actually called a parametrix. I'm out of time, but Kevin will continue next time. Thanks for listening.

2. Si Li, deformation quantization, renormalization, and index theorem

This is qoint with Qin Li. This will be a case study of Costello's renormalization method.

Basically it'll be a one dimensional example, the loop space, and we'll look at $S \mapsto (X, \omega)$, the S will be a circle and (X, ω) will be a symplectic manifold.

In my first lecture, I'm going to build up a relation, an equivalence between this deformation quantization and the perturbative renormalization, the perturbative BV-quantization in this particular field theory.

This lets us use some geometric method, developed by Fedosov, which allows us to compute the quantum corrections to all orders exactly.

This example is very nice, we can do the calculations for all loops using geometry. That will be my first talk, and in this situation, the quantum observables in this formalism, if you go back to the deformation quantization side, this is the star product. This is a very nice picture.

The next talk will be a global version of this correspondence, where if you have a trace map in the deformation quantization, this is the same thing as correlation functions. In particular, by this correspondence, there are some symmetries of the deformation quantization that allow us to get some equations for our correlation functions. A rescaling symmetry of the equations for the correlation functions gives a calculation of the partition function and on the other side, well, this is a nice analysis method which gives you an algebraic index theorem.

Remark 2.1. *When $X = T^*M$, this is Gwilliam-Grady. This can be viewed as a generalization of their theory.*

Okay, so let me start with the deformation quantization in a very classical sense. I work with a symplectic manifold. I have a Poisson structure on smooth functions. In classical deformation quantization, I deform this to a star product on $C^\infty(X)[[\hbar]]$.

The leading term is the product of functions and I have higher order corrections with $\hbar$, with properties. For instance, the next term is given by the Poisson bracket, $\lim_{\hbar\to 0}(f*g-g*f)/\hbar=\{f,g\}$.

A basic example is a flat space, for example, $\mathbb{R}^{2n}$, with a symplectic form $\omega = \omega_{ij}dx^i\wedge dx^j$, with constant ω_{ij}.

We have a Moyal product $(f*g)(x) = e^{\hbar\omega^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}}|_{y=z=x}f(y)g(z)$ Then the Fedosov idea is to work as if we are in the flat situation in each fiber.

Precisely this formalization will be equivalent to the BV formalism in this picture. We have the Weyl bundle, where the fiber is, well, the bundle is $\widehat{Sym}(T^*X)[[\hbar]]$. Let me specify some gradings, where the degree of T^*X is 1 and the degree of $\hbar$ is 2. Then you get a fiberwise product on this Weyl bundle. This is a weight, not a degree, really.

So we have a fiberwise product $\mathcal{W}*\mathcal{W}\mapsto\mathcal{W}$. For example, a local section σ can always be written

$$\sum_{k,I}\sigma_{k,I}xy^I\hbar^k$$

where x^i is a local coordinate on X and y^i is the same on the fiber.

This grading is preserved by this product, so this is a symmetry of this deformation quantization. It's precisely because of this grading, you can view this as a supersymmetric localization.

So what I want to do is the following, well, to make these things work, Fedosov introduced a very interesting method. Let ∇ be a symplectic connection on the cotangent bundle, so it preserves the symplectic form and is torsion free. So it induces a connection on $\mathcal{W}$. You can always make some quantum connections to make this flat. Let's call $\wedge^p\otimes\mathcal{W}$ the $\mathcal{W}$-valued p-forms. We want to correct ∇ to a flat connection.

This is similar to what you want to do for BV quantization.

Theorem 2.1. *(Fedosov)*
Given any set of closed two-forms $\{w_k\}$, there exists a unique element in the Weyl bundle, $I\in\wedge^1\otimes W$ which is

$$\sum_{k\geq 1}I^{(k)}$$

satisfying the following properties:

1. *the initial condition,* $I^{(1)}=w_{ij}dx^idy^j$, $I^{(2)}=0$
2. $\nabla I+\frac{1}{2\hbar}[I,I]+\tilde{R}=\omega_\hbar$.
3. *some guage-fixing condition.*

So there's a bunch of data involved. First, the bracket is the barkcet for the fiberwise Moyal star product. The $\tilde{R}$ is a two-form, given by the curvature of the symplectic connection, and in formulas it's something like

$$\sum\omega_{ij}R^j_{k\ell m}y^iy^kdx^\ell\wedge dx^m.$$

The final piece $\omega_\hbar=\omega+\omega_1\hbar+\omega_2\hbar^2\in\wedge 2$. This is like solving the master equation modulo a constant.

Once you have this equation, you can describe a quantum connection from I, which is $D=\nabla+\frac{1}{\hbar}[I,\quad]$ which is a new connection on $\wedge^*\otimes\mathcal{W}$ and then $D^2=\frac{1}{\hbar}[\omega_\hbar,\quad]$. This is a flat connection on the Weyl-bundle.

Theorem 2.2. *(Fedosov) D-flat sections are in correspondence with $C^\infty(X)[[t]]$ via $y = 0$.*

Then you have a natural star product on flat sections which induces a star product on $C^\infty[[t]]$.

Now let's relate this to a one-dimensional Chern Simons type theory, looking at $S \mapsto (X, \omega)$. So our fields in this story will be basically smooth functions on S with coefficients in a particular Lie algebra specified by the target. So this is

$$\mathcal{A}^*(S) \otimes \mathfrak{g}_X[1]$$

where the left factor is smooth forms on S and the right factor is $\wedge^* \otimes T_X$. The leading bosonic component will describe an element in $C^\infty(S) \otimes T_X$. This is some sort of deformation near a constant map.

So basically the field describes a σ-model in a neighborhood of constant maps. It's a very particular low energy limit.

I'll tell you the classical and quantum action functionals together.

Let $I \in \wedge^p \otimes \mathcal{W}$ then $I : Sym^k(\mathcal{E}) \to \mathcal{A}(S) \otimes \wedge^*$ where $\alpha_1, \ldots, \alpha_k \to I(\alpha_1, \ldots, \alpha_k)$ which is by pairing with TX and T^*X. This will give me a functional $\tilde{I}$ which is defined on fields by

$$\tilde{I}(\alpha) = \sum \frac{1}{k!} \int_N I(\alpha, \ldots, \alpha).$$

Now my action functional will be of this form. You can view this as a sheaf of interactions on the circle. My action functional will be of the following form. You have $\frac{1}{2}\int_S \omega(\alpha, d_S \alpha) + \tilde{I}(\alpha)$ for $\tilde{I} \in \wedge^1 \otimes \mathcal{W}$.

The I is the interaction part which has a genus expansion $\sum_0 \tilde{I}_g \hbar^k$. Then $\tilde{I}_0$ will be the classical interaction and the higher part will be the quantum corrections.

Now let me tell you the I that I choose. It will come from the Fedosov equation. Here I need a notion of quantum master equation. You need some kind of picture from quantization. To model some kind of path integral from this kind of data perturbatively by Feynman diagrams, you have to do this in a clever way to avoid the divergence and be compatible with gauge symmetry. First I need a BV operator from d_S. Let's choose the standard flat metric on the circle and call d_S^* the adjoint of d_S. Then let Δ_S be the Laplacian. Call $e^{-L\Delta_S}$ the heat kernel. This will become singular as $\hbar \to 0$.

Define P_ϵ^L, the propagator, as $\int_\epsilon^L d_S^* e^{-t\Delta_S} dt$. If we take the small ϵ and large L limit, $d_S^* \frac{1}{\Delta_S}$ models the "$\frac{1}{d_S}$ behavior"

Given a gramp Γ, we get a map $\omega_\Gamma(P_E^L, \tilde{I})$.

You put the propagator on internal edges, $\tilde{I}$ on the vertices and α on the external edges.

Theorem 2.3.

$$\lim_{\epsilon \to 0} \omega_\Gamma(P_\epsilon^L, \tilde{I})$$

exists.

Let's collect the data. Call this one $\tilde{I}[L]$. If you don't know this, I refer you to Kevin's talk this afternooon. You take the sum over connected graphs of $\frac{\omega_\Gamma(P^L P_\epsilon, \tilde{I})}{|Aut\ \Gamma|}$, and take the limit as $\epsilon \to 0$. So you can rewrite this as

$$e^{\hat{I}[L]/\hbar} = \lim e^{\hbar partial_{P_\epsilon^L}} e^{\frac{\tilde{I}}{\hbar}}$$

Now we say that $\tilde{I}(L)$ satisifies the quantum master equation if $Q + \hbar\Delta_L + \{\tilde{I}, \quad\}_L$ squares to zero.

[something about effectivity]

So for example

$$\{\Phi_1, \Phi_2\}_L = \Delta_L(\Phi_1\Phi_2) - \Delta_L(\Phi_1)\Phi_2 - (-1)^{|\Phi_1|}\Phi_1\Delta_L(\Phi_2)$$

Theorem 2.4. *$\tilde{I}(L)$ satisfies the quantum master equation if and only if $\nabla I + \frac{1}{2\hbar}[I, I] + \tilde{R} = \omega$.*

So here our classical interaction $\tilde{I}_0$ gives a curved Lie algebra on $g_X[1]$. This is in Costello and Grady-Gwilliam.

There's a nice obstruction theory for the space of quantizations, and the nice space of BV quantizations (modulo gauge equivalence) exactly is given by these two-forms specified by the data $\{\omega_k\}$. I'll come back to this one.

That's the identification.

I have five more minutes, let me say just a few words. So somehow you can say something about quantum observables, let me describe the local case. Next time I'll do a global thing. You can look at some kind of classical observables in my field. In my case it'll be some small open set U, and then the classical observables on U, with $Q + \{\tilde{I}_0, \}$, the cohomology gives you smooth functions. If you take the quantum observables, using $Q + \hbar\Delta_L + \{\tilde{I}(L), \}$ then the cohomology is D-flat sections of $\mathcal{W}$.

So now you get a factorization product on quantum observables, this is the same thing as the Fedosov star product.

I hope I gave you some idea of what this looks like. Next time I'll explain these global observables. In the quantum $L \to \infty$ limit you'll get correlations plus some kind of index from analysis of the circle. Thanks very much.

3. Jae-Suk Park:QFT algebra and quantum correlations

I want to thank the organizers for this invitation, they were able to overcome my fear of visiting Stony Brook.

The same old question is, "what is quantum field theory?" and "when are two quantum field theories physically equivalent." I'll focus today on quantum correlations. Whatever the path integral is, the data could be put into two parts, something "transcendental" and something "algebraic." The transcendental part I don't know, it's some kind of integral that I don't know how to define.

If I have an equivalence class of quantum observable, I can put $\langle\mathcal{O}\rangle$, that's transcendental, but the two point, three point, correlations, those are algebraic and determinable.

Let me give a stupid example. Let $A = \mathbb{R}[x]$ with the classical action $S = \frac{x^2}{2}$. Here the partition function is $\int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx = \langle 1\rangle$. Normally I'd normalize this and get

$$\langle p(x)\rangle = \int p(x)e^{-\frac{x^2}{2}} dx / \int e^{-\frac{x^2}{2}} dx$$

It's easy to evaluate this without doing any integrals. So if $\rho = \frac{d}{dx} - x$ then $\langle\rho p(x)\rangle = 0$.

So it's easy to figure out that $x^{2k-1} \sim 0$ by ρ and $x^{2k} \sim (2k - 1)!!1$.

When you learn to compute the path integral, your model is the Gaussian or S^2 or something. Let me give one more example, also maybe stupid. Let A be a complex valued polynomials in y and $x_0, \ldots, x_n$, and $S = yG(x)$, where G is homogeneous. I can imagine $\rho_\mu = \frac{\partial}{\partial z^\mu} - \frac{\partial S}{\partial z^\mu}$ for $z^\mu = y, \underline{x}$.

Then the equivalence classes are the primitive parts of the middle dimensional cohomology of X_G, a hypersurface in $\mathbb{CP}^n$. Of course this is closely related to BV quantization.

Then we, anyway, the BV setup is basically, we extend this algebra into, adding antifields, $\mathbb{R}[x, \eta]$, where η is a degree -1 element squaring to zero and commuting with x, so this is $\mathcal{A}^{-1} \oplus \mathcal{A}^0$. Then starting with ρ you can define a differential, you can say $K = \rho\frac{\partial}{\partial\eta} = \frac{\partial^2}{\partial x\partial\eta} - x\frac{\partial}{\partial\eta}$.

Then we consider $(\mathcal{A}, K) \to (\mathbb{R}, 0)$, a cochain map to the ground field, so $\mathcal{A}^0 = \mathbb{R}[x] \to x$ it's just the integration $\int_{-\infty}^{\infty}(\quad)e^{\frac{-x^2}{2}}dx/\int e^{\frac{-x^2}{2}}dx$ then this is a cochain map. This is how BV quantization works, roughly speaking. We can regard this as a classical action, but the ρ is some kind of quantum symmetry. You can do exactly the same thing and for c, this path integral, you can construct this for any middle dimensional cycle. This turns out to be the period integral for the hypersurface.

Usually to compute periods, you consider deformations of this action functional and decide how the expectation value changes as you deform it. You have some version of the Gauss-Manin connection and that gives you a Picard-Fuchs equation that you solve to evaluate the integral.

Most of the problem is that the measure is not really defined. How do you define K? That's really hard, I think Kevin will explain. [Laughter].

As a physicist, people used BV-quantization to ensure that things were independent of gauge choice. But I think it's much more. We can use it to figure out things without doing any path integral.

So that's the idea. But the other thing we know is that if X and Y are observables, and this is a map to the ground field $\mathcal{A} \to ?$, and we know $\langle X \cdot Y\rangle = \langle X\rangle\langle Y\rangle + \hbar\cdots$. This is not an algebra homomorphism c, we never deal with algebra homomorphisms. Secondly, if you take $\hbar = 0$ then the correlations go away and we're dealing with algebra maps, then the morphism is evaluating some function. If the ground field is $\mathbf{k}$, then this is a $\mathbf{k}$-point in the underlying space. Quantum field theory seems to suggest that we change the definition of the point. Perhaps we should should change geometry to a quantum correlated space. I don't know how to do that.

So I'd like to talk more about correlation. Let me start with classical correlation and then move to quantum correlation. There's a notion of algebraic probability space. So A is a unital associative algebra (commutative today) with a $\mathbf{k}$-linear map to $\mathbf{k}$ which preserves the unit. An element of A is a random variable and ι is the expectation map. Then they want to know a couple of types of things. I want to know the distribution, $\iota(e^{tx})$, so this is the expectation value of x^n. They also like the notion of independence.

They have for this purpose the notion of cumulants, there are $\varphi_n^\iota : S^nA \to \mathbf{k}$. Instead of writing combinatorial formulas, let me write some examples:

$$\varphi_1(X)) - \iota(X)$$

$$\varphi_2(X, Y) = \iota(XY) - \iota(X)\iota(Y)$$

So we say X, Y are independent if $\varphi_n^\iota(X+Y,\dots,X+Y) = \varphi_n^\iota(X,\dots,X) + \varphi_n^1(Y,\dots,Y)$.

So let me define for $\gamma = \sum t_i x_i$, we can write $\iota(e^\gamma) = e^F$ where $F = \sum \frac{1}{n!} t_{\alpha_1} \dots t_{\alpha_n} \varphi_n^\iota(x_{\alpha_1} \dots x_{\alpha_n})$.

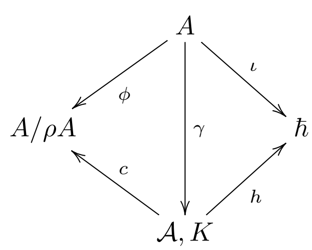
So we should interpret this as an L_∞ map. So look at $A \to \mathbf{k}$. Then say we have a Lie algebra and representation, faithful, $\rho : g \to End_{\mathbf{k}}(A)$ such that $\iota\rho = 0$. This is something we can consider. I want this ρ to act on A as a linear differential operator of finite order.

So in this case, what is the order of this differential operator? I can look at the iterated commutation

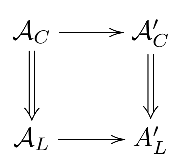
$$[\dots[[\rho(g), L_{X_1}], L_{X_2}], \dots,], L_{X_n}]1_A$$

which i'll call $\ell_n^\rho(x_1,\dots,x_n)$. We say this is order $\mathbf{k}$ if $\ell_{k+1}^{\rho(g)}$ vanishes but $\ell_k^{\rho(g)}$ does not.

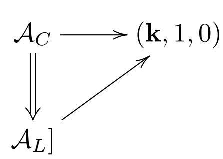
So this is a symmetry and I'll extned the notion of probability space and I'll extend this to commutative homotopy probability space, which really involves some, it's a unital graded algebra $A, K, 1_A$, and $K^2 = 0$ and $K^2 = 0$, and $K1_A = 0$. Then I consider a map c to the ground field $(\mathbf{k}, 0, 1)$, and let's say this is a chain map, and $cK = 0$ and $c(1) = 1$, defined up to homotopy. Using this symmetry, if you have this original problem with a symmetry in (g, ρ), I'd like to replace A with, consider a projection $A/\rho(A)$. Then replace this with a graded piece of the differential, I get A embedded in (A, K) as an algebra homomorphism and h is a cochain quasiisomprhism such that this diagram commutes:



Then I'd like to form a category of these guys, the objects are graded algebras with unit and differential. This is an object. A morphism should be a unit preserving cochain map. Then a morphism should have 1 goes to 1 and preserve the differential. First of all, K does not need to be a derivation of the product. Another thing is that f does not need to be an algebra homomorphism. So you get



So we can do this for our example



We have no obstructions to going to the homology so this is

$$\begin{array}{ccccc} & & \mathcal{A}_C & \longrightarrow & (\mathbf{k},1,0) \\ & & \Downarrow & \nearrow & \\ (H,1_H,0) & \xrightarrow{\phi^H} & \mathcal{A}_L] & & \end{array}$$

This functor is a homotopy functor, which depends only on the L_∞ homotopy type of ϕ_H and ϕ^c.

So we write down $\kappa_n^H = (\phi^c \bullet \phi^H)n$ The entire partition function is just

$$Z(t) = \sum e^{\frac{1}{n!}\kappa_n^H(\gamma,\cdots,\gamma)}$$

where $\gamma = \sum t_i e_i$.

We may say that ϕ^c is from ι, but a big problem is the homotopy type of φ^H. That'll be a problem in quantum field theory.

So now I want to go to the quantum correlaton. By the way, from the previous example, it's clear that c should not be an algebra homomorphism, but why should K not be a derivation? Being zero moment doesn't mean that X^2 has a zero moment. In quantum field theory, if my observables are classically independent, then we expect that the map will be an algebra homomorphism and the differential to a derivation in the classical limit.

For the quantum, the QFT-algebra, more precisely binary CQFT algebra, define this in $(\mathcal{A}[[\hbar]], 1_A, K)$ where $K = \sum K^{(i)}\hbar^i$ where $K^{(n)} : A \to A$ of degree 0. We should have $K^2 = 0$ and $K(1) = 0$.

We can make $\ell_m^K : S^n(\mathcal{A}[\hbar]) \to \mathcal{A}[\hbar]$ and this is an L_∞ structure but we also want this to be divisible by $\hbar^{n-1}$.

So the failure of being a derivation is divisible by $\hbar$, et cetera.

This is the CQFT algebra, a natural derivation of BV algebras. In many examples you get $K^0 + \hbar K^{(1)}$ and that's just BV.

So now what's a morphism, we want the descendent L_∞ map which is divisible by Φ_m^f to be divisible by $\hbar^{n-1}$.

When we divide, this $\nu_n = \frac{1}{\hbar^n}\ell_n^k$.

Let me write down some important parts of this. We have $B[[\hbar]]e^{\frac{-S[[\hbar]]}{\hbar}}$, then we have

$$f(B[[\hbar]]e^{\frac{-S[[\hbar]]}{\hbar}}) = B'[[\hbar]]B'[[\hbar]]e^{-S'[[\hbar]]/\hbar}.$$

We'd like to determine the quantum correlations. I'd like to mention, in this remaining time, make one remark and suggestion, which is that, if you statr with a QFT algebra, then the algebra I've defined is an L_∞ algebra. This is only picking up *some* L_∞ morphisms, the special ones which are giving special choices of coordinates, related to the special coordinates in geometry. [missed some]

Another remark is about the quantum master equation. What is this? Of course, $K^2 = 0$ we solved, so what's a quantum observable, it's something annihilated by K and $x \sim x'$ if $x' = x + K\lambda$. This doesn't mean that the two point functions of these two are the same.

You'd rather solve this equation:

$$Ke^{tX/\hbar} = 0.$$

We can't solve this, we try to find some corrections, but anyway, we want $Ke^{-\frac{\gamma}{k}} = 0$. This is parameterized by a parameter ring. This is the same as $K(\gamma) + \frac{1}{2}\nu_2(\gamma,\gamma) + \ldots$

Then an algebra homomorphism of parameter rings sends solutions to solutions. It turns out that this L_∞ algebra, because it's coming from something deeper, well, you can build then in a different way, the deformations are not just governed by an L_∞ algebra. Now you get more morphisms. So the same remark the, if you reduce to the cohomology you get a map to the ground ring, but it doesn't need to be a homomorphism. One should develop a version of this, maybe we'll even do a quantum space, the points are quantum correlated. Of course, you can have multiple arity version. By the way, the analysis, you can do this only for anomaly free quantum field theory, whatever it means. Okay, sorry for this completely disorganized talk.

4. Kevin Costello: the details no one else wanted to talk about

I'll start with a continuation of Owen's talk, say some things about the quantum master equation, I haven't been thinking about this for the past several years, so this will be pedagogical. I won't say that it's homological integration in a lengthy way. If $M \xrightarrow{f} \mathbb{R}$, then $\mathcal{O}(T^*[-1]M) = C^\infty(M)$ in degree 0, then $C^\infty(M, TM)$ in degree -1, and then $C^\infty(M, \wedge^2 TM)$ in degree -2. Then $d\mu \in \Omega^{\text{top}}(M)$ with $d\mu_f = e^{f/\hbar} d\mu$ and divergence with respect to $d\mu_f$ goes $Vect(M) \to C^\infty(M)$ and is $\frac{1}{\hbar}\iota_{df} + Div_{d\mu}$. So this operation, $\hbar Div_{d\mu_f} = \iota_{df} + \hbar Div_{d\mu}$ gives a differential on polyvector fields $PV(M)[[\hbar]] = \mathcal{O}(T^*[-1]M)$ and $H^0(PV(M)[[\hbar]])$ is functions modulo divergence.

One more subtlety, you could have $f \in \mathcal{O}(T^*[-1]M)$ as before you can consider this bracketing with f plus $\hbar Div_\mu$, where you notice that $\{f, \} = \iota_{df}$. So what I wanted to get to was the quantum master equation, which is that $(\{f, \quad\} + \hbar Div_{d\mu})^2 = 0$. This is necessary to make sense of this complex. My goal is to explain how to do this in infinite dimension and what are the problems.

So Owen gave a definition of the classical BV field theory. I'll give some examples. Take a scalar field theory on a manifold X. Then in BV, the fields are smooth functions on X in degree 0 and top forms on X in degree 1. This is, if X is compact, then you can pair these, this is more or less $T^*[-1]C^\infty(X)$, this is the kind of thing you might want to consider for integrals of smooth functions.

Owen explained what functions, this complex, let's call it $\mathcal{E}$, Owen explained that $\mathcal{O}(\mathcal{E}) = \prod_n Hom_{\text{cts}}(E^{\times n}, \mathbb{R})_{S^n}$. For example, in degree zero, we might have, well, let's see. If I take $\omega \in \Omega^{\text{top}}(X)$, I get $F : \mathcal{E} \to \mathbb{R}$ by $F(\phi, \psi) = \int \phi\omega$. Here ϕ is in degree 0 and ψ is in degree 1. This is a linear function of the ϕ field. More generally, I could have some g, a smooth function, and define $G(\phi, \psi)$ as $\int_X g\psi$. These are basic linear functions on fields.

More generally, what kind of things can I pair with smooth functions? I can use distributional coefficients and a distribution to pair with top forms. So g can be a distribution. Let me give a slightly more general example.

If I have $B \in \Omega^{\text{top}}(X \times X)$ then $F_B(\phi, \psi) = \int_{X\times X} \phi(x_1)\phi(x_2)B(x_1, x_2)$. More generally, B can be distributional.

An important ingredient in all of these things is the Poisson bracket on these spaces of functions. Let's try to compute some Poisson brackets explicitly and see why it's a bit of a tricky beast. What I'll do is write down the Poisson kernel.

This will be some tensor, inside a completion of $\mathcal{E} \otimes \mathcal{E}$. This $\mathcal{E}$ is smooth sections of a trivial bundle plus top forms, $C^\infty(M, \mathbb{R} \oplus \wedge^{\text{top}} T^*M[-1]) =: C^\infty(M, E)$. So $\mathcal{E} \hat{\otimes} \mathcal{E} = C^\infty(M^2, E \boxtimes E)$. If I have $\bar{\mathcal{E}}$ (meaning distributional sections, very general singularities) I'll write

$$\bar{\mathcal{E}}\hat{\otimes}\bar{\mathcal{E}}$$

, distributional sections of $E \boxtimes E$ on M^2. Expanding the tensor using linear algebra rules, I get $\bar{\mathcal{E}}\hat{\otimes}\bar{\mathcal{E}} =$

$$\underset{0}{\bar{\Omega}^0(M \times M)} \quad \underset{1}{\bar{\Omega}^{n,0}(M \times M) \oplus \bar{\Omega}^{0,n}(M \times M)} \quad \underset{2}{\bar{\Omega}^{n,n}(M \times M)}$$

where I'm using the subscript to say which factor of the product the forms come from, the first or second factor of $M \times M$. Okay, so $\Pi = \delta_{\text{diag}} \in (\bar{\mathcal{E}}\hat{\otimes}\bar{\mathcal{E}})_1$.

Let's compute some brackets. If I take some ω, then $F_\omega(\phi) = \int \phi\omega$. Later on, if I take $g \in C^\infty(X)$ then $F_g(\psi) = \int \psi g$. Then

$$\{F_\omega, F_g\} = \int_X \omega g$$

Why is this? I just pair the tensors together. Note that ω and g could be distributions, they were defined perfectly well, but in that case, this Poisson bracket would be ill-defined.

This is what Owen referred to earlier in his talk.

The next thing we might like to understand is the BV Laplacian. So as we learned in Jae-Suk's talk, I have my Poisson tensor, in some completion, if we have a function $F \in \widehat{Sym}^* \mathcal{E}^*$, in $\prod Hom(\mathcal{E}^{\times n}, \mathbb{R})_{S_n}$, then ΔF is going to be contracting F and Π. So let's do a simple example to see what this looks like. Suppose I take $B \in \Omega^{n,n,0}(X^3)$, then I get $F_B(\phi, \psi) = \int_{X^3} B(x_1, x_2, x_3)\phi(x_1)\phi(x_2)\psi(x_3)$. Then $(\Delta F_B)(\phi, \psi)$, I just put my Poisson kernel in two of these guys, this is

$$\int_{X \times X} \phi(x_1) B(x_1, x_2, x_3)$$

so now we see why there might be a problem.

Problem 4.1. *This is very often ill-defined*

We want a quantum master equation for a local functional S but ΔS will involve multiplication of distributions. So if B is a delta function of a small diagonal, so that the corresponding functional F_B is local, this implies that ΔF_B is ill defined. It involves restricting this distribution and setting the variables to be the same. But this involves multiplying distributions which is very bad.

Next I want to say how to solve this problem using homotopical things.

So first let's assume our action functional can be written as some kind of kinetic plus some kind of interaction term and that the kinetic term, for simplicity is the usual term $S_{\text{kin}}(\varphi) = \int \varphi \underbrace{D}_{\text{Laplacian}} \varphi$ and $S(\varphi) = S_{\text{kin}}(\varphi) + I(\varphi)$. Then

$$\Pi = \delta_{\text{diag}} \in \bar{\mathcal{E}}\hat{\otimes}\bar{\mathcal{E}}$$

and we can consider a mollified version $\Pi_L = (e^{-LD} \otimes 1)\Pi \in \mathcal{E}\hat{\otimes}\mathcal{E}$, hit this with the heat kernel and smooth it, it's no longer distributional. Now we let the Laplacian of L and the Poisson bracket of L be the Poisson bracket and BV operator coming from Π_L. Now it's easy to see that then Δ_L and $\{\ ,\ \}_L$ are always well-defined because the issues previously were multiplying distributions, now we have something smooth, I can multiply anything with that and I have no problem.

You might wonder why you're allowed to do this. I can always change the Poisson bracket by a homotopy. The desingularized guy is homotopic to the original one. That is, Π_L is cochain homotopic to the original one Π, and what's the cochain homotopy? It's the propagator $\int_0^L K_t dt \in D(X \times X) \in \bar{\mathcal{E}} \hat{\otimes} \bar{\mathcal{E}}$ of degree 0, integration with the heat kernel.

Now I see my disorganization catching up to me, I forgot to tell you the differential, but Owen did. It's a homotopy with respect to the differential $C^\infty(X) \xrightarrow{D} \Omega^{\text{top}}(X)$ on $\mathcal{E}$ coming from the quadratic term in the action.

So why is this a homotopy? I apply my differential to this, and $P(0, L)$ is the kernel for the operator of degree -1 from $\Omega^{\text{top}} \to C^\infty(X)$ which sends $\omega \mapsto \int_0^L \frac{e^{-tD}\omega dt}{dVol_g}$.

It is an exercise in integration by parts to check that this is the homotopy.

Before we wanted a local functional I which satisfies

$$QI + \frac{1}{2}\{I, I\} + \hbar \Delta I = 0.$$

We've realized some of the terms are badly defined, so now we want a functional $I[L]$ which satisfies

$$QI[L] + \frac{1}{2}\{I[L], I[L]\}_L + \hbar \Delta_L I[L] = 0.$$

We haven't gotten to the end of the story. Notice there's a word here up top which isn't there below, which is "local." Our kernels are all homotopic for different L. The BV algebras are all homotopic so I should be able to translate solutions for the scale L quantum master equation to any other scale. Then we'll be able to think of locality.

The homotopy given by $\Pi_L \sim \Pi_{L'}$ is $\int_L^{L'} K_t dt$, which should imply that we can transfer solutions to the quantum master equation for Π_L to those for $\Pi_{L'}$.

There's a very simple formula for it, and this formula is something you can discover just by thinking about homological algebra. So if $I[L]$ satisfies the Π_L quantum master equation, we let $W(P(L, L'), I[L])$ be

$$\sum_{\gamma,\text{conn}} \frac{1}{|Aut\ \Gamma|}\Gamma$$

where I put $I[L]$ on each vertex and put P on each edge. So then if the first one satisfies it for Π_L, the second one does for $\Pi_{L'}$. I'm sure homological people can see easily why this is true. Then you apply your differential and the terms all cancel.

If you use the formula $e^{\hbar \partial_{P(L,L')}} e^{I[L]/\hbar}$ it's much cleaner, I thought this notation would take too long to explain.

Instead of a solution $I[L]$ for a single L, we want $I[L]$ for all $L > 0$ related by the equation $W(P(L, L'), I[L]) = I[L']$. Why am I bothering with this if I can get one from the other? But let's think about what we were missing, we wanted an action function that was *local* and satisfied the quantum master equation. Now locality should be easy to guess, which is that as L tends to zero, this becomes more and more local. So $I[L]$ is approximately local as $L \to 0$. You have to be careful, the $L \to 0$ limit will not exist, but we can still say there is an asymptotic expantion, or there's exponentially fast decay near the diagonal, or instead of scaling, you use a parametrix.

That's nice, it's a definition, but can you construct these? Let $\mathcal{O}_\ell(\mathcal{E})$ be the local functionals. It's a cochain complex with differential $Q + \{I^{cl}, \quad\}$. Then term by term in $\hbar$ there is an obstruction $\mathcal{O} \in H^1(\mathcal{O}_\ell(\mathcal{E}))$ to quantizing to the next order and if this obstruction is equal to zero, the set of quantizations to the next order is a torsor for $H^0(\mathcal{O}_\ell(\mathcal{E}))$. So, what's it saying? Given the classical action functional. This will tell you whether you can quantize and if you have many choices, in nice cases.

So next time I want to explain what this is good for and how you can use it to recover the operator product and correlation functions. If you take Yang-Mills theory on $\mathbb{R}^4$, there is a one parameter family of quantizations which are $Iso(4)$-invariant and "compatible in a certain way" with scaling $\mathbb{R}^4$. The proof is that computing the cohomology groups, H^1 vanishes and H^0 is one dimensional. In physics people care about renormalization. Typically a scale invariant classical theory doesn't quantize to a scale invariant theory, but you want it to be true up to a logarithmic term.

Let me stop there unless there are questions.

5. September 30: Vasily Dolgushev, Operads, homotopy algebras, and all that jazz

Thank you very much, I'd like to thank the organizers for the opportunity to give lectures. I'll also thank my in-laws for taking care of my children while I am here, and they will also watch the video.

The goal of these three lectures is to answer the question "what do homotopy algebras form?" The lectures will be partially based on three papers. One is with Chris Rogers, "On enhancement of..." The second, also with Chris, is "A version of the Goldman-Millson theorem..." The third, with Alex Hoffnung and Chris Rogers is "What do homotopy algebras form?"

The reason I put Chris in blue is because he's looking for a tenure track job.

Erdös had this idea of the book, where mathematics was written in ideal form. I feel like this material is from the book. These lectures, though, may be quite basic and uninteresting to a specialist. If they are interesting, I do not imply that you are not a specialist.

I will let $\mathbf{k}$ be a field of characteristic zero and I will often be talking about cochain complexes of $\mathbf{k}$-vector spaces, on which I will assume no boundedness condition. By Hom I mean the inner hom in the category of cochain complexes. Hom and $\otimes$ will be over $\mathbf{k}$. Then s and s^{-1} will be shift operators. If I have a cochain complex or graded vector space, then $(sV)^{\cdot} = sV^{\cdot-1}$. So think of s as a variable of degree 1.

Sometimes I will use $|v|$ to mean the degree of v, provided it is homogeneous. Then S_n will denote the symmetric group on n letters, the group of permutations. If I have a vector space and S_n acts, then W^{S_n} will be the space of invariants and W_{S_n} the quotient space of coinvariants. Let me apologize in advance, sometimes I will lie, they will be technical lies, I'll tell you, "I am lying here," I hope I won't forget to tell you.

Let me start with an example. Let (A, ∂) be a cochain complex. I'll form another cochain complex

$$\mathcal{L}_A = \prod_{n\geq 2} Hom((s^{-1}A)^{\otimes n}, s^{-1}A)$$

I claim that this is naturally a differential graded Lie algebra. The differential is inherited from A, and the bracket, let me not write a precise formula, but let me write vaguely, I don't change the order of arguments, but I insert the second operator at some point, and then change the order:

$$[P,Q]_G(a_1,a_2,\ldots) = \sum_i \pm P(a_1,\ldots,a_{i-1},Q(a_i,\ldots)\ldots) - (-1)^?(P \leftrightarrow Q)$$

Definition 5.1. *An A_∞ structure on a cochain complex (A,∂) is a degree one element in $\mathcal{L}_A$ such that*

$$\partial m + \frac{1}{2}[m,m]_G = 0.$$

If I unfold this definition, I have a collection of operations. $m_2 : A\otimes A \to A$ will have degree 0, then $m_3 : A^{\otimes 3} \to A$ will be degree -1, and so on. These will satisfy a complicated sequence of quadratic equations.

The first equation tells us that m_2 is compatible with the differential. The second equation tells us that m_2 is almost associative. It's associative up to homotopy. Then m_3 is the corresponding homotopy operator. Then every differential graded associative algebra is an A_∞ algebra with $m_3 = m_4 = \cdots = 0$.

This is the first incidence when I'm lying a little bit. In this case I'm lying a little bit, m_2 is actually multiplication up to a sign.

Let me give an example of geometric origin. Let M be a compact manifold without boundary. Consider exterior forms with the de Rham differential. This is a great differential graded algebra, even (graded) commutative, so it's an A_∞ algebra. If we choose a metric on M, then we can talk about harmonic forms, let me denote their graded vector space by $\mathcal{H}^\cdot(M)$. Now Hodge theory tells us that the graded vector space of harmonic forms is isomorphic to the real cohomology of M. But it gives us more than that.

So first of all, $(\mathcal{H}^\cdot(M),0)$ is a subspace of $(\Omega^\cdot(M),d)$ (via i). Hodge theory also gives us a projection operator p. Using Green's function we can build a homotopy $\chi : \Omega^\cdot(M) \to \Omega^{\cdot-1}(M)$ such that $i\circ p + d\circ\chi + \chi\circ d = id_{\Omega^\cdot(M)}$ and $p\circ i = id_{\mathcal{H}^\cdot(M)}$.

One problem with dealing with harmonic forms is if I multiply two harmonic forms, usually it's not a harmonic form. It seems that I'm losing the product here. I can take the harmonic forms, multiply them, and project, so $a_1 \bullet a_2 = p(a_1\cdot a_2)$. This is associative and commutative. So we have this algebra $(\mathcal{H}^\cdot(M),\bullet)$ which is nice but we lose information about M.

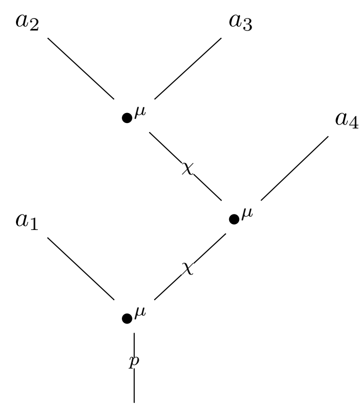
I want to tell you that this $\bullet$ is part of an A_∞ structure.

Claim 5.1. *Using p and χ one can construct an A_∞ structure on $\mathcal{H}^\cdot(M)$ with the zero differential such that $m_2 = \bullet$*

So for $n \geq 3$ I need to construct $m_n(a_1,\ldots,a_n)$ in $\mathcal{H}^\cdot(M)$. This is constructed as a sum over planar rooted trees with n leaves as

$$\sum \pm m_T(a_1,\ldots a_n)$$

So here is what I have to do for m_T: I take my tree



and I put χ on each internal edge, μ on each trivalent vertex, the arguments on the incoming leaves, and p on the root. Then I read downward. If you do this right, you get an m_n that makes an A_∞ structure which remembers more information.

If you want examples, let me give references, Babenko-Taimanov or Fernadez-Muñoz, "The geography of non-formal manifolds."

If you change the metric, you get the same homotopy class of A_∞ algebra.

So what I showed you is a baby example of a transfer theorem. Every homotopy algebra should have a transfer theorem. I can mention another motivation. When I read papers about Chern-Simons invariants, whatever it means, it seems that constructing these is applying some kind of homotopy transfer theorem. In the quantum sense it's some version of homotopy transfer.

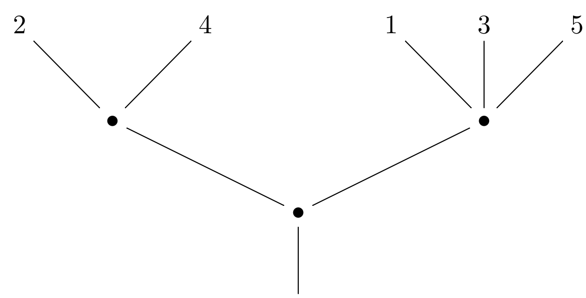
When you mention a name, there may be someone in the audience who will say "I did it earlier and much better."

So now I can actually start my lectures. I'll need to talk about such things as operads, cooperads, cobar construction, things like this.

Let me say

Definition 5.2. *A* collection *is a collection* $\{P(n)\}_{n\geq 0}$ *so that each* $P(n)$ *carries a left action of* S_n.

The example, if V is a vector space, let $End_V(n)$ be $Hom(V^{\otimes n}, V)$. Clearly S_n acts. But we can also compose and let's see what we have. So we also have a special element id in $Hom(V,V)$. Then to every labeled planar tree t we get a composition.



So this gives a map $\mu_t : End_V(2) \otimes End_V(2) \otimes End_V(3) \to End_V(5)$ by inserting the maps on the vertices and composing.

This is an example of an operad.

Definition 5.3. *An* operad $\mathcal{O}$ *is a collection* $\{\mathcal{O}(n)\}_{n\geq 0}$ *with* $e \in O(1)$ *and every labeled planar tree* t *gives* $\mu_t : \mathcal{O}(n_1) \otimes \cdots \mathcal{O}(n_r) \to \mathcal{O}(n)$ *where* n *is the number of leaves of* t *and* n_i *is the number of edges terminating at the* i*th vertex.*

This data should satisfy axioms. There are two types, one about compatibility with the S_n *action. There is also an associativity law which can be explained by saying that it doesn't matter what order we contract edges in. The special element should play the role of the identity.*

Okay. so End_V is called the endomorphism operad of V.

You may wonder about morphisms of operads, $\mathcal{P}$ and $\mathcal{O}$. These are maps of collections compatible with the structure, so $\mathcal{P}(n) \overset{\varphi_n}{\to} \mathcal{O}(n)$ that are compatible with multiplications which send the identity to the identity.

Definition 5.4. *Given an operad* $\mathcal{O}$ *and a vector space* V*, we say that* V *is an* algebra over $\mathcal{O}$ *if we have an operad morphism* $\varphi : \mathcal{O} \to End_V$*.*

Many algebraic structures are algebras over operads, like associative algebras, Lie algebras, and so on. If I consider $\{As(n)\}_{n\geq 0}$ where I let $As(0) = 0$ and $As(n)$ the regular representation of S_n, $\mathbf{k}[S_n]$, then As is naturally an operad and As-algebras are associative algebras without unit.

Here is another collection $\{Com(n)\}_{n\geq 0}$. Then $Com(n)$ is $\mathbf{k}$ with the trivial S_n action (with $Com(0) = 0$). Then Com-algebras are commutative associative algebras without unit.

There is also a Lie operad. What is $Lie(n)$. Take the free Lie algebra on n generators $Lie(a_1, \ldots, a_n)$. Look inside this at Lie words satisfying the property that every generator appears exactly once. For example, $Lie(a_1, a_2, a_3)$, then I can't use the word $[a_1, a_2]$ or the word $[[a_1, a_2], a_2]$. But $[[a_1, a_2], a_3]$ is allowed. Take the span of such "admissible" Lie words and you get $Lie(n)$.

I started a little late. Let me go on a little bit. The category of operads has an initial object $*$, which is $*(1) = \mathbf{k}$ and $*(n) = 0$ otherwise. Then $*$ is initial in operads. Since every operad has the identity element, this is initial.

Definition 5.5. *We say an operad is augmented if we have a morphism* $\mathcal{O} \to *$*.*

For the purposes of my lectures, I'll assume my operads are *reduced* in the sense that $\mathcal{O}(0) = 0$ and $\mathcal{O}(1) = \mathbf{k}$. Then all such operads are augmented.

6. Kevin Costello, Part II

What I'm going to talk about is joint work with my esteemed collaborator Owen, so I'd like to start with classical observables in a field theory and then I'll talk about quantizing.

Consider again the simple case of scalar field theory, where fields are φ in $C^\infty(M)$, and the action functional might be, for example, $S(\varphi) = \int \varphi\Delta\varphi + \varphi^4$ Consider an open subset $U \subset M$. Then we want to define observables on U to be $\mathcal{O}(C^\infty(U))$, but where we divide by $V(S)$, the ideal defined by applying a vector field V to S. At the quantum level, we want something that looks like, this is too naive, but we want $\mathcal{O}(C^\infty(U))$ modulo the subspace which looks like $Div_{e^{-S/\hbar}dLeb}(V)$. This is the functional measure in the subscript.

Let's see how this works classically first.

Problem 6.1. *If $\varphi \in C^\infty(U)$, then $S(\varphi)$ is not really defined because $\int_U \varphi\Delta\varphi + \varphi^4$ might not converge.*

We'll have to modify our approach. But if I have α a compactly supported function, then $\frac{\partial S}{\partial \alpha}$ is a function on $C^\infty(U)$. We learned this in high school, the Euler-Lagrange equations. So why is that? If I take this function, and I look at $\alpha\Delta\varphi$ and $\alpha\varphi^3$, then this has compact support so the integral converges nicely.

Because $S(\varphi)$ is not really a function we can't apply a vector field to it. This tells us, well, S is not a function but dS is a closed one-form on $C^\infty(U)$ along the leaves of the foliation given by $C_c^\infty(U) \subset C^\infty(U)$.

If we write $Vect_c(C^\infty(U))$ for the vectro fields along the foliation, then we *do* have $Vect_c(C^\infty(U)) \overset{\vee dS}{to} \mathcal{O}(C^\infty(U))$ makes sense. So $Vect_c(C^\infty(U))$ looks like $\prod Hom_{\mathrm{cts}}(C^\infty(U^n), C_c^\infty(U))_{S^n}$.

We can actually go a little bit further. Suppose instead of having a compactly supported smooth function, this was a compactly supported distribution. Then $\frac{\partial S}{\partial \alpha}(\varphi)$ makes sense for all $\varphi \in C^\infty(U)$.

Again, if you replace one φ with an α everywhere, I get a compactly supported distribution, which I can integrate. So there's an extra continuity property, meaning it's well defined on some sort of bigger thing.

So if $\overline{Vect}_c(C^\infty(U)) = \prod Hom(\mathcal{C}^\infty(U)^{\times n}, D_c(U))_{S_n}$, this is a nonlinear map from smooth functions to compactly supported distributions, these are vector fields with compact support and singularities. Bar means distributional.

Okay, so with these more singular vector fields, the operation of contracting with dS with these guys, $\iota_{dS} : \overline{Vect}_c(C^\infty(U)) \to \mathcal{O}(C^\infty(U))$ makes sens and $H^0(Obs^{cl}(U))$ is the cokernel.

Let's recall that in BV, we said our fields are $\mathcal{E}(U)$ is $C^\infty(U)$ in degree 0 and $\Omega^n(U)$ in degree 1.

So consider functions on $\mathcal{E}(U)$. In degree 0, it will be $\mathcal{O}(C^\infty(U))$. In degree -1 it will be $\mathcal{O}(C^\infty(U) \times \Omega^n(U))$ which are linear in $\Omega^n(U)$. We can write this as

$$\prod_{k\geq 0} Hom(C^\infty(U)^{\times k} \times \Omega^n(U), \mathbb{R})_{S_k} = \prod_{k\geq 0} Hom(C^\infty(U)^{\times k}, D_c(U))_{S_k} = \overline{Vect}_c(C^\infty(U))$$

where Hom means multilinear maps.

So now what we find, in these functions on fields on U, with the differential bracketing with the action functional, $(\mathcal{O}(\mathcal{E}(U)), \{S, \quad\})$, this looks like

$$\overline{PV}_c(C^\infty(U)) \to \overline{Vect}_c(C^\infty(U)) \overset{\{S,\}}{\to} \mathcal{O}(C^\infty(U))$$

If α is a compactly supported distribution in $D_c(U) \subset \overline{Vect}_c(C^\infty(U))$, then as an example of the simplest bracket, $\{S, \alpha\} = \frac{\partial S}{\partial \alpha}$.

Now let's imagine we live in a fantasy world and see what we'd expect and then make it rigorous. Naively, at the quantum level we expect that quantum observables should be functions on $\mathcal{E}(U)$ with the differential $\{S, \quad\} + \hbar\Delta$.

Last time we saw this doesn't make much sense. But let's for the moment proceed in this naive way and see what structure we would find. Recall that divergence with respect to $e^{S/\hbar}dLeb$ is $\frac{1}{\hbar}\iota_{dS} + \Delta$ so $H^0 Obs^q(U) = \mathcal{O}(C^\infty(U))$/divergence for functional measure. So normally for a compact manifold you expect the functions to be one-dimensional.

But for us, typically $H^0(Obs^q(U))$ is infinite dimensional unless $U = M$ is a compact manifold.

So the simplest thing we find is, we suppose that $H^0(Obs^q(M))$ is one dimensional over $\hbar$ (this happens every time your theory is massive, for example), then the map $H^0(Obs^q(U)) \to H^0(Obs^q(M))$ *is* the functional integral. This is true up to scale because they have the same kernel.

So what properties would we expect this to have. If $U \subset W$ then we expect a cochain map $Obs^q(U) \to Obs^q(W)$. We just view our function on one as a function on the other.

As before, we expect $Obs^q(W)$ to be commutative algebras but where the differential is not a derivation. So we have a map, if $U, V \subset W$ are disjoint, we have a map $Obs^q(U) \times Obs^q(V) \to Obs^q(W)$ which takes a pair of observables $\mathcal{O}_1, \mathcal{O}_2$ to $\mathcal{O}_1\mathcal{O}_2$.

The key point is that we expect that this map is a cochain map. In the language of Jae-suk's talk, [unintelligible].

Let's apply the differential to see,

$$(\{S, \quad\} + \hbar\Delta)(f_1 f_2) = \{S, f_1 f_2\} + \hbar\Delta(f_1 f_2) = \{S, f_1\}f_2 + f_1\{S, f_2\} + \Delta(f_1)f_2 + f_1\Delta(f_2) + \hbar\{f_1, f_2\}$$

So if f_1 and f_2 have disjoint support, then Π the kernel for $\{\ ,\ \}$ is δ_{diag}. and so disjoint support means that $\{f_1, f_2\} = 0$ so then the differential really is a derivation.

It's important to realize that the commutative product of observables is only really defined if they are disjoint.

So we've sketched why Obs^q should be a factorization algebra. For every U we have a cochain complex $Obs^q(U)$ and for every inclusion $U \subset W$ a cochain map $Obs^q(U) \to Obs^q(W)$. We also have for every disjoint union of U and V in W we have a cochain map $Obs^q(U) \times Obs^q(V) \to Obs^q(W)$. This should have an associativity property.

We were pretty sloppy, we ignored the issues, Δ is badly behaved, things like that, we have to make sense of all these things. What I want to sketch is how to construct all of this rigorously, that is, $Obs^q(U)$ and the factorization structure we expect.

Last time we defined a quantum field theory to be some functionals $I[L] \in \mathcal{O}(\mathcal{E}(M))$ which satisfy renormalization group flow and a quantum master equation built from Π_L which is $(e^{-L\Delta} \otimes 1)\Pi$ where $\Pi = \delta_{\text{diag}}$.

Let's define this for $U = M$ first and then realize them for other things as subspaces.

$$Obs^q(M) = \mathcal{O}(\mathcal{E}(M))[[\hbar]]$$

with differential $d + \hbar\Delta_L + \{I[L], \quad\}_L$. Last time we noted that all of these dg BV algebras are homotopic for different L. Last time we translated solutions of the quantum master equation from one L to another, we'll do something similar here.

If I have $\mathcal{O} \in \mathcal{O}(\mathcal{E}(M))[[\hbar]]$, it's closed, that's true if nad only if $I[L] + \epsilon\mathcal{O}$ satisfies the quantum master equation modulo ϵ^2. The idea is that we'll let $\mathcal{O}$ depend on L so that when L is small it lives in a neighborhood of [unintelligible].

So an observable on M is a family $\mathcal{O}(\mathcal{E}(M))[[\hbar]]$ for all $L > 0$ related by the equation $W(P(L, L'), I[L] + \epsilon\mathcal{O}[L]) = I[L'] + \epsilon\mathcal{O}[L']$ modulo ϵ^2. Because this operation is invertible, then quantum observables on M is a vector space and RG flow intertwines the differentials $d + \hbar\Delta_L + \{I[L], \quad\}_L$. Say defining

$$Q\mathcal{O}[L] = d\mathcal{O}[L] + \hbar\Delta_L\mathcal{O}[L] + \{I[L], \mathcal{O}[L]\}_L$$

makes $Obs^q(M)$ into a cochain complex.

We say that $\mathcal{O}$ is supported on U if $\mathcal{O}[L]$ is supported on U for $L \to 0$ (maybe to be precise this is something like exponential decay away from U).

The last thing we need is the factorization product map. If U and V are disjoint, and $\mathcal{O}_U$ and $\mathcal{O}_V$ are observables, then at scale L, we say that $\mathcal{O}_U \cdot \mathcal{O}_V$ is defined $(\mathcal{O}_U \cdot \mathcal{O}_V)[L] = \mathcal{O}_U[L]\mathcal{O}_V[L]$ as $L \to 0$.The key point for this to be a cochain map is that if L is very small, then $(d + \hbar\Delta_L + \{I[L], \quad\}_L)(\mathcal{O}_U[L] \cdot \mathcal{O}_V[L])$ is the derivation terms plus $\{\mathcal{O}_U[L], \mathcal{O}_V[L]\}_L$ where as $L \to 0$, $\mathcal{O}_U$ is supported on U, $\mathcal{O}_V$ is supported on V, and $\{\ ,\ \}$ is supported near the diagonal.

I'm out of time so thank you very much.

7. Ryan Grady: Moduli/β-functions for BV theories

[Note: I did a particularly bad job taking notes for this talk. My apologies!]

I want to start with a sort of risky cartoon, hopefully not meant to be funny but still a bit of a cartoon. Let's pretend we have a reasonable notion of a space of theories, and pretend there's a flow on this space. Let me choose some units, so let's use the parameter ℓ, which has units length which is the inverse of energy.

Let's suppose that I have a point in that space, and consider a flow line through here, okay, and so I might hope that as ℓ goes to zero, I go to some fixed point. There's no a priori reason to assume that this is the only thing flowing from such a fixed point, and I could have a whole space of flow lines out of my fixed point. What might I ask?

I could ask for a flow to a fixed point as $\ell \to 0$ and I could ask that the unstable manifold is finite dimensional. The point is that if we are, if we know the fixed points, to say which flow line I'm on, I only need to specify a finite number of conditions. The idea of point one is to relate measurements at a nonzero value of ℓ and relate this to the high energy limit.

This is only a not-wonderful motivation, but I want to extract a couple of things that are going on.

We have a flow here, which is (local) renormalization group flow. and we have an infinitesimal generator, which I'll call a β-function. That's great if you're fixated on this cartoon, but what is the output?

Theorem 7.1. *(Friedan, 80s); (G.-Li); (Nguyen 2014)*
The two dimensional non-linear sigma model, maps from Σ to a Riemannian manifold X has local RG flow given by Ricci flow on X modulo $\hbar^2$.

So this is some way of saying you can have a clue of your geometry from baking this cartoon precise.

So as before, let $(\mathcal{E}, S)$ be a space of fields and an action functional. The setup I want is that the space of fields admits an action of the semigroup of positive reals. So $\mathcal{E}$ has an action of $\mathbb{R} > 0$. So here $R_\lambda : \mathcal{E} \to \mathcal{E}$ induces an action on $\mathcal{O}(\mathcal{E}) \ni \Phi$. So

$$(R_\lambda \Phi)\alpha = \Phi R_{\lambda^{-1}}\alpha.$$

We'll say Φ has weight m if $R_\lambda\Phi = \lambda^m\Phi$.

One important caveat we'll ask here is that this classical action functional have weight zero. It's a good exercise to see what the weight of your favorite functional is over your bowl of lucky charms tomorrow.

I've only talked about the classical theory here. I want to point out that there's a way to go from local action functionals to effective theories via RG-flow.

Theorem 7.2. *(Costello)*
Let me not be too precise, but for a suitable class of theories where S is a quadratic part plus an interaction term $Q + I$, there exists a BV theory, with $I[L] = \lim_{\epsilon \to 0} W(P_\epsilon^L, I + I^{ct}(\epsilon))$.

Theorem two, let me say in words, there's an obstruction theory for this being quantizable.

Definition 7.1. *Suppose that we have a collection $\{I[L]\}$ of effective interactions and we want to define a new operator on these action functionals, define local RG-flow $(RG_\lambda I)[L]$. What is this? It's $R_\lambda I[\lambda^{-2}L]$. I set this up to also scale the length scale. This is my local RG-flow.*

You should check that if these guys satisfy the non-local RG-flow and the quantum master equation, then under local RG flow these conditions will be preserved.

Proposition 7.1. *If $\{I[L]\}$ satisfies standard flow and the quantum master equation then so does the family $\{RG_\lambda I[L]$*

How do we pick off our beta function?

Definition 7.2. *Define $\beta[L]$ to be the limit as $\lambda \to 1$ of $\lambda \frac{\partial}{\partial \lambda}(RG_\lambda I)[L]$.*

It's not clear that this contains interesting information but I hope to convince you that it does.

This has higher loop information. It's indexed by the genus arising in RG flow. [missed]

Now $\beta[L] = \beta_0 + \hbar\beta_1[L] + \hbar^2\beta_2[L]$. So modulo $\hbar$ there is nothing there. I'm really only interseted the [missed].

I don't want to discuss existence, but maybe I'll work with one guy and ask that the limit exists as $L \to 0$. I'll define β_1 as the limit of $\beta_1[L]$. You can think of this as following from this proposition

Proposition 7.2.

$$[\beta_1] \in H^0(\mathcal{O}_{loc}|\mathcal{E}, \{S, \quad\})$$

This homology group is basically the tangent space.

Okay next we'd like a way to pick off this function in terms of our quantization. Suppose that we've obtained our effective action by taking $W(P_\epsilon^L, I - I^{ct}(\mathcal{E}))$. This is our BV theory. Let's assume that our counter terms have the form I_ϵ^{ct}.

So uh, in good situations, let's assume that our counter terms look like

$$I^{ct}(\mathcal{E}) = I^{\log} + \frac{1}{\epsilon}A + \frac{1}{\epsilon^2}B.$$

Then $\beta_1 = I^{\log}$.

Let's do this in an example, the example of the non-linear sigma model. It's convenient because Si introduced some notation yesterday. Let Σ be a curve (for us $\mathcal{E}$). I want to study the maps to X but that's too hard.

Yesterday Si intorduced this L_∞ algebra. You could say that its what [missed] looks like.

My space of fields is, it'll have two bits, it's the de Rham complex on $\Sigma \otimes gGg_X[1]$. I'll add to that $\Omega^2_\Sigma \otimes \mathfrak{g}[x]$. Now I should tell you what the algebra is. There will be a

$$S(\alpha, \gamma) = \int_\Sigma \ell(e^\alpha)\gamma + \int_\Sigma h(e^\alpha, \partial_{z^\alpha}, \partial_{\bar{z}^\alpha})dzd\bar{z}$$

where $\ell(e^\alpha) = \sum e_n \alpha^{\otimes n} n!$ with ℓ_n defining the L_∞ algebra.

[Some discussion].

So I'll rescale the target with the $\mathbb{R}_{>0}$ action on $\mathcal{E}$. I should also take compact support.

So this is our action, it induces an action on functionals. You should check, I'm not going to do this, that S is invariant.

Hopefully if I put things in with the correct number of derivatives, everything works out and it has to be invariant.

So let's try to compute β_1. What contributes to the $\log_\epsilon$ business.

You need one-loop Feynman diagrams.

[Picture]

So what sort of diagrams can show up, I told you that the propagator goes on internal edges [Missed some]. By polarizing with γ and α I figure out what needs to be split, $\beta_1 = \beta_g + \beta_h$.

Proposition 7.3. $[\beta_1] = [\beta_h] \in H^0(Q_{loc}(\mathcal{E}), \{S, \quad\})$

The point is then we need to compute what $\beta_{\mathfrak{g}}$ and β_h look like.

I want to compute the β_h term. The trick is to do it in terms of geodesic normal coordinates. Let me just sort of, some of you will protest, let me finish by roughly telling you the relevant Feynman diagrams. There's a loop with many strands and the involutive graph. The former is a multiple of the Ricci tensor and the latter is zero in normal coordinates.

That's where I'd like to stop.

8. Si Li, Part II

Last time we had (X, W, ∇) where W was $\widehat{Sym}(T^*X)[[\hbar]]$ and we were considering $S \to X$ with $\mathcal{E} = \mathcal{A}^*(S) \otimes \mathfrak{g}_X[1]$ where this $\mathfrak{g}_X[1] = \wedge^\cdot \otimes T_X$.

You have $I \in \wedge^1 \otimes W$ which leads to $\tilde{I} = \sum_k \frac{1}{k!} \int_S I(\alpha, \ldots, \alpha)$ We tried to look at a Fedosov type problem, solving $\nabla I + \frac{1}{2\hbar}[I, I] + R = W_\hbar$. This leads to a quantum master equation for $\hat{I}[L]$

Today we want to look at quantum observables as $L \to \infty$

Let me start with the global observables $Ob^q(S)$. So $\mathcal{O}[L]$ is in there. This has a differential $Q + \hbar\Delta_L + \{\tilde{I}_L, \quad\}_L$ where $Q = \nabla + d_S$. One nice thing about this global thing is, that at $L = \infty$, the Δ_L is contraction with the heat kernel so it becomes purely harmonic. So then this is quasiisomorphic to something smaller and somehow we can restrict our discussions to something smaller given by harmonic inputs. In our example it's the circle so it's extremely simple, $\mathbb{C}[d\theta]$. In a similar way, $\mathcal{O}(\mathcal{E})$ can be reduced to something smaller, functions on this data, $\mathcal{O}(\mathbb{H}(S) \otimes \mathfrak{g}_X[1])$.

You somehow, well, let me say what this looks like. Let me introduce a bit of terminology. Let Ω_W^{-*} be the sheaf of de Rham of W. So this is $\bigoplus \widehat{Sym}(T^*X) \otimes$

$\wedge^p T^* X[p][[\hbar]]$. So this is the sheaf of de Rham. So functions on this harmonic piece is the same thing as this kind of data valued in forms, $\mathcal{O}(\mathbb{H}(S)\otimes\mathfrak{g}_X[1]) = \wedge^*\otimes\Omega_W^{-*}$.

Let me start with an observable closed under the differential (satisfying the quantum master equation) $\mathcal{O}[L]$. Let me write this like, as $L\to\infty$ we have $(Q+\hbar\Delta_\infty+\tilde{R}/\hbar)\mathcal{O}[\infty]e^{-\tilde{I}[\infty]/\hbar}$.

Let $[\mathcal{O}[\infty]e^{\tilde{I}/\hbar}]$ be the restriction to harmonic input in $\mathcal{O}(\mathbb{H}(S)\otimes\mathfrak{g}_X[1])$. These functions can be identified with $\wedge^*\otimes\Omega_W^{-*}$. My goal is to integrate these observables on this side.

I want to ge a number in $\mathbb{C}[[\hbar]]$ (probably with some negative powers in $\hbar$). First, integrate over the fiber to get a map $\wedge^*\otimes\Omega_W^{-*}\to\wedge^*$. The second one will be integration over X which everyone knows.

How does this work? The toy model is like, let's first work on fiberwise integration. Let's look at this simple algebra $A=\mathbb{C}[[x_i,\eta_i]]$ with degree of x_i zero and η_i one. We have a natural BV Laplacian $\sum\frac{\partial}{\partial x^i}\frac{\partial}{\partial\eta_i}$. We have $H^*(A,\Delta)=\mathbb{C}\eta_1\wedge\cdots\wedge\eta_n$. If you get some information from Owen's talk, we should replace integration with taking $H^*(\Delta)$. So $\int f(x,\eta)=\frac{\partial}{\partial\eta_1}\cdots\frac{\partial}{\partial\eta_n}f|_{x=0}$.

We need to choose a good volume form. Let $\mathcal{R}_{\omega^{-1}}:\Omega_W^p\to\Omega_W^{p-2}$ be the contraction with $\omega^{ij}(x)\frac{\partial}{\partial y^i}\wedge\frac{\partial}{\partial y^j}$.

Definition 8.1.

$$\int_W : \wedge^*\otimes\Omega_W^{-*}\to\wedge^*$$

is given by

$$\alpha\mapsto\frac{\hbar^n}{n!}R^n_{\omega^{-1}}\alpha|_{y=0}$$

This is compatible with the differential, where $\nabla+\hbar\Delta_\infty+\tilde{R}/\hbar\mapsto d$.

Now let $\mathcal{O}[L]\in H^0(Ob^q(S))$. Define the following function

$$\langle\mathcal{O}[L]\rangle=\int_X\int_W[\mathcal{O}[\infty]e^{\tilde{I}[\infty]/\hbar}].$$

So this is

$$\int_X e^{\hbar R_{\omega^{-1}}}[\mathcal{O}[\infty]e^{\tilde{I}[\infty]/\hbar}]|_{y=0}.$$

I need to say one more thing to connect this to the trace map.

I can do this for a bunch of local observables. If you have U_i disjoint, you get the map described in Kevin's talk from $Ob^q(U_1)\otimes\cdots\otimes Ob^q(U_n)\to Ob^q(S)$ where $\mathcal{O}_1\cdots\mathcal{O}_n\mapsto\mathcal{O}_1*\cdots*\mathcal{O}_n$. Then $\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle=\langle\mathcal{O}[L]\rangle$.

Locally we have that $H^*(Ob^q(U))\cong C^\infty(X)[[\hbar]]$ so $Tr:C^\infty(M)\to\hbar^{-dim\ X/2}\mathbb{C}[[\hbar]]$, so $Tr(f*g)=Tr(g*f)$. This is discussed in Owen and Kevin's book.

This kind of trace map is unique up to rescaling, this is a theorem by Nest-Tsygan. In particular, what we're interested in is calculating partition functions, so for instance what is the trace of 1. This will tell you something about the index theorem.

Problem 8.1.

$$Tr(1)=\langle 1\langle$$

(the partition function)

So basically we'll use two symmetries, One is a cyclic symmetry on the circle, let's call this one θ, the coordinate on the circle. So $\frac{\partial}{\partial\theta}$ and then ι_θ, contraction with $\frac{\partial}{\partial\theta}$ which induces ι_θ^* by duality $\mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$, so this plays the role of the cyclic differential in this story.

Now if you look at this basic fact, if you represent this one on our model of de Rham on the Weyl algebra, this is nothing but the usual de Rham differential on Ω_W^{-*}, $[\iota_\theta^*\alpha] = d_W[\alpha]$. Let me make it very precise. This goes

$$Sym^p T^*X \otimes \wedge^q T^*X \to Sym^{p-1}(T^*X) \otimes \wedge^{q+1} T^*X$$

Last time we described one particular form

$$\tilde{I}(\alpha) = \sum_k \frac{1}{k!} \int_S I(\alpha, \dots \alpha).$$

This should have one one-form as input. I also have the following where all the input should be zero forms:

$$\bar{I}(\alpha) = \sum_k \frac{1}{k!} \int_S d\theta I(\alpha, \dots, \alpha).$$

Note that $\tilde{I} = \iota_\theta^* \bar{I}$ so in particular $\iota_\theta^* \tilde{I} = 0$. Note the

Fact 8.1. *renormalization group flow and the quantum master equation is compatible with ι_θ^*.*

Then $\iota_\theta^* \hat{I}[L] = 0$.

The second symmetry, the rescaling symmetry, is the other. There is a natural grading on $W = \widehat{Sym} T^*(X)[[\hbar]]$, say, give T^*X degree 1 (call a variable y^i) and $\hbar$ degree 2.

Fact 8.2. *The Fedosov equation respects the W-grading.*

So

$$\nabla I/\hbar) + \frac{1}{2}[I/\hbar, I/\hbar] + R/\hbar = \omega_\hbar/\hbar.$$

Let E be the Euler vector field $\sum y^i \frac{\partial}{\partial y^i} + 2\hbar \frac{\partial}{\partial \hbar}$. Then

$$\nabla \beta_I + \frac{1}{\hbar}[I, \beta_I] = 2\hbar \partial_\hbar(\omega_\hbar/\hbar)$$

Where $\beta_I = E(I/\hbar)$ and the quantum observable is

$$\hat{\beta}_I[L] e^{\hat{I}[L]/\hbar} = \lim_{\epsilon \to 0} e^{\hbar P_\epsilon^L} (\hat{\beta}_I e^{\hat{I}/\hbar})$$

Similarly,

$$\bar{\beta}_I[L] e^{\hat{I}[\infty]/\hbar} := \lim_{\epsilon \to 0} e^{\hbar P_\epsilon^L} \bar{\beta}_I e^{\hat{I}[\infty]/\hbar}.$$

The lemmas you need are

Lemma 8.1.

$$\hat{\beta}_I[L] = \iota_\theta^* \bar{\beta}_I[L]$$

Lemma 8.2.

$$(Q + \hbar\Delta_L + R/\hbar)(\bar{\beta}_I[L] e^{\tilde{I}[L]/\hbar}) = 2\hbar\partial_\hbar(\omega_\hbar/\hbar) e^{\hat{I}[L]/\hbar}.$$

[Sketch of proof, missed.] Anyway, let's try to work with this information, let's get the partition function. To calculate this partition function in this example, it's

$$\langle 1\rangle = \int_X e^{\hbar R_{\omega^{-1}}}[e^{\tilde{I}[\infty]/\hbar}]|_{y=0}.$$

This is hard to do, but let's see how it changes with $\hbar$. So consider $2\hbar\partial_\hbar$ of the integrand. This equals, basically, the same data

$$e^{\hbar R_{\omega^{-1}}}[\tilde{\beta}_I[\infty e^{\tilde{I}[\infty]/\hbar}]|_{y=0}$$

which is equal to

$$e^{\hbar}R_{\omega^{-1}}d\omega[]$$

Well, first, let me say a little bit, basically, $\tilde{\beta}_I[L]e^{\tilde{I}[L]/\hbar} - 2\hbar\partial_\hbar(\omega_\hbar/\hbar)e^{\tilde{I}[L]/\hbar}$ which is

$$Q + \hbar\Delta_L + R/\hbar + \iota_\theta^*](\bar{\beta}_I[L]e^{\hat{I}[L]/\hbar})$$

So above it's equal to

$$(2\hbar\partial_\hbar\omega_\hbar/\hbar)(e^{\hbar}R_{\omega^{-1}}[e^{\tilde{I}[\infty]/\hbar}] + e^{\hbar R_{\omega^{-1}}}(\nabla + \hbar\Delta_\infty + R/\hbar + d\omega)(—)$$

So

Lemma 8.3.

$$2\hbar\partial_\hbar(e^{-\omega_\hbar/\hbar}(e^{\hbar R_{\omega^{-1}}}[e^{\tilde{I}[\infty]/\hbar}])$$

is exact.

So there are two parts, there's the $\hat{A}$-genus and then higher correction terms. This follows from Grady-Gwilliam. so

$$\langle 1\rangle = \int_X e^{\omega_\hbar/\hbar}\hat{A}(X)$$

This is an algebraic index theorem (Fedosov, Nest-Tsygan). I think I'll stop here.

9. October 1: Katarzyna Rejzner, BV algebras in perturbative algebraic quantum field theory

Thank you so much and thanks for giving me a chance to talk. I find it really inspiring to listen to all these talks and hear the connections between what Kevin and Owen did and what I'm doing.

I want to give a flavor of the Lorentzian field theory. One hour is probably not enough time to really explain it but I'll make an attempt. So my talk, I'll start with some history, a very short introduction. This all is based on the ideas of Epstein-Glaser, from 73, so quite old stuff. The modern approach to this dates back to the 90s, I should mention my collaborators and friends who basically came up with this idea, K. Fredenhagen, R. Brunetti, and M. Dutsch. The idea that there are BV algebras, what I'll focus on, is 2011 work of mine with Fredenhagen.

Let me start with the basic setup to see how the objects I'm considering are related to the rest of the workshop. So I start with a spacetime in Lorentzian signature, a Lorentzian manifold. I'll assume it's four dimensional. We also impose that it has a Cauchy surface and we call such manifolds globally hyperbolic. This means that topologically our manifold M is just $\Sigma \times \mathbb{R}$. An example is

Minkowski space-time, which is $\mathbb{R}^4$ with the metric which is the diagonal matrix
$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

Now the field configuration space $\mathcal{E}(M)$ is a space of sections of some vector bundle over M, $\Gamma(E \xrightarrow{\pi} M)$. I'll focus on the very simple example of a scalar field. Here $\mathcal{E}(M)$ is $C^\infty(M, \mathbb{R})$, a scalar field. On the space of fields I want to have observables. In order to have these I'll first introduce functionals on $\mathcal{E}(M)$, these are elements of $C^\infty(\mathcal{E}(M), \mathbb{R})$, so I require them to be smooth. It relates very much to Owen's talk because if I take a Taylor expansion, I will get elements of $\prod_{n=0}^\infty Hom_{\text{cont}}(\mathcal{E}(M)^{\otimes n}, \mathbb{R})$.

Good. Now this is basically it about the objects of the theory. Now one needs some restrictions on the functionals, one cannot do anything with such a big space. So in particular we have

Definition 9.1. *A* local functional $\mathcal{F}_{\text{loc}}$ *is of the form*
$$F(\phi) = \int_M f(j^k(\phi))d\mu$$
where $j_x^k(\phi) = (x, \phi(x), \partial\phi(x), \ldots)$ *up to order* k*.*

I want products so I also take

Definition 9.2. Multilocal functionals $\mathcal{F}$ *are finite sums of finite products of local functionals.*

Fact 9.1. *The derivative of a multilocal functional is a compactly supported smooth function, not a distribution.*

There are a couple more definitions before we set off. We can be in a very nice situation where the functional derivative is actually a smooth function. Then we can define something like the support of a functional.

Definition 9.3. *A* support *of* $F \in \mathcal{F}$ *is the closure of the union of all supports of the derivatives of* F *:*
$$\overline{\cup_\phi supp(F^{(1)}(\phi))}$$

Definition 9.4. *A functional is* regular *if all its derivatives are smooth.*

Are there any questions at this point before I move on? Let me show you how the classical BV formalism arises here.

The first thing we need to define is an action. I'll use a certain version of the Lagrangian formalism. You can maybe already see a potential problem. If I take a Lagrangian density and try to integrate it, it won't make sense because it's not compact. To solve this, we introduce a spacetime cutoff for the functional. So we switch on the interaction in some finite region and we just have to choose this region large enough.

Definition 9.5. *An* action *is a map* $S : \mathcal{D} \to \mathcal{F}_{\text{loc}}$*.*

An example is
$$S_0 f(\phi) = \frac{1}{2}\int(-\nabla_\mu\phi\nabla^\mu\phi + m^2\phi^2)f d\mu.$$

This is an action. With this I want to define some dynamics. I say the Euler Lagrange derivative is some kind of limit of this action,

Definition 9.6. *The* Euler-Lagrange derivative *is defined as*

$$dS(\phi) = \lim_{f\to 1} S(f)^{(1)}(\phi).$$

More concretely, $\langle dS(\phi), h\rangle = \langle S(f)^{(1)}(\phi), h\rangle$ where $f = 1$ on the support of h.

What is dS? As you might expect, it's a section of the cotangent bundle to $\mathcal{E}$. I'm not really cheating. With quite natural definitions, you can introduce a topology on $\mathcal{E}$ such that $T\mathcal{E}$ is $\mathcal{E} \times \mathcal{D}$, and in this case $\Gamma(T^*\mathcal{E}) \cong C^\infty(\mathcal{E}, \mathcal{D}')$. So dS can be seen as a smooth function from the configuration space to $\mathcal{D}'$. Now if you want to go back to Kevin and Owen's framework, you just have to look at Taylor expansions of elements of such spaces.

Now I can finally define the equation of motion. When a talk has more definitions than theorems it's suspicious, but we do actually prove some things.

Definition 9.7. *The* equation of motion *is the condition $dS(\phi) = 0$. The space of solutions is $\mathcal{E}_S \subset \mathcal{E}$.*

Now I want to understand multilocal functionals on this space of solutions. So I'm interested in $\mathcal{F}_S = \mathcal{F}/\mathcal{F}_0$ where $\mathcal{F}_0$ are functionals which vanish on $\mathcal{E}_S$. There are some obvious candidates for such functionals. If I pair a vector field with the one-form dS I get an element of $\mathcal{F}_0$. So if I take $X \in \Gamma(T\mathcal{E})$, then $\langle dS, X\rangle$ vanishes on $\mathcal{E}_S$. This is the pairing between one-forms and vector fields. I need to make sure I obtain something in $\mathcal{F}$.

Let me define $V \in \Gamma(T\mathcal{E})$ which is the space of multilocal vector fields and essentially V is derivations of $\mathcal{F}$. In very lucky situations all of my $\mathcal{F}_0$ is of that form. So let's assume we're there. So let $\delta_S = \langle dS, \quad\rangle$, called ι_{dS} in Owen's talk, and this, well, I have vector fields and I have $V \to \mathcal{F}$ via δ_S. The image is $\mathcal{F}_0$. I can extend this with exterior powers of V. From my assumption it follows that $\mathcal{F}_S = H_0(\wedge V, \delta_S)$.

Right, so why is it good? If I want to quantize the theory, I want to somehow keep track of the equations of motion. Working with this quotient is messy, so it's easier to work with the elements of this differential graded algebra. Instead of quantizing $\mathcal{F}_S$ I'll quantize this differential graded algebra.

So the first question is, what's the kernel of δ_S in degree -1. I want to know which elements map to 0 under δ_S. So these are vector fields such that $\langle dS, X\rangle = 0$. Obviously the image of δ_S are in this kernel. I'll call such elements local symmetries and if all the elements in the kernel are of the form $\delta_S()$ of something in $\wedge^2 V$ then I'll say they're all symmetries. It remains a question if there is something from V which is not trivial.

In the standard example, Yang-Mills theory or even easier, electrodynamics, I have a Lie algebra on $\mathcal{F}$ and this induces me, well, in such a way that the action is invariant under this, and in this case I have elements in V that are not in the image but are in the kernel of δ_S.

I'll maybe summarize, non-trivial $H_1(\delta_S)$ means that there are non-trivial local symmetries. Let me start with the trivial case where there are no symmetries and this complex is already a resolution so the higher cohomology vanishes. For the scalar field $(\wedge V, \delta_S)$ is a resolution meaning that $H_k(\wedge V, \delta_S) = 0$ [for $k > 0$].

So $\wedge V$ is the space of polyvector fields. I have the Schouten bracket $\{\cdot,\cdot\}$ and finally arrive at a differential Gerstenhaber algebra, so $(\wedge V, \{\cdot,\cdot\}, \delta_S)$ is a differential Gerstenhaber algebra. Here the BV operator is the divergence of a vector field.

The problem is that if I try to define the divergence of a local vector field, I realize this is not well-defined. The naive approach, pretending everything is finite dimensional, I immediately run into trouble. But let me be optimistic and try to get as far as I can, forgetting the functional analysis I love so much.

For now I will only consider well-behaving funcitonals, so only $\mathcal{F}_{\text{reg}}$ because I want to show you some algebraic data. You guys aren't functional analysts, so I won't show you distributions. On this space I'll introduce certain products. For this I need some input from PDEs, dS_0 as an operator, where

$$S_0 f(\phi) = \frac{1}{2}\int(-\nabla_\mu\phi\nabla^\mu\phi + m^2\phi^2) f d\mu.$$

So this has distinguished Green's functions and bisolutions. I have the so-called Feynman propagator D_F which is some special symmetric Green's function. We also have the two-point function D_+ (the bisolution, not symmetric) and both of these objects are distributions on M^2.

I need these to, now I want to do some deformation quantization. So I deform the pointwise product in two ways. First I define a star product

$$(F \star G)\phi = \sum_{n=0}^{\infty} \langle F^{(n)}\phi, D +^{\otimes n} G^{(n)}\phi\rangle \frac{\hbar^n}{n!}$$

and the other thing is the time ordered product

$$(F \cdot_T G)\phi = \sum_{n=0}^{\infty} \langle F^{(n)}\phi, DF^{\otimes n} G^{(n)}\phi\rangle \frac{\hbar^n}{n!}.$$

Now define $\mathcal{T} = e^{\langle i\hbar D_F, \frac{\delta^2}{\delta\phi^2}\rangle}$; then

$$F \cdot_T G = \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G).$$

SO now the S-matrix is $S(V) = e_T^{\frac{iV}{\hbar}} = \sum(\frac{i}{\hbar}V)\cdot_T \cdots \cdot_T (\frac{i}{\hbar V})$. My problem is that $F \cdot_T G$ is well defined only for F and G with disjoint support. So the problem is to extend $\mathcal{T}$ and the time-ordered product to elements of $\mathcal{F}_{\text{loc}}$ with overlapping support. This is the Epstein-Glaser [Fire drill]

Let me not bother about minimization. In the classical theory I had δ_{S_0}. Now I can try to see how the S-matrix is behaving with respect to this differential. My idea is to extend from regular functionals to local functionals. One of the properties I want to keep is that $\partial_{S_0} e_T^{\frac{iV}{\hbar}} = 0$.

For $V \in \mathcal{F}_{\text{reg}}$ I can do this by computation, rewrite the exponential as

$$e_T^{\frac{iV}{\hbar}} \cdot_T \left(\frac{1}{2}\{V + S_0, V + S_0\}_T - i\hbar\Delta V\right)$$

This is a bit of a surprise. Here Δ is the BV Laplacian, a divergence operator on regular vector fields. You recognize that this is the quantum master equation. I don't want to go through the computation which is messy. Our approach is that this could be ill-defined extending to local functions, both the bracket and the product. Instead of trying to make sense of the structures on the right hand side, instead I work on the left hand side, treating $\delta_{S_0} e_T^{\frac{iV}{\hbar}}$ as the renormalization condition and define the right hand side by using this form.

Now when I want to go to the renormalized theory, I'll have to give you a definition of T on local functionals, but this automatically gives meaning to the right hand side.

Maybe to write one last formula, after the renormalization, I obtain $\mathcal{T}_r$ and a quantum master equation of the form $\frac{1}{2}\{S_0 + V, S_0 + V\}_{T_r} - i\hbar\Delta_{V,T_r}(V)$. This is still local, it plays essentially the same role as the original BV operator, but this one is well-defined on local functionals, but it depends on the choice of T_r, the way we renormalized our theory. This is to be expected because we have renormalization freedom and the non-uniqueness is governed by the renormalization group.

I guess this is a good moment to stop and thanks for the extra time.

10. Mikhail Movshev

I do not take notes at slide talks.

11. Maxime Zabzine, Localization for 5D Yang-Mills theory

I'd like to thank the organizers. I'm the first physics speaker of this workshop and will do things non-rigorously. It depends on your point of view. I assume there are more math people than physicists. I will try to explain the main idea. I hope that tomorrow Vasily will tell you the rest. First of all, let's again, so far, many people discussed perturbative quantum field theory. I'm preoccupied with something else. I want to consider non-perturbative stuff.

What we are trying to mimic from the finite dimensional setting is the Atiyah-Bott fixed point theorem. Imagine I have a $U(1)$ action on M, so $U(1)$ is realized by V and I have an equivariant differential $d_V = d + \xi\imath_V$. I'd like to use α with $d_V\alpha = 0$ and $\alpha = \alpha^{top} + \cdots + \xi^{\#}\alpha^0$.

I'd like to calculate $\int_M \alpha = \int_M \alpha^{top}$ and that is $\sum \frac{\alpha^0(p)}{\sqrt{det\ L_p}}$ where the sum is over the fixed points. This is just the determinant of a differential operator even if the left hand side is not defined. I'm not trying to prove Atiyah-Bott in an infinite dimensional setting, then you want to understand the idea to know what operator to use.

Let me sketch out how the proof goes. I have a coordinate x on my manifold M and then let me introduce θ^μ, which transforms as dx^μ, so basically I'm looking at the shifted tangent bundle $T[1]M$. If I look at the functions $C^\infty T[1]M$, this is $\Omega^\bullet M$, so this is $\alpha(x, \theta) = \sum \alpha_{\mu_1\cdots\mu_k}(x)\theta^{\mu_1}\cdots\theta^{\mu_k}$.

So $\delta x^\mu = \theta^\mu$ and $\delta\theta^\mu = \xi V^\mu(x)$.

Then $\delta\alpha$ is d_V, the equivariant differential on forms on M.

So say that $\delta\alpha = 0$. Then I would like to calculate $\int d^n x d^n\theta\alpha(x, \theta)$. I'll put here $e^{t\delta V}$ and call this thing $Z[t]$.

Theorem 11.1.

$$\frac{dZ}{dt} = 0 \leftrightarrow \delta^2 V = 0$$

by Stokes' theorem.

So if $\delta^2 V = 0$ then $Z[0] = \lim_{t\to\infty} Z[t]$.

The most convenient thing is to choose $V = -\theta^\mu g_{\mu\nu}(x)V^\nu(x)$. Then $\delta^2 V = 0$ if and only if $\mathcal{L}_V g = 0$.

So my leading terms are when $V^\mu(x) = 0$. The proof says, let me assume that the fixed point is zero, I'll have quadratic terms and higher terms. Now if I translate by t, then higher terms will be cancelled by higher powers of $\frac{1}{t}$. This is the proof of Atiyah-Bott.

The next thing, I can do some nice toy models. For example, my goal is to try to move to the infinite dimensional setting. Let me do a three dimensional example. Let me consider $S = \frac{k}{4\pi}\int_M Tr(AdA + \frac{2}{3}A^3)$. I'd like to calculate $Z = \int_{A/G} e^S DA$. So I can wrtie $\delta A = \Psi$. Then $\delta\Psi$ will go to some symmetries that I just have to invent. Let me assume I have some vector field, so I have $\mathcal{L}_V A + d_A\Phi$, the Lie derivative plus a gauge transformation. If I write $\Phi = i\sigma - i_V A$ then $\delta\Psi = i_V F + id_A\sigma$ and $\delta\sigma = -ii_V\Psi$.

Now, in principle, I don't have enough fields. Some fields are missing. My fermions in three dimensions should have four components. So I should introduce $\delta\chi = H$ and I introduce $\delta H = \mathcal{L}_V\chi - i[\sigma, \Psi]$.

If I'm doing this thing then $S_{SCS} = S_{CS}(A - ik\sigma) - \frac{k}{4\pi}\int Tr\kappa \wedge \Psi \wedge \Psi$.

Then $\delta S_{SCS} = 0$ if and only if $i_V\kappa = 1$, $i_V(d\kappa) = 0, \kappa \in \Omega^1(M)$.

I need $\mathcal{V}$ so that $\delta\mathcal{V}$ looks like $F \wedge *F + d_A\sigma \wedge *d_A\sigma + \cdots$ and this term will look like $\delta(\Psi \wedge *\overline{\delta\Psi} + \chi \wedge *(\frac{F\wedge\kappa}{\kappa d\kappa} - H))$.

This will always work if $\mathcal{V}$ is a Killing field, $\delta^2\mathcal{V} = 0$ and $\mathcal{L}_V(g) = 0$.

For $M = S^3$, $F = 0, d_A\sigma = 0$ we get $A = 0$ and σ a constant.

I have just one fixed point and look at the action.

$$Z_{S^3} = \int d\sigma e^{\frac{ik}{4\pi}} Tr\ \sigma^3 \frac{(det_{\Omega^0}\mathcal{R}_V + ad_\sigma)^{3/2}}{\sqrt{det\ \mathcal{L}_V + ad\sigma / \sqrt{det\ \mathcal{L}_V + ad\sigma r}}}.$$

[Too physical in tone for me. I made it halfway through, that's an improvement!]

12. Vasily Dolgushev, Part II

I promised I would give you a cooperad. So what's that, it's a collection $\{C(n)\}$ and instead of a multiplication it's equipped with a comultiplication

$$\Delta_t : C(n) \to C(n_1) \otimes \cdots \otimes C(n_r)$$

and a counit map $C(1) \to \mathbf{k}$. The easiest way to get cooperads is by dualizing operads. Take an operad in vector spaces for which $O(n)$ is finite dimensional for every n. Then the dual is naturally a cooperad.

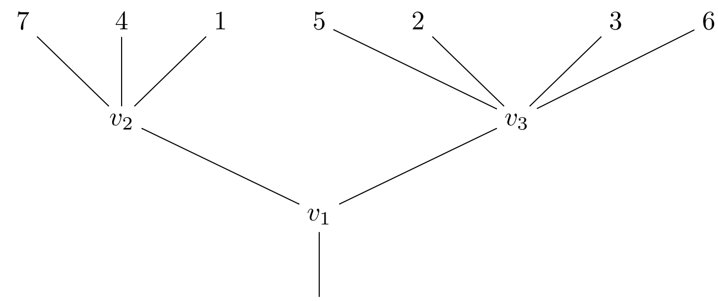
Now we can similarly define what a coalgebra over a cooperad is and many types of coalgebra are coalgebras over a cooperad. For instance, if you define $coAs = As^*$, then $coAs$-coalgebras are precisely coassociative coalgebras. If you take $coCom := Com^*$, then coalgebras are precisely cocommutative coassociative coalgebras. Similarly, *coLie* is obtained by dualizing Lie.

Remember we also had $*$ which was naturally an operad, it's also a cooperad, terminal in the category of cooperads. This allows us to define coaugmentation. We say that a cooperad is coaugmented if we have a morphism of cooperads $* \to C$, going *from* the terminal object to C. We still have the assumption that $C(0) = 0$ and $C(1) = \mathbf{k}$ (for simplicity during the lectures). Then the cooperad will naturally be coaugmented. We will also need the cokernel of the coaugmentation $coker(* \to C)$. This is just killing $C(1)$. For example if I take the cooperad $coCom$ and do this, I get $\mathbf{k}$ starting in arity 2.

When I do this, I get a cooperad but without counit. It's possible to define Δ_t.

So you can think of C_0 as a non-unital cooperad. If I have an operad O or a cooperad C and have a cochain complex V, then I'll denote by $O(V)$ the free O-algebra generated by V and by $C(V)$ the cofree C-coalgebra cogenerated by V. As graded vector spaces, $O(V) = \bigoplus (O(n) \otimes V^{\otimes n})_{S_n}$ and $C(V) = \bigoplus (C(n) \otimes V^{\otimes n})_{S_n}$.

I'd like to have free operads, so $\mathbb{OP}(P)$ will be the free operad generated by P. I'll think of $\mathbb{OP}(P)$ in arity n, represent these by labeled planar trees whose vertices are labeled by elements of P of the appropriate arity.

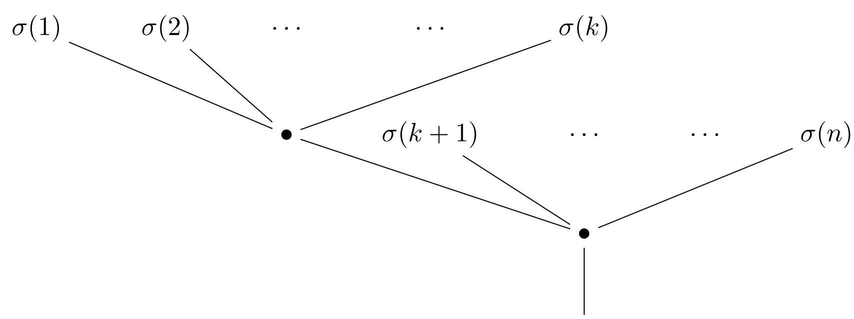


It's a pleasure to make maps out of a free operad because maps from $\mathbb{OP}(P)$ to O are determined uniquely by their image on the generators.

Now I'd like to remind you of the cobar functor. It's a functor from coaugmented dg cooperads to dg operads. If you forget differentials, it is $\mathbb{OP}(sC_0)$ (in the category of graded vector spaces). Now I have to tell you about the differential.

Let me assume for simplicity that there is no differential on C. The interesting part comes from comultiplication.

Let's say I have $2 \leq k \leq n-1$ and a $k, n-k$-shuffle σ, meaning $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(n)$. Then I make a tree t_σ:



Then the differential of sV is

$$-\sum_{k=2}^{n-1} \sum_{\sigma} (s \otimes s) \Delta_{t_\sigma} V$$

and it's an exercise to see that this squares to zero.

I'll need Λ, the endomorphisms of $s^{-1}\mathbf{k}$ and Λ^{-1}, which is $End_{s\mathbf{k}}$, so it's one dimensional with weird degrees and sign factors. The former has $\Lambda(n) = s^{n-1} \otimes sgn_{S_n}$ and $\Lambda^{-1}(n) = s^{1-n} \otimes sgn_{S_n}$. Both of these are operads and cooperads. I can tensor any operad with Λ and get another one. Similarly, I can tensor a cooperad and get a cooperad.

Theorem 12.1. *The natural maps $Cobar\Lambda^{-1}coAs \to As$, $Cobar\Lambda^{-1}coLie \to Com$, and $\Lambda^{-1}coCom \to Lie$ are quasiisomorphisms of differential graded operads.*

Here I'm thinking of the targets as operads with zero differential. It allows me to say that these are free resolutions of the operads governing associative, commutative associative, and Lie algebras. This allows me to say that A_∞-algebras are governed by $Cobar(\Lambda^{-1}coAs)$. Similarly, Com_∞-algebras are governed by $Cobar(\Lambda^{-1}coLie)$ and L_∞-algebras are governed by $Cobar(\Lambda^{-1}coCom)$.

Let me recall a proposition about $Cobar$

Proposition 12.1. *(Getzler-Jones; Ginzburg-Kapranov)*
Let V be a cochain complex. Then $Cobar(C)$-algebra structures on V are in bijection with degree one coderivations Q of $C(V)$ satisfying two conditions: restricted to V I get zero and $\partial Q + \frac{1}{2}[Q,Q] = 0$, the Maurer-Cartan equation.

Instead of proving this proposition, let me say *why* it should be true. What does it mean? We have a morphism $Cobar(C) \to End_V$. If we forget about differentials for a moment, we have an operad morphism $\mathbb{OP}(sC_0) \to End_V$, which is a degree zero map $(sC_0(n) \otimes V^{\otimes n})_{S_n} \to V$ or a degree one map $(C_0(n) \otimes V^{\otimes n})_{S_n} \to V$. On the other hand a coderivation is determined by its projection onto V:

$$\begin{array}{ccc} Q : C(V)\circ Q & \longrightarrow & C(V) \\ & \searrow^{p} & \downarrow{\scriptstyle p} \\ & & V \end{array}$$

Then the differential condition gives me the Maurer-Cartan equation.

[argument with Bruno about pedagogy]

This proposition tells me that every $Cobar(C)$-algebra A gives me a dg C-coalgebra $(C(A), \partial + Q)$, and this motivates the definition of ∞-morphism.

Definition 12.1. *An ∞-morphism U from A to B is a homomorphism of C-coalgebras which is compatible with these differentials.*

Since U lands on a cofree coalgebra, it is uniquely determined by its projection onto B:

$$C(A) \xrightarrow{U} C(B) \xrightarrow{p} B$$

and the *linear term* $(p \circ U)|_A$ is a chain map. We say that an ∞-morphism $U : A \to B$ is an ∞-quasi-isomorphism if the linear term is a quasiisomorphism of chain complexes.

If you write down compatibilities with the differentials, you see that U_1 is almost compatible with the operad structure, it's compatible up to homotopies from the other terms.

Let me just say that with all these examples, it justifies the definition that given a cooperad C we say that A is a homotopy algebra of type C if A is a $cobar(C)$-algebra. For my lectures I will use this pedestrian idea of homotopy algebra, that they are algebras over cobar of something.

This is also partially motivated by the homotopy transfer theorem.

Theorem 12.2. *I'll still have C and let $A \xrightarrow{\varphi} B$ be a quasiisomorphism of cochain complexes. Let B be a $cobar(C)$ algebra. Then there exists a $Cobar(C)$-structure on A and an ∞-morphism $U : A \to B$ such that the linear term of U is φ.*

Then you can ask if this is unique and it's unique up to homotopy.

Let me talk a little bit about shifted L_∞ algebras.

Definition 12.2. *A shifted L_∞-algebra structure on a cochain complex L is a $Cobar(coCom)$-structure.*

Due to the facts I already explained to you, this is the same as a coderivation of degree one $\underline{S}(L) \to \underline{S}(L)$ satisfying the Maurer-Cartan equation.

Here I didn't say that $\underline{S}(L)$ is the truncated symmetric algebra, viewed though as a symmetric *coalgebra*. This is the same thing as a collection of maps, I have a unary bracket $\{v\}$ which is the differential. I have higher brackets $\{\ldots\}L^{\otimes n} \to L$ which are graded symmetric and satisfy the relation

$$\sum_{k=1}^{n} \sum_{\sigma \in Sh_{k,m-k}} \pm \{\{v_{\sigma(1)}, \ldots, v_{\sigma(k)}\}_k, v_{\sigma(k+1)}, \ldots, v_{\sigma(m)}\}_{m-k+1} = 0$$

The simplest example is $s^{-1}\mathfrak{g}$ for a Lie algebra $\mathfrak{g}$.

From now on I won't use the word shifted.

Definition 12.3. *I will say that an L_∞-algebra L is* filtered *if it has a descending filtration*

$$L = \mathcal{F}_1 L \supset \mathcal{F}_2 L \supset \cdots$$

so that $L = \lim L/\mathcal{F}_n L$.

Let me assume that my multibrackets are compatible with the filtration, so that $\{\mathcal{F}_{m_1} L, \ldots, \mathcal{F}_{m_r} L\} \subset \mathcal{F}_{m_1 + \cdots + m_r} L$.

Now when I'm talking about ∞-morphisms of L_∞-algebras, I'm talking about coalgebra homomorphisms compatible with the differentials. All such maps are determined by projections to the cogenerators. You may ask what I require with respect to the filtration. Sometimes I want to require it to be strict like it is for the brackets and sometimes only continuous with respect to the filtration.

I want to define a Maurer-Cartan element and I don't want to worry about convergence, that's why I want to use this kind of filtration.

Definition 12.4. *A* Maurer-Cartan element *of L is a degree zero (sorry about the shift) element α such that*

$$\partial\alpha + \sum_{m=2}^{\infty} \frac{1}{m!}\{\alpha, \ldots, \alpha\}_m = 0.$$

If I have a filtered L_∞ algebra and a Maurer-Cartan element α, then I can form another L_∞ algebra L^α which has the same underlying wector space and

$$\partial^\alpha(v) = \partial(\alpha) + \sum_{m=1}^{\infty} \frac{1}{m!}\{\alpha, \ldots, \alpha, v\}_{m+1}$$

and

$$\{v_1, \ldots, v_m\}_m^\alpha = \sum_{k=0}^{\infty} \frac{1}{k!}\{\alpha, \ldots, \alpha, v_1, \ldots, v_m\}_{k+m}$$

This is the L_∞ algebra L *twisted by α*. So if I have a map $U : L \to \tilde{L}$ and a Maurer-Cartan element $\alpha \in L$, then $U(\alpha)$ is generally not a Maurer-Cartan element, but I can define

$$U_*(\alpha) = \sum_{m=1}^{\infty} \frac{1}{m!} p \circ U(\alpha^m)$$

and *this* is a Maurer-Cartan element.

Now let's say I have a map $U : L \to \tilde{L}$ and $\alpha \in L$. Can we modify U and get an ∞-morphism $L^\alpha \to \tilde{L}^{U_*\alpha}$. I say

$$p \circ U^\alpha(v_1, \dots v_m) = \sum_{k=0}^{\infty} \frac{1}{k!} p \circ U(\alpha^k v_1, \dots, v_k).$$

Let's continue tomorrow.

13. October 2: Vasily Dolgushev, Part III

For every filtered L_∞ algebar L we have a functor $L \mapsto MC(L)$ from L_∞ algebras to simplicial sets. Let me attribute this to Getzler and Hinich.

Now to talk about this, let me introduce the de Rham algebra of polynomial forms on Δ_n. This is $\Omega^\cdot_n = \mathbf{k}[t_0, \dots, t_n, dt_0, \dots dt_n]$ where this is *graded* commutative, with the degrees of t_i zero and degrees of dt_i one, modulo the ideal generated by $t_0 + \dots + t_n - 1, dt_0 + \dots + dt_n$. So this has a differential where $d(t_i) = dt_i$ and $d(dt_i) = 0$. So for instance $\Omega^\cdot_0 = \mathbf{k}$. Or $\Omega^\cdot_1 = \mathbf{k}[t, dt]$ with the differential sending t to dt.

Now when I have a filtered L_∞ algebra, then for every n I form another filtered L_∞ algebra by tensoring with Ω_n. I need to use the completed tensor product, since L has a filtration. Now we can define the simplicial set $\{MC_n(L)\}$ which is a simplicial set, $MC_n(L) := MC(L \hat{\otimes} \Omega_n)$.

This upgrades the assignment of Maurer Cartan elements.

I should have told you that the collection of Ω_n is a simplicial object in the category of differential graded commutative algebras.

If you have an L_∞ morphism $F : L \to \tilde{L}$ then it is easy to construct $F_* : MC_*(L) \to MC_*(\tilde{L})$, using the formula from last time $\alpha \in L \mapsto F_*(\alpha) = \sum_{m \geq 1} \frac{1}{m!} p \circ F(\alpha^m)$. So we have a functor to simplicial sets.

Zero simplices are precisely Maurer-Cartan elements of L. What about one-simplices? $MC_1(L)$ consists of expressions $\beta = \beta_0(t) + dt\beta_1(t)$ where $\beta_0(t) \in L^0 \hat{\otimes} \mathbf{k}[t]$ and $\beta_1(t) \in L^{-1} \hat{\otimes} \mathbf{k}[t]$.

So the Maurer-Cartan equation implies two things: first $\beta_0(t)$ is Maurer-Cartan, so

$$\partial\beta_0(t) + \frac{1}{2}\{\beta_0(t), \beta_0(t)\} + \dots = 0$$

and

$$\frac{d}{dt}\beta_0(t) = \partial\beta_1(t) + \sum_{k=1}^{\infty} \frac{1}{k!}\{\beta_0(t), \dots, \beta_0(t), \beta_1(t)\} = \partial^{\beta_0(t)}\beta_1(t)$$

So we say that α and α' in $MC(L)$ are connected by a one-cell if there is a $\beta \in MC_1(L)$ such that $\beta_0(0) = \alpha$ and $\beta_0(1) = \alpha'$.

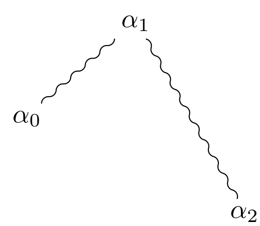
If Maxim is listening he will think that this is a very perverse way of thinking about gauge-equivalent flat connections.

There are a couple of important theorems about this simplicial set.

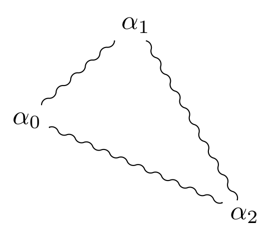
Theorem 13.1. *(Getzler, Hinich, Chris Rogers*[1]*, D.) For every filtered L_∞-algebra L, the simplicial set $MC_*(L)$ is a Kan complex, a fibrant object in the category of simplicial sets.*

This is nice because now we can think about $MC_*(L)$ as an infinity groupoid. I say DGH ∞-groupoid for Deligne-Getzler-Hinich.

If you have never heard about the Kan condition on simplicial sets, let me mention a consequence. The relation $\alpha \sim \alpha'$ is an equivalence relation. It wasn't obvious that this was an equivalence relation. This is because if I have two one cells



then there is a 2-cell spanning these so you have a one-cell between α_0 and α_2:



Now let me give another theorem, a version of a Goldman-Millson theorem

Theorem 13.2. *If $F : L \to \tilde{L}$ is an ∞-quasiisomorphism of filtered L_∞ algebras compatible with filtrations in an honest way (strictly), then $F_* : MC_*(L) \to MC_*(\tilde{L})$ is a weak equivalence of simplicial sets*

This means that $\pi_0 MC_*(L)$ is in bijection with $\pi_0 MC_*(\tilde{L})$ and all higher homotopy groups are isomorphic.

What I particularly like about the proof we wrote is that it is in some sense constructive. Say you have $F : L \to \tilde{L}$ and you have $\beta \in MC(\tilde{L})$, then you wonder if there is α such that $\beta \rightsquigarrow F_*(\alpha)$ by a one-cell. We construct it iteratively step by step on $L/\mathcal{F}_n L$.

So for example, let A be an associative $\mathbf{k}$-algebra and ϵ a formal deformation parameter. Then I can consider $L_A = C^*(A, A)$. Well, then I will adjoin ϵ and consider $L_A = \epsilon C^*(A, A)[[\epsilon]]$. So then $MC(L_A)$ are formal associative deformations of A and if α is connected to α' by a one-cell, then, well, it's if and only if the corresponding formal deformations are equivalent.

To finally answer the question, what do homotopy algebras form, I need to remind you that L_∞ algebras form a symmetric monoidal category. If I have L and $\tilde{L}$, then $L \times \tilde{L} = L \oplus \tilde{L}$. The bracket is defined in the most obvious way:

$$\{v_1 \oplus \tilde{v}_1, \dots v_n \oplus \tilde{v}_n\} = \{v_1, \dots v_n\} \oplus \{\tilde{v}_1, \dots, \tilde{v}_n\}$$

[1] I'll put a green color so he'll have a green light for a job

So if I have two L_∞ morphisms F, G then I should be able to construct $F \oplus G$. To do this, note that $S(L_1 \oplus L_2) = S(L_1) \otimes S(L_2)$. Extend using $F(1) = 1$ and $G(1) = 1$.

I need an enhanced version of a morphism from L to $\tilde{L}$. Let me consider instead of just an L_∞ morphism, let's consider pairs, a morphism in $Lie_\infty^{\mathrm{Enh}}$ is a pair (F, α) where α is a Maurer-Cartan element of $\tilde{L}$ and $F : L \to \tilde{L}^\alpha$.

You may wonder how to compose. If I have L_1, L_2, and L_3 and I have α and β Maurer-Cartan elements in L_2 and L_3. Then I have an L_∞ morphism F from $L_1 \to L_2^\alpha$ and another $G : L_2 \to L_3^\beta$. Then I can twist to get $L_2^\alpha \overset{G^\alpha}{\rightsquigarrow} L_3^{\beta+G_*(\alpha)}$. Then the resulting pair is $G^\alpha \circ F, \beta + G_*(\alpha)$.

So $G_*(\alpha)$ is a Maurer-Cartan element in L_3^β, so

$$\partial^\beta G_*(\alpha) + \sum_{m=2}^{\infty} \frac{1}{m!} G_*(\alpha), \ldots G_*(\alpha)\}^\beta = 0$$

and if you take this apart, it's the Maurer-Cartan equation for the sum.

You can upgrade this to a symmetric monoidal category. This is based on the observation that if you have α and α' Maurer-Cartan elements in L and L', then $\alpha \oplus \alpha'$ is Maurer-Cartan in $L \oplus L'$.

Theorem 13.3. *(Chris Rogers, D.) Every category enriched over $Lie_\infty^{\mathrm{Enh}}$ can be integrated to a simplicial category (enriched over simplicial sets)*

So $map(A, B)$ is a shifted L_∞ algebra so we just take $map(A, B)$ to $MC_*(map(A, B))$.

So let me now fix a cooperad C, it can be dg. I'll still assume that $C(0) = 0$ and $C(1) = \mathbf{k}$.

Theorem 13.4. *(Alex Hoffman, Chris Rogers, D.)*
The category of $cobar(C)$-algebras is enriched over $Lie_\infty^{\mathrm{Enh}}$.

That's not a nice way to say it.

Let me tell you about the mapping spaces. Say A and B are $cobar(C)$ alegbras, so Q_A is a coderivation on $C(A)$ and Q_B is a coderivation on $C(B)$, I'd like to define a mapping space. The maps are $Hom(C(A), B)$. What are the multibrackets? If $f_i : C(A) \to B$, then, well $\Delta_m(X) = (C(M) \otimes C(A)^{\otimes m})^{S_m}$ and my bracket is

$$\{f_1, \ldots, f_n\}(X) = Q_B((1 \otimes f_1 \otimes \cdots \otimes f_m)\Delta_m(X)).$$

The composition is defined by a nice formula. So say I have A_1, A_2, and A_3 three $cobar(C)$-algebras, then I have to define a composition U_{comp}, a map

$$\underline{S}(map(A_2, A_3) \oplus map(A_1, A_2)) \to map(A_1, A_3).$$

Let me remind you that $map(A_2, A_3) = Hom(C(A_2), A_3)$ and $map(A_1, A_2) = Hom(C(A_1), A_2)$. So I sholud be able to go from

$$\underline{S}(Hom(C(A_2), A_3) \oplus Hom(C(A_1), A_2))$$

to $Hom(C(A_1), A_3)$. So if I have $g \in Hom(C(A_2), A_3)$ and $f_i \in Hom(C(A_1), A_2)$, with $X \in C(A_1)$, then I comultiply X with Δ_n, and then I apply $1 \otimes f_1 \otimes \cdots f_n$, and then I apply g to the result.

How is this related to the ∞-morphisms we discussed before? Well

Theorem 13.5. *(Dotsenko, Poncin, S.M.-B.V.?)*

For each pair of ∞-algebras A and B, we have $MC(Hom(C(A), B))$ are in bijection with ∞-morphisms.

Two ∞-morphisms are homotopy equivalent if and only if there is a one-cell connecting them. After this definition I feel like there are no questions.

You can ask what happens if I apply Getzler-Hinich to the category of all of these.

Theorem 13.6. *(Alex Hoffman, Chris Rogers, D.) We have that π_0 of the simplicial category is the homotopy category of homotopy $(Cobar(C))$-algebras.*

Any questions?

So if we look at the category of ∞-cooperads, then for every object of this category, every zero-cell we have a simplicial category of the corresponding homotopy algebras. It's interesting to understand the corresponding categorical structure, and that's what Chris Rogers and Brian Paljug are working on. So you see Brian is green here too because I want him to have a green light for a postdoc.

14. Claudia Scheimbauer, Factorization homology as a fully extended TFT

So first of all, thank you for the invitation to talk here, it is a pleasure. Today I'll talk about the main results of my thesis, to say that Factorization homology is a fully extended TFT. I'll be using some of the tools we've heard so far, so hopefully it won't be too far off. This is joint work with my advisor Damien Calaque.

Let me remind you what a TFT is.

Definition 14.1. *An n-dimensional TFT is a symmetric monoidal functor $\mathcal{F}$ from $Cob_n \to \mathcal{C}$*

This means we take a disjoint union to some tensor product. What is Cob_n? The objects are $(n-1)$-dimensional closed manifolds and morphisms M, N are n-dimensional manifolds with boundary, which has two components, incoming identified with M and outgoing with N. We have to take diffeomorphism classes to make composition work and then we have a category.

The first question mathematicians ask is are there examples, and the second question is for a classification.

A classification for $n = 1$ is easy. We're looking at one-dimensional manifolds and we have classification theorems there. Usually, or most commonly, $\mathcal{C}$ is $Vect_{\mathbf{k}}$. The classification tells us that $\mathcal{C}$ is determined by $F(*)$. This should be a finite dimensional vector space and every such finite dimensional vector space gives us a TFT. An object in $\mathcal{C}$ has finiteness conditions.

For $n = 2$ there is also a classification, but as soon as we start going up in dimension, you want a classification but you don't have one for manifolds. These were decomposition theorems for manifolds. We cut up our manifold into elementary pieces. For $n > 2$ there is no real decomposition theorem. You might also want to cut into smaller pieces, but you'll need not just $n-1$ dimensional guys but also $n-2$ and so on, getting smaller things as well.

Originally we only included data of $(n-1)$-dimensional guys but it becomes apparent that you will need higher codimension and now you need higher categories.

This was introduced by Freed and formalized in a more precise way by Baez-Dolan, who made the cobordism hypothesis, that there should be a classification of these using higher category theory. But n-categories are not easy objects. Things become difficult very fast. The big step, done by Lurie, was to introduce homotopy into the game, use ∞, n-categories.

Let me give a definition first.

Definition 14.2. *Let $Bord_n^{fr}$ be an $(\infty - n)$-category of bordisms. A* fully extended *n-dimensional TFT is a symmetric monoidal functor $F : Bord_n^{fr} \to \mathcal{C}$ of (∞, n)-categories.*

The framed could be other structures, you could get oriented, unoriented, whatever you want.

For this version you can write a classification theorem.

Theorem 14.1. *(Lurie, cobordism hypothesis) $ev : Fun^{\otimes}(Bord_n^{fr}, \mathcal{C}) \to \mathcal{C}$ where $F \mapsto F(*)$. That is, $Fun^{\otimes}(Bord_n^{fr}, \mathcal{C}) \to \mathcal{C}$ factors through a category of "fully-dualizable objects" C^{fd}, and this is an equivalence of ∞-groupoids.*

So identify you target category and pick a fully dualizable object, that's your TFT. If you pick any bordism and want to know what your TFT assigns to it, you want to know what it does, that's maybe not easy.

So one might ask for examples without using the cobordism hypothesis. So the goal is to write down a TFT and explicitly construct it without recourse to this theorem.

So what do we have to do to construct such a thing. We need a bordism category, a target, and then the functor and show that it's symmetric monoidal.

I'll start by talking about the target. As the target, we want to have something which is a category of E_n-algebras. It's an (∞, n)-"Morita"-category of E_n-algebras.

My objects are E_n-algebras. For $n = 1$ these are A_∞ algebras, for $n = 2$ braided categories are prominent examples. So I have E_n-algebras in a ground category S.

My morphisms are bimodules E_n-algebras, my 2-morphisms, to be bimodules between bimodules, and so on. What we will want, we have a point goes to an E_n-algebra. To a one-dimensional guy bimodules between the E_n-algebras, and then a 2-morphism, here is the central picture.

For $n = 1$ we have algebra with bimodules. I say Morita because the homotopy category will be the Morita category.

You can think of an E_2 algebra as an E_1 algebra in E_1-algebras. The bimodule between bimodules will be a bimodule in the second structure.

Okay, so so far this is rather mysterious, but this is where somehow for me, I know operadic people will object, I should mention E_n is the little disk operad. A very useful tool for modeling this target category is factorization algebras. Using these you can get a very clean description of this target category.

Our ground category will be chain complexes. We've seen, explained by Kevin, that factorization algebras are the structure that a field theory has, so it's not surprising that such things should appear.

Definition 14.3. *Let X be a topological space. A factorization algebra with values in S is a functor from open sets of X to S, and for every disjoint union of open sets, I have maps $\mathcal{F}(U_1) \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$, where V contains the union, such that, there are coherence conditions so that the composition of structure maps of little guys into middle guys followed by structure maps from the middle guys into the big* *ones should be the same as the map straight from the little guys to the big guys:*

$$\begin{array}{ccc} \prod\prod \mathcal{F}(U_i) & \longrightarrow & \prod \mathcal{F}(V_i) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}(W) & \end{array}$$

This is a prefactorization algebra, and the gluing condition that I won't go into says that the value on larger open sets can be recovered from the value on an open cover.

For example, let $X = \mathbb{R}$ and A an associative algebra. For an interval I assign A and for a bigger interval I also assign A. If I include U and V into a bigger interval, then I need $\mathcal{F}(U) \otimes \mathcal{F}(V) \to \mathcal{F}(W)$ and this is $A \otimes A \to A$ and that's multiplication. By this definition you can extend this (I claim) to a factorization algebra. This is an example of a type called *locally constant*. This means that if I have an inclusion of U into V which is a weak equivalence then their values under $\mathcal{F}$ are a weak equivalence. This is locally constant.

Conversely, I can start with a locally constant factorization algebra on $\mathbb{R}$ and define A to be $\mathcal{F}(\mathbb{R})$ and now by reading the same diagram, I see that an open interval sits inside $\mathbb{R}$ via a weak equivalence, so this means I have this A by local constancy. By a homotopy transfer theorem, this induces an A_∞ structure.

So what I've sketched for you is that (E_1)-algebras and factorization algebras on $\mathbb{R}^1$ are the same thing. Lurie showed that this is an equivalence of $(\infty, 1)$-categories. That holds replacing 1 with n, and I'll just work with locally constant factorization algebras.

Example 14.1. *For A, B associative algebras and M an (A, B)-bimodule let me take $X = \mathbb{R}$ and $p \in \mathbb{R}$. On an open interval including p I put M, on an open interval left of p I put A, and on an open interval to the right of M I put B. This is not locally constant, but it's very special, namely what's called* locally constant with respect to the stratification

Like we did before, you can start with a structure which is locally constant with respect to a stratification and read upwards to get a homotopy bimodule. So homotopy bimodules are roughly in bijection with factorization algebras locally constant with respect to this stratification. We also have these bimodules as *pointed*.

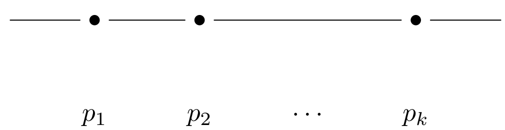
Now we have all the incredients to model this ∞, n-category for $n = 1$.

Let's look at what we want to do for $n = 1$. If we look above we want objects to be E_1-algebras and morphisms bimodules. So how do we model these? We chose the model of complete Segal spaces. For technical reasons, I will choose a diffeomorphism between $(0, 1)$ and $\mathbb{R}$.

What does a complete Segal space look like? We start with a space of objects $(Alg_1)_0$ which are algebras. These are locally constant factorization algebras on $\mathbb{R}$. I also need a space of 1-morphisms, which will be factorization algebras locally constant with respect to the stratification.

Now I get source and target maps here. A one-morphism induces a locally constant factorization algebra left of the point and another one right of the point. So $\mathcal{F}|_{(0,p)}$ is my source and $\mathcal{F}|_{(p,1)}$ is my target.

A Segal space consists of more, we have a whole simplicial space of guys. What will my kth guy be? I have factorization algebras locally constant with respect to this stratification



You compose, I have two maps from $(Alg_1)_2$ to $(Alg_1)_1$ by forgetting. I compose by rescaling down an interval, and you take A, M, B, N, C to $A, M \otimes_B N, C$.

This turns out to be a complete Segal space. You can come up with pictures for the higher versions but because of a lack of time I won't do that.

Remember that our goal was to come up with a functor from $Bord_n^{fr}$ to Alg_n, a factorization algebra on $(0,1)^n$. The tool is factorization homology, John Francis, David Ayala, many other people. What we want to do here is, we start with a manifold, an E_n-algebra, we take an n-manifold and I want to think of this as a locally constant factorization algebra on $\mathbb{R}^n$, I think of a patch where I'm gluing in my E_n algebra. Then I will patch them together to give a factorization algebra on M. The framing tells you how you can patch them together.

Formally, you have an E_n-algebra, I said this was locally constant, so it's a functor from $Open(\mathbb{R}^n) \to S$, so from this functor you can extract a functor from $Disk_n^{fr} \to S$. This is a topological category with objects disjoint unions of $\mathbb{R}^n$ and morphisms framed embeddings. You can say an E_n-algebra is a functor out of this. On the other hand, starting with a framed manifold, I can produce a functor $(Disk_n^{fr})^{op}$ to spaces, where $\sqcup \mathbb{R}^n$ goes to $Emb^{fr}(\sqcup \mathbb{R}^n, M)$.

I need one more (technical) condition, which is that S is "tensored over spaces." Then I can define factorization homology to be $\int_M A$, the coend of these two functors $\underline{M} \otimes_{Disk_n^{fr}} \mathcal{A}$.

The assignment $U \to \int_U A$ is a locally constant factorization algebra on M.

But my target was factorization algebras on $(0,1)^n$. This is where the bordism category comes in. Let's look at our source $Bord_n$. Look at $M, I_0 \subset \cdots \subset I_k$ such that M is a closed one dimensinal submanifold of $\mathbb{R}^\infty \times (0,1)$ such that the projection π to $(0,1)$ is proper. Then that projection should also have only regular values in $\cup I_i$. If I cut my $(0,1)$, I get a cobordism between the preimage of 0 and the preimage of 1. I choose collars where I cut and this gives a bordism. This can be made into a complete Segal space and without telling you much what this is, the essential part is the map π. I have an embedded submanifold of $\mathbb{R}^\infty \times (0,1)^n$ and a projection to $(0,1)^n$ I can push forward along π the factorization algebra A, $\pi_* \int_M A$ will not have the nice properties I want, though.

For example, for A associative, looking at what happens over a point with a projection from a circle, I get $A \otimes A^{op}$ when my preimage is two points and I get A as a bimodule and over the other parts I get $\mathbf{k}$. This isn't locally constant or anything, but I can do my rescaling thing to make it locally constant. In the middle is where the actual thing is happening. So we pushforward again along ρ. This really is a well-defined element in Alg_n, everything works well and this is the functor.

I'm already over time, but in this example, we take the global sets $\mathbf{k}$ over the ends and over the middle we get global sections of the factorization algebra, and this is $A \otimes_{A \otimes A^{op}} A^{op}$.

15. Vasily Pestun, Supersymmetric localization and equivariant cohomology of quantum field theories

I did not attend this talk.

16. Chris Elliott, Nonperturbative Descriptions for Twists of Classical Field Theories

Thank you very much for inviting me to speak. This is based on joint work in progress with Philsang Yoo.

I'm mostly going to concentrate on the perturbative, at least for the first half. I'm going to talk first in a formal neighborhood. If you prefer I can write local and global.

We want to understand twists of $N = 4$ theories, that's our motivation.

For example, $N = 4$ super Yang-Mills in 4d admits a family of topological twists, a $\mathbb{CP}^1$ family parameterized by $(\lambda Q \mu)$ and the germs of the solutions to the equations of motion in $(1 : 0)$ nad $(0 : 1)$ twists near $\Sigma \times S^1$ form, say,

$$EOM_A(\Sigma \times S^1) = T^*(L\ Bun_G(\Sigma))_{dR}$$

(holomorphic bundles, de Rham stack) and

$$EOM_B(\Sigma \times S^1) = T^* Loc_G(\Sigma \times S^1).$$

This is motivated by some work on geometric Langlands. This is a proof of concept, a sanity check for some relationship there.

I'll get much less technical and abstract now. Let me say some concrete things about what I mean by a twist. The idea goes back to Witten in the eighties, where he recovers Donaldson from $n = 2$ theories. You're cooking up a nice fermionic symmetry and looking at the Q-cohomology.

The kind of data that we'll need. Let me say the data of a classical field theory. This is the local or formal version, I'd say perturbative but I've been put off once already.

A formal classical field theory on a manifold X is a sheaf of elliptic L_∞ algebras on X, with some technical condition that I won't write, equipped with a pairing $E \otimes E[3] \to Dens_X$, nondegenerate, invariant, symmetric, and bilinear.

This is like the classical BV-complex. This is the complex, start with some Φ, and then look at polyvector fields equipped with $d_\Phi + \iota_{dS}$.

Alternatively, you could consider the derived critical locus of S. I mean you take the intersection in the cotangent bundle of Φ between the zero section and the graph of dS in the derived sense.

Take the shifted tangent complex $T_p[-1]dCrit(S)$.

The data we need to get a twist is a [unintelligible] supergroup.

Define a supergroup $H = \mathbb{C}^\times \ltimes \prod \mathbb{C}$ where the right $\mathbb{C}$ has weight 1

Definition 16.1. Twisting data *for a (formal) classical field theory is an action of H.*

There's an equivalence between super vector spaces with H-action and super cochain-complexes.

If we have a SUSY action, we can get an H-action from the following two things: an odd symmetry Q such that $Q^2 = \Gamma(Q, Q) = 0$. A $\mathbb{C}^\times \subset G_R$, this preserves the bosonic piece. We find a $\mathbb{C}^*$ inside there so that Q has weight one. This gives an action.

[missed some]

So what does it mean to twist? I can describe a functor, this equivalence will go from a super cochain complex $\mathcal{E}$, we can consider $F(\mathcal{E})$ (F is the functor to super cochain comlexes) and then we can take the total complex with respect ot the two complex cochain structures.

Definition 16.2. *A twist $\mathcal{E}^Q$ of $\mathcal{E}$ is the sheaf obtained by applying $T \circ t \circ Ft \circ \mathcal{E}(U)$ locally on X, at least.*

Claim 16.1. *This is still a clasical field theory.*

Our first idea is just to take the $\Pi\mathbb{C}$ invariants.

These exact symmetries act trivially. We can say what these invariants are in general. Assuming Q acts linearly, this looks like

$$\mathcal{E} \otimes \mathbb{C}[[t]],_\epsilon + tQ$$

Instead, we could try to just extract the fiber. This lives over the odd formal disk. What we can do is restrict to the formal punctured disk and then take $\mathbb{C}^*$-invariants.

There's a very good reason to try to do this. One reason to do this, if we took $\mathbb{C}^*$-invariants without doing this restriction, what we'd get is we'd get elements of $\mathcal{E}$ of the form ϕt^k where ϕ has weight $-k$ and the total thing is invariant. So in particular we'd be throwing away everything of positive weight.

In particular, we find that anything q-exect [[unintelligible]].

I have time so let me do an example. The simplest example is something coming from $N = 1$ gauge theory in four dimensions. This will be the theory that Kevin discusses. There are no topological twists in this sense but there's a holomorphic twist. One could write down [missed]. We could look at $N = 1$ holomorphic Chern Simons on the twistor space.

[example].

The classical BV complex for ASD Yang-Mills is obtained by pushing forward the classical formula for holomorphic Chern Simons:

$$\Omega^{0,*}(\mathbb{PT}'; \mathfrak{g}_P)\bar{\partial}]$$

What's more,

There's a supersymmetric version of this too. For the $N = 1$ theory, that version, instead, work with $\mathbb{CP}^3$, well,

$$\Omega^{0,*}(\mathbb{PT})'; \Pi\mathcal{O}(-1) \oplus \mathcal{O}) \otimes \mathfrak{g}_{\oplus}\mathcal{O} \oplus \Pi\mathcal{O}(1) \times K_{P\mathbb{T}} \otimes \mathfrak{g}_P$$

. [Missed some] The twisted complex is $\Omega^{0,*}(\mathbb{PT}'; \mathcal{O}(-1)[-1] \to 0) \otimes \mathfrak{g}_P \oplus (\mathcal{O} \to \mathcal{O}(1)[-1]) \otimes k \otimes \mathfrak{g}_P \cong \omega^{0,*}(\mathbb{PT}'; \mathcal{O}_{Z(Q)} \otimes \mathbf{k} \otimes \mathfrak{g}_P$. [missed] We want to define the twist of the derived critical locus given an H-action compatible with these tangent complexes.

I should say exactly what objects I'm talking about. Let X and Y be derived schemes. These should basically be functors from nonpositively graded commutative dgas to Set.

I don't have a better name but then we can talk about preschemes.

Given a preschemed there is a classical truncation $\tau : cRing = toSet$. In particular, if we start with a nice scheme, it's a nice scheme.

[missed some]

Assume X and Y had nice shifted tangent complexes $T[-1]X$ sheaves of L_∞ algebras.

Theorem 16.1. *If there exist isomorphisms from the underlying classical spaces*

$$f_X : X \to C \leftarrow Y : f_Y$$

where C is a scheme and there is a quasiisomorphism

$$\begin{array}{ccc} f_X^* T[-1]X & \longrightarrow & f_Y^* T[-1]Y \\ & \searrow \quad \swarrow & \\ & C & \end{array}$$

as sheaves of L_∞ algebras, then X and Y are canonically isomorphic. Further, if X and Y have an action of G such that everything is G-equivariant. Then this descends to $X/G \to Y/G$.

[Comment about local to global]

It's five minutes early but I think that's what I wanted to say.

17. October 3: Qin Li, Perturbative Rozansky-Witten theory

Thank you, I want to thank the organizers for the invitation. I'll basically talk about perturbative Rozansky-Witten theory, this is joint with Kwokwai Chan and Conan Leung.

So Rozansky-Witten theory is a sigma-model $M^3 \to (X, \omega)$ where this is a holomorphic symplectic manifold. I'll talk about the setup of the classical theory and then quantize. This is a case study of Kevin's formalism. Although this has been studied by Rozansky-Witten, there are things that are not clear, a conjecture, so on, so part of the motivation is to study quantization of this theory, so we can get Rozansky-Witten invariants rigorously, and we can study structure of the quantum observables.

Let me start from the theory setup. The space of fields is differential forms $\mathcal{A}_M^* \otimes \mathfrak{g}_X[1]$. Here $\mathfrak{g}_X$ is the L_∞ algebra encoding the complex symplectic geometry of X.

I can write my classical action, similar to the one-dimensional Chern-Simons theory in Si's talk, so $S(\alpha) = \int_M \langle d_M \alpha, \alpha \rangle_\omega + \langle e^\ell(\alpha), \alpha \rangle_\omega$.

Now let me write it more clearly, so

$$\int_M \langle d_M \alpha, \alpha \rangle + \langle \ell_0, \alpha \rangle + \frac{1}{2!} \langle \ell_1(\alpha), \alpha \rangle + \frac{1}{3!} \langle \ell^2(\alpha^{\otimes 2}), \alpha \rangle + \cdots$$

So the physical Rozansky-Witten functional is $S = L_1 + L_2$, I'm ignoring the L_1 part and the L_2 part is

$$\frac{1}{\sqrt{k}} \epsilon^{\mu\nu\ell} (\varepsilon_{IJ} \chi_\mu^U \nabla_\nu \chi^J + \frac{1}{3} \omega_{IJ} \Omega^J_{kLM} \chi^I_\mu \chi^k_\mu \chi^\ell_p J^M).$$

These two actions, one is a reformulation of the other. We can see that this is really Chern-Simons like, this is really $AdAandA \wedge A \wedge A$. It's even more clear in our model because you have the L_∞ identity which is like a Jacobi identity.

Lemma 17.1. *The action functional satisfies the classical master equation.*

We have a classical theory satisfying the classical master equation so we can talk about quantization. This is a theory on a three dimensional spacetime. I'm using some old technique to do this quantization stuff, the technique of compactification of configuration spaces.

Let me tell you what I'm going to do. By definition a quantization of a classical theory I is a family $I[L]$ of interaction functionals parameterized by real numbers satisfying

(1) renormalization group flow equations
(2) quantum master equation
(3) $\lim_{L\to 0} I^{(0)}[L] = I_{\mathrm{cl}}$

We can do the naive thing, try to take the limit

$$\lim_{\epsilon\to 0} e^{\hbar\partial_{P_\epsilon^L}} e^{I_{\mathrm{cl}}/\hbar} =: e^{I[L]/\hbar}.$$

This has singularities, that is, $\lim_{\epsilon\to 0} P_\epsilon^L$ is only smooth on $M\times M\backslash\Delta$.

Definition 17.1. $M^0(n) := \{(m_, \ldots m_n) | m_i \neq m_j\}$ *is the configuration space, and there is a compactification of this,* $M[n]$ *as a manifold with corners.*

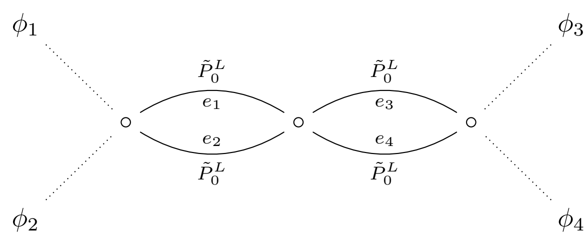
Let me give you an example of this, if you look at $M[2]$, there is no corner but only boundary, the interior is $M\times M\backslash\Delta$. The boundary is isomorphic to the unit sphere bundle of the tangent bundle.

The technical thing, after a lengthy calculation, you can show:

Proposition 17.1. *There exists a smooth lifting* $\tilde{P}_0^L \in \mathcal{A}^2(M[2])$*.*

Now I will use this smooth lifting to define the quantization

$$e^{I[L]/\hbar} = e^{\hbar\partial_{\tilde{P}_0^L}} e^{I_{\mathrm{cl}}/\hbar}.$$



So

$$\int_{M[3]} \prod_e \prod^{*} (\tilde{P}_0^L)\phi_1\phi_2\phi_3\phi_4 = W(P_0^L, I_{\mathrm{cl}})(\phi_1,\phi_2,\phi_3,\phi_4)$$

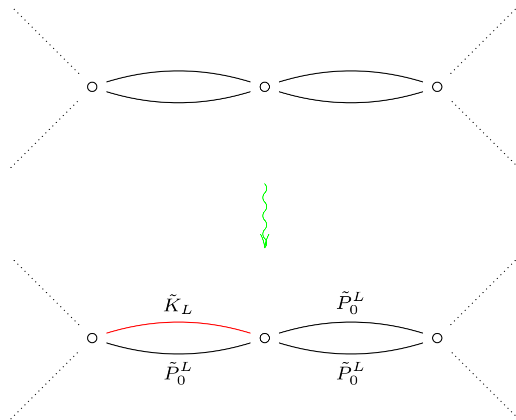
Theorem 17.1. $\{I[L]\}_{L>0}$ *satisfies the quantum master equation*

$$QI[L] + \{I[L], I[L]\}_L + \hbar\Delta_L I[L] = 0.$$

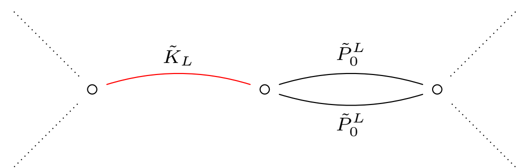
Q is the de Rham differential on the inputs.

(1) $d_M \tilde{P}_0^L = \tilde{K}_L$
(2) classical master eqution.

So you'll see this kind of graph where black edges are propagators and the red one is the heat kernel.



[Then somehow by a way that I didn't understand this diagram leads to:]



So local classical observables. Let $H_g \subset M$ be an open subset, the classical observables are

$$(\widehat{Sym}(\epsilon^\vee(H_g)), \underbrace{Q}_{=d_M+\iota_1} + \{I_{\mathrm{cl}}, \quad \}) \cong (\widehat{Sym}(H^*_{\mathrm{cpt}}(H_g) \otimes \mathfrak{g}_X[1]), \iota_1 + \{I_{\mathrm{cl}}, \quad \})$$

What you can do is further see some quasiisomorphism and find something interesting,

$$(\mathcal{A}^*_X \otimes_{\mathcal{O}_X} Jet^{\mathrm{hol}}_X(\wedge^* T_X)^{\otimes g}, d_{D_X})$$

and this is $H^*(X, \wedge^* T_X)^{\otimes g}$ so this is really cohomology on the target. It's not really canonical because the tensor g comes from a handlebody decomposition so you need to choose H_1 generators. The correlators are integration over these circles. The point is, this is basic classical observable stuff. If you look at quantum observables, there are no essential quantum corrections, so

Proposition 17.2.

$$H^*(Obs^{\mathrm{q}}(H_g)) \cong H^*(X, (\wedge^* T_X)^{\otimes g})[[\hbar]]$$

and this is the same as the Hilbert space in the conjectured [unintelligible] of the Rozansky-Witten structure. That's not a surprise, you are attaching handlebodies to open manifolds to get closed things, and then the local quantum observables are supposed to give us a hint of the field theory structure.

Now let's talk about the global quantum field theory and partition function.

$$\widehat{Sym}(\mathcal{E})((\hbar)), Q + \{I[\infty], \quad \} + \hbar\Delta_\infty) \cong \widehat{Sym}(\mathbb{H})((\hbar)), Q + \{I[\infty], \quad \} + \hbar\Delta_\infty)$$

For every global quantum observable O, to compute the correlation functions of O is to "take the coefficients of the top fermions." Let me say a little more.

$$Obs^{\mathrm{q}}(M) \cong \mathcal{A}_X^* \otimes Jet_X^{\mathrm{hol}}(\bigotimes_i \widehat{Sym}^*(\mathbb{H}^i \otimes T_X)^\vee)$$

I'm doing the symmetric tensor in the $\mathbb{Z}/2$-graded case.

Now there is a canonical "top fermion," the fermions are in odd degree, now $\pi_R^*(\omega^n)^{1+b_1(M)}$, this is my canonical top fermion for this path integral stuff.

What we want to do to compute the correlation function or partition function of our observables, the top fermion, I should put $O \cdot e^{I[\infty]/\hbar}$, that's where you take the coefficients of top fermions.

We find that this is exactly the same as,

Theorem 17.2. *the partition function of our theory gives the Rozansky-Witten invariant.*

Okay, I'll just stop.

18. Theo Johnson-Freyd, Poisson AKSZ theories

Thank you to the organizers, it's been a wonderful week so far, I learned a lot from all the talks. My goal will be to explain some version of the AKSZ construction and it's gonna be kind of a Poisson version. I emphasize Poisson because the usual requires that everyone be symplectic. You might as well learn my version. AKSZ's paper is good, you should read it, but if you haven't read it just pretend that I came up with it.

I'll tell that at the end, I want to start at a different part of the story. I want to remind you of an algebraic structure. This plays a role in many examples. He had a name—well, I will call it—so—I'm going to call it commutative homotopy BV algebra. In the literature, often they say sub infinity, but I want subspace to be something else, so I won't do that.

Definition 18.1. *A commutative homotopy BV-algebra is the following data. It's a graded commutative algebra (this always means $\mathbb{Z}$-graded), it has a differential on it $\Delta : A \to A[[\hbar]]$. I'll try to keep my talk without deciding whether I'm homological or cohomological. It makes this into an E_0-algebra, which is something that Claudia talked about.*

I know no examples but I have no objection to making A an algebra over $\mathbb{R}[[\hbar]]$. There's one other condition, in normal BV it's a second order operator. This relaxes to the condition that if you look at Δ modulo $\hbar^n$, this is an nth order differential operator.

I won't remind you why Jae-Suk cares about these [something about] perturbative oscillating integrals.

I should also give some references, many people have noticed this structure. In mathematics, the earliest place I know of where this was written down was Kravchenko 1998. In physics the earliest I know was Batalin-Bering-Dangaard 1996.

The classical limit, look at the principal symbols, I'm too close to the board to spell principal, you can take Δ mod $\hbar^n$ and take these data, this is what I think Jae-Suk was calling the descendents are a shifted L_∞ algebra on the commutative algebra A that acts by derivations. In fact, these are "commutative homotopy P_0-algebras."

This is all well and good. These are like what you might want to find in the perturbative classical limit of a quantum theory.

Now the question is how to find examples, interesting examples. These were perturbative in Planck's constant, let me focus on examples that are also perturbative in the geometric data as well. I'll focus on $A = \widehat{Sym}(V)$ for V some graded vector space.

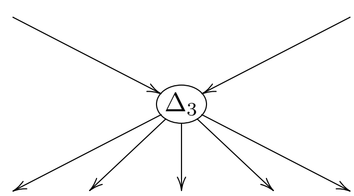
How should you think about these? There's also, I should emphasize, the point is that you should think of $Spec(A)$ as an infinitesimal manifold. So this is supposed to be the algebra of functions on this infinitesimal manifold. All geometric structures are described by their Taylor coefficients. Gabriel, I don't even capitalize "taylor." Do you capitalize "narcissist?" I'll assert, by geometric structures, I've chosen V, I've chosen a linear chart. I'm only going to think about examples of that type.

Let me reformulate, the data of a commutative homotopy BV algebra on $\widehat{Sym}(V)$ is, look at Δ_3, the third term, this is a fourth order differential operator that annihilates 1. I can write this as

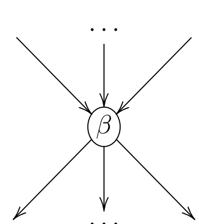
$$\underbrace{\Delta_{3,(1)}}_{\text{“linear part”}} + \underbrace{\Delta_{3,(2)}}_{\text{“pure second order”}} + \underbrace{\Delta_{3,(3)}}_{\text{“pure third order”}} + \cdots$$

So This is $V \to A$, $V \otimes V \to A$, and so on.

I can also expand these in terms of outputs, so I can write $\Delta_{3,(2)} = \sum_{n \geq 0} \Delta_{3,(2)}^{(n)}$, so $\Delta_{3,(2)}^{(5)}$ is a map $V^{\otimes 2} \to V^{\otimes 5}$ like this:

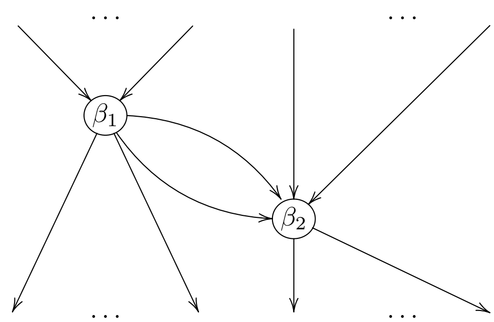


I can index this data by β (some linear combination of these three numbers) which have some number of inputs, some number of outputs, so on.

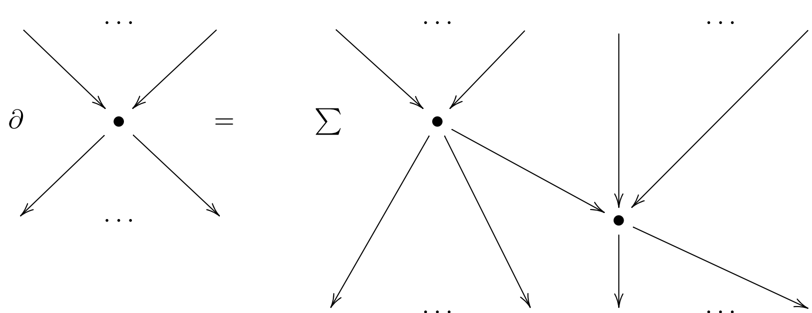


I should have asked all structures to vanish at the origin. It starts with a map $V \to \mathbb{R}$ so it starts with $\Delta_{0,(1)}^{(1)}$, if you work out $\Delta^2 = 0$, this is a linear differential on V. Let me declare this to be the differential on V. Then the rest of this can be written, so that

$[\partial_V, \beta]$, you sum over all graphs with the same number of inputs and outputs.



So this sum should have $\beta = \beta_1 + \beta_2 + b_1$ of the graph. In the classical limit, a commutative homotopy P_0 structure breaks up a corolla into tree-like compositions with 0 on the vertices.



We've had these nice lectures from Dolgushev on operads and Koszul duality, well operads are to rooted trees as properads are to connected directed-acyclic graphs and dioperads to directed trees.

So if you know operads, this is a bar or cobar,

Theorem 18.1 (Drummond-Cole-Terilla-Tradler)**.** *Commutative homotopy BV structures on $\widehat{Sym}(V)$ are in correspondence with $cobar(coFrobenius)$ structures on V. This is cobar in the category of properads.*

Exactly the same proof says, I'll do this in another color, that commutative homotopy P_0-structures on $\widehat{Sym}(V)$ are in correspondence with $cobar(coFrob)$ structures on V in the category of dioperads.

I'm going to give some examples, I should say finite dimensional in a bunch of places and I'll leave it out.

Given a properad P with finiteness, I will say $\mathbb{D}P = cobar(P^*)$, and ther's a canonical map $\mathbb{D}Frob \to \mathbb{D}P \otimes P$, this is a duality, and Frobenius is a unit for tensor products.

If I have any two properads, I can tensor the representations together and you can pull back along homomorphisms, and what this tells you is that if you can give X a P-structure and Y a $\mathbb{D}P$ structure then $\widehat{Sym}(X \otimes Y)$ is a commutative homotopy BV algebra. If I take the dual in dioperads, it's a commutative homotopy P_0 algebra. You might be able to do one and not the other in some case.

So for my example, I want to say a commutative homotopy P_d structure is an L_∞ structure on $A[1-d]$ acting by multiderivations, and the same exercise as before says that, this combinatorics, says that when A is a completed symmetric algebra, these are dual in the sense of dioperads to $Frob_d$, so $P_d = \mathbb{D}^{\mathrm{di}} Frob_d$, meaning comultiplication is degree $\pm d$. The example is homology of a d-dimensional compact manifold. You could put involutive to make this work if you want to say properads.

This is supposed to be a field theory and I haven't said field, so let me say field. The example was going to say, to get these kind of structures, I could use a P_d-structure on something and a Frobenius structure on—

I'd like interesting examples that are field-theoretic. We defined quantum and now we'll define field theory, which

Definition 18.2. Field theory *has something to do with mapping spaces.*

So mapping spaces are way too hard, so I'll use the only one I understand, given M and some target X, which will be an infinitesimal manifold, I don't understand smooth maps from M to X but I do understand locally constant maps M to X. Then these might as well be $Maps^{\mathrm{lc}}(M, \mathbb{R}) \otimes X$, and the locally constant maps $M \to \mathbb{R}$, and $Maps^{\mathrm{lc}}(M, \mathbb{R}) = \Omega^\bullet_{dR}(M)$. Many of you guys are better at analysis. The PDE d is the only one I understand. To say I have a classical or quantum field theory, I should give this mapping space one of these structures. I want to have "classical" or "quantum" field theory with target X this is topological in M if X has a P-structure for some P and $\Omega^\bullet_{dR}$ has a $\mathbb{D}P$-structure.

Well, let me be specific to the P_d structure. There's a universal way to write down resolutions in algebras. Generally, we have that $\mathbb{D}\mathbb{D}$(thing) is a cofibrant resolution of thing. So this is true in properads or dioperads. So I know that, let me work with compact manifolds, I know that $H^*_{dR}(M)$ is $Frob_d$ so I could ask, is $\Omega^*(M)$ "homotopy $Frob_d$." There is a stupid way that it is, you could take the structure and transfer it. The output doesn't look physical. You should ask for some restriction on this, you could ask that all structures, all maps, the operations involved, are, you can't make them strictly local, but you could make them local in the sense that Kevin gave. Their support is near a diagonal. This means local in the sense that supports can be made arbitrarily close to diagonals. You can make that precise. Then I'll tell you the answer, which is yes for dioperads (canonically).

Theorem 18.2 (J.-F.)**.** *The space of choices is contractible.*

So this is Frobenius in the dioperad sense, you always get a P_0 structure for X if it's homotopy Poisson.

I'll tell you the answer for properads and then stop.

Theorem 18.3. *For properads, the answer is yes when $d = 0$, when $d = 1$ the answer is no. It's just no. There is no chain level structure. There's an obstruction in genus 2. It's the same obstruction in Merkulov's work in a different guise. When $d > 2$ it's yes for the wrong reason. There's a theoremification functor to apply, there's a lot of work to do that, I'm sure I can do that but. The space of such structures, well, let me do d at least 2 and I want to do a non-compact manifold $\mathbb{R}^d$, framed manifolds, the space of such structures is homotopy equivalent to the space of E_d-formality quasiisomorphisms. So it's a complicated space that by hard theorems is nonempty.*

19. Pavel Mnev, An example of a cellular topological quantum field theory in BV-BFV formalism, with Segal-like gluing.

Thanks a lot, this is joint with Cattaneo and Reshetikhin, and Losev also influenced these discussions. I'll discuss this formalism, and the point of our discussions is to put together two of our favorite toys, effective actions in BV formalism, this somehow stems from the old idea of effective action by Wilson, we've seen some examples in the talks of Kevin. The other toy is having field theories on manifolds with boundaries, so when you're gluing you're composing operations.

So let me, I don't want to dwell on the generalities of the formalism, I'll give a rough outline. The classical picture is as follows. The BV-BFV, the B and V in both stand for Batalin and Vilkovisky. The F, the BFV is quite different. This involves rings of reductions on coisotropic submanifolds. For manifolds with boundary, for gauge theories, the general structure is like that, okay, let me summarize it for a closed manifold. There I should have a space of fields $\mathcal{F}$, a graded supermanifold, an odd symplectic structure in degree -1, my BRST differential, $Q \in \mathcal{X}(\mathcal{F})_{+1}$ and $Q^2 = 0$. The action S is a function on $\mathcal{F}$ of degree 0 and it satisfies the structural equation $\iota_Q \omega = \delta S$, which implies $\{S, S\}_\omega = 0$. I want in the boundary case to do something a little different. I will put $\mathcal{F}_\partial$, the "phase space" on the boundary, an exact 2-form of degree 0, we have $\omega_\partial = \delta\alpha_\partial$, a degree $+1$ square zero operator Q_∂, and an odd action S_∂, so $\iota_{Q_\partial}\omega_\partial = \delta S_\partial$.

Now on the bulk I get $(\mathcal{F}, \omega, Q, S)$ and what I want is a restriction map to the phase space which is pulling back fields to the boundary. The other relation is that Q is projectible and projects to Q_∂. I want to deform the equation relating Q, ω, and S by terms involving α. So $\iota_Q\omega = \partial S + \pi^*\alpha_\partial$. There is this correction, it's not Hamiltonian at all. This means that $\mathcal{L}_Q\omega = \pi^*\omega_\partial$ and $Q(S) = \pi^*(2S_\partial - \iota_{Q_\partial}\alpha_\partial)$. A more symmetric way to write it is $\frac{1}{2}\iota_Q\iota_Q\omega = \pi^* S_\partial$.

[Theo: is this like a Lagrangian in the symplectic thing you get for the boundary? In derived geometry?]

I cannot answer you in derived geometry, I can answer in Russian if you like. [Laughter]

Let us go on. I don't want to be precise here, I'll give a very general outline, the idea is that first you choose the datum of the geometric quantization of this phase space. I want to pick a fibration over a manifold B. You can work more generally. I want this polarization to be compatible with α_∂ so that restricted to fibers this is zero. This means functions on the sections of the [unintelligible] can be identified with functions on B.

This space of states comes equipped with an odd operator $\hat{S}_\partial$. If things work well for me, I can quantize it to square to zero. This is an operator on the space of states of degree 1. The physical name is quantum [unintelligible] charge. Then my space of states is a complex and the cohomology is the "reduced space of states."

For the bulk I should get a vector in the space of states (partition function). To M I associate a vector in this space which is a function on B. Then

$$Z_M(b) = \int_{\mathcal{L} \subset \pi^{-1}\{L\}}$$

The point is, I'm thinking of my space of fields as fibered over B. Then on the other side, I want to choose a subspace, a Lagrangian corresponding gauge fixing. I want ω restricted to the fibers, I'll call this ω_B, Lagrangian with respect to ω_B. This is a choice, considerations which are theorems in finite dimensions and a principle for infinite dimensional integrals imply it's independent of choice.

[Missed] Generally it's good to leave something from integration here. This might sound unmotivated for you. We'll split out some part of the fiber and leave it as a parameter in the partition funciton. We want some bundle over B, call it $\mathbb{Z}$ with a fiber $\mathbb{F}$ and the partition function should be a function on the total space.

I would like to say that $Z_M(b, \varphi_{Z_M})$ is $\int_L e^{\frac{1}{\hbar}S}$ where L is the fiber over (b, φ).

I wanted to say, where does this partition function take values? In the space of states for the boundary. So $Z_m \in H_\partial \hat{\otimes} Fun(\mathcal{F}_{ZM})$.

$$i\hbar\Delta_{ZM} + \frac{i}{\hbar}\hat{S}_\partial)Z =$$

.

[missed some]

For gluing, one step is Segal-like and the other has to do with zero modes. Cut into two manifolds, then the idea is that first I construct, I cut along some codimension one submanifold Σ. So first I take $\int_b Z_I(b, \varphi^I_{ZM}) Z_{II}(b, \varphi^{II}_{ZM})$ and then [for reasons that i missed] you integrate out the redundant zero mode, get from Z' to Z_M.

This is the general—[Questions]

So I want to discuss a very simple gauge theory, the so-called Abelian BF theory, there's no way to go simpler. In the continuum setting, think of a manifold of dimension n, you endow it with a flat bundle of rank m, the holonomy should be orthogonal, but it works more generally with a $SL_\pm(m, R)$-local system on M. Okay, so the space of fields for this model contains two parts, essentially two copies of $\Omega^\bullet(M, E)[1] \oplus \Omega^\bullet(M, E^*)[n-2]$.

The action $S = \int_M \langle B^\wedge, d_E A\rangle$. So this, one of these, these, well, A and B are $\mathcal{F} \to \Omega^\bullet(M, E)$ (degree 1) and $\mathcal{F} \to \Omega^\bullet(M, E)$ (degree $n-2$).

Let me say something about the space of fields of a cellular model. I want a manifold with boundary with a cellular decomposition X and a local system E. What I want is $C^\bullet(X, E)[1] \oplus C^\bullet(X^\vee, E^*)[n-2]$.

For the dual to make sense at all, I need this to be a ball complex, what is the dual decomposition? If M is closed, start with the cell composition of the circle as a triangle. For every cell I denote $e(X)$, I have a cell $\cup(e)X^\vee$, the dimension of a cell in the dual is comlementary.

If you want to go to boundary, there are two candidates, and I'd like to resort to pictures. Think of a cell decomposition of a disk, [pictures]

I have two duals, one I do the thing you'd expect and the other you eliminate the boundary cells. Neither of these is an involution but they are inversese.

What I need is to go to the language of cobordisms. I color with *in* and *out*. Then if I'm given a manifold with in boundary and out boundary, what does it mean to go to the dual? It means that I take X^{V_+} going in and X^{V_-} coming out.

I wrote the space of fields, the action is $\langle B, dA\rangle_X = \langle B, A\rangle_{X_in}$.

Now Q is the lifting of the coboundary operator on cchains to $Fun(\mathcal{F})$ and the boundary, the kinetic form is obvious, the point is that the intersection pairing is degenerate, there are cells, well [missed]

Now $\mathcal{F}_\partial = C^*(X_\partial)[1] \oplus C(X_2)[n-2]$

What's important here? [missed, I have to stop.]

20. Bruno Vallette: Everything you always wanted to know about the HTT but were afraid to ask

Bruno requested that I not take notes during this talk.