PROPER BASE CHANGE FOR LCC ÉTALE SHEAVES OF SPACES

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Abstract. The goal of this paper is to prove the proper base change theorem for locally constant constructible étale sheaves of spaces, generalizing the usual proper base change in étale cohomology. As applications, we show that the profinite étale homotopy type functor commutes with finite products and the symmetric powers of proper algebraic spaces over a separably closed field, respectively. In particular, the commutativity of the étale fundamental groups with finite products is extended to all higher homotopy groups.

Contents

1. Introduction

1.1. One of the fundamental results of étale cohomology theory is the proper base change theorem [17, Exposé XII.5.1]; consider a cartesian square

(1.1.1)
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

in the category of schemes. Let F be a torsion étale sheaf of abelian groups on X. If f is proper, then for each integer $n \geq 0$, the canonical base change morphism

$$g^* R^n f_* F \to R^n f'_*(g'^* F)$$

is an isomorphism in $D^+(\mathcal{A}\mathrm{b}(Y'_{\text{ét}}))$.

In this paper, we study a non-abelian version of the proper base change theorem; in the simple case of set-valued étale sheaves, the push-pull transformation

$$g^* f_* \to f'_* g'^*$$

is an isomorphism of functors from the category of set-valued étale sheaves on X to the category of set-valued étale sheaves on Y'. Extending to the ∞-categories of étale sheaves which take values in the ∞-category $\mathcal{S}$ of spaces, there is a commutative diagram of ∞-topoi

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}_{\text{ét}}(X') \arrow[r, "g'_*"] \arrow[d, "f'_*"] & \mathcal{S}\mathrm{hv}_{\text{ét}}(X) \arrow[d, "f_*"] \\ \mathcal{S}\mathrm{hv}_{\text{ét}}(Y') \arrow[r, "g_*"] & \mathcal{S}\mathrm{hv}_{\text{ét}}(Y). \end{tikzcd} \tag{1.1.2}$$

The main result of this paper is the following:

Theorem 1.2. *Suppose we are given a cartesian square (1.1.1) of quasi-compact and quasi-separated schemes. Let F be a locally constant constructible object of $\mathcal{S}\mathrm{hv}_{\text{ét}}(X)$. If f is proper, then the push-pull morphism*

$$\theta : g^* f_* F \to f'_* g'^* F$$

is an equivalence in $\mathcal{S}\mathrm{hv}_{\text{ét}}(Y')$.

Remark 1.3. Suppose (1.1.1) is a pullback square of topological spaces. If f is a proper and separated, then the canonical base change morphism is an isomorphism in $D^+(\mathcal{A}\mathrm{b}(Y'))$ [16, Exposé Vbis.4.1.1]. Likewise, in the simple case, the push-pull transformation is an isomorphism of functors.

These proper base change theorems were generalized to a non-abelian case by Jacob Lurie in [9, 7.3.1.18]; suppose we are given a fibered square (1.1.1) of locally compact Hausdorff spaces. If f is proper, then the push-pull transformation is an isomorphism of functor between the ∞-categories of $\mathcal{S}$-valued sheaves.

1.4. However, a non-abelian version in the algebro-geometric setting is subject to some restrictions; if it were true for every $\mathcal{S}$-valued sheaves, we would have the proper base change for any (not necessarily torsion) étale sheaves of abelian groups by the same argument as in [9, 7.3.1.19], which is not the case (see [17, Exposé XII.2]).

So, we need to impose some finiteness condition on étale sheaves of spaces. Indeed, we will show that the non-abelian proper base change theorem holds for those étale sheaves of spaces which are locally constant constructible [10, E.2.5.1]. Its proof will be parallel to the proof of the classical one: use passage to limit argument to reduce to the case when Y is a strictly henselian local ring and g is the inclusion of the closed point (cf. [17, Exposé XII.6.1]).

Remark 1.5. The proof cannot be reduced to the case of 0-truncated sheaves—the usual proper base change for set-valued sheaves. This is because 0-truncation functors do not necessarily commute with pushforward functors. So, as in the proof of Lurie's generalization for topological spaces, a new approach is required. In our case, the key observation from [10, E.2.3.3] is that for geometric morphisms of ∞-topoi, the associated morphisms of locally constant constructible objects are completely determined by the maps of profinite shapes of the ∞-topoi.

1.6. Our applications of the non-abelian proper base change theorem 1.2 come from the étale homotopy theory. Let $f : X \to \operatorname{Spec} k$ and $g : Y \to \operatorname{Spec} k$ be morphisms of schemes, where k is a separably closed field. Assume f is proper and g is locally of finite type. It is well-known from [13, Exposé X.1.7] that the natural map

$$\pi_1(X \times_k Y) \to \pi_1(X) \times \pi_1(Y)$$

is an isomorphism of profinite groups (here we omit the base points). This raises a natural question of whether it can be extended to higher homotopy groups. Or, more precisely, if the natural map

$$\widehat{\mathrm{h}}(X \times_k Y) \to \widehat{\mathrm{h}}(X) \times \widehat{\mathrm{h}}(Y) \tag{1.6.1}$$

is an equivalence of profinite spaces, where $\widehat{\mathrm{h}}$ denotes the profinitely completed étale homotopy type functor. By cohomological criteria [1, 4.3], it suffices to show that for each integer $n \geq 0$ and for each local coefficient system of finite abelian groups on $\widehat{\mathrm{h}}(X) \times \widehat{\mathrm{h}}(Y)$, the associated map of n-th cohomology groups is an isomorphism. The difficulty here is that we cannot simply apply the Künneth formula in étale cohomology [17, Exposé XVII.5.4.3] because we do not know how local systems on the product of profinite spaces look like in general.

To this end, we shift our perspective from étale homotopy types [1, p.114] to (∞-)shapes (see [2, 6.0.5] for the equivalence of these two approaches and [5, 4.1.5] for the equivalence of profinite completions in ∞-category theory and in model category theory). For each scheme T and the ∞-topos $\mathcal{S}\mathrm{hv}_{\text{ét}}(T)$ of $\mathcal{S}$-valued étale sheaves on T, the associated shape $\mathrm{Sh}(T)$ is defined to be the composite $\pi_* \circ \pi^* : \mathcal{S} \to \mathcal{S}$, where $\pi_* : \mathcal{S}\mathrm{hv}_{\text{ét}}(T) \to \mathcal{S}$ denotes the essentially unique geometric morphism of ∞-topoi. The advantage of this ∞-categorical perspective is an intermediate object $\mathrm{Sh}(X) \circ \mathrm{Sh}(Y)$—the composition of pro-spaces—which helps us to understand the homotopy type of $X \times_k Y$; there is a commutative triangle of pro-spaces

$$\begin{tikzcd} & \mathrm{Sh}(X \times_k Y) \arrow[dr] & \\ \mathrm{Sh}(X) \circ \mathrm{Sh}(Y) \arrow[ur] \arrow[rr] & & \mathrm{Sh}(X) \times \mathrm{Sh}(Y). \end{tikzcd} \tag{1.6.2}$$

The bottom line is that the bottom map is under our control and that the left diagonal map is obviously very closely related to the non-abelian proper base change theorem 1.2 by the definition of shapes: in the setting of (5.3.1), the left diagonal is nothing but the canonical map

$$p_* p^* q_* q^* \to p_* {q'}_* {p'}^* q^*.$$

Using this idea, we will show in 5.3 that (1.6.1) is an equivalence.

Remark 1.7. The intermediate object with the maps in (1.6.2) are naturally defined in the ∞-categorical setup, but hardly seen in the classical étale homotopy theory or in its model categorical refinement.

1.8. A second application concerns symmetric powers of algebraic spaces. Using the qfh topology of schemes, Marc Hoyois proved that for a quasi-projective schemes X over a separably closed field k and a prime ℓ different from the characteristic of the field k, the natural map

$$\operatorname{Sym}^n \mathrm{h}(X) \to \mathrm{h}(\operatorname{Sym}^n X) \tag{1.8.1}$$

is a $\mathbb{Z}/\ell$-homological equivalence (here h denote the étale homotopy type functor); see [7, 5.4]. One of the main steps in the proof is that the map (1.6.1) before taking the profinite completions induces a $\mathbb{Z}/\ell$-homological equivalence for finite type k-schemes, which can be reduced to the Künneth formula in étale cohomology (see [7, 5.1]). As mentioned above, however, this is not the case for its profinite version 5.3. Using 5.3 as an essential piece, we prove in 6.12 that (1.8.1) is a profinite equivalence for proper k-algebraic spaces X.

Remark 1.9. Under the extra assumption that X is geometrically normal, the commutativity of the profinite étale homotopy types and the symmetric powers 6.12 recovers [18, Theorem 1] of Arnav Tripathy. In contrast with Tripathy's proof involving a concrete discussion of the étale fundamental groups of symmetric powers, our approach is very formal.

1.10. Conventions. We assume that the reader is familiar with the basic theory of ∞-categories as developed in [9]. We follow the set-theoretic convention of [9].

Let $*$ denote the final object of an ∞-category $\mathcal{C}$, if exists.

1.11. Acknowledgements. This work was supported by IBS-R003-D1. The author is grateful to Marc Hoyois for helpful comments on profinite étale homotopy types.

2. Preliminaries on Shapes and Profinite Completion

2.1. Let us give a quick review of pro-categories in ∞-category theory (see [10, A.8.1] for more details). Let $\mathcal{C}$ be an accessible ∞-category which admits finite limits. Let $\mathrm{Pro}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$ denote the full subcategory spanned by those functors $\mathcal{C} \to \mathcal{S}$ which are accessible and preserve finite limits. We refer to it as the *∞-category of pro-objects of* $\mathcal{C}$. It has the expected universal property: let $\mathcal{D}$ be an ∞-category which admits small cofiltered limits, and let $\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \subseteq \mathrm{Fun}(\mathrm{Pro}(\mathcal{C}), \mathcal{D})$ denote the full subcategory spanned by those functors which preserve small cofiltered limits. By virtue of [10, A.8.1.6], the Yoneda embedding induces an equivalence of ∞-categories

$$\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

2.2. Recall from [10, E.0.7.8] that an object $X \in \mathcal{S}$ is defined to be *π-finite* if it satisfies the following conditions:

(i) The space X is n-truncated for some integer $n \geq -2$.
(ii) The set $\pi_0 X$ is finite.
(iii) For each $x \in X$ and each integer $m \geq 1$, the group $\pi_m(X, x)$ is finite.

Let $\mathcal{S}_\pi \subseteq \mathcal{S}$ denote the full subcategory spanned by the π-finite spaces. The associated pro-category $\mathrm{Pro}(\mathcal{S}_\pi)$ is referred to as the *∞-category of profinite spaces*.

2.3. The universal property of pro-categories applied to the fully faithful embedding $i : \mathcal{S}_\pi \hookrightarrow \mathcal{S}$ extends it to a fully faithful embedding $\mathrm{Pro}(i) : \mathrm{Pro}(\mathcal{S}_\pi) \to \mathrm{Pro}(\mathcal{S})$ of pro-categories. Meanwhile, composition with i induces a forgetful functor $\mathrm{Pro}(\mathcal{S}) \to \mathrm{Pro}(\mathcal{S}_\pi)$ which is left adjoint to $\mathrm{Pro}(i)$ by [10, E.2.1.3]. We refer to it as the *profinite completion functor*.

2.4. Let $\mathcal{X}$ be an ∞-topos. For an essentially unique geometric morphism $\pi_* : \mathcal{X} \to \mathcal{S}$, the composition $\pi_*\pi^* : \mathcal{S} \to \mathcal{S}$ is an object of $\mathrm{Pro}(\mathcal{S})$, which is referred to as the *shape of* $\mathcal{X}$ and denoted by $\mathrm{Sh}(\mathcal{X})$. In fact, [10, E.2.2.1] supplies a left adjoint functor $\mathrm{Sh} : \infty\mathcal{T}\mathrm{op} \to \mathrm{Pro}(\mathcal{S})$, where $\infty\mathcal{T}\mathrm{op}$ denotes the ∞-category of ∞-topoi. Composing with the forgetful functor $\mathrm{Pro}(\mathcal{S}) \to \mathrm{Pro}(\mathcal{S}_\pi)$, one obtains a functor

$$\mathrm{Sh}_\pi : \infty\mathcal{T}\mathrm{op} \to \mathrm{Pro}(\mathcal{S}_\pi).$$

We refer to the image $\mathrm{Sh}_\pi(\mathcal{X})$ of an ∞-topos $\mathcal{X}$ under this functor as the *profinite shape of* $\mathcal{X}$.

Definition 2.5. Let $f_* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. We say that f_* is a *profinite shape equivalence* if it induces an equivalence $\mathrm{Sh}_\pi(\mathcal{X}) \to \mathrm{Sh}_\pi(\mathcal{Y})$ of profinite spaces.

2.6. Recall from [10, E.2.5.1] that an object F of an ∞-topos $\mathcal{X}$ is *locally constant constructible* if there exists a finite collection of objects $\{X_i \in \mathcal{X}\}_{1 \le i \le n}$ such that the map $\coprod X_i \to *$ is an effective epimorphism, a collection of π-finite spaces $\{Y_i\}_{1 \le i \le n}$, and equivalences $\pi_i^* Y_i \simeq F \times X_i$ in the ∞-topos $\mathcal{X}_{/X_i}$ for $1 \le i \le n$, where ${\pi_i}^* : \mathcal{S} \to \mathcal{X}_{/X_i}$ is the essentially unique geometric morphism. Let $\mathcal{X}^{\mathrm{lcc}} \subseteq \mathcal{X}$ denote the full subcategory spanned by the locally constant constructible objects.

If $f^* : \mathcal{Y} \to \mathcal{X}$ is a geometric morphism of ∞-topoi, then it carries locally constant constructible objects to locally constant constructible objects. In what follows, it is of the utmost importance that locally constant constructible objects completely determine profinite shape equivalences; according to [10, E.2.3.3], the pushforward f_* is a profinite shape equivalence if and only if the restriction functor $f^* : \mathcal{Y}^{\mathrm{lcc}} \to \mathcal{X}^{\mathrm{lcc}}$ is an equivalence of ∞-categories.

2.7. According to [2, 6.0.5], for locally noetherian schemes, the ∞-categorical counterparts of Artin-Mazur's étale homotopy types (or their model categorical reformulations) are the shapes of the hypercompletions of the ∞-topoi of $\mathcal{S}$-valued étale sheaves on the schemes; see also [5, 4.1.5] for the compatibility of profinite completions in model category theory and in ∞-category theory.

In this paper, we will mainly work with the shapes of ∞-topoi rather than of the hypercompletions of ∞-topoi. When applying our results to étale homotopy types, there is no harm in doing so because profinitely completed étale homotopy types are of interest to us:

Lemma 2.8. *Let $\mathcal{X}$ be an ∞-topoi. Then the natural map*

$$\mathrm{Sh}_\pi(\mathcal{X}^\wedge) \to \mathrm{Sh}_\pi(\mathcal{X})$$

of profinite spaces is an equivalence, where $\mathcal{X}^\wedge$ denotes the hypercompletion of the ∞-topoi $\mathcal{X}$.

Proof. By evaluating at π-finite spaces, the statement follows immediately from [9, 6.5.2.9] that $\tau_{\le n}\,\mathcal{X} \subseteq \mathcal{X}^\wedge$ for every integer $n \ge -2$. □

3. Limits with respect to Shapes and Lcc sheaves

Throughout this section, we fix a diagram of ∞-topoi satisfying the following condition:

($*$) Let $\mathcal{I}$ be a filtered ∞-category and let $p : \mathcal{I}^{\mathrm{op}} \to \infty\mathcal{T}\mathrm{op}$ be a diagram of ∞-topoi $\{\mathcal{X}_i\}$. Assume that each $\mathcal{X}_i$ is coherent [10, A.2.1.6] and that for each of the transition morphisms ${p_{ij}}_* : \mathcal{X}_j \to \mathcal{X}_i$, the restriction of ${p_{ij}}_*$ to $\tau_{\le n-1}\,\mathcal{X}_j$—the full subcategory of $\mathcal{X}_j$ spanned by the $(n-1)$-truncated objects—commutes with filtered colimits for all integers $n \ge 0$.

Remark 3.1.

(i) In fact, for the rest of this section, we may assume henceforth that $\mathcal{I}$ is a directed partially ordered set I; see [9, 5.3.1.18].
(ii) The condition $(*)$ is met by schemes of interest in this paper, as we will see in a moment (see 3.5 for more examples).

3.2. Let X be a scheme. The usual small étale topology on X induces a Grothendieck topology on the nerve of the category $\text{Ét}(X)$ of étale X-schemes. Let $\mathcal{S}\mathrm{hv}_{\text{ét}}(X)$ (resp. $\mathcal{P}_{\text{ét}}(X)$) denote the ∞-category of sheaves (resp. presheaves) of spaces on $\mathrm{N}(\text{Ét}(X))$. Remark from [9, 6.4.5.7] that the ∞-topos $\mathcal{S}\mathrm{hv}_{\text{ét}}(X)$ is 1-localic.

Now assume that X is quasi-compact and quasi-separated. Let $\text{Ét}(X)^{\mathrm{fp}} \subseteq \text{Ét}(X)$ denote the full subcategory consisting of those étale X-schemes which are of finite presentations. With respect to the induced Grothendieck topology on $\mathrm{N}(\text{Ét}(X)^{\mathrm{fp}})$, the associated geometric morphism of ∞-topoi is an equivalence since the ∞-topoi are 1-localic and it is the case for the 0-truncations. Specifically, one may assume henceforth that $\mathcal{S}\mathrm{hv}_{\text{ét}}(X)$ is induced by a finitary Grothendieck topology [10, A.3.1.1], and thus is coherent by virtue of [10, A.3.1.3].

3.3. Let I be a filtered 1-category. Let $\{X_i\}_{i \in I^{\mathrm{op}}}$ be a compatible family of quasi-compact and quasi-separated schemes with affine transition maps. Set $X := \lim X_i$. There is an equivalence $\mathcal{S}\mathrm{hv}_{\text{ét}}(X) \simeq \lim_{i \in I^{\mathrm{op}}} \mathcal{S}\mathrm{hv}_{\text{ét}}(X_i)$ in $\infty\mathcal{T}\mathrm{op}$ as the ∞-topoi are all 1-localic and it is the case for the usual 1-topoi by virtue of [16, Exposé VII.5.6] and [16, Exposé VI.8.2.3].

One deduces from the following result that the diagram $\{\mathcal{S}\mathrm{hv}_{\text{ét}}(X_i)\}$ of ∞-topoi satisfies the condition $(*)$:

Theorem 3.4. (cf. [16, Exposé VI.5.1]) *Let $\mathcal{C}$ and $\mathcal{D}$ be small ∞-categories which admit finite limits and which are equipped with finitary Grothendieck topologies. Let $f : \mathcal{C} \to \mathcal{D}$ be a continuous functor which commutes with finite limits. Let $p_* : \mathcal{S}\mathrm{hv}(\mathcal{D}) \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ denote the induced geometric morphism of ∞-topoi. Then for each integer $n \geq 0$, the restriction of p_* to $\tau_{\leq n-1} \mathcal{S}\mathrm{hv}(\mathcal{D})$ commutes with filtered colimits.*

Proof. Let $\mathcal{J}$ be a filtered ∞-category, and let $\{F_\alpha\}_{\alpha \in \mathcal{J}}$ be a compatible family of $(n-1)$-truncated sheaves on $\mathcal{D}$. We claim that the canonical map

$$\operatorname*{colim}_{\alpha \in \mathcal{J}} p_* F_\alpha \to p_* \operatorname*{colim}_{\alpha \in \mathcal{J}} F_\alpha$$

is an equivalence of sheaves on $\mathcal{D}$. Using [9, 1.2.4.1], it suffices to show that for each $G \in \mathcal{S}\mathrm{hv}(\mathcal{C})$, the induced map

$$\mathrm{Map}_{\mathcal{S}\mathrm{hv}(\mathcal{C})}(G, \operatorname{colim} p_* F_\alpha) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}(\mathcal{C})}(G, p_* \operatorname{colim} F_\alpha)$$

is a weak homotopy equivalence. It follows from [10, A.3.1.3] that the finitary assumption guarantees the composition of the Yoneda embedding $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ with the sheafification $L : \mathcal{P}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ carries each object $C \in \mathcal{C}$ to a coherent object $L(j(C))$ of $\mathcal{S}\mathrm{hv}(\mathcal{C})$. After applying [10, A.2.3.1] to the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})_{/L(j(C))}$, one deduces that the restriction of $\mathrm{Map}_{\mathcal{S}\mathrm{hv}(\mathcal{C})}(L(j(C)), -)$ to $\tau_{\leq n-1} \mathcal{S}\mathrm{hv}(\mathcal{C})$ commutes with filtered colimits. Since $\mathcal{P}(\mathcal{C})$ is generated under small colimits by the Yoneda embedding, one can then reduce to establishing that the canonical map

$$\operatorname{colim} \mathrm{Map}_{\mathcal{S}\mathrm{hv}(\mathcal{C})}(L(j(C), p_* F_\alpha) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}(\mathcal{C})}(L(j(C), p_* \operatorname{colim} F_\alpha)$$

is an equivalence for each $C \in \mathcal{C}$. By the adjunction of (p^*, p_*), the desired equivalence is a consequence of [10, A.2.3.1] applied to $\mathcal{S}\mathrm{hv}(\mathcal{D})_{/Lj(f(C))}$. $\square$

Remark 3.5. The previous result also implies that the diagrams of ∞-topoi satisfying the condition $(*)$ exist in great abundance. Recall from [10, A.7.5.3] that there is an equivalence of ∞-categories from the ∞-category of bounded ∞-pretopoi to the opposite of the full subcategory of the ∞-category of coherent ∞-topoi spanned by the bounded coherent ∞-topoi, given on objects by $\mathcal{C} \mapsto \mathcal{S}\mathrm{hv}(\mathcal{C})$ (here $\mathcal{C}$ is equipped with the effective epimorphism topology [10, A.6.2.4]): see [10, A.7.4.1] and [10, A.7.5.2] for the definitions of these ∞-categories, and [10, A.7.1.2] for the definition of bounded ∞-topoi.

If $\mathcal{C} \to \mathcal{D}$ is a morphism of bounded ∞-pretopoi (equipped with the effective epimorphism topology), then it satisfies the assumptions of 3.4 (in a larger universe), thereby the equivalence of ∞-categories guarantees that every cofiltered diagram of bounded coherent ∞-topoi satisfies the condition $(*)$.

3.6. Let $\mathcal{X}$ denote the cofiltered limit of the diagram p in $(*)$. Let $I^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be the diagram of $\mathcal{X}_i$ obtained via the embedding $\infty\mathcal{T}\mathrm{op} \subseteq \widehat{\mathcal{C}\mathrm{at}}_\infty$, where $\widehat{\mathcal{C}\mathrm{at}}_\infty$ denote the ∞-category of (not necessarily small) ∞-categories. Then this diagram classifies a Cartesian fibration $q : Z \to I$. Using [9, 3.3.3.2] and [9, 6.3.3.1], the underlying ∞-category of the ∞-topos $\mathcal{X}$ can be identified with the ∞-category of cartesian sections of q.

3.7. Let $\pi_* : \mathcal{X} \to \mathcal{S}$ and $\pi_{i*} : \mathcal{X}_i \to \mathcal{S}$ denote the unique (up to homotopy) geometric morphisms, respectively. Let $p_{i*} : \mathcal{X} \to \mathcal{X}_i$ denote the geometric morphism associated to the limit ∞-topos $\mathcal{X}$.

The virtue of the condition $(*)$ is that one can describe p_i^* as in the case of 1-categories (see, for example, [11, Lemma 2]):

Lemma 3.8. *Fix $i \in I$ and an integer $n \geq 0$. Let F be an $(n-1)$-truncated object of $\mathcal{X}_i$. Then the section of q, given by*

$$j \mapsto \operatorname*{colim}_{k \geq i,j} p_{jk_*} p_{ik}^* F,$$

is equivalent to $p_i^* F$.

Proof. The condition $(*)$ guarantees that this section is cartesian, so it can be viewed as an object of $\mathcal{X}$. Note that the pushforward p_{i*} sends an object $\{G_j\}$ of $\mathcal{X}$ to its i-th component G_i. Therefore, it suffices to observe that there is a chain of equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathcal{X}}(\{\operatorname*{colim}_{k \geq i,j} p_{jk_*} p_{ik}^* F\}, \{G_j\}) &\simeq \lim_{j \geq i} \lim_{k \geq j} \mathrm{Map}_{\mathcal{X}_j}(p_{jk_*} p_{ik}^* F, p_{jk_*} G_k) \\
&\simeq \lim_{k \geq i} \mathrm{Map}_{\mathcal{X}_k}(p_{ik}^* F, G_k) \\
&\simeq \mathrm{Map}_{\mathcal{X}_i}(F, G_i).
\end{aligned}$$

$\square$

3.9. The following result shows that profinite shapes preserve limits of ∞-topoi (under the condition $(*)$):

Proposition 3.10. *The canonical map*

$$\mathrm{Sh}_\pi(\mathcal{X}) \to \lim_{i \in I^{\mathrm{op}}} \mathrm{Sh}_\pi(\mathcal{X}_i)$$

is an equivalence of profinite spaces.

Proof. This amounts to the assertion that for each π-finite space A, the canonical map

$$\operatorname*{colim}_{i\in I^{\mathrm{op}}} \pi_{i*}\pi_i^* A \to \pi_*\pi^* A$$

is an equivalence of spaces. Fix $i \in I$. Then there is a chain of equivalences

$$\pi_*\pi^* A \simeq \pi_{i*}p_{i*}p_i^*\pi_i^* A \simeq \pi_{i*} \operatorname*{colim}_{k\geq i} p_{ik*}\pi_k^* A \simeq \operatorname*{colim}_{k\geq i} \pi_{k*}\pi_k^* A,$$

where the second and third equivalences follow from 3.8 and [10, A.2.3.1], respectively. □

3.11. We mimic the proof of [10, E.2.7.1] to investigate the relationship between limits of ∞-topoi and locally constant constructible objects:

Theorem 3.12. *The canonical map*

$$\operatorname*{colim}_{i\in I} \mathcal{X}_i^{\mathrm{lcc}} \to \mathcal{X}^{\mathrm{lcc}}$$

is an equivalence of ∞-categories.

Proof. Let p^* denote the canonical map in the statement. According to [10, E.2.3.2], the ∞-categories $\mathcal{X}^{\mathrm{lcc}}$ and $\mathcal{X}_i^{\mathrm{lcc}}$ are ∞-pretopoi [10, A.6.1.1]. Since small filtered colimits of ∞-pretopoi can be computed in $\widehat{\mathcal{C}\mathrm{at}}_\infty$ [10, A.8.3.1], one can view p^* as a morphism of ∞-pretopoi [10, A.6.4.1]. To prove that p^* is an equivalence, by virtue of [10, A.9.2.1], it suffices to show that p^* satisfies the following conditions:

(i) The functor p^* is essentially surjective.
(ii) For every (-1)-truncated morphism $u : F' \to F$ in $\operatorname{colim} \mathcal{X}_i^{\mathrm{lcc}}$, if p^*u is an equivalence in $\mathcal{X}^{\mathrm{lcc}}$, then u is an equivalence.

To establish the condition (i), let $H \in \mathcal{X}^{\mathrm{lcc}}$. Then one can choose a full subcategory $\mathcal{K} \subseteq \mathcal{S}$ spanned by finitely many π-finite spaces such that H is $\mathcal{K}$-constructible; see [10, E.2.7.2]. Then [10, E.2.7.7] supplies an essentially unique geometric morphism $g_* : \mathcal{X}^{\mathrm{lcc}} \to \mathrm{Fun}(\mathcal{K}^{\simeq}, \mathcal{S})$ such that $H \simeq g^*\iota$, where $\iota \in \mathrm{Fun}(\mathcal{K}^{\simeq}, \mathcal{S})$ denotes the inclusion functor (here $\mathcal{K}^{\simeq}$ denotes the largest Kan complex contained in $\mathcal{K}$). Since $\mathcal{K}^{\simeq} \in \mathcal{S}_\pi$, it follows from 3.10 that g_* factors through some geometric morphism $g_{i*} : \mathcal{X}_i^{\mathrm{lcc}} \to \mathrm{Fun}(\mathcal{K}^{\simeq}, \mathcal{S})$. Therefore, $H \simeq g^*\iota \simeq p_i^* g_i^* \iota$ is in the essential image of the functor p^*.

It remains to verify the condition (ii). Fix such a morphism u. One can find i such that u is the image of some morphism $u_i : F_i' \to F_i$ in $\mathcal{X}_i^{\mathrm{lcc}}$. Enlarging i if necessarily, one can assume that u_i is a (-1)-truncated morphism in $\mathcal{X}_i^{\mathrm{lcc}}$. Now [10, E.2.6.7] supplies a complement $G \in \mathcal{X}_i^{\mathrm{lcc}}$ of F_i' as a subobject of F_i. The assumption that p^*u is an equivalence implies $p_i^*G \simeq \emptyset$, where $\emptyset$ denote the initial object of $\mathcal{X}$. Applying [10, E.2.6.7] to the (-1)-truncated object $\tau_{\leq -1}G$ of $\mathcal{X}_i^{\mathrm{lcc}}$, one obtains a decomposition $* \simeq \tau_{\leq -1}G \coprod G'$, where $*$ denote the final object of $\mathcal{X}_i^{\mathrm{lcc}}$. Let T denote the topological space $\{t, t'\}$ with the discrete topology. Using [10, A.6.4.4], one can find a geometric morphism $g_i{}^* : \mathcal{S}\mathrm{hv}(T) \to \mathcal{X}_i^{\mathrm{lcc}}$ which classifies the decomposition of $*$ (here t and t' correspond to $\tau_{\leq -1}G$ and G', respectively). Note that $p_i^*\tau_{\leq -1}G$ is initial in $\mathcal{X}$, so that p_i^*G' is final in $\mathcal{X}$. Therefore, [10, A.6.4.4] implies that the composite $\mathcal{X} \xrightarrow{p_{i*}} \mathcal{X}_i \xrightarrow{g_{i*}} \mathcal{S}\mathrm{hv}(T)$ factors through the inclusion $i_* : \mathcal{S}\mathrm{hv}(\{t'\}) \hookrightarrow \mathcal{S}\mathrm{hv}(T)$. Since

$\mathcal{S}\mathrm{hv}(\{t'\}) \simeq \mathcal{S}$, one can find a dotted arrow which makes the left triangle commute in the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "p_{i*}"] \arrow[dr] & \mathcal{X}_i \arrow[r, "g_{i*}"] \arrow[d, dotted] & \mathcal{S}\mathrm{hv}(T) \\ & \mathcal{S}\mathrm{hv}(\{t'\}). \arrow[ur, "i_*"'] & \end{tikzcd}$$

Remark that the right triangle does not necessarily commute; nonetheless, since $\mathcal{S}\mathrm{hv}(T)$ is a profinite ∞-topos [10, E.2.4.3], 3.10 guarantees that enlarging i if necessary, one can assume that the right triangle also commutes. Then, $\tau_{\leq -1}G$ is initial in $\mathcal{X}_i$ and thus so is G. Consequently, u is an equivalence, thereby completing the proof. □

4. Proper Base Change for Lcc Étale Sheaves of Spaces

4.1. Suppose we are given a commutative diagram of ∞-categories

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "q'_*"] \arrow[d, "p'_*"] & \mathcal{C} \arrow[d, "p_*"] \\ \mathcal{D}' \arrow[r, "q_*"] & \mathcal{D}\,. \end{tikzcd}$$

Assume that q_* and q'_* admit left adjoints denoted by q^* and q'^*, respectively. Consider the composition

$$q^*p_* \to q^*p_*q'_*q'^* \xrightarrow{\sim} q^*q_*p'_*q'^* \to p'_*q'^*,$$

where the first and third arrows are induced by a unit for the adjunction (q'^*, q'_*) and a counit for the adjunction (q^*, q_*), respectively. This composite is referred to as the *push-pull transformation* (sometimes called the *Beck-Chevalley transformation*).

4.2. The purpose of this section is to prove the non-abelian proper base change theorem 1.2 whose proof is deferred until the end of the section. As in the case of the usual proper base change theorem, we will show that the morphism θ in 1.2 is an equivalence by looking at stalks.

4.3. So, we begin with the special case of 1.2 where the morphism g in (1.1.1) is a point of Y; consider a pullback diagram of quasi-compact and quasi-separated schemes

$$\begin{tikzcd} X_{\overline{s}} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} k \arrow[r, "\overline{s}"] & S \end{tikzcd}$$

where f is proper and k is a separably closed field. There is a commutative diagram of ∞-topoi

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}_{\text{ét}}(X_{\overline{s}}) \arrow[r, "p'_*"] \arrow[d, "f'_*"] & \mathcal{S}\mathrm{hv}_{\text{ét}}(X) \arrow[d, "f_*"] \\ \mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k) \arrow[r, "p_*"] & \mathcal{S}\mathrm{hv}_{\text{ét}}(S). \end{tikzcd}$$

4.4. For each object $F \in \mathcal{S}\mathrm{hv}_{\text{ét}}(S)$, let $F_{\overline{s}}$ denote the pullback p^*F. To give an explicit description of the pullbacks, consider a continuous functor

$$\delta : \text{Ét}(S) \to \text{Ét}(\operatorname{Spec} k) : S' \mapsto \operatorname{Spec} k \times_S S'.$$

Then the restriction functor

$$\hat{p}_* : \mathcal{P}_{\text{ét}}(\operatorname{Spec} k) \xrightarrow{\circ \mathrm{N}\,\delta^{\mathrm{op}}} \mathcal{P}_{\text{ét}}(S)$$

admits a left adjoint $\hat{p}^*$ given by the left Kan extension along $\mathrm{N}\,\delta^{\mathrm{op}}$ [9, 4.3.3.7], given on objects by the formula

$$(\hat{p}^*F)(Y) = \operatorname*{colim}_{(S',\rho)\in I_Y^{\mathrm{op}}} F(S'). \tag{4.4.1}$$

Here I_Y is the 1-category whose objects are pairs (S', ρ), where $S' \in \text{Ét}(S)$ and $\rho : Y \to \delta(S')$ is a morphism in $\text{Ét}(\operatorname{Spec} k)$, and for which a morphism

$$(S'', \rho') \to (S', \rho)$$

is a morphism $a : S'' \to S'$ in $\text{Ét}(S)$ such that $\delta(a) \circ \rho' = \rho$.

Note that if $Y = \operatorname{Spec} k$, then one can identify I_Y with the category $I_{\overline{s}}$ of étale neighborhoods of the geometric point $\overline{s} : \operatorname{Spec} k \to S$.

Lemma 4.5. *Let G be an object of $\mathcal{P}_{\text{ét}}(\operatorname{Spec} k)$. Then the canonical map*

$$\alpha_G : \operatorname{Map}_{\mathcal{P}_{\text{ét}}(\operatorname{Spec} k)}(*, G) \to \operatorname{Map}_{\mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k)}(*, LG)$$

is a weak homotopy equivalence, where $L : \mathcal{P}_{\text{ét}}(\operatorname{Spec} k) \to \mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k)$ denotes the sheafification functor.

Proof. Let Γ and γ denote the global section functors on $\mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k)$ and $\mathcal{P}_{\text{ét}}(\operatorname{Spec} k)$, respectively. We prove by induction that the map

$$\alpha_G : \gamma G \to \Gamma LG,$$

which is induced by the unit for the adjunction between $\mathcal{P}_{\text{ét}}(\operatorname{Spec} k)$ and $\mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k)$, is n-connective for every $n \geq 0$. The case $n = 0$ follows from the property of sheafification that every section of LG locally comes from that of G (up to equivalence) and that every étale covering of the final object of $\text{Ét}(\operatorname{Spec} k)$ admits a refinement by the identity on the final object.

Suppose that $n > 0$. It suffices to prove that for every pair of maps $\eta, \eta' : * \to \gamma G$, the induced map of fiber products

$$\beta_G : * \times_{\gamma G} * \to * \times_{\Gamma LG} *$$

is $(n-1)$-connective. Let $H = * \times_G * \in \mathcal{P}_{\text{ét}}(\operatorname{Spec} k)$. Using the fact that γ, Γ, and L all commute with finite limits, one can identify β_G with α_H; the desired result now follows from the inductive hypothesis. $\square$

4.6. We will need the following lemmas to prove the special case of proper base change theorem:

Lemma 4.7. *Let $k \subset K$ be an extension of separably closed fields and let X be a proper scheme over* $\operatorname{Spec} k$*. Let $X_{k'} := X \times_{\operatorname{Spec} k} \operatorname{Spec} K$. For each object $F \in \mathcal{S}\mathrm{hv}^{\mathrm{lcc}}_{\text{ét}}(X)$, the canonical map*

$$\operatorname{Map}_{\mathcal{S}\mathrm{hv}_{\text{ét}}(X)}(*, F) \to \operatorname{Map}_{\mathcal{S}\mathrm{hv}_{\text{ét}}(X_K)}(*, F')$$

is a weak homotopy equivalence, where F' denotes the pullback of F to $\mathcal{S}\mathrm{hv}_{\text{ét}}(X_{k'})$.

Proof. According to [17, Exposé XII.5.4], the projection $X_k \to X$ induces a weak equivalence on the profinite étale homotopy types, which in turn implies that the natural map

$$\mathrm{Sh}_\pi(\mathcal{S}\mathrm{hv}_{\text{ét}}(X_K)^\wedge) \to \mathrm{Sh}_\pi(\mathcal{S}\mathrm{hv}_{\text{ét}}(X)^\wedge)$$

is an equivalence of profinite spaces by [5, 4.2.2], where $\mathcal{S}\mathrm{hv}_{\text{ét}}(X_K)^\wedge$ and $\mathcal{S}\mathrm{hv}_{\text{ét}}(X)^\wedge$ denote the hypercompletions of the ∞-topoi $\mathcal{S}\mathrm{hv}_{\text{ét}}(X_K)$ and $\mathcal{S}\mathrm{hv}_{\text{ét}}(X)$, respectively. By virtue of 2.8, the natural map $\mathrm{Sh}_\pi(\mathcal{S}\mathrm{hv}_{\text{ét}}(X_K)) \to \mathrm{Sh}_\pi(\mathcal{S}\mathrm{hv}_{\text{ét}}(X))$ is also an equivalence of profinite spaces. Now [10, E.2.3.3] guarantees that the functor $\mathcal{S}\mathrm{hv}^{\mathrm{lcc}}_{\text{ét}}(X) \to \mathcal{S}\mathrm{hv}^{\mathrm{lcc}}_{\text{ét}}(X_K)$ is an equivalence of ∞-categories, which completes the proof. $\square$

4.8. Using [17, Exposé XII.5.5] in lieu of [17, Exposé XII.5.4], an analogous statement holds:

Lemma 4.9. *Let R be a henselian local ring and let X be a proper scheme over* $\operatorname{Spec} R$ *with closed fiber X_k. For each object $F \in \mathcal{S}\mathrm{hv}^{\mathrm{lcc}}_{\text{ét}}(X)$, the canonical map*

$$\operatorname{Map}_{\mathcal{S}\mathrm{hv}_{\text{ét}}(X)}(*, F) \to \operatorname{Map}_{\mathcal{S}\mathrm{hv}_{\text{ét}}(X_k)}(*, F_k)$$

is a weak homotopy equivalence, where F_k denotes the pullback of F to $\mathcal{S}\mathrm{hv}_{\text{ét}}(X_k)$.

4.10. Note that the essentially unique geometric morphism

$$\mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k) \to \mathcal{S} : F \mapsto \operatorname{Map}_{\mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k)}(*, F)$$

is an equivalence of ∞-categories as $\mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k)$ is an 1-localic ∞-topos and it is the case for the usual 1-topoi. Under this equivalence, the pushforward functor f'_* in 4.3 can be identified with the global section functor on the ∞-topos $\mathcal{S}\mathrm{hv}_{\text{ét}}(X_{\overline{s}})$.

We now formulate and give a proof of the special case of the proper base change theorem under this identification:

Proposition 4.11. *In the situation of 4.3, for each object $F \in \mathcal{S}\mathrm{hv}^{\mathrm{lcc}}_{\text{ét}}(X)$, the push-pull morphism*

$$\theta : (f_*F)_{\overline{s}} \to \operatorname{Map}_{\mathcal{S}\mathrm{hv}_{\text{ét}}(X_{\overline{s}})}(*, p'^*F)$$

is an equivalence in the ∞-category $\mathcal{S}$.

Proof. Recall the strict henselization of S at $\overline{s}$:

$$\mathcal{O}_{S,\overline{s}} := \operatorname*{colim}_{(U,\overline{u}) \in I^{\mathrm{op}}_{\overline{s}}} \Gamma(U, \mathcal{O}_U) \tag{4.11.1}$$

where $I_{\overline{s}}$ is the category of étale neighborhoods of $\overline{s}$, which is cofiltered. Fix a separable closure $k(s)^{\mathrm{sep}}$ of $k(s)$ in k. Consider a commutative diagram of schemes

$$\begin{tikzcd}
X_{\overline{s}} \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X_{(\overline{s})} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec}(k(s)^{\mathrm{sep}}) \arrow[r] & \operatorname{Spec}(\mathcal{O}_{S,\overline{s}}) \arrow[r] & S
\end{tikzcd}$$

where all squares are cartesian. There are canonical maps

$$(f_*F)_{\overline{s}} \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_{(\overline{s})})}(*, F_{(\overline{s})}) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X')}(*, F') \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_{\overline{s}})}(*, p'^*F)$$

where $F_{(\overline{s})}$ and F' denote the pullbacks of F to $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_{(\overline{s})})$ and $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X')$, respectively. Since the second and third arrows are equivalences by 4.9 and 4.7, respectively, it remains to show that the first arrow is an equivalence.

Under the identification $\mathcal{S}\mathrm{hv}_{\acute{e}t}(\operatorname{Spec} k) \simeq \mathcal{S}$, the stalk $(f_*F)_{\overline{s}}$ is identified with its global section which is equivalent to the global section of the presheaf $\hat{p}^*(f_*F)$ by 4.5. Using the formula (4.4.1) and the fact that colimits are preserved under the change of cofinal map of index categories [9, 4.1.1.8], there is an identification

$$(f_*F)_{\overline{s}} \simeq \operatorname*{colim}_{(U,\overline{u}) \in I_{\overline{s}}^{\mathrm{aff,op}}} F(X \times_S U)$$

where $I_{\overline{s}}^{\mathrm{aff}} \subseteq I_{\overline{s}}$ is the full subcategory spanned by the affine étale neighborhoods of $\overline{s}$. On the other hand, applying the cofinality argument to (4.11.1), there is an isomorphism of schemes

$$X_{(\overline{s})} \simeq \lim_{(U,\overline{u}) \in I_{\overline{s}}^{\mathrm{aff}}} X \times_S U.$$

Since the projection $X \times_S U \to X$ is étale, one can identify $F(X \times_S U)$ with $\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X \times_S U)}(*, F_{X \times_S U})$, where $F_{X\times_S U}$ is the restriction of F to $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X \times_S U)$. Consequently, the desired equivalence will follow if the canonical map

$$\operatorname*{colim}_{(U,\overline{u}) \in I_{\overline{s}}^{\mathrm{aff,op}}} \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X \times_S U)}(*, F_{X \times_S U}) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_{(\overline{s})})}(*, F_{(\overline{s})})$$

is an equivalence; this is guaranteed by 3.12, thereby completing the proof. □

4.12. We are now ready to prove the proper base change theorem 1.2:

Proof of 1.2. Every π-finite space is a truncated object of $\mathcal{S}$, and the pullbacks and pushforwards of geometric morphisms preserve n-truncated objects for every integer n. So, we may work with the hypercompletions of ∞-topoi and the associated geometric morphisms in (1.1.2). Since $\mathcal{S}\mathrm{hv}_{\acute{e}t}(Y')$ is locally coherent by 3.2 and [10, A.3.1.3], so is its hypercompletion $\mathcal{S}\mathrm{hv}_{\acute{e}t}(Y')^\wedge$ by [10, A.2.2.2]. Consequently, the ∞-categorical Deligne Completeness Theorem [10, A.4.0.5] guarantees that $\mathcal{S}\mathrm{hv}_{\acute{e}t}(Y')^\wedge$ has enough points. Therefore, one can check that θ is an equivalence after pulling back along a geometric morphism $p_* : \mathcal{S} \to \mathcal{S}\mathrm{hv}_{\acute{e}t}(Y')$ (given by a point of $\mathcal{S}\mathrm{hv}_{\acute{e}t}(Y')^\wedge$ followed by the fully faithful geometric morphism $\mathcal{S}\mathrm{hv}_{\acute{e}t}(Y')^\wedge \to \mathcal{S}\mathrm{hv}_{\acute{e}t}(Y')$). Since $\mathcal{S}\mathrm{hv}_{\acute{e}t}(Y')$ is an 1-localic ∞-topos and points of the étale 1-topos of Y' are determined by geometric points $\overline{s} : \operatorname{Spec} k \to Y'$, where k is a separably closed field [16, Exposé VIII.7.9], one can assume that the point p of $\mathcal{S}\mathrm{hv}_{\acute{e}t}(Y')$ is induced by a geometric point $\overline{s}$ of Y'. Consider the commutative diagram of ∞-topoi

$$\begin{tikzcd}
\mathcal{S}\mathrm{hv}_{\acute{e}t}(X'_{\overline{s}}) \arrow[r, "p'_*"] \arrow[d, "f''_*"] & \mathcal{S}\mathrm{hv}_{\acute{e}t}(X') \arrow[r, "g'_*"] \arrow[d, "f'_*"] & \mathcal{S}\mathrm{hv}_{\acute{e}t}(X) \arrow[d, "f_*"] \\
\mathcal{S}\mathrm{hv}_{\acute{e}t}(\operatorname{Spec} k) \arrow[r, "p_*"] & \mathcal{S}\mathrm{hv}_{\acute{e}t}(Y') \arrow[r, "g_*"] & \mathcal{S}\mathrm{hv}_{\acute{e}t}(Y),
\end{tikzcd}$$

where the left square is associated to the cartesian square defining $X'_{\bar{s}} := \operatorname{Spec} k \times_{Y'} X'$. In that case, $p^*\theta$ fits into a commutative triangle

$$\begin{tikzcd} & p^* f'_* g'^* F \arrow[dr, "\theta'"] & \\ p^* g^* f_* F \arrow[ur, "p^*\theta"] \arrow[rr, "\theta''"] & & f''_* p'^* g'^* F, \end{tikzcd}$$

where θ' and θ'' are equivalences by virtue of the special case of proper base change theorem 4.11. Therefore, $p^*\theta$ is an equivalence as desired. $\square$

5. Application: Profinite Shapes of Products

5.1. For each scheme X, let $\mathrm{Sh}(X)$ and $\mathrm{Sh}_\pi(X)$ denote the shape and profinite shape of the ∞-topos $\mathcal{S}\mathrm{hv}_{\text{ét}}(X)$, respectively.

5.2. Let X be a locally noetherian scheme, and let $\overline{x} : \operatorname{Spec} K \to X$ be a geometric point of X. For each integer $n \geq 0$, the n-th homotopy (pro-)group $\pi_n(X, \overline{x})$ of the pointed scheme $(X, \overline{x})$ is defined to be the n-th homotopy pro-group of the étale homotopy type of X with the associated base point (here the étale homotopy type of X can be replaced by the shape of the hypercompletion of the ∞-topoi $\mathcal{S}\mathrm{hv}_{\text{ét}}(X)$; see [2, 6.0.5]).

It follows from [4, 3.4.2] that the fundamental groups of locally noetherian schemes are isomorphic to the enlarged étale fundamental groups in the sense of [15, Exposé X.6]. In particular, their profinite completions are isomorphic to the étale fundamental groups in the sense of [13, Exposé V.7].

By the universal property of profinite completions, there is a natural map

$$\widehat{\pi}_n(X, x) \to \pi_n(\widehat{\mathrm{h}}(X), x) \tag{5.2.1}$$

of profinite groups, where $\widehat{\pi}_n(X, x)$ denotes the profinite completion of $\pi_n(X, \overline{x})$ and $\widehat{\mathrm{h}}(X)$ denotes the profinite étale homotopy type of X.

When $n = 1$, (5.2.1) is an isomorphism by virtue of [1, 3.7]. From this point of view, the following theorem generalizes the commutativity of étale fundamental groups (or, equivalently, fundamental groups of profinite étale homotopy types) with finite products of proper schemes over a separably closed field [13, Exposé X.1.7] to all higher homotopy groups of profinite étale homotopy types of such schemes:

Theorem 5.3. *Let $f : X \to \operatorname{Spec} k$ be a separated and finite type morphism of schemes, and let $g : Y \to \operatorname{Spec} k$ be a proper morphism of schemes, where k is a separably closed field. Then the canonical map*

$$\mathrm{Sh}_\pi(X \times_k Y) \to \mathrm{Sh}_\pi(X) \times \mathrm{Sh}_\pi(Y)$$

is an equivalence of profinite spaces.

Proof. There is a commutative diagram of ∞-topoi

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}_{\text{ét}}(X \times_k Y) \arrow[r, "p'_*"] \arrow[d, "q'_*"] & \mathcal{S}\mathrm{hv}_{\text{ét}}(Y) \arrow[d, "q_*"] \\ \mathcal{S}\mathrm{hv}_{\text{ét}}(X) \arrow[r, "p_*"] & \mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k). \end{tikzcd} \tag{5.3.1}$$

Recall the commutative triangle (1.6.2) of pro-spaces, which is induced by canonical maps. Using the fact that profinite completions commute with finite limits, the desired equivalence will follow if one can show that both the left diagonal and the bottom maps in (1.6.2) are equivalences after applying the profinite completion functor. For the left diagonal map, we need to show that for each π-finite space A, the canonical map

$$p_* p^* q_* q^* A \to p_* {q'}_* {p'}^* q^* A$$

is an equivalence of spaces; since g is proper and the sheaf $q^* A$ is locally constant constructible, this is a consequence of the proper base change for lcc étale sheaves of spaces 1.2.

For the bottom map, choose cofiltered systems $\{A_i \in \mathcal{S}\}_{i \in \mathcal{I}}$ and $\{B_j \in \mathcal{S}\}_{j \in \mathcal{J}}$ in such a way that $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$ are equivalent to the cofiltered limits of the functors corepresented by A_i and B_j, respectively. Let A be a π-finite space. The bottom map evaluated at the final object of $\mathcal{S}$ is clearly an equivalence, so we may assume that A is n-truncated for some $n \geq -1$. There is a chain of equivalences

$$\begin{aligned} (\mathrm{Sh}(X) \circ \mathrm{Sh}(Y))(A) &\simeq \mathrm{Sh}(X)(\operatorname*{colim}_{j \in \mathcal{J}^{\mathrm{op}}} \mathrm{Map}_{\mathcal{S}}(B_j, A)) \\ &\simeq \operatorname*{colim}_{j \in \mathcal{J}^{\mathrm{op}}} \mathrm{Sh}(X)(\mathrm{Map}_{\mathcal{S}}(B_j, A)) \\ &\simeq \operatorname*{colim}_{j \in \mathcal{J}^{\mathrm{op}}} \operatorname*{colim}_{i \in \mathcal{I}^{\mathrm{op}}} \mathrm{Map}_{\mathcal{S}}(A_i, \mathrm{Map}_{\mathcal{S}}(B_j, A)) \\ &\simeq (\mathrm{Sh}(X) \times \mathrm{Sh}(Y))(A), \end{aligned}$$

where the second equivalence follows from [10, A.2.3.1] that the pro-space $\mathrm{Sh}(X)$ commutes with filtered colimits of n-truncated objects of $\mathcal{S}$. $\square$

5.4. It is usually not true that (5.2.1) is an isomorphism for higher homotopy groups. Nevertheless, in some cases 5.3 can be stated in terms of the homotopy groups of (étale homotopy types of) schemes rather than the homotopy groups of the profinite étale homotopy types of the schemes:

Corollary 5.5. *In the situation of 5.3, assume further that X and Y are geometrically normal. Let $\overline{(x,y)} : \operatorname{Spec} K \to X \times_k Y$ be a geometric point of $X \times_k Y$. Let $\overline{x}$ and $\overline{y}$ denote the induced geometric points on X and Y, respectively. Then for each integer $n \geq 0$, the canonical map*

$$\pi_n(X \times_k Y, \overline{(x,y)}) \to \pi_n(X, \overline{x}) \times \pi_n(Y, \overline{y})$$

is an isomorphism of profinite groups.

Proof. Since X and Y are geometrically normal, $X \times_k Y$ is normal. Then one deduces from [1, 11.1] that the étale homotopy types of X, Y, and $X \times_k Y$ are all profinite. Equivalently, the shapes of the hypercompletions of the ∞-topoi $\mathcal{S}\mathrm{hv}_{\text{ét}}(X)$, $\mathcal{S}\mathrm{hv}_{\text{ét}}(Y)$, and $\mathcal{S}\mathrm{hv}_{\text{ét}}(X \times_k Y)$ are profinite spaces by combining [2, 6.0.5] with [5, 4.1.5] which guarantees that profinite completions in model category theory and in ∞-category theory are equivalent. Using 2.8, the statement follows from 5.3. $\square$

6. Application: Profinite Shapes of Symmetric Powers

6.1. Let X be a connected CW-complex. For each integer $n \geq 0$, the *n-th symmetric power* $\operatorname{Sym}^n X$ is defined as the quotient space X^n/S_n of the natural action of the symmetric group S_n on the n-fold product of X with itself. The classical Dold-Thom theorem [6, 6.10] shows that for each integer $i > 0$, the natural map

$$H_i(X;\mathbb{Z}) \to \pi_i(\operatorname*{colim}_{n\geq 0} \operatorname{Sym}^n X)$$

is an isomorphism.

In the setting of algebraic geometry, Arnav Tripathy proved in [18, Theorem 1] that for a proper, normal, and geometrically connected algebraic space X over a separably closed field, the natural map (1.8.1) is a $\natural$-isomorphism [1, 4.2] of pro-objects in the homotopy category of those connected pointed CW-complexes whose homotopy groups are all finite; in particular, the Dold-Thom theorem in the algebro-geometric setting holds.

6.2. The purpose of this section is to generalize [18, Theorem 1] by removing the assumption that X is normal. In contrast with Tripathy who made a detailed study of the étale fundamental group of $\operatorname{Sym}^n X$ and used Deligne's computation of the cohomology of $\operatorname{Sym}^n X$, we will use the qfh topology where the behavior of symmetric powers of algebraic spaces becomes categorical. Remark that the idea of using the qfh topology in the study of étale homotopy types appears in the work of Marc Hoyois [7].

As we will use various topologies associated to algebraic spaces, let us make it clear what the associated shapes mean:

Definition 6.3. Let $\mathcal{X}$ be an ∞-topos. The *shape of an object* $F \in \mathcal{X}$ is the shape of the ∞-topos $\mathcal{X}_{/F}$. The *profinite shape of* $F \in \mathcal{X}$ is the profinite completion of the shape of F. Let $\operatorname{Sh}(F)$ and $\operatorname{Sh}_\pi(F)$ denote the shape and the profinite shape of F, respectively.

6.4. Let T be a usual 1-topos, and let F be an object of T. The author defined the topological type $\mathrm{h}(F)$ and the profinite topological type $\widehat{\mathrm{h}}(F)$ of F in [4, 2.3.2] and [4, 4.1.7], respectively; they are compatible with the definitions above in the following sense:

Lemma 6.5. *Let $\mathcal{X}$ be a 1-localic ∞-topos, and let F be an object of $\mathcal{X}$. Then the shape $\operatorname{Sh}((\mathcal{X}_{/F})^\wedge)$ of the hypercomplete ∞-topos $(\mathcal{X}_{/F})^\wedge$ is equivalent to the topological type $\mathrm{h}(F)$ of F as an object of the 1-topos $\operatorname{Disc}(\mathcal{X})$ under the equivalence of the model categorical and the infinity categorical pro-spaces [2, 6.0.1]. Moreover, the profinite shape $\operatorname{Sh}_\pi(F)$ of $F \in \mathcal{X}$ is equivalent to the profinite topological type $\widehat{\mathrm{h}}(F)$ of $F \in \operatorname{Disc}(\mathcal{X})$ under the equivalence of the model categorical and the infinity categorical profinite spaces [2, 7.4.9].*

Proof. Combine [2, 6.0.4] with [4, 2.3.16] that $\mathrm{h}(F)$ is equivalent to the topological type of the 1-topos $\operatorname{Disc}(\mathcal{X})_{/F}$. For profinite completions, use [5, 4.1.5] and 2.8. $\square$

6.6. Let (Sch/S) denote the category of S-schemes. Let $\mathcal{T}_{\text{ét}}$ and $\mathcal{T}_{\text{qfh}}$ denote the ∞-category of $\mathcal{S}$-valued sheaves with respect to the big étale and the big qfh topologies on (the nerve of) (Sch/S), respectively. The identity functor $(\mathrm{Sch}/S)_{\text{ét}} \to (\mathrm{Sch}/S)_{\text{qfh}}$ is continuous and commutes with finite limits, thereby inducing a geometric morphism of ∞-topoi

$$i_* : \mathcal{T}_{\text{qfh}} \to \mathcal{T}_{\text{ét}} .$$

Let X be an algebraic space over a scheme S. Regarding X as a 0-truncated object of $\mathcal{T}_{\text{ét}}$, one can define its shape $\mathrm{Sh}(X)$ and profinite shape $\mathrm{Sh}_\pi(X)$ (see 6.3). In the case where X is a scheme, these definitions do not conflict with 5.1 (cf. [4, 3.2.3]).

6.7. Fix an integer $n \geq 0$. Let S_n denote the symmetric group on n letters. Let $\mathrm{Sub}(S_n)$ denote the partially ordered set of subgroups of S_n. For each subgroup $H \leq S_n$, let $o(H)$ denote the set of orbits of the induced action of H on the n letters.

Let $\mathcal{C}$ be an ∞-category which admits colimits and finite limits. According to Hoyois [7, p.3], the n-th symmetric power functor $\mathrm{Sym}^n : \mathcal{C} \to \mathcal{C}$ is defined by the formula

$$\mathrm{Sym}^n X = \operatorname*{colim}_{H \in \mathrm{Sub}(S_n)} X^{o(H)},$$

where $X^{o(H)}$ denotes the $|o(H)|$-fold product of X with itself; when $\mathcal{C} = \mathcal{S}$, this is compatible with the usual definition of symmetric powers of spaces, and when $\mathcal{C}$ is a 1-category, the symmetric powers behave like categorical quotients in a sense that $\mathrm{Sym}^n X$ is equivalent to the coequalizer of the diagram

$$S_n \times X^n \rightrightarrows X^n,$$

where the two maps are the natural action of S_n on the n-fold product X^n of X with itself and the projection onto the second factor.

6.8. Let X be a locally of finite type and separated algebraic space over a scheme S. Associated to the natural action of S_n on the n-fold product X^n of X over S is the groupoid $S_n \times X^n \rightrightarrows X^n$ in algebraic spaces over S. It follows from [14, 5.3] that a GC quotient $q : X^n \to \mathrm{Sym}^n X$ of the groupoid exists (see [14, 3.17] for the definition of a GC quotient).

The advantage of the qfh topology over the étale topology is that the GC quotient and the diagonal morphism

$$q : X^n \to \mathrm{Sym}^n X$$

$$j : S_n \times X^n \to X^n \times_{\mathrm{Sym}^n X} X^n$$

are qfh coverings of algebraic spaces. In other words, the pullbacks i^*q and i^*j are epimorphisms in the category of qfh sheaves of sets on S.

Remark 6.9. The virtue of q and j being qfh coverings is that the pullback of the symmetric power $\mathrm{Sym}^n X$ becomes a categorical quotient in the category of qfh sheaves of sets on S, even if $\mathrm{Sym}^n X$ itself is not a categorical quotient in the category of big étale sheaves on S:

Proposition 6.10. *Let X be a locally of finite type and separated algebraic space over a scheme S. Then the canonical map*

$$\mathrm{Sym}^n(i^*X) \to i^*(\mathrm{Sym}^n X)$$

is an equivalence in $\mathcal{T}_{\text{qfh}}$.

Proof. It follows from [7, 2.2] that $\tau_{\leq 0}\mathcal{T}_{\mathrm{qfh}} \subseteq \mathcal{T}_{\mathrm{qfh}}$ is stable under symmetric powers. Combining this with the fact that the pullback i^* preserves 0-truncated objects, it amounts to proving the equivalence in the usual 1-topos of qfh sheaves of sets on S. Since i^* preserves finite limits, it suffices to show that the diagram of big étale sheaves of sets on S

$$S_n \times X^n \rightrightarrows X^n \longrightarrow \mathrm{Sym}^n X$$

defining $\mathrm{Sym}^n X$ pulls back to a coequalizer of qfh sheaves of sets on S. Invoking the fact that i^*q is an (effective) epimorphism, i^* applied to the diagram

$$X^n \times_{\mathrm{Sym}^n X} X^n \rightrightarrows X^n \longrightarrow \mathrm{Sym}^n X$$

which is induced by the fiber product $X^n \times_{\mathrm{Sym}^n X} X^n$ is a coequalizer of qfh sheaves of sets on S. We complete the proof by observing that one still has a coequalizer diagram after replacing $i^*(X^n \times_{\mathrm{Sym}^n X} X^n)$ of the coequalizer by $i^*(S_n \times X^n)$ using the epimorphism i^*j. $\square$

Lemma 6.11. *Let X be a quasi-compact, quasi-separated, and locally noetherian algebraic space over a scheme S. Then the natural map*

$$\mathrm{Sh}_\pi(i^*X) \to \mathrm{Sh}_\pi(X)$$

is an equivalence of profinite spaces.

Proof. Using 6.5 and the profinite hypercover descent [4, 4.2.11], we may assume that X is a scheme. Consider a morphism of 1-topoi induced by the identity functor $(\mathrm{Sch}/X)_{\text{ét}} \to (\mathrm{Sch}/X)_{\mathrm{qfh}}$ which is continuous and commutes with finite limits. By virtue of [8, Theorem 1] and its proof, the morphism of topoi induces isomorphisms on non-abelian first cohomology groups for locally constant sheaves of finite groups, and on global sections for locally constant sheaves of finite sets. On the other hand, [19, 3.4.4] guarantees that it induces isomorphisms on abelian cohomology groups for locally constant sheaves of finite abelian groups, which completes the proof. $\square$

Theorem 6.12. *Let X be a proper algebraic space over a separably closed field k. Then the natural map*

$$\mathrm{Sym}^n(\mathrm{Sh}_\pi(X)) \to \mathrm{Sh}_\pi(\mathrm{Sym}^n X)$$

is an equivalence of profinite spaces.

Proof. Consider a commutative square of profinite spaces

$$\begin{tikzcd}
\mathrm{Sym}^n(\mathrm{Sh}_\pi(i^*X)) \arrow[r] \arrow[d] & \mathrm{Sh}_\pi(i^*\,\mathrm{Sym}^n X) \arrow[d] \\
\mathrm{Sym}^n(\mathrm{Sh}_\pi(X)) \arrow[r] & \mathrm{Sh}_\pi(\mathrm{Sym}^n X).
\end{tikzcd}$$

By virtue of 6.11, the vertical arrows are equivalences. To complete the proof, it suffices to show that the top horizontal arrow is an equivalence. The commutativity of profinite shapes and products 5.3 (which also holds for algebraic spaces by virtue of 6.5 and [4, 4.2.11]) combined with 6.11 supplies that for each $H \leq S_n$, the canonical map

$$\mathrm{Sh}_\pi((i^*X)^{o(H)}) \to (\mathrm{Sh}_\pi(i^*X))^{o(H)}$$

is an equivalence of profinite spaces. Passing to the colimit over $H \in \mathrm{Sub}(S_n)$, the left diagonal arrow in the following commutative diagram of profinite spaces is an equivalence:

$$\begin{tikzcd} & \mathrm{Sym}^n(\mathrm{Sh}_\pi(i^*X)) \arrow[dr] & \\ \operatorname{colim} \mathrm{Sh}_\pi(\mathcal{T}_{\mathrm{qfh}_{/i^*X^{o(H)}}}) \arrow[ur] \arrow[rr] & & \mathrm{Sh}_\pi(\mathrm{Sym}^n i^*X). \end{tikzcd}$$

Invoking the fact that the profinite shape functor preserves colimits (because it is a left adjoint) and that the functor $\mathcal{T}_{\mathrm{qfh}}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ which carries each object $F \in \mathcal{T}_{\mathrm{qfh}}$ to the ∞-category $\mathcal{T}_{\mathrm{qfh}_{/F}}$ preserves small limits [9, 6.1.3.9], the bottom map is an equivalence and therefore so is the right diagonal. The desired equivalence now follows from 6.10. □

Remark 6.13.

(i) Assume further that X is connected and geometrically normal. By virtue of [1, 11.1], $\mathrm{h}(X)$ and $\mathrm{h}(\mathrm{Sym}^n X)$ are pro-objects in the homotopy category of those connected CW-complexes whose homotopy groups are all finite. As Hoyois pointed out to the author, it is not clear that they are profinite in the sense of [12, §2.7]. Nevertheless, both a $\natural$-isomorphism of profinite spaces in the sense of Artin-Mazur and a weak equivalence of profinite spaces in the sense of Gereon Quick [12, 2.6] are precisely those maps which induce isomorphisms on all homotopy groups. Therefore, using [12, 2.33] and 6.5, one can recover [18, Theorem 1].

(ii) In comparison with [3, 10.0.4]—which gives an alternative proof of [18, Theorem 1]—in the thesis of the author, the theorem above holds without the assumption that X is geometrically normal, and does not depend on the computation of the étale fundamental group of $\mathrm{Sym}^n X$ by Indranil Biswas and Amit Hogadi (see [3, 9.3.5]). It also fills in the gap in the proof of [3, 9.2.11] via 5.3.

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