AN EQUIVALENCE OF PROFINITE COMPLETIONS

CHANG-YEON CHOUGH

Abstract. The goal of this paper is to provide an equivalence of profinite completions of pro-spaces in model category theory and ∞-category theory. As an application, we show that the author's comparison theorem for algebro-geometric objects in model category theory recovers that of David Carchedi in ∞-category theory.

Contents

1. Introduction

1.1. Statement of the main results.

1.1.1. There are two existing notions of profinite completions of pro-simplicial sets. In model category theory, the profinite completion functor on pro-simplicial sets

$$\mathrm{Pro}(\mathbf{SSet}) \to \widehat{\mathbf{SSet}}$$

is defined to be the limit of level-wise profinite completion of simplicial sets; see [14, §2.7]. On the other hand, in ∞-category theory, the profinite completion functor on pro-spaces

$$\mathrm{Pro}(\mathcal{S}) \to \mathrm{Pro}(\mathcal{S}_\pi)$$

is defined as a left adjoint to the fully faithful embedding $\mathrm{Pro}(\mathcal{S}_\pi) \hookrightarrow \mathrm{Pro}(\mathcal{S})$; see [12, 3.6.1].

The goal of this paper is to prove that these two profinite completions are compatible with each other. Recall from [5, 4.2.3, 4.2.5] that the model categorical profinite completion admits a right adjoint to provide an adjunction

$$\mathrm{Pro}(\mathbf{SSet}) \rightleftarrows \widehat{\mathbf{SSet}},$$

which is, according to [5, 4.2.6], a Quillen adjunction with respect to the strict model category structure on pro-simplicial sets and Quick's model category structure on profinite spaces. In particular, one can consider the underlying ∞-functor

$$\mathrm{Pro}(\mathbf{SSet})_\infty \to \widehat{\mathbf{SSet}}_\infty$$

of the profinite completion of pro-simplicial sets. Recall also that the underlying ∞-category of $\mathrm{Pro}(\mathbf{SSet})$ (resp. $\widehat{\mathbf{SSet}}$) is equivalent to $\mathrm{Pro}(\mathcal{S})$ (resp. $\mathrm{Pro}(\mathcal{S}_\pi)$) by [2, 6.0.1] (resp. [2, 7.4.9]). All things considered, there is a diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Pro}(\mathbf{SSet})_\infty \arrow[r] \arrow[d, "\wr"] & \widehat{\mathbf{SSet}}_\infty \arrow[d, "\wr"] \\
\mathrm{Pro}(\mathcal{S}) \arrow[r] & \mathrm{Pro}(\mathcal{S}_\pi)
\end{tikzcd}$$

with the model categorical and ∞-categorical profinite completions in the top and bottom arrows respectively. By analyzing the vertical equivalences with respect to the Quillen adjunction of [5, 4.2.6], we show that this diagram commutes:

Theorem 1.1.2. (Theorem 4.1.5 in the text) *The underlying ∞-functor associated to the profinite completion on* $\mathrm{Pro}(\mathbf{SSet})$ *is equivalent to the profinite completion on* $\mathrm{Pro}(\mathcal{S})$.

Remark 1.1.3. This compatibility serves as a key for connecting profinite completion of topological types and that of shapes. Indeed, as an application, the comparison theorem for algebraic stacks via shapes will be deduced from the one via topological types.

1.1.4. Recall from [5, 4.3.18] that for a locally of finite type algebraic stack $\mathcal{X}$ over $\mathbb{C}$ and the associated topological stack $\mathcal{X}^{\mathrm{top}}$, the map of profinitely completed topological types

$$\widehat{\mathrm{h}}(\mathcal{X}^{\mathrm{top}}) \to \widehat{\mathrm{h}}(\mathcal{X})$$

is a weak equivalence of profinite spaces.

While the work of Ilan Barnea, Yonatan Harpaz, and Geoffroy Horel in [2] provides a connection between topological types and shapes, the compatibility between the model categorical and ∞-categorical profinite completions enables to connect their profinite completions. In particular, one obtains:

Theorem 1.1.5. (Theorem 4.2.4 in the text) *Let $\mathcal{X}$ be a locally of finite type algebraic stack over $\mathbb{C}$. Then the map of profinitely completed shapes*

$$\widehat{\mathrm{Sh}}(\mathcal{X}^{\mathrm{top}}) \to \widehat{\mathrm{Sh}}(\mathcal{X})$$

is an equivalence in $\mathrm{Pro}(\mathcal{S}_\pi)$.

Remark 1.1.6. This recovers [4, 4.14] of David Carchedi in his independent work of the étale homotopy type of higher stacks [4]. Compared to his proof involving the reformulation of local systems in ∞-category theory, we obtain 4.2.4 as a formal consequence of [5, 4.3.18] by the compatibility of profinite completions 4.1.5 and the work of Barnea-Harpaz-Horel. Note that [5, 4.3.18] was also obtained formally from Artin-Mazur's classical theorem [1, 12.9] under the machinery developed in [5]. Consequently, 4.1.5 fills in the missing piece as one tries to obtain the comparison theorem for algebraic stacks in ∞-category theory formally from the geometry of schemes–the study on étale coverings and cohomology.

1.2. **Motivation.**

1.2.1. In 1960s, Michael Artin and Barry Mazur developed the étale homotopy theory of schemes [1]. Associated to a locally noetherian scheme X is its pro-homotopy type, a pro-object in the homotopy category of simplicial sets, which is referred to as the étale homotopy type of X. It recovers the étale cohomology and enlarged étale fundamental group of X and. Moreover, it provides a homotopy theory for schemes.

Artin-Mazur's classical comparison theorem [1, 12.9], which generalizes the Riemann existence theorem, says that for a connected finite type scheme X over $\mathbb{C}$, there is a map from the singular complex associated to the underlying topological space of analytification of X to the étale homotopy type of X, which induces an isomorphism on the profinite completion.

Following Artin-Mazur's seminal work, Eric Friedlander extended it to the étale topological types of simplicial schemes [6]. Recently, Ilan Barnea and Tomer Schlank rediscovered Artin-Mazur's étale homotopy types using model category theory as they constructed new model category structures on pro-categories [3].

1.2.2. Based on the work of Barnea-Schlank, the author extended the previous theories to simplicial algebraic spaces and algebraic stacks [5]. In fact, the theory of topological types of topoi was developed in [5] under model category theory and applied to algebro-geometric objects so that it generalizes the previous theories. In particular, the comparison theorem for algebraic stacks [5, 4.3.18], which generalizes Artin-Mazur's comparison theorem for schemes, was proved in a formal model categorical way.

On the other hand, David Carchedi proved independently the comparison theorem for algebraic stacks using ∞-category theory [4, 4.14] as he studies the étale homotopy type of higher stacks.

To provide a connection between these two different approaches to the comparison theorem, it is necessary to compare the profinite completion of pro-simplicial sets in model category theory to the profinite completion of pro-spaces in ∞-category theory.

As a preliminary step, we review the work of Ilan Barnea, Yonatan Harpaz, and Geoffroy Horel on the equivalence between the model categorical and ∞-categorical approaches to pro-categories [2], which connects topological types of topoi to shapes of associated ∞-topoi. After then, based on Barnea-Harpaz-Horel's model for the ∞-category of profinite spaces, the equivalence between the profinite completions will be proved by the characterization of profinite completion of pro-simplicial sets that it is a left adjoint [5, 4.2.3].

1.3. **Convention.**

1.3.1. In this paper, an *algebraic space* X over a scheme S is a functor $X : (\mathbf{Sch}/S)^{\mathrm{op}} \to \mathbf{Set}$ such that the following holds:

(i) X is a sheaf with respect to the big étale topology.
(ii) The diagonal
$$\Delta : X \to X \times_S X$$
is representable by schemes.
(iii) There exists a S-scheme U and an étale surjection $U \to X$.

An *algebraic stack* $\mathcal{X}$ over a scheme S is a stack in groupoids over the big étale site $(\mathbf{Sch}/S)_{\acute{e}t}$ of S-schemes such that the following holds:

(i) The diagonal
$$\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$$

is representable by algebraic spaces.

(ii) There exists a S-scheme X and a smooth surjection $\pi : X \to \mathfrak{X}$.

Remark 1.3.2. These definitions only assume the minimum conditions compared to those in the literature. For example, we do not assume the quasi-compactness on the diagonal.

1.3.3. An ∞-category in this paper refers to an $(\infty, 1)$-category. Among several equivalent models of $(\infty, 1)$-categories, we take quasi-categories throughout the paper; our basic reference is [11].

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2. The comparison theorem via topological types

In this section we recall from [5] topological types of algebro-geometric objects and their comparison theorems.

2.1. Review on topological types and profinite completions.

2.1.1. Let us briefly recall the topological types of topoi [5, §2.3]. Let T be a topos and consider the (2-categorical) unique morphism of topoi

$$\Gamma = (\Gamma^*, \Gamma_*) : T \to \mathbf{Set}.$$

Then for the pro-categories of simplicial objects, Γ^* admits a left adjoint L_{Γ^*} and, moreover, the adjunction

$$L_{\Gamma^*} : \mathrm{Pro}(T^{\Delta^{\mathrm{op}}}) \rightleftarrows \mathrm{Pro}(\mathbf{SSet}) : \Gamma^* \tag{2.1.1.1}$$

is a Quillen adjunction with respect to Barnea-Schlank's model category structures [3, §7.3].

Recall from [5, 2.3.2] that for the left derived functor $\mathbf{L}L_{\Gamma^*} : \mathbf{Ho}(\mathrm{Pro}(T^{\Delta^{\mathrm{op}}})) \to \mathbf{Ho}(\mathrm{Pro}(\mathbf{SSet}))$ the *topological type* of a topos T is defined to be the pro-simplicial set

$$\mathrm{h}(T) := \mathbf{L}L_{\Gamma^*}(*)$$

where $*$ is a final object of T. More generally, the *topological type* $\mathrm{h}(F_\bullet)$ of a simplicial object $F_\bullet$ in T is defined to be the pro-simplicial set

$$\mathbf{L}L_{\Gamma^*}(F_\bullet).$$

Remark 2.1.2.

(i) Topological types of stacks [5, 2.3.6] is a bit more involved as it is necessary to deal with the 2-categorical aspects of stacks. It will be reviewed in 2.2.2 with the comparison theorem for algebraic stacks.

(ii) Various properties of topological types can be found in [5, §2.3, §2.4]. Especially, there are the descent theorems [5, 2.3.23, 2.3.32, 2.3.36] obtained in a very categorical manner, compared to other approaches in the literature.

2.1.3. Let us recall from [14] profinite spaces and profinite completions. The category $\widehat{\mathcal{E}}$ of compact, Hausdorff, and totally disconnected topological spaces is equivalent to the pro-category of finite sets. The forgetful functor $\widehat{\mathcal{E}} \to \mathbf{Set}$ admits a left adjoint. Denote the category of simplicial objects in $\widehat{\mathcal{E}}$ by $\widehat{\mathbf{SSet}}$ and refer to its objects as *profinite spaces.* There is an adjunction

$$\widehat{\ }: \mathbf{SSet} \rightleftarrows \widehat{\mathbf{SSet}} : |\ \ |$$

where the right adjoint is the induced forgetful functor. The left adjoint is referred to as the *profinite completion* of simplicial sets.

Following [14, §2.7], the *profinite completion* functor on $\mathrm{Pro}(\mathbf{SSet})$ is the composition of the functor on the pro-categories associated to the profinite completion of simplicial sets followed by the limit functor on $\mathrm{Pro}(\widehat{\mathbf{SSet}})$:

$$\mathrm{Pro}(\mathbf{SSet}) \longrightarrow \mathrm{Pro}(\widehat{\mathbf{SSet}}) \xrightarrow{\lim} \widehat{\mathbf{SSet}} : (X_i)_{i\in I} \mapsto \lim_{i\in I} \widehat{X_i}.$$

Notation 2.1.4. Denote by $\widehat{\mathrm{h}}(T)$ the profinite completion of the topological type $\mathrm{h}(T)$ which is a priori a pro-simplicial set. Likewise for $\mathrm{h}(F_\bullet)$.

2.1.5. One of the key technical ingredients for the comparison theorem for algebraic stacks via topological types is the fact that the profinite completion on pro-simplicial sets admits a right adjoint [5, 4.2.3]; it implies that the profinite completion commutes with homotopy colimits (see [5, 4.2.8]). For later use, let us recall how the proof works.

Recall from [2, 7.2.3] that a simplicial set X is τ_n*-finite* if it is a level-wise finite sets and the canonical map $X \to \operatorname{cosk}_n \tau_n X$ is an isomorphism. Also, X is τ*-finite* if it is τ_n-finite for some $n \geq 0$. For the full subcategory $\mathbf{SSet}_\tau \subset \mathbf{SSet}$ of τ-finite simplicial sets, there is a natural inclusion $\mathbf{SSet}_\tau \to \widehat{\mathbf{SSet}}$ and the functor

$$\mathrm{Pro}(\mathbf{SSet}_\tau) \to \widehat{\mathbf{SSet}}.$$

induced by the universal property of pro-categories is an equivalence of categories by [2, 7.4.1].

There is an adjunction

$$\mathrm{Pro}(\mathbf{SSet}) \rightleftarrows \mathrm{Pro}(\mathbf{SSet}_\tau). \tag{2.1.5.1}$$

whose right adjoint is induced by the natural inclusion $\mathbf{SSet}_\tau \to \mathbf{SSet}$. It then follows from [5, 4.2.3] that the left adjoint Ψ is equivalent to the profinite completion on pro-simplicial sets under the equivalence of categories of [2, 7.4.1]. Therefore, there is an adjunction

$$\mathrm{Pro}(\mathbf{SSet}) \rightleftarrows \widehat{\mathbf{SSet}} \tag{2.1.5.2}$$

whose left adjoint is the profinite completion.

2.2. The comparison theorem for algebraic stacks via topological types.

2.2.1. In [5], Artin-Mazur's comparison theorem for schemes [1, 12.9] was generalized to simplicial algebraic spaces [5, 4.3.14] and algebraic stacks [5, 4.3.18]. To set up the comparison theorems via topological types, note from [5, 4.3.1] that the big étale site Ét is the category of complex analytic spaces equipped with the Grothendieck topology that a collection of morphisms $\{X_i \to X\}$ is a covering of X if each morphism $X_i \to X$ is étale and the map $\coprod_{i\in I} X_i \to X$ is surjective. On the other hand, recall from [5, 3.2.1] that LFÉ/$\mathbb{C}$ is the site whose underlying category is the full subcategory of the category of $\mathbb{C}$-schemes consisting of locally of finite type $\mathbb{C}$-morphisms with the coverings induced by coverings in the big étale topology on $\operatorname{Spec}\mathbb{C}$.

The topological types of locally of finite type (simplicial) schemes/algebraic spaces over $\mathbb{C}$ are defined to be the topological types of the corresponding (simplicial) sheaves in the topos $(\text{LFÉ}/\mathbb{C})^{\sim}$; see [5, 3.2.6]. Likewise for (simplicial) complex analytic spaces.

For a locally of finite type simplicial algebraic space $X_\bullet$ over $\mathbb{C}$, its comparison theorem [5, 4.3.14] provides that the map of profinitely completed topological types

$$\widehat{\mathrm{h}}(X_\bullet^{\mathrm{an}}) \to \widehat{\mathrm{h}}(X_\bullet)$$

is a weak equivalence of profinite spaces with respect to Quick's model category structure [14, 2.12].

2.2.2. To obtain the topological types of algebraic stacks, recall from [5, 3.3.1] that for an algebraic stack $\mathcal{X}$ that is locally of finite type over $\mathbb{C}$, one defines the site $\text{LFÉ}(\mathcal{X})$ as following. An object is a pair (Y, y), where $y : Y \to \mathcal{X}$ is a locally of finite type morphism over $\mathbb{C}$ with Y an algebraic space. A morphism

$$(Y, y) \to (Z, z)$$

is a pair (h, h^b) where $h : Y \to Z$ is a morphism of algebraic spaces and $h^b : y \to z \circ h$ is a 2-morphism of functors. A collection of maps

$$\{(h_i, h_i^b) : (Y_i, y_i) \to (Y, y)\}$$

is a covering if the underlying collection of morphisms of algebraic spaces $\{y_i : Y_i \to Y\}$ is an étale covering: each y_i is étale and $\coprod Y_i \to Y$ is surjective.

Following [5, 3.3.3], the topological type $\mathrm{h}(\mathcal{X})$ of $\mathcal{X}/\mathbb{C}$ is defined to be the topological type of the associated topos $(\text{LFÉ}/\mathcal{X})^{\sim}$.

To set up the comparison theorem for algebraic stacks, note that the category of topological spaces with the usual open coverings gives rise to the big topological site Top. Recall from [13, §20] that there is a functor from the category of locally of finite type algebraic stacks over $\mathbb{C}$ to the category of stacks over Top. Denote by $\mathcal{X}^{\mathrm{top}}$ the image of $\mathcal{X}$ under this functor. The comparison theorem for algebraic stacks [5, 4.3.18] provides that there is a weak equivalence

$$\widehat{\mathrm{h}}(\mathcal{X}^{\mathrm{top}}) \to \widehat{\mathrm{h}}(\mathcal{X}).$$

3. Topological types and shapes

In this section we review the original work of Ilan Barnea, Yonatan Harpaz, and Geoffroy Horel [2] to provide the relationship between topological types and shapes [2, 6.0.4].

3.1. **Preliminaries on pro-categories.**

3.1.1. Let us briefly review pro-categories in ∞-category theory. We will recall pro-categories associated to accessible ∞-categories which admit finite limits. In fact, for the purpose of connecting topological types to shapes, we should consider more general case of locally small ∞-categories. To avoid too much technical details, let us restrict our attention to the case of accessible ∞-categories; see [2, §3.2] for the technicality involving locally smallness.

Let $\mathcal{C}$ be an accessible ∞-category which admits finite limits. Recall from [12, 3.1.1] that a *pro-object* of $\mathcal{C}$ is a functor $\mathcal{C} \to \mathcal{S}$ which is accessible and preserves finite limits. Denote by $\mathrm{Pro}(\mathcal{C})$ the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$ spanned by the pro-objects of $\mathcal{C}$. We refer to $\mathrm{Pro}(\mathcal{C})$ as the ∞-category of pro-objects of $\mathcal{C}$. The pro-category $\mathrm{Pro}(\mathcal{C})$ enjoys a universal property. For this, let $\mathcal{D}$ be an ∞-category which admits small cofiltered limits, and let $\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \subset \mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D})$ denote the full subcategory spanned by those functors which preserve small cofiltered limits. It then follows from [12, 3.1.6] that the Yoneda embedding induces an equivalence of ∞-categories

$$\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

3.1.2. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between accessible ∞-categories which admit finite limits. By the universal property of pro-categories, there is an induced functor on the pro-categories

$$\mathrm{Pro}(f) : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{D}),$$

which commutes with small cofiltered limits. Assuming further that f is accessible and commutes with finite limits, there is a functor

$$\mathrm{Pro}(\mathcal{D}) \to \mathrm{Pro}(\mathcal{C})$$

induced by the composition with f. It can be checked this functor is a left adjoint to $\mathrm{Pro}(f)$.

3.1.3. The ∞-category of spaces $\mathcal{S}$ is a final object in the ∞-category of ∞-topoi; see [11, 6.3.4.1]. In other words, for an ∞-topos $\mathcal{X}$, there exists a unique geometric morphism $\pi_* : \mathcal{X} \to \mathcal{S}$ up to equivalence. The composition $\pi_* \circ \pi^* : \mathcal{S} \to \mathcal{S}$ is a pro-object of $\mathcal{S}$, which is referred to as the *shape* of $\mathcal{X}$ and denoted by $\mathrm{Sh}(\mathcal{X})$; see [11, 7.1.6.3].

Applying 3.1.2 to $\pi^* : \mathcal{S} \to \mathcal{X}$, one sees that $\mathrm{Pro}(\pi^*)$ admits a left adjoint

$$\pi_! : \mathrm{Pro}(\mathcal{X}) \to \mathrm{Pro}(\mathcal{S}).$$

It can be checked that for each object $X \in \mathcal{X}$ and its overcategory $\mathcal{X}/X$, there is an equivalence of pro-spaces

$$\pi_!(X) \simeq \mathrm{Sh}(\mathcal{X}/X).$$

In particular, for a final object $*$ of $\mathcal{X}$, there is an equivalence of pro-spaces $\pi_!(*) \simeq \mathrm{Sh}(\mathcal{X})$.

3.2. **Comparison between topological types and shapes.**

3.2.1. Let $(\mathcal{C}, \mathcal{W})$ be a relative category. There is a natural functor of ∞-categories

$$\mathrm{N}\mathcal{C} \to \mathcal{C}_\infty$$

from the nerve of $\mathcal{C}$ to the underlying ∞-category of the relative category. Associated to a simplicial set S is its homotopy category $\mathrm{h}S$; see [11, 1.1.5.14]. Considering the canonical isomorphism $\mathcal{C} \xrightarrow{\sim} \mathrm{hN}\mathcal{C}$, one obtains a functor

$$\mathcal{C} \to \mathrm{h}\mathcal{C}_\infty. \tag{3.2.1.1}$$

Definition 3.2.2. Let $(\mathcal{C}, \mathcal{W})$ be a relative category. For an object $x \in \mathcal{C}$, the *underlying ∞-object* x_∞ is the image of x under the functor (3.2.1.1).

Remark 3.2.3. In the literature, the underlying ∞-category $\mathcal{C}_\infty$ is often chosen in such a way that the underlying map on objects is the identity. We do not make such a choice so as to avoid any confusion in the case we use a specific model for $\mathcal{C}_\infty$ whose underlying map on objects is not the identity.

3.2.4. There is a concrete description of the underlying ∞-objects for simplicial model categories. Indeed, let $\mathcal{M}$ be a simplicial model category with its underlying ∞-category $\mathcal{M}_\infty = \mathrm{N}(\mathcal{M}^{\mathrm{cf}})$ which is the simplicial nerve [11, 1.1.5.5] of the full subcategory $\mathcal{M}^{\mathrm{cf}} \subset \mathcal{M}$ of fibrant-cofibrant objects. For an object $X \in \mathcal{M}$, its underlying object is X^{cf} which is a cofibrant fibrant approximation [7, 8.1.2] of a fibrant cofibrant approximation of X in $\mathcal{M}$.

For a simplicial Quillen adjunction between simplicial model categories

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : G,$$

there is an induced adjunction of underlying ∞-categories

$$F_\infty : \mathcal{M}_\infty \rightleftarrows \mathcal{M}_\infty : G_\infty .$$

It can be checked that for the left derived functor $\mathbf{L}F : \mathbf{Ho}(M) \to \mathbf{Ho}(N)$ applied to an object $A \in \mathcal{M}$, there is an equivalence

$$(\mathbf{L}F(A))_\infty \simeq F_\infty(A_\infty).$$

Notation 3.2.5. Denote the underlying ∞-object of the topological type $\mathrm{h}(T)$ by $\mathrm{h}_\infty(T)$ and that of its profinite completion $\widehat{\mathrm{h}}(T)$ by $\widehat{\mathrm{h}}_\infty(T)$. Likewise for $\mathrm{h}(F)$.

3.2.6. Applying the previous discussion of adjunctions and underlying ∞-objects to the simplicial Quillen adjunction (2.1.1.1), one obtains an adjunction of underlying ∞-categories

$$(L_{\Gamma^*})_\infty : \mathrm{Pro}(T^{\Delta^{\mathrm{op}}})_\infty \rightleftarrows \mathrm{Pro}(\mathbf{SSet})_\infty : \mathrm{Pro}(\Gamma^*)_\infty . \tag{3.2.6.1}$$

Moreover, for an object $F \in T^{\Delta^{\mathrm{op}}}$, there is an equivalence in $\mathrm{Pro}(\mathbf{SSet})_\infty$

$$\mathrm{h}_\infty(F) \simeq (L_{\Gamma^*})_\infty(F_\infty).$$

3.2.7. The main bridge that connects topological types and shapes is the equivalence between the model categorical and ∞-categorical approaches to pro-categories by Barnea-Harpaz-Horel [2]. Recall from [3, 7.13] that for a topos T, there is a weak fibration category structure [3, 1.2] on $T^{\Delta^{\mathrm{op}}}$, from which one can associate the underlying ∞-category of the underlying relative category. According to [2, 6.0.1], there is a natural equivalence of ∞-categories

$$\mathrm{Pro}(T^{\Delta^{\mathrm{op}}})_\infty \to \mathrm{Pro}(T_\infty^{\Delta^{\mathrm{op}}}).$$

In fact, this equivalence is functorial with respect to right derived functors [2, 5.2.5]. In the case of our interest, note that the underlying ∞-functor Γ^*_∞ of the constant sheaf functor $\Gamma^* : \mathbf{SSet} \to T^{\Delta^{\mathrm{op}}}$ induces a functor

$$\mathrm{Pro}(\Gamma^*_\infty) : \mathrm{Pro}(\mathcal{S}) \to \mathrm{Pro}(T_\infty^{\Delta^{\mathrm{op}}})$$

by the universal property of pro-categories [2, 3.2.19]. It then follows from [2, 5.2.5] that this functor is equivalent to the right adjoint $\mathrm{Pro}(\Gamma^*)_\infty$ of (3.2.6.1) under the equivalence of [2, 6.0.1]. That is, there is a commutative diagram of ∞-categories

$$\begin{tikzcd} \mathrm{Pro}(\mathbf{SSet})_\infty \arrow[r, "\mathrm{Pro}(\Gamma^*)_\infty"] \arrow[d, "\wr"] & \mathrm{Pro}(T^{\Delta^{\mathrm{op}}})_\infty \arrow[d, "\wr"] \\ \mathrm{Pro}(\mathbf{SSet}_\infty) \arrow[r, "\mathrm{Pro}(\Gamma^*_\infty)"] & \mathrm{Pro}(T_\infty^{\Delta^{\mathrm{op}}}). \end{tikzcd} \tag{3.2.7.1}$$

Remark 3.2.8. There is an equivalence of ∞-categories $\mathbf{SSet}_\infty \simeq \mathcal{S}$. Let us identify them throughout the rest of this paper.

3.2.9. Let $\mathcal{C}$ be a site whose associated topos is T. According to [11, 6.5.2.14], for Jardine's model category [9, 2.3] of simplicial presheaves on $\mathcal{C}$, its underlying ∞-category is an ∞-topos. On the other hand, notice that $T_\infty^{\Delta^{\mathrm{op}}}$ can be thought as the underlying ∞-category of Joyal's model category [10] of $T^{\Delta^{\mathrm{op}}}$. It then follows from [9, 2.8] and [11, A.3.1.12] that these underlying ∞-categories are equivalent, and thus $T_\infty^{\Delta^{\mathrm{op}}}$ is an ∞-topos.

Let us describe the shape of the ∞-topos $T_\infty^{\Delta^{\mathrm{op}}}$. With respect to Joyal's model category structure on $T^{\Delta^{\mathrm{op}}}$, the adjunction $\Gamma^* : \mathbf{SSet} \rightleftarrows T^{\Delta^{\mathrm{op}}} : \Gamma_*$ becomes a Quillen adjunction. By 3.2.4, there is an adjunction of underlying ∞-categories

$$\Gamma^*_\infty : \mathcal{S} \rightleftarrows T_\infty^{\Delta^{\mathrm{op}}} : (\Gamma_*)_\infty .$$

Since Γ^*_∞ commutes with finite limits by [2, 2.4.13], one sees that

$$(\Gamma^*_\infty, (\Gamma_*)_\infty) : T_\infty^{\Delta^{\mathrm{op}}} \to \mathcal{S}$$

is the unique geometric morphism of the ∞- topos $T^{\Delta^{\mathrm{op}}}$. Therefore, there is an equivalence of pro-spaces

$$\mathrm{Sh}(T_\infty^{\Delta^{\mathrm{op}}}) \simeq (\Gamma_*)_\infty \circ \Gamma^*_\infty.$$

3.2.10. The functor

$$\Gamma^*_\infty : \mathcal{S} \to T_\infty^{\Delta^{\mathrm{op}}}$$

is accessible and commutes with finite limits, and hence $\mathrm{Pro}(\Gamma^*_\infty)$ admits a left adjoint $\Gamma_!$. Considering the left adjoints to the horizontal functors in (3.2.7.1), one obtains a commutative diagram of ∞-categories

(3.2.10.1)
$$\begin{tikzcd} \mathrm{Pro}(T^{\Delta^{\mathrm{op}}})_\infty \arrow[r, "(L_{\Gamma^*})_\infty"] \arrow[d, "\wr"] & \mathrm{Pro}(\mathbf{SSet})_\infty \arrow[d, "\wr"] \\ \mathrm{Pro}(T_\infty^{\Delta^{\mathrm{op}}}) \arrow[r, "\Gamma_!"] & \mathrm{Pro}(\mathcal{S}). \end{tikzcd}$$

3.2.11. Finally, one can state [2, 6.0.4] in terms of underlying ∞-objects of topological types. Let F be an object in $T^{\Delta^{\mathrm{op}}}$. Applying F_∞ to (3.2.10.1), one sees from 3.1.3 and 3.2.6 that there is an equivalence of pro-spaces

$$\mathrm{h}_\infty(F) \simeq \mathrm{Sh}(T_\infty^{\Delta^{\mathrm{op}}}/F_\infty)$$

under the right vertical equivalence of (3.2.10.1). In particular, there is an equivalence of pro-spaces $\mathrm{h}_\infty(T) \simeq \mathrm{Sh}(T_\infty^{\Delta^{\mathrm{op}}})$.

4. An equivalence of profinite completions and the comparison theorem via shapes

In this section we use the key statement–the profinite completion is a left adjoint–and the model for ∞-categorical profinite spaces by Barnea-Harpaz-Horel to provide an equivalence of profinite completions in model category theory and ∞-category theory. As an application, we obtain the comparison theorem for algebro-geometric objects in terms of shapes.

4.1. **An equivalence of profinite completions.**

4.1.1. Let $X \in \mathcal{S}$ be a pro-space. According to [12, 2.3.2], X is defined to be π-*finite* if it satisfies the following conditions:

(i) X is n-truncated for some $n \geq -2$.
(ii) $\pi_0 X$ is finite.
(iii) For each $x \in X$ and each $m \geq 1$, $\pi_m(X, x)$ is finite.

Deonte by $\mathcal{S}_\pi$ the full subcategory of $\mathcal{S}$ spanned by the π-finite spaces. The associated pro-category $\mathrm{Pro}(\mathcal{S}_\pi)$ is referred to as the ∞-category of profinite spaces; see [12, 3.6.1].

4.1.2. It follows from 3.1.2 that the fully faithful embedding $\mathrm{Pro}(\mathcal{S}_\pi) \to \mathrm{Pro}(\mathcal{S})$ admits a left adjoint

$$\mathrm{Pro}(\mathcal{S}) \to \mathrm{Pro}(\mathcal{S}_\pi)$$

which is referred to as the *profinite completion functor*. For a pro-space X, its image under this functor is denoted by $\widehat{X}$ and referred to as the *profinite completion* of X.

4.1.3. In order to obtain an equivalence of profinite completions in model category theory and ∞-category theory, we review [2, 7.4.9] of Barnea-Harpaz-Horel, which provides a connection between the model categorical profinite homotopy theory in the sense of Quick [14] and the ∞-categorical one. Recall from [8, 2.2] that for a set K of Kan complexes, one can localize the strict model category structure on $\mathrm{Pro}(\mathbf{SSet})$ to obtain a new model category structure denoted by $L_K \mathrm{Pro}(\mathbf{SSet})$; cofibrations are the cofibrations in the strict model category structure and a map $X \to Y$ is a weak equivalence if and only if

$$\mathrm{Map}_{\mathrm{Pro}(\mathbf{SSet})}(Y, A) \to \mathrm{Map}_{\mathrm{Pro}(\mathbf{SSet})}(X, A)$$

is a weak equivalence for each $A \in K$.

Choosing K to be K^π (see [2, 7.2.8]), $L_{K^\pi}\mathrm{Pro}(\mathbf{SSet})$ can be a model for $\mathrm{Pro}(\mathcal{S}_\pi)$. Namely, it follows from [2, 7.2.12] that there is an equivalence of ∞-categories

$$L_{K^\pi}\mathrm{Pro}(\mathbf{SSet})_\infty \xrightarrow{\sim} \mathrm{Pro}\,(\mathcal{S}_\pi).$$

On the other hand, the localized model category $L_{K^\pi}\mathrm{Pro}(\mathbf{SSet})$ is Quillen equivalent to Quick's model category structure on $\widehat{\mathbf{SSet}}$. To see this, let us view $\mathrm{Pro}(\mathbf{SSet}_\tau)$ as a model category through the equivalence of categories $\mathrm{Pro}(\mathbf{SSet}_\tau) \simeq \widehat{\mathbf{SSet}}$ [2, 7.4.1]. In this regard, the adjunction (2.1.5.1) becomes a Quillen adjunction (see [2, 7.4.5]) which induces a Quillen equivalence (see [2, 7.4.8])

$$L_{K^\pi}\,\mathrm{Pro}(\mathbf{SSet}) \rightleftarrows \mathrm{Pro}(\mathbf{SSet}_\tau).$$

All in all, one obtains an equivalence $\widehat{\mathbf{SSet}}_\infty \simeq \mathrm{Pro}(\mathcal{S}_\pi)$ [2, 7.4.9].

4.1.4. To compare the profinite completions, recall from [5, 4.2.6] that the adjunction (2.1.5.2) is a Quillen adjunction with respect to the strict model category structure on pro-simplicial sets (cf. [5, 2.2.6]) and Quick's model category structure on profinite spaces.

Theorem 4.1.5. *The underlying ∞-functor associated to the profinite completion on* $\mathrm{Pro}(\mathbf{SSet})$ *is equivalent to the profinite completion on* $\mathrm{Pro}(\mathcal{S})$*. That is, there is a commutative diagram of ∞-categories*

$$\begin{tikzcd} \mathrm{Pro}(\mathbf{SSet})_\infty \arrow[r] \arrow[d, "\wr"] & \widehat{\mathbf{SSet}}_\infty \arrow[d, "\wr"] \\ \mathrm{Pro}(\mathcal{S}) \arrow[r] & \mathrm{Pro}(\mathcal{S}_\pi). \end{tikzcd} \tag{4.1.5.1}$$

Proof. The Quillen adjunction

$$\mathrm{id} : \mathrm{Pro}(\mathbf{SSet}) \rightleftarrows L_{K^\pi}\mathrm{Pro}(\mathbf{SSet}) : \mathrm{id}$$

of the localization induces an adjunction

$$\mathrm{Pro}(\mathbf{SSet})_\infty \rightleftarrows L_{K^\pi}\mathrm{Pro}(\mathbf{SSet})_\infty.$$

It then follows from [5, 4.2.3] and [2, 7.4.8] that one can replace the top arrow of the square by the left adjoint of the induced adjunction.

Now, it suffices to show the commutativity of the square after replacing the horizontal arrows by their right adjoints. Yet such a diagram commutes by the construction of the equivalence $L_{K^\pi}\mathrm{Pro}(\mathbf{SSet})_\infty \simeq \mathrm{Pro}(\mathcal{S}_\pi)$ (see the proof of [2, 7.1.2] and [2, 7.2.11]). $\square$

4.2. **The comparison theorem via shapes.**

Theorem 4.2.1. *Under the equivalence of ∞-categories $\widehat{\mathbf{SSet}}_\infty \simeq \mathrm{Pro}(\mathcal{S}_\pi)$, for each object $F \in T^{\Delta^{\mathrm{op}}}$, there is an equivalence of profinite spaces*

$$\widehat{\mathrm{h}}_\infty(F) \simeq \widehat{\mathrm{Sh}}(T_\infty^{\Delta^{\mathrm{op}}}/F_\infty).$$

In particular, there is an equivalence of profinite spaces $\widehat{\mathrm{h}}_\infty(T) \simeq \widehat{\mathrm{Sh}}(T_\infty^{\Delta^{\mathrm{op}}})$.

Proof. Since the profinite completion on pro-simplicial sets preserves weak equivalences (cf. [14, 2.14. 2.28]), it follows from 3.2.4 that the image of $\mathrm{h}_\infty(F)$ under the top arrow of (4.1.5.1) is equivalent to $\widehat{\mathrm{h}}_\infty(T)$ which is the underlying ∞-object of $\widehat{\mathrm{h}}(T)$ (see 3.2.5 for the notation). Then considering the equivalence of topological types and shapes 3.2.11, the statement follows from the equivalence of profinite completions 4.1.5. $\square$

Remark 4.2.2. The equivalence in 4.2.1 is functorial in the following sense. Let $f : T' \to T$ a morphism of topoi. For each $F \in T^{\Delta^{\mathrm{op}}}$,

$$\widehat{\mathrm{h}}_{T'}(f^*F) \to \widehat{\mathrm{h}}_T(F)$$

is a weak equivalence in $\widehat{\mathbf{SSet}}$ if and only if

$$\widehat{\mathrm{Sh}}(T'^{\Delta^{\mathrm{op}}}_\infty/f^*_\infty F_\infty) \to \widehat{\mathrm{Sh}}(T^{\Delta^{\mathrm{op}}}_\infty/F_\infty)$$

is an equivalence in $\mathrm{Pro}(\mathcal{S}_\pi)$. In particular, $\widehat{\mathrm{h}}(T') \to \widehat{\mathrm{h}}(T)$ is a weak equivalence in $\widehat{\mathbf{SSet}}$ if and only if $\widehat{\mathrm{Sh}}(T'^{\Delta^{\mathrm{op}}}_\infty) \to \widehat{\mathrm{Sh}}(T^{\Delta^{\mathrm{op}}}_\infty)$ is an equivalence in $\mathrm{Pro}(\mathcal{S}_\pi)$.

Notation 4.2.3. For a locally of finite type algebraic stack $\mathcal{X}$ over $\mathbb{C}$, denote by $\mathrm{Sh}(\mathcal{X})$ the shape of the ∞-topos $T^{\Delta^{\mathrm{op}}}_\infty$ where $T = (\mathrm{LF\acute{E}}/\mathcal{X})^\sim$. Likewise for the associated topological stack (see [5, 4.3.15, 4.3.17]).

Theorem 4.2.4. (cf. [4, 4.14]) *Let $\mathcal{X}$ be a locally of finite type algebraic stack over $\mathbb{C}$. Then the map of profinitely completed shapes*

$$\widehat{\mathrm{Sh}}(\mathcal{X}^{\mathrm{top}}) \to \widehat{\mathrm{Sh}}(\mathcal{X}).$$

is an equivalence in $\mathrm{Pro}(\mathcal{S}_\pi)$.

Proof. Under the compatibility of the profinitely completed topological types and shapes 4.2.1 (cf. 4.2.2), the statement is reduced to the comparison theorem for algebraic stacks via topological types [5, 4.3.18]. □

Remark 4.2.5.

(i) Similarly, one can translate the comparison theorem for simplicial algebraic spaces [5, 4.3.14] to the one described by shapes.
(ii) While the author developed the theory of topological types under model category theory and, in particular, proved the comparison theorem for algebraic stacks [5, 4.3.18], David Carchedi showed independently the comparison theorem for algebraic stacks under ∞-category theory [4, 4.14]. Here we recover his comparison theorem by establishing the equivalence of profinite completions.

References

[1] M. Artin and B. Mazur. *Etale homotopy*. Lecture Notes in Mathematics, No. 100. Springer-Verlag, Berlin-New York, 1969, pp. iii+169.

[2] Ilan Barnea, Yonatan Harpaz, and Geoffroy Horel. "Pro-categories in homotopy theory". In: *Algebr. Geom. Topol.* 17.1 (2017), pp. 567–643. ISSN: 1472-2747. DOI: `10.2140/agt.2017.17.567`. URL: `http://dx.doi.org/10.2140/agt.2017.17.567`.

[3] Ilan Barnea and Tomer M. Schlank. "A projective model structure on pro-simplicial sheaves, and the relative étale homotopy type". In: *Adv. Math.* 291 (2016), pp. 784–858. ISSN: 0001-8708. DOI: `10.1016/j.aim.2015.11.014`. URL: `http://dx.doi.org/10.1016/j.aim.2015.11.014`.

[4] David Carchedi. "On the\'etale homotopy type of higher stacks". In: *arXiv preprint arXiv:1511.07830* (2015).

[5] Chang-Yeon Chough. *Topological types of Algebraic stacks*. 2016. URL: `http://cgp.ibs.re.kr/~chough/chough_toptypes.pdf`.

[6] Eric M. Friedlander. *Étale homotopy of simplicial schemes*. Vol. 104. Annals of Mathematics Studies. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982, pp. vii+190. ISBN: 0-691-08288-X; 0-691-08317-7.

[7] Philip S. Hirschhorn. *Model categories and their localizations*. Vol. 99. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003, pp. xvi+457. ISBN: 0-8218-3279-4.

[8] Daniel C. Isaksen. "Completions of pro-spaces". In: *Math. Z.* 250.1 (2005), pp. 113–143. ISSN: 0025-5874. DOI: `10.1007/s00209-004-0745-x`. URL: `http://dx.doi.org/10.1007/s00209-004-0745-x`.

[9] John Frederick Jardine. "Simplical presheaves". In: *Journal of Pure and Applied Algebra* 47.1 (1987), pp. 35–87.

[10] André Joyal. *A letter to Grothendieck*. 1983.

[11] Jacob Lurie. *Higher topos theory.* Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: 10.1515/9781400830558. URL: http://dx.doi.org/10.1515/9781400830558.

[12] Jacob Lurie. "Derived Algebraic Geometry XIII: Rational and p-adic Homotopy Theory". In: *Preprint* (2011). URL: http://www.math.harvard.edu/~lurie/papers/DAG-XIII.pdf.

[13] Behrang Noohi. "Foundations of topological stacks I". In: *arXiv preprint math/0503247* (2005).

[14] Gereon Quick. "Profinite homotopy theory". In: *Doc. Math.* 13 (2008), pp. 585–612. ISSN: 1431-0635.

Center for Geometry and Physics, Institute for Basic Science (IBS), Pohang 37673, Republic of Korea

E-mail address: chough@ibs.re.kr