THE PRO-ÉTALE TOPOLOGY FOR ALGEBRAIC STACKS

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Abstract. The pro-étale topology by Bhargav Bhatt and Peter Scholze allows us to reconstruct the derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves on a topologically noetherian scheme in the usual derived manner. We extend this result from schemes to algebraic stacks.

Contents

1. Introduction

1.1. Let X be a separated scheme of finite type over a field k of characteristic p and let ℓ be a prime different from p. In practice, the "derived category" $\mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ of constructible $\overline{\mathbb{Q}}_\ell$-sheaves on X is defined to be the limit of the derived categories of constructible $\mathbb{Z}/\ell^n\mathbb{Z}$-sheaves tensored with $\overline{\mathbb{Q}}_\ell$ over $\mathbb{Z}_\ell$:

$$\mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell) := \lim_n \mathrm{D}^b_c(X, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell.$$

This indirect approach was refined by the work of Bhargav Bhatt and Peter Scholze on the pro-étale topology; see [2]. The pro-étale topology is finer than the étale topology, so that one can construct a sheaf associated to the topological ring $\overline{\mathbb{Q}}_\ell$ which captures the topological information on the ring (see [2, 4.2.12]), but not too coarse, so that one can redefine $\mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ as the usual derived category of constructible sheaves of modules over the sheaf of rings associated to $\overline{\mathbb{Q}}_\ell$ (see [2, 6.8.14]). In particular, as Bhatt–Scholze pointed out, the $\overline{\mathbb{Q}}_\ell$-homotopy type of X can be constructed directly, unlike the work of Deligne in [3, 5.2]. The motivation for this paper is to give such a direct construction of $\overline{\mathbb{Q}}_\ell$-homotopy types for algebraic stacks.

The main result of this paper is to generalize [2, 6.8.14] from schemes to algebraic stacks:

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Theorem 1.2. *Let S be a noetherian scheme and let $\mathcal{X}$ be an algebraic stack of finite type over S. Then:*

(i) The natural functor

$$\operatorname*{colim}_{F \subseteq E} \mathrm{D}_{\mathrm{cons}}((\mathrm{Sch}\,/\,\mathcal{X})_{\text{proét}}, \mathcal{O}_F) \to \mathrm{D}_{\mathrm{cons}}((\mathrm{Sch}\,/\,\mathcal{X})_{\text{proét}}, \mathcal{O}_E)$$

is an equivalence of triangulated categories, where the colimit is indexed by the finite extensions F of $\mathbb{Q}_\ell$ contained in E.

(ii) The natural functor

$$\mathrm{D}_{\mathrm{cons}}((\mathrm{Sch}\,/\,\mathcal{X})_{\text{proét}}, \mathcal{O}_E)[\ell^{-1}] \to \mathrm{D}_{\mathrm{cons}}((\mathrm{Sch}\,/\,\mathcal{X})_{\text{proét}}, E)$$

is an equivalence of triangulated categories.

Remark 1.3. Here the coefficient sheaves of rings and the derived categories are defined as in 4.4 and 4.17, respectively, by using the pro-étale topology for algebraic stacks of 2.5.

1.4. Our strategy for proving the main result 1.2 is to reduce to the case of schemes of [2, 6.8.14] by supplying an equivalence between the (derived) category of constructible sheaves on algebraic stacks and the (derived) category of constructible sheaves on simplicial algebraic spaces; see 4.24 and 4.27. We will make use of many arguments developed in the work of Martin Olsson (see [4]). However, there are some notable aspects that are new in our work. First, we make an observation that a big étale (resp. pro-étale) sheaf on a scheme is locally constructible in the sense of [1, VII.1.1.9] (see 3.4) if and only if it is cartesian in the sense of 3.1; see 3.5. This simple observation (which might not be new but does not appear in the literature to the extent of the author's knowledge) turns out to be surprisingly useful since it establishes a close connection between the study of small étale (resp. pro-étale) sheaves on schemes and the study of big étale (resp. pro-étale) sheaves on algebraic stacks, and therefore will be used repeatedly throughout the paper. We also note that our treatment of cartesian sheaves on algebraic stacks is simpler than it is in [4]. This is due to our new result 4.7, which guarantees that the coefficient sheaves of rings we work with in this paper (see 4.4) are cartesian in the sense of 3.1; it not only simplifies our work, but also provides some generalization of the work of Bhatt–Scholze; see 4.30.

1.5. **Conventions.** We follow the theory of algebraic spaces and stacks as developed in [5].

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2. Pro-Étale Topology for Algebraic Stacks

In this section we define the pro-étale topology for algebraic stacks. Throughout the rest of this paper, we fix a base scheme S and an algebraic stack $\mathcal{X}$ over S.

2.1. Let X be a scheme. Let $\mathrm{Sch}\,/X$ denote the category of schemes over X. It is equipped with a Grothendieck topology, where a collection of X-morphisms $\{f_i : Y_i \to Y\}_{i \in I}$ is a covering if each f_i is weakly étale (that is, f_i and $\Delta_{f_i} : Y_i \to Y_i \times_Y Y_i$ are flat; see [2, 1.2]) and the collection is a fpqc covering of Y. We refer to it as the *big pro-étale topology* on X and denote by $(\mathrm{Sch}/X)^{\sim}_{\text{proét}}$ the associated topos. Note that the (small) pro-étale topology on X of [2, 4.1.1] is the induced topology on the full subcategory $\mathrm{Proét}(X) \subseteq \mathrm{Sch}/X$ spanned by the weakly étale X-schemes. Let $X_{\text{proét}}$ denote the associated topos.

2.2. The following lemma (whose proof is immediate from the fpqc descent for flat morphisms; see, for example, [6, Tag 02L2]) shows that the property of being a weakly étale morphism is well-defined for morphisms of algebraic spaces, or more generally for representable morphisms of algebraic stacks:

Lemma 2.3. *The property of being a weakly étale morphism is stable with respect to the fpqc topology. Moreover, it is stable and local on domain with respect to the étale topology.*

2.4. We can extend the big pro-étale topology on schemes (see 2.1) to algebraic stacks:

Definition 2.5. Let $\mathrm{AS}/\mathcal{X}$ denote the category whose objects are pairs (T,t), where T is an algebraic space over S and $t : T \to \mathcal{X}$ is a morphism over S, and whose morphisms $(T',t') \to (T,t)$ are pairs (f, f^b) where $f : T' \to T$ is a morphism of algebraic spaces over S and $f^b : t' \to t \circ f$ is an isomorphism of functors. It is equipped with a Grothendieck topology, where a collection of maps $\{(f_i, f_i^b) : (T_i,t_i) \to (T,t)\}$ is a covering if each f_i is weakly étale (resp. étale) and the underlying collection $\{f_i : T_i \to T\}$ is a fpqc (resp. étale) covering. We refer to it as the *big pro-étale (resp. étale) topology* on $\mathcal{X}$ and denote by $(\mathrm{AS}/\mathcal{X})^{\sim}_{\text{proét}}$ (resp. $(\mathrm{AS}/\mathcal{X})^{\sim}_{\text{ét}}$) the associated topos.

Remark 2.6. Let $\mathrm{Sch}/\mathcal{X} \subseteq \mathrm{AS}/\mathcal{X}$ denote the full subcategory spanned by those objects (T,t) for which T is a scheme. With respect to the induced topology, the inclusion functor induces an equivalence of topoi.

Remark 2.7. In the special case where $\mathcal{X}$ is equivalent to a scheme X, we can recover the usual big étale topology on X (see, for example, [5, 2.1.13]).

2.8. For each object $(T,t) \in \mathrm{Sch}/\mathcal{X}$, the functor

$$\mathrm{Proét}(T) \to (\mathrm{Sch}/\mathcal{X})_{\text{proét}} : (f : T' \to T) \mapsto (T', t \circ f)$$

is cocontinuous (see 2.1), and therefore induces a morphism of topoi $i_{t_{\text{ét}}} : T_{\text{proét}} \to (\mathrm{Sch}/\mathcal{X})^{\sim}_{\text{proét}}$ for which the pullback functor carries a big pro-étale sheaf F on $\mathcal{X}$ to its restriction to $T_{\text{proét}}$, which we denote by $F_{(T,t)}$. If $(f, f^b) : (T',t') \to (T,t)$ is a morphism in $\mathrm{Sch}/\mathcal{X}$, then there is a canonical morphism $\rho_{(f,f^b)} : f^{-1}F_{(T,t)} \to F_{(T',t')}$ in $T'_{\text{proét}}$ satisfying the following conditions:

(i) For each composition $(T'',t'') \xrightarrow{(f',f'^b)} (T',t') \xrightarrow{(f,f^b)} (T,t)$ in $\mathrm{Sch}/\mathcal{X}$, the following diagram commutes:

$$\begin{tikzcd}
f'^{-1}f^{-1}F_{(T,t)} \arrow[r, "\rho_{(f,f^b)}"] \arrow[d, "\simeq"'] & f'^{-1}F_{(T',t')} \arrow[d, "\rho_{(f',f'^b)}"] \\
(f \circ f')^{-1}F_{(T,t)} \arrow[r, "\rho_{(f,f^b)\circ(f',f'^b)}"] & F_{(T'',t'')}.
\end{tikzcd}$$

(ii) If $(f, f^b) : (T',t') \to (T,t)$ is a morphism for which f is weakly étale, then the map $\rho_{(f,f^b)}$ is an isomorphism.

As in the case of lisse-étale sheaves on $\mathcal{X}$, the big pro-étale sheaf F can be recovered from the data $(\{F_{(T,t)}\}, \{\rho_{(f,f^b)}\})$; see [5, 9.1.12] for details.

2.9. For later reference, we record a useful diagram. Let $f : Y \to X$ be a morphism of schemes and let $f_{\text{ét}} : Y_{\text{ét}} \to X_{\text{ét}}$ denote the associated morphism of topoi. Then there is a commutative diagram of topoi (and similarly for the pro-étale case):

$$\begin{tikzcd} & (\mathrm{Sch}\,/X)^{\sim}_{\text{ét}} \arrow[d, "(i_!, i^{-1})"] \\ Y_{\text{ét}} \arrow[ur, "i_{f_{\text{ét}}}"] \arrow[r, "f_{\text{ét}}"'] & X_{\text{ét}}. \end{tikzcd}$$

3. Cartesian Sheaves on Algebraic Stacks

In this section we define the notion of a *cartesian sheaf* on $(\mathrm{Sch}\,/\,\mathcal{X})_{\text{proét}}$ and describe those sheaves via hypercovers, as in the case of lisse-étale sheaves of [4, §4]. We fix a smooth surjection $p : X \to \mathcal{X}$ for which X is a scheme throughout this section.

Definition 3.1. Let $\tau \in \{\text{ét}, \text{proét}\}$.

(i) A big τ-sheaf of sets F on $\mathcal{X}$ is *cartesian* if for each morphism $(f, f^b) : (T', t') \to (T, t)$ in $\mathrm{Sch}\,/\,\mathcal{X}$, the canonical map $f^{-1}F_{(T,t)} \to F_{(T',t')}$ of small τ-sheaves on T' is an isomorphism (see 2.8).

(ii) Let Λ be a big τ-sheaf of rings on $\mathcal{X}$. A big τ-sheaf of Λ-modules F on $\mathcal{X}$ is *cartesian* if for each morphism $(f, f^b) : (T', t') \to (T, t)$ in $\mathrm{Sch}\,/\,\mathcal{X}$, the canonical map of small τ-sheaves of $\Lambda_{(T',t')}$-modules on T'

$$f^*F_{(T,t)} := f^{-1}F_{(T,t)} \otimes_{f^{-1}\Lambda_{(T,t)}} \Lambda_{(T',t')} \to F_{(T',t')}$$

is an isomorphism. Let $\mathrm{Mod}^{\mathrm{cart}}_{\Lambda}((\mathrm{Sch}\,/\,\mathcal{X})_\tau) \subseteq \mathrm{Mod}_{\Lambda}((\mathrm{Sch}\,/\,\mathcal{X})_\tau)$ denote the full subcategory spanned by the cartesian τ-sheaves of Λ-modules.

3.2. We now give an alternative characterization of big cartesian sheaves on schemes. Let X be a scheme. The restriction functor $i^{-1} : (\mathrm{Sch}\,/X)_{\text{ét}} \to X_{\text{ét}}$ from the big étale topos to the small one admits both left and right adjoints, which we denote by $i_!$ and i_*, respectively. Note that the adjoint pairs $(i_!, i^{-1})$ and (i^{-1}, i_*) form morphisms of topoi. Similarly, we have morphisms of topoi $(j_!, j^{-1})$ and (j^{-1}, j_*) in the case of the pro-étale topology.

Remark 3.3. In the situation of 3.1, if $\mathcal{X}$ is equivalent to a scheme X, then a big étale sheaf of sets F on X is cartesian if and only if the canonical map $t^{-1}_{\text{ét}} i^{-1}F \to F_{(T,t)}$ is an isomorphism for each object $(t : T \to X) \in \mathrm{Sch}\,/X$ (and similarly for the pro-étale case).

3.4. According to [1, VII.1.1.9], a big étale sheaf of sets F on a scheme X is *locally constructible* if the adjunction map $i_! i^{-1}F \to F$ is an isomorphism (see 3.2). We define a locally constructible big pro-étale sheaf similarly.

The following assertion (which is a consequence of 2.9 and 3.3) might be known to the experts, but the author could not find it in the literature, so record it here:

Proposition 3.5. *Let $\tau \in \{\text{ét}, \text{proét}\}$. A big τ-sheaf of sets F on a scheme X is locally constructible if and only if it is cartesian. In particular, there is an equivalence of categories between X_τ and the full subcategory of $(\mathrm{Sch}\,/X)^{\sim}_\tau$ spanned by the cartesian sheaves.*

3.6. We recall a bit of terminology. Let Δ denote the category of combinatorial simplices. According to [7, $\mathrm{V}^{\mathrm{bis}}$.1.2.1], a Δ-*topos* consists of a topos T_n for each object $[n] \in \Delta$ and a morphism of topoi $f_\delta : T_m \to T_n$ for each morphism $(\delta : [n] \to [m]) \in \Delta$ such that for each composition $[n] \xrightarrow{\delta} [m] \xrightarrow{\epsilon} [k]$, there is an isomorphism $f_\delta \circ f_\epsilon \simeq f_{\epsilon\circ\delta}$. We refer to it as the *simplicial topos*. We will often abuse notation by denoting the Δ-topos and the morphism of topoi f_δ by $T_\bullet$ and δ, respectively. The *total topos* associated to the simplicial topos $T_\bullet$, which we denote by $\mathrm{Tot}(T_\bullet)$, is a category whose objects are collections $(\{F_n\}_{[n]\in\Delta}, \{F_\delta\}_{(\delta:[n]\to[m])\in\Delta})$ where $F_n \in T_n$ and $F_\delta : \delta^{-1}F_n \to F_m$ is a morphism in T_m such that for each composition $[n] \xrightarrow{\delta} [m] \xrightarrow{\epsilon} [k]$, the diagram

$$\begin{tikzcd}
f_\epsilon^{-1} f_\delta^{-1} F_n \arrow[r, "F_\delta"] \arrow[d, "\simeq"'] & f_\epsilon^{-1} F_m \arrow[d, "F_\epsilon"] \\
(f_{\epsilon\circ\delta})^{-1} F_n \arrow[r, "F_{\epsilon\circ\delta}"] & F_k
\end{tikzcd}$$

commutes, and whose morphisms $(\{F_n\}, \{F_\delta\}) \to (\{G_n\}, \{G_\delta\})$ are a collection of morphisms $\{d_n : F_n \to G_n\}_{[n]\in\Delta}$ such that $d_m \circ F_\delta = G_\delta \circ \delta^{-1}(d_n)$ for each morphism $(\delta : [n] \to [m]) \in \Delta$. We refer to an object of the total topos as a *sheaf on the simplicial topos* $T_\bullet$.

3.7. Let $X_\bullet$ denote the 0-coskeleton of the smooth surjection $p : X \to \mathcal{X}$, so that $X_n = X \times_{\mathcal{X}} X \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} X$ is the $(n+1)$-fold fiber product of X over $\mathcal{X}$. There is an associated simplicial topos, which we denote by $X_{\bullet,\text{proét}}$ (resp. $(\mathrm{Sch}\,/X_\bullet)^\sim_{\text{proét}}$), consisting of the small pro-étale topos $X_{n,\text{proét}}$ (resp. big pro-étale topos $(\mathrm{Sch}\,/X_n)^\sim_{\text{proét}}$) for each object $[n] \in \Delta$, and the morphism of topoi $\delta : X_{m,\text{proét}} \to X_{n,\text{proét}}$ (resp. $\delta^{\text{big}} : (\mathrm{Sch}\,/X_m)^\sim_{\text{proét}} \to (\mathrm{Sch}\,/X_n)^\sim_{\text{proét}}$) induced by $\delta : X_m \to X_n$ for each morphism $(\delta : [n] \to [m]) \in \Delta$.

Remark 3.8. If Λ is a sheaf of rings on $(\mathrm{Sch}\,/\,\mathcal{X})_{\text{proét}}$, then its restrictions to the small and big pro-étale topoi of X_n for every n induce sheaves of rings on $X_{\bullet,\text{proét}}$ and $(\mathrm{Sch}\,/X_\bullet)^\sim_{\text{proét}}$, respectively. We will generally abuse notation by denoting these sheaves by Λ.

3.9. Throughout the rest of this section, we fix a cartesian sheaf of rings Λ on $(\mathrm{Sch}\,/\,\mathcal{X})_{\text{proét}}$.

Definition 3.10.

(i) A sheaf of Λ-modules $F_\bullet$ on $X_{\bullet,\text{proét}}$ is *cartesian* if for each morphism $(\delta : [n] \to [m]) \in \Delta$, the map $\delta^* F_n \to F_m$ is an isomorphism. Let $\mathrm{Mod}^{\mathrm{cart}}_\Lambda(X_{\bullet,\text{proét}}) \subseteq \mathrm{Mod}_\Lambda(X_{\bullet,\text{proét}})$ denote the full subcategory spanned by the cartesian sheaves of Λ-modules.

(ii) A sheaf of Λ-modules $F_\bullet$ on $(\mathrm{Sch}\,/X_\bullet)^\sim_{\text{proét}}$ is *cartesian* if for each n, F_n is cartesian in the sense of 3.1 and for each morphism $(\delta : [n] \to [m]) \in \Delta$, the map $\delta^{\text{big}*} F_n \to F_m$ is an isomorphism. Let $\mathrm{Mod}^{\mathrm{cart}}_\Lambda((\mathrm{Sch}\,/X_\bullet)^\sim_{\text{proét}}) \subseteq \mathrm{Mod}_\Lambda((\mathrm{Sch}\,/X_\bullet)^\sim_{\text{proét}})$ denote the full subcategory spanned by the cartesian sheaves of Λ-modules.

3.11. Let $\mathrm{Des}(X/\,\mathcal{X}, \Lambda)$ denote the category whose objects are pairs (F, σ) where F is a small pro-étale sheaf of Λ_X-modules and $\sigma : \mathrm{pr}_1^* F \to \mathrm{pr}_2^* F$ is an isomorphism of small pro-étale sheaves of $\Lambda_{X\times_{\mathcal{X}}X}$-modules such that $\mathrm{pr}_{23}^*(\sigma) \circ \mathrm{pr}_{12}^*(\sigma) = \mathrm{pr}_{13}^*(\sigma)$ on $X \times_{\mathcal{X}} X \times_{\mathcal{X}} X$, and whose morphisms $(F,' \sigma') \to (F, \sigma)$ are morphisms $\phi : F' \to F$ of sheaves of Λ_X-modules such that $\sigma \circ \mathrm{pr}_1^*(\phi) = \mathrm{pr}_2^*(\phi) \circ \sigma'$ (here $p_i : X \times_{\mathcal{X}} X \to X$ and $p_{ij} : X \times_{\mathcal{X}} X \times_{\mathcal{X}} X \to X \times_{\mathcal{X}} X$ denote the projections).

3.12. There is a tautological functor $\mathrm{Mod}^{\mathrm{cart}}_{\Lambda}((\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{pro\acute{e}t}}) \to \mathrm{Mod}^{\mathrm{cart}}_{\Lambda}(X_{\bullet,\mathrm{pro\acute{e}t}})$. There is also a functor $\mathrm{Mod}^{\mathrm{cart}}_{\Lambda}(X_{\bullet,\mathrm{pro\acute{e}t}}) \to \mathrm{Des}(X/\,\mathcal{X}, \Lambda)$ which carries a collection $(\{F_n \in X_{n,\mathrm{pro\acute{e}t}}\}, \{F_\delta\})$ to the pair $(F_0, F_{d_1}^{-1} \circ F_{d_0})$. By mimicking the proof of [4, 4.4, 4.5] for the lisse-étale case, we obtain the following:

Proposition 3.13. *The functors* $\mathrm{Mod}^{\mathrm{cart}}_{\Lambda}((\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{pro\acute{e}t}}) \to \mathrm{Mod}^{\mathrm{cart}}_{\Lambda}(X_{\bullet,\mathrm{pro\acute{e}t}}) \to \mathrm{Des}(X/\,\mathcal{X}, \Lambda)$ *are equivalences of categories.*

3.14. There is an evident extension functor $j_! : \mathrm{Mod}_{\Lambda}(X_{\bullet,\mathrm{pro\acute{e}t}}) \to \mathrm{Mod}_{\Lambda}((\mathrm{Sch}\,/X_\bullet)^{\sim}_{\mathrm{pro\acute{e}t}})$. For later use, we show that $\mathrm{Mod}^{\mathrm{cart}}_{\Lambda}((\mathrm{Sch}\,/X_\bullet)^{\sim}_{\mathrm{pro\acute{e}t}})$ is also equivalent to $\mathrm{Mod}^{\mathrm{cart}}_{\Lambda}((\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{pro\acute{e}t}})$:

Lemma 3.15. *The functor $j_!$ restricts to an equivalence of categories*

$$\mathrm{Mod}^{\mathrm{cart}}_{\Lambda}(X_{\bullet,\mathrm{pro\acute{e}t}}) \to \mathrm{Mod}^{\mathrm{cart}}_{\Lambda}((\mathrm{Sch}\,/X_\bullet)^{\sim}_{\mathrm{pro\acute{e}t}}).$$

Proof. There is a well-defined restriction functor $\mathrm{Mod}^{\mathrm{cart}}_{\Lambda}((\mathrm{Sch}\,/X_\bullet)^{\sim}_{\mathrm{pro\acute{e}t}}) \to \mathrm{Mod}^{\mathrm{cart}}_{\Lambda}(X_{\bullet,\mathrm{pro\acute{e}t}})$. By virtue of 3.5, it is a quasi-inverse to the restriction of $j_!$, thereby completing the proof. □

3.16. We close this section with a discussion of the relationship between big pro-étale and étale sheaves on $\mathcal{X}$. The identity functor $(\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{pro\acute{e}t}} \to (\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{\acute{e}t}}$ is cocontinuous, and therefore induces a morphism of topoi $\nu : (\mathrm{Sch}\,/\,\mathcal{X})^{\sim}_{\mathrm{pro\acute{e}t}} \to (\mathrm{Sch}\,/\,\mathcal{X})^{\sim}_{\mathrm{\acute{e}t}}$ (cf. [2, §5]).

Proposition 3.17. *Let F be a cartesian sheaf of abelian groups on* $(\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{\acute{e}t}}$. *Then the derived adjunction map $F \to R\nu_*\nu^{-1}F$ is an isomorphism.*

Proof. Recall that we fixed a smooth surjection $p : X \to \mathcal{X}$, where X is a scheme. There is a commutative diagram of topoi

$$\begin{tikzcd}
(\mathrm{Sch}\,/X)^{\sim}_{\mathrm{pro\acute{e}t}} \arrow[r, "\nu_X"] \arrow[d, "r_{\mathrm{pro\acute{e}t}}"'] & (\mathrm{Sch}\,/X)^{\sim}_{\mathrm{\acute{e}t}} \arrow[d, "r_{\mathrm{\acute{e}t}}"] \\
(\mathrm{Sch}\,/\,\mathcal{X})^{\sim}_{\mathrm{pro\acute{e}t}} \arrow[r, "\nu_{\mathcal{X}}"'] & (\mathrm{Sch}\,/\,\mathcal{X})^{\sim}_{\mathrm{\acute{e}t}},
\end{tikzcd}$$

where $r_{\mathrm{pro\acute{e}t}}^{-1}$ and $r_{\mathrm{\acute{e}t}}^{-1}$ are the restriction functors. Note that the restriction of F to the big étale topos of X is cartesian, hence locally constructible by virtue of 3.5. Combining this with 2.9 and the fact that the canonical map of functors $r_{\mathrm{\acute{e}t}}^{-1} \circ \nu_{\mathcal{X}*} \to \nu_{X*} \circ r_{\mathrm{pro\acute{e}t}}^{-1}$ is an isomorphism, we are reduced to the case of small étale sheaves on schemes, in which case the desired result follows from [2, 5.1.6]. □

4. Constructible Sheaves on Algebraic Stacks

In this section we generalize some of the results about constructible pro-étale sheaves on schemes (see [2, §6]) to algebraic stacks.

4.1. One of the special features of the pro-étale topology is the notion of a "constant " sheaf associated to a topological space; see [2, 4.2.12]. The following extension to algebraic stacks is immediate:

Lemma 4.2. *Let T be a topological space. Then the presheaf* $(\mathrm{Sch}\,/\,\mathcal{X})^{\mathrm{op}}_{\mathrm{pro\acute{e}t}} \to \mathcal{S}\mathrm{et}$ *which carries an object* $(U, u) \in (\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{pro\acute{e}t}}$ *to* $\mathrm{Map}_{\mathrm{cont}}(U, T)$ *is a sheaf for the pro-étale topology.*

Remark 4.3. In the special case where $\mathcal{X}$ is equivalent to a scheme X, the restriction of the sheaf to the small pro-étale topos agrees with $\mathcal{F}_T$ of [2, 4.2.12].

4.4. Let us introduce our coefficient rings for constructible sheaves. Let E be an algebraic extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}_E$. Let $E_{\mathcal{X}}$ and $\mathcal{O}_{E,\mathcal{X}}$ denote the big pro-étale sheaves on $\mathcal{X}$ associated to the topological rings E and $\mathcal{O}_E$ in the sense of 4.2, respectively. Note that in the special case of schemes, this is an abuse of notation (see [2, 6.8.1]). However, we will see in 4.7 that there is little risk of confusion; see also 3.5 and 4.3.

4.5. The following assertion guarantees that our results in Section 3 can be applied to $\Lambda \in \{E_{\mathcal{X}}, \mathcal{O}_{E,\mathcal{X}}\}$ (see 3.9):

Proposition 4.6. *Let $f : X \to Y$ be a morphism of schemes. Then the canonical maps $f^{-1}E_Y \to E_X$ and $f^{-1}\mathcal{O}_{E,Y} \to \mathcal{O}_{E,X}$ are isomorphisms of small pro-étale sheaves.*

Proof. By virtue of the second part of [2, 6.8.2], it will suffice to consider the case of $\mathcal{O}_E$, where E is a finite extension of $\mathbb{Q}_\ell$. In this case, the first part of [2, 6.8.2] guarantees that $\mathcal{O}_E$ is the limit of constant sheaves. Then we can regard it as a limit of representable sheaves, where each of the transition maps is affine. Consequently, it is representable and the desired result follows from the fact that f^{-1} preserves representable sheaves. □

Corollary 4.7. *The big pro-étale sheaves $E_{\mathcal{X}}$ and $\mathcal{O}_{E,\mathcal{X}}$ are cartesian in the sense of* 3.1.

Proof. In view of 4.6, this follows immediately from the fact that for each object $(T,t) \in \mathrm{Sch}/\mathcal{X}$, the restrictions of $E_{\mathcal{X}}$ and $\mathcal{O}_{E,\mathcal{X}}$ to $T_{\text{proét}}$ are isomorphic to E_T and $\mathcal{O}_{E,T}$, respectively. □

4.8. We devote the remainder of this section to defining constructible pro-étale sheaves on algebraic stacks and studying their properties, generalizing the case of schemes of [2]. For the rest of this section, we often abuse notation by denoting the sheaves on an algebro-geometric object which are associated to the topological rings E and $\mathcal{O}_E$ in the sense of 4.2 by Λ, and assume the following:

($*$) The base scheme S is noetherian and the algebraic stack $\mathcal{X}$ is of finite type over S.

4.9. Let X be a topologically noetherian scheme (that is, the underlying topological space is noetherian; see [2, 6.6.9]). Let us recall the notion of a *constructible sheaf* on X; see [2, 6.8]. A *lisse Λ-sheaf* is a small pro-étale sheaf of Λ-modules L on X such that L is locally free of finite rank. We say that a small pro-étale sheaf of E-modules F on X is *constructible* if there exists a finite stratification $\{X_i \to X\}$ such that each restriction F_{X_i} is lisse and that a complex $K \in D(X_{\text{proét}}, E)$ is *constructible* if it is bounded and all cohomology sheaves are constructible. Let $\mathrm{D}_{\mathrm{cons}}(X_{\text{proét}}, E) \subseteq \mathrm{D}(X_{\text{proét}}, E)$ denote the full subcategory spanned by the constructible complexes. Finally, a small pro-étale sheaf of $\mathcal{O}_E$-modules F on X is *constructible* if there exists a finite stratification $\{X_i \to X\}$ such that each restriction F_{X_i} is locally isomorphic to $P \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}$ for some finitely presented $\mathcal{O}_E$-module P. Let $\mathrm{Cons}_X(\mathcal{O}_E)$ denote the category of the constructible sheaves.

The property of being a constructible sheaf is local for the étale topology:

Lemma 4.10. *Let X be a topologically noetherian scheme and let F be a small pro-étale sheaf of Λ-modules. Let $\{f_i : X_i \to X\}$ be an étale cover. If each restriction F_{X_i} is constructible, then so is F.*

Proof. By noetherian induction, it will suffice to show that for any irreducible closed subset $Y \subseteq X$, there exists a non-empty open subset $U \subseteq Y$ such that the restriction F_U is locally free of finite rank. Suppose we are given such a subset Y. Let us choose some X_i for which its image under f_i contains the generic point of Y. Since F_{X_i} is constructible, we can choose a locally closed subset $X_i' \subset X_i$ such that $F_{X_i'}$ is a locally free E-module of finite rank in the case where $\Lambda = E$, and is locally isomorphic to $P \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X_i'}$ for a finitely presented $\mathcal{O}_E$-module P in the case where $\Lambda = \mathcal{O}_E$. Writing X_i' as an intersection of an open subset $U \subseteq X_i'$ and a closed subset $C \subseteq X_i'$, we complete the proof by observing that $U \cap Y$ is the desired non-empty open subset of Y. □

Proposition 4.11. *Let F be a cartesian sheaf of Λ-module on $(\mathrm{Sch}/\mathcal{X})_{\text{proét}}$. Then the following conditions are equivalent:*

- (*i*) *For every object $(T,t) \in \mathrm{Sch}/\mathcal{X}$ for which T is noetherian, the restriction $F_{(T,t)}$ is constructible.*
- (*ii*) *There exists a smooth surjection $t : T \to \mathcal{X}$ such that T is a quasi-compact scheme and that $F_{(T,t)}$ is constructible.*

Proof. This is an immediate consequence of 4.10 and that for a morphism of topologically noetherian schemes, the pullback of a constructible Λ-sheaf is constructible. □

Remark 4.12. The existence of a smooth cover $T \to \mathcal{X}$ as in the second condition of 4.11 is guaranteed by our assumption on $\mathcal{X}$ that it is of finite type over the noetherian base scheme (see 4.8). Note that in this case, the scheme X is noetherian.

Definition 4.13. A sheaf of Λ-modules F on $(\mathrm{Sch}/\mathcal{X})_{\text{proét}}$ is *constructible* if it is cartesian (see 3.1) and if the equivalent conditions of 4.11 are satisfied. Let $\mathrm{Cons}_{\mathcal{X}}(\Lambda) \subseteq \mathrm{Mod}_{\Lambda}^{\mathrm{cart}}((\mathrm{Sch}/\mathcal{X})^{\sim}_{\text{proét}})$ the full subcategory spanned by the constructible Λ-sheaves.

Remark 4.14. In the special case where $\mathcal{X}$ is equivalent to a scheme X, a cartesian sheaf of Λ-modules F on $(\mathrm{Sch}/X)_{\text{proét}}$ is constructible in the sense of 4.13 if and only if its restriction to $X_{\text{proét}}$ is constructible in the sense of [2, 6.8.6, 6.8.10]. Consequently, even if the notations of 4.13 and [2, 6.8.10] conflict with one another, there is little danger of confusion.

4.15. The next result enables us to define constructible complexes on $\mathcal{X}$:

Lemma 4.16. *The category $\mathrm{Cons}_{\mathcal{X}}(\Lambda)$ is abelian.*

Proof. Choose a smooth surjection $p : X \to \mathcal{X}$ for which X is a quasi-compact scheme (see 4.12). We have a morphism of topoi $(p^{-1}, p_*) : X_{\text{proét}} \to (\mathrm{Sch}/\mathcal{X})^{\sim}_{\text{proét}}$ for which the pullback functor p^{-1} is the restriction functor. It then follows from the proof of 4.7 that there is an induced morphism of ringed topoi $(X_{\text{proét}}, \Lambda_X) \to ((\mathrm{Sch}/\mathcal{X})^{\sim}_{\text{proét}}, \Lambda_{\mathcal{X}})$. Since the functor $p^* : \mathrm{Mod}_{\Lambda_{\mathcal{X}}}((\mathrm{Sch}/\mathcal{X})_{\text{proét}}) \to \mathrm{Mod}_{\Lambda_X}(X_{\text{proét}})$ can be identified with p^{-1}, it is exact. Combining this observation with 4.11, we can reduce to the case of schemes, in which case the desired result follows from [2, 6.8.7, 6.8.11]. □

Definition 4.17. A complex of Λ-modules on $(\mathrm{Sch}/\mathcal{X})_{\text{proét}}$ is *constructible* if it is bounded and all cohomology sheaves are constructible in the sense of 4.13. Let $\mathrm{D}_{\mathrm{cons}}((\mathrm{Sch}/\mathcal{X})_{\text{proét}}, \Lambda) \subseteq \mathrm{D}((\mathrm{Sch}/\mathcal{X})_{\text{proét}}, \Lambda)$ denote the full subcategory spanned by the constructible complexes.

4.18. Our strategy for proving the main result 1.2 is to study sheaves on the simplicial topos $(\mathrm{Sch}\,/X_\bullet)^\sim_{\mathrm{proét}}$ to reduce to the case of schemes. To this end, we start by defining constructible sheaves on the simplicial topos:

Definition 4.19.

(i) A sheaf of Λ-modules $F_\bullet$ on $X_{\bullet,\mathrm{proét}}$ is *constructible* if it is cartesian (see 3.10) and each F_n is constructible in the sense of [2, 6.8.6, 6.8.10]. Let $\mathrm{Cons}_{X_\bullet}(\Lambda) \subseteq \mathrm{Mod}^{\mathrm{cart}}_\Lambda(X_{\bullet,\mathrm{proét}})$ denote the full subcategory spanned by the constructible sheaves.

(ii) A sheaf of Λ-modules $F_\bullet$ on $(\mathrm{Sch}\,/X_\bullet)^\sim_{\mathrm{proét}}$ is *constructible* if it is cartesian (see 3.10) and each restriction of F_n to $X_{n,\mathrm{proét}}$ is constructible. Let $\mathrm{Cons}_{\mathrm{Sch}\,/X_\bullet}(\Lambda) \subseteq \mathrm{Mod}^{\mathrm{cart}}_\Lambda((\mathrm{Sch}\,/X_\bullet)^\sim_{\mathrm{proét}})$ denote the full subcategory spanned by the constructible sheaves.

Remark 4.20. The equivalence of 3.15 restricts to an equivalence of categories $\mathrm{Cons}_{X_\bullet}(\Lambda) \to \mathrm{Cons}_{\mathrm{Sch}\,/X_\bullet}(\Lambda)$.

Remark 4.21. If $F_\bullet$ is cartesian sheaf of Λ-modules on $X_{\bullet,\mathrm{proét}}$ (resp. $(\mathrm{Sch}\,/X_\bullet)^\sim_{\mathrm{proét}}$), then it is constructible if and only if F_0 is constructible.

4.22. There is a natural morphism of simplicial topoi $(\mathrm{Sch}\,/X_\bullet)^\sim_{\mathrm{proét}} \to (\mathrm{Sch}\,/\,\mathcal{X})^\sim_{\mathrm{proét}}$, where we regard $(\mathrm{Sch}\,/\,\mathcal{X})^\sim_{\mathrm{proét}}$ as a constant simplicial topos, inducing a morphism of ringed topoi $\pi : (\mathrm{Tot}((\mathrm{Sch}\,/X_\bullet)^\sim_{\mathrm{proét}}), \Lambda) \to ((\mathrm{Sch}\,/\,\mathcal{X})^\sim_{\mathrm{proét}}, \Lambda)$ (see 3.6 and 3.8). We have an adjunction

$$\pi^* : \mathrm{Mod}_\Lambda((\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{proét}}) \rightleftarrows \mathrm{Mod}_\Lambda((\mathrm{Sch}\,/X_\bullet)^\sim_{\mathrm{proét}}) : \pi_*$$

for which the pullback functor π^* carries a cartesian sheaf on $(\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{proét}}$ to a cartesian sheaf on $(\mathrm{Sch}\,/X_\bullet)^\sim_{\mathrm{proét}}$. Moreover, the restriction of π^* to $\mathrm{Mod}^{\mathrm{cart}}_\Lambda((\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{proét}})$ is isomorphic to the composition of the functor $\mathrm{Mod}^{\mathrm{cart}}_\Lambda((\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{proét}}) \to \mathrm{Mod}^{\mathrm{cart}}_\Lambda(X_{\bullet,\mathrm{proét}})$ of 3.12 with the equivalence $j_! : \mathrm{Mod}^{\mathrm{cart}}_\Lambda(X_{\bullet,\mathrm{proét}}) \to \mathrm{Mod}^{\mathrm{cart}}_\Lambda((\mathrm{Sch}\,/X_\bullet)^\sim_{\mathrm{proét}})$ of 3.15. Note that π^* is exact because $\pi^* = \pi^{-1}$ (see 3.8).

4.23. Throughout the rest of this section, we assume that the scheme X appearing in our fixed cover $p : X \to \mathcal{X}$ of Section 3 is noetherian (see 4.12).

Lemma 4.24. *The functor π^* restricts to an equivalence of categories $\mathrm{Cons}_{\mathcal{X}}(\Lambda) \to \mathrm{Cons}_{\mathrm{Sch}\,/X_\bullet}(\Lambda)$.*

Proof. By virtue of 3.13 and 3.15, the pullback functor π^* restricts to an equivalence of categories $\mathrm{Mod}^{\mathrm{cart}}_\Lambda((\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{proét}}) \to \mathrm{Mod}^{\mathrm{cart}}_\Lambda((\mathrm{Sch}\,/X_\bullet)^\sim_{\mathrm{proét}})$. Invoking 4.23, the desired equivalence follows from 4.11 (see also 4.21). □

4.25. Combining 4.16 with 4.24, we see that $\mathrm{Cons}_{\mathrm{Sch}\,/X_\bullet}(\Lambda)$ is an abelian category. In particular, the following definition makes sense:

Definition 4.26. A complex of Λ-modules on $(\mathrm{Sch}\,/X_\bullet)^\sim_{\mathrm{proét}}$ is *constructible* if it is bounded and all cohomology sheaves are constructible in the sense of 4.19. Let $\mathrm{D}_{\mathrm{cons}}((\mathrm{Sch}\,/X_\bullet)^\sim_{\mathrm{proét}}, \Lambda) \subseteq \mathrm{D}((\mathrm{Sch}\,/X_\bullet)^\sim_{\mathrm{proét}}, \Lambda)$ denote the full subcategory spanned by the constructible complexes.

Proposition 4.27. *The functor π^* of 4.22 induces an equivalence of triangulated categories*

$$\mathrm{D}_{\mathrm{cons}}((\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{proét}}, \Lambda) \to \mathrm{D}_{\mathrm{cons}}((\mathrm{Sch}\,/X_\bullet)^\sim_{\mathrm{proét}}, \Lambda).$$

Proof. We first observe that the category $\mathrm{Cons}_{\mathcal{X}}(\Lambda)$ is closed under extensions by reducing to the case of schemes; see the proof [2, 6.8.9] and [2, 6.8.11]. Combining this observation with 4.16 and 4.24, we deduce that the subcategories $\mathrm{D}_{\mathrm{cons}}((\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{pro\acute{e}t}}, \Lambda)$ and $\mathrm{D}_{\mathrm{cons}}((\mathrm{Sch}\,/X_\bullet)^{\sim}_{\mathrm{pro\acute{e}t}}, \Lambda)$ are triangulated. We next show that the derived unit map $\mathrm{id} \to R\pi_* \circ \pi^*$ for the adjunction (π^*, π_*) of 4.22 is an isomorphism of functors on $\mathrm{D}^+((\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{pro\acute{e}t}}, \Lambda)$. Since the unique morphism from the big pro-étale sheaf on $\mathcal{X}$ represented by the object $(X, p) \in \mathrm{Sch}\,/\,\mathcal{X}$ to a final object of $(\mathrm{Sch}\,/\,\mathcal{X})^{\sim}_{\mathrm{pro\acute{e}t}}$ is an epimorphism, the desired isomorphism is a formal consequence of cohomological descent (see, for example, [5, 2.4.16] and its proof). To complete the proof, it will suffice to show that for each object $K \in \mathrm{D}_{\mathrm{cons}}((\mathrm{Sch}\,/X_\bullet)^{\sim}_{\mathrm{pro\acute{e}t}}, \Lambda)$, the derived counit map $\pi^* R\pi_* K \to K$ is an isomorphism. For this, we may assume that K is a constructible sheaf on $(\mathrm{Sch}\,/X_\bullet)^{\sim}_{\mathrm{pro\acute{e}t}}$ concentrated in degree 0, in which case the desired result follows from 4.24. □

4.28. We are now ready to prove the main result 1.2 of this paper.

Proof of 1.2. To show the fully faithfulness of both functors, as in the proof of [2, 6.8.14], it will suffice to prove that for each object $K \in \mathrm{D}_{\mathrm{cons}}((\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{pro\acute{e}t}}, \mathcal{O}_E)$, the internal derived Hom functor $\underline{\mathrm{RHom}}(K, \bullet)$ commutes with direct sums in $\mathrm{D}^{\geq 0}((\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{pro\acute{e}t}}, \mathcal{O}_E)$. Using the fully faithfulness of the functor $\pi^* : \mathrm{D}^+((\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{pro\acute{e}t}}, \mathcal{O}_E) \to \mathrm{D}^+((\mathrm{Sch}\,/X_\bullet)^{\sim}_{\mathrm{pro\acute{e}t}}, \mathcal{O}_E)$ (see the proof of 4.27), we are reduced to the case of schemes, in which case the desired result follows from [2, 6.8.12] (see also the proof of [2, 6.3.14]). To verify essential surjectivity, we note that the fully faithfulness guarantees that we can reduce to the case of constructible sheaves on $(\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{pro\acute{e}t}}$ concentrated in degree 0. Combining 3.13 with 4.20 and 4.24, we deduce that the category $\mathrm{Cons}_{\mathcal{X}}(\Lambda)$ is equivalent to the full subcategory of $\mathrm{Des}(X/\,\mathcal{X}, \Lambda)$ spanned by those objects (F, σ) for which F is constructible (see also 4.21). Using this observation and the fully faithfulness, the desired results follow from [2, 6.8.11] and [2, 6.8.13] for the first and second functors, respectively. □

4.29. We conclude this section with a few remarks about the behavior of the derived category of constructible sheaves with respect to morphisms of algebraic stacks. Suppose we are given a morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ satisfy condition $(*)$ of 4.8. We have a morphism of topoi $(f^{-1}, f_*) : (\mathrm{Sch}\,/\,\mathcal{X})^{\sim}_{\mathrm{pro\acute{e}t}} \to (\mathrm{Sch}\,/\,\mathcal{Y})^{\sim}_{\mathrm{pro\acute{e}t}}$ for which the pullback functor f^{-1} is the restriction functor. Using the natural map $f^{-1}\Lambda_{\mathcal{Y}} \to \Lambda_{\mathcal{X}}$, we obtain a derived adjunction

$$Lf^* : \mathrm{D}((\mathrm{Sch}\,/\,\mathcal{Y})_{\mathrm{pro\acute{e}t}}, \Lambda_{\mathcal{Y}}) \rightleftarrows \mathrm{D}((\mathrm{Sch}\,/\,\mathcal{X})_{\mathrm{pro\acute{e}t}}, \Lambda_{\mathcal{X}}) : Rf_*.$$

Since the natural map is an isomorphism, we conclude that the derived pullback functor Lf^* coincides with the usual pullback functor $f^* = f^{-1}$.

Remark 4.30. Let $f : X \to Y$ be a morphism of schemes and let $(f^{-1}, f_*) : X_{\mathrm{pro\acute{e}t}} \to Y_{\mathrm{pro\acute{e}t}}$ be the associated morphism of small pro-étale topoi. We have a natural map $f^{-1}\Lambda_Y \to \Lambda_X$ and it induces a derived adjunction

$$Lf^* : \mathrm{D}(Y_{\mathrm{pro\acute{e}t}}, \Lambda_Y) \rightleftarrows \mathrm{D}(X_{\mathrm{pro\acute{e}t}}, \Lambda_X) : Rf_*.$$

It then follows from [2, 6.8.15] that Lf^* can be identified with f^{-1} provided that f is étale or a closed immersion. By virtue of 4.6, we conclude that the assertion on the derived pullback functor holds for any morphism of schemes.

4.31. Assume now that f is proper. Arguing as in the proof of [4, 9.14], we see that if F is a constructible sheaf of $\Lambda_{\mathcal{X}}$-modules on $(\mathrm{Sch}\,/\,\mathcal{X})_{\text{proét}}$, then $R^i f_* F$ is a constructible sheaf of $\Lambda_{\mathcal{Y}}$-modules on $(\mathrm{Sch}\,/\,\mathcal{Y})_{\text{proét}}$. Consequently, the derived adjunction of 4.29 restricts to an adjunction

$$f^{-1} : \mathrm{D}_{\mathrm{cons}}((\mathrm{Sch}\,/\,\mathcal{Y})_{\text{proét}}, \Lambda_{\mathcal{Y}}) \rightleftarrows \mathrm{D}_{\mathrm{cons}}((\mathrm{Sch}\,/\,\mathcal{X})_{\text{proét}}, \Lambda_{\mathcal{X}}) : Rf_* .$$

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