Equidistribution of positive closed currents

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E. Schröder first used Newton’s method to study complex roots in one complex variable: finding a root of \( f(x) = 0 \) by iteration where \( f \) is a polynomial.

\[
x_{n+1} = P(x_n) = \cdots = P^n(x_0) \quad \text{where} \quad P(x) := x - g(x)/g'(x)
\]
The study of the good initial values and bad initial values leads to the definitions of the Fatou-Julia set.

**Definition**

A point \( p \in X \) belongs to the Fatou set \( F \) if there exists an open neighborhood \( U_p \ni p \) where \( \{f^n\} \) is an equicontinuous family. The Julia set \( J \) is its complement \( X \).

**Remark**

In general, the Julia set has a very complicated structure, known as fractal.
Examples of the Julia Set
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How do we study these complicated set?
Brolin’s equidistribution theorem

**Theorem (Brolin, 1965)**

Let \( f(z) = z^d + \cdots \) be a given polynomial of degree \( d \geq 2 \). Then, there exists a subset \( \mathcal{E} \subset \mathbb{C} \) such that \( \# \mathcal{E} \leq 1 \) such that if \( a \in \mathbb{C} \setminus \mathcal{E} \), then

\[
\frac{1}{d^n} \sum_{f^n(z) = \alpha} \delta_z \to \mu \text{ as } n \to \infty
\]

where \( \mu \) is a harmonic measure on the filled Julia set of \( f \). The limit is independent of the choice of \( a \in \mathbb{C} \setminus \mathcal{E} \). The exceptional set \( \mathcal{E} = \emptyset \) unless \( f \) is affinely conjugate to \( z \to z^d \). In this case, the set \( \mathcal{E} = \{0\} \) is totally invariant.

Such convergence towards a unique measure is called *equidistribution*. 
Example. Consider $f(z) = z^2$ and $\delta_1$. when $n = 0$
Example. Consider $f(z) = z^2$ and $\delta_1$. when $n = 1$
Example. Consider $f(z) = z^2$ and $\delta_1$.

when $n = 2$
Example. Consider $f(z) = z^2$ and $\delta_1$.

when $n = 3$
Example. Consider $f(z) = z^2$ and $\delta_1$. when $n = 4$
Example. Consider $f(z) = z^2$ and $\delta_1$.

In this case, the measure $\mu$ is the Lebesgue measure on the unit circle and the exceptional set $\mathcal{E}$ is $\{0\}$. 
Generalization of Brolin’s Theorem

In one dimensional case,
Lyubich, Freire-Lopes-Mañé

In higher dimensional case, when codimension = 1,
Favre-Jonsson, Dinh-Sibony, Fornæss-Sibony, Guedj,
Russakovskii-Shiffman, Sibony, Parra, Taflin,

when codimension = k, i.e, the measure case.
Briend-Duval, Fornæss-Sibony, Dinh-Sibony
Roughly speaking, a positive \((p, p)\)-current is a \((p, p)\)-form with measure coefficients. A positive closed current can be understood as a (analytic) generalization of analytic subsets.

Let \(A\) be an analytic subset of pure dimension \(k - p\). Then, the current of integration \([A]\) on \(A\) is defined by

\[
\langle [A], \varphi \rangle = \int_{\text{Reg} A} \varphi \quad \text{for} \quad \varphi \in \mathcal{D}^{k-p}.
\]

Also, smooth forms are currents in an obvious way:

\[
\langle \psi, \varphi \rangle = \int_X \psi \wedge \varphi \quad \text{for} \quad \varphi \in \mathcal{D}^{k-p}.
\]
On the other hand, according to a theorem of Siu, we have

**Theorem (Siu, 1978)**

Let $S$ be a positive closed $(p,p)$-current of $\mathbb{P}^k$. If the support of $S$ is an analytic subset of $\mathbb{P}^k$ of pure dimension $k - p$, then, $S = \sum_j c_j [V_j]$ where $c_j > 0$ is a constant and $V_j$'s are analytic subsets of $\mathbb{P}^k$ of pure dimension $k - p$.

In what follows, $\mathcal{C}_p$ denotes the space of positive closed $(p,p)$-currents of unit mass of $\mathbb{P}^k$. 
Generalization of Brolin’s Theorem: Equidistribution of Analytic Subsets

Conjecture (Dinh-Sibony)

Let \( f \) be a holomorphic endomorphism of \( \mathbb{P}^k \) of algebraic degree \( d \geq 2 \) and \( T \) its Green current.

Then \( d^{-p_n}(f^n)^*[H] \) converge to \( sT^p \) for every analytic subset \( H \) of \( \mathbb{P}^k \) of pure codimension \( p \) and of degree \( S \) which is generic.

Here \( H \) is generic if either \( H \cap E = \emptyset \) or \( \text{codim} H \cap E = p + \text{codim} E \) for any irreducible component \( E \) of every totally invariant analytic subset of \( \mathbb{P}^k \).
Theorem (Dinh-Sibony)

Let $f$ be a holomorphic endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Let $\mu$ be the equilibrium measure of $f$ and $\mathcal{E}$ a maximal proper analytic subset of $\mathbb{P}^k$ which is totally invariant under $f$, i.e. $f^{-1}(\mathcal{E}) = f(\mathcal{E}) = \mathcal{E}$. Then

$$d^{-kn}(f^n)^*(\delta_a)$$

converges to $\mu$ if and only if $a \notin \mathcal{E}$. 

Measure case: $p = k$
Theorem (Dinh-Sibony, Taflin)

Let $f$ be a holomorphic endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ and $T$ its Green current. If $H$ is a hypersurface and of degree $s$ which is generic in the Zariski sense, then the sequence $d^{-n}(f^n)^*[H]$ converges to $T$ exponentially fast.
A quasi-potential of $S$ is a quasi-plurisubharmonic function $u$ on $\mathbb{P}^k$ such that

$$S - \omega = \text{dd}^c u$$

If we impose a normalizing condition, for example, $\sup_{\mathbb{P}^k} u = 0$, then $u$ is unique. Note that

$$\langle S - \omega, \varphi \rangle = \int_{\mathbb{P}^k} u \wedge \text{dd}^c \varphi$$
Let $S \in \mathcal{C}_p$. Then, we have a $(p-1, p-1)$-current $U_S$ such that

$$S - \omega^p = \ddbar U_S.$$

However, there are two difficulties:
1) $U_S$ is no more a function and
2) there is no canonical choice of $U_S$. 
Super-potentials

**Definition (Dinh-Sibony, 2009)**

Let $S \in \mathcal{C}_p$ be smooth. Then, we define the super-potential $\mathcal{U}_S$ of mean $m$ by

$$\mathcal{U}_S(R) = \langle S, \mathcal{U}_R \rangle$$

for $R \in \mathcal{C}_{k-p+1}$ where $\mathcal{U}_R$ denotes a quasi-potential of $R$ of mean $m$. For general $S \in \mathcal{C}_p$, we define by

$$\mathcal{U}_S = \lim_{\theta \to 0} \mathcal{U}_{S_{\theta}}.$$  

The mean of quasi-potential $\mathcal{U}_R$ is defined by $\langle \mathcal{U}_R, \omega^p \rangle$. Then, we have

$$\langle S - \omega^p, \varphi \rangle = \langle \mathcal{U}_S, M\omega^{k-p+1} + \text{dd}^c \varphi \rangle - \langle \mathcal{U}_S, M\omega^{k-p+1} \rangle.$$
Known results on equidistribution of positive closed $(p, p)$-currents with $1 < p < k$

**Theorem (Dinh-Sibony, 2009)**

Let $\mathcal{H}_d(\mathbb{P}^k)$ denote the set of holomorphic endomorphisms of degree $d \geq 2$ on $\mathbb{P}^k$.

There is a Zariski dense open set $\mathcal{H}^*_d(\mathbb{P}^k)$ in $\mathcal{H}_d(\mathbb{P}^k)$ such that, if $f$ is in $\mathcal{H}^*_d(\mathbb{P}^k)$, then $d^{-pn}(f^n)^*(S)$ converges to $T^p$ uniformly with respect to $S \in \mathcal{C}_p$.

In particular, for $f$ in $\mathcal{H}^*_d(\mathbb{P}^k)$, $T^p$ is the unique current in $\mathcal{C}_p$ which is $f^*$-invariant.
Known results on equidistribution of positive closed $(p, p)$-currents with $1 < p < k$

**Theorem (A-, 2016)**

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of degree $d \geq 2$. Let $T^p$ denote the Green $(p, p)$-current associated with $f$ on $\mathbb{P}^k$.

Then, there is a proper (possibly empty) invariant analytic subset $E$ for $f$ such that $d^{-p n} (f^n)^* S$ converges to $T^p$ exponentially fast in the sense of currents for every $S \in \mathcal{C}_p$ smooth on $E$. 
In the case of $p = 1$, a quasi-potential $u$ of $S \in C_1$, that is a q-psh function $u$ such that $\text{dd}^c u = S - \omega$ and $\sup_{\mathbb{P}^k} u = 0$ is a function. It is very convenient to talk about the regularity and/or singularity.

1. The Lelong number of $S$ is 0 or equivalently, the Lelong number of $u$ is 0.

2. $u$ is locally bounded.
Super-potentials continuous/bounded near an analytic subset

Let $D_{k-p+1}(W)$ for an open subset $W$ of $\mathbb{P}^k$ denote the space of closed $(k-p+1, k-p+1)$-currents $R$ on $\mathbb{P}^k$ such that $\text{supp} R \subseteq W$ and $R$ can be written as $R = R_+ - R_-$ where $R_\pm$ are positive closed $(k-p+1, k-p+1)$-currents of the same mass.

**Definition**

A super-potential $\mathcal{U}_S$ of mean $m$ is continuous in $W$ if $\mathcal{U}_S(\cdot)$ is a continuous function on $D_{k-p+1}$ with respect to the topology of $D_{k-p+1}$. A super-potential $\mathcal{U}_S$ of mean $m$ is bounded in $W$ if there exists an constant $C_{S,m} > 0$ such that

$$\mathcal{U}_S(R) \leq C_{S,m}$$

where $R \in D_{k-p+1}(W)$. 
Theorem (A.-, preprint)

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of algebraic degree $d \geq 2$. Then, there exists a proper (possibly empty) invariant analytic subset $E$ for $f$ such that if $S \in \mathcal{C}_p$ is a current with its super-potential $U_S$ of mean 0 bounded near $E$, then we have

$$d^{-pn}(f^n)^*S \to T^p$$

exponentially fast in the sense of currents where $\mathcal{C}_p$ denotes the set of positive closed $(p, p)$-currents of unit mass on $\mathbb{P}^k$ and $T$ the Green current associated with $f$. 
Proposition

Let $E$ be a proper analytic subset of $\mathbb{P}^k$. If a positive closed $(p, p)$-current $S$ on $\mathbb{P}^k$ of unit mass has singularities such that

$$S \leq M_S \sum S_1 \wedge \cdots \wedge S_p \quad \text{near } E$$

in the sense of currents where the sum is a finite sum, $M_S$ is a non-negative constant and every $S_i$ is a positive closed $(1, 1)$-current of 1, whose quasi-potential $q_i$ is $(K, \alpha)$-Hölder continuous near $E$. Then, the super-potential $U_S$ of $S$ of mean 0 is PB near $E$. 
Let \( R \) be a positive closed \((p, p)\)-current on an open set \( U \) of \( \mathbb{C}^k \). Let \( z \) denote the coordinates in \( \mathbb{C}^k \) and \( B_a(r) \) the ball of center \( a \) and of radius \( r \). Define for \( a \in U \)

\[
\nu(R, a, r) := \frac{\| R \wedge (dd^c \|z\|)^2\|^{k-p} \|_{B_a(r)}}{\pi^{k-p} r^{2(k-p)}}.
\]

When \( r \) decreases to 0, \( \nu(R, a, r) \) is decreasing and the Lelong number of \( R \) at \( a \) is the limit

\[
\nu(R, a) := \lim_{r \to 0} \nu(R, a, r).
\]

In particular, when \( p = 1 \) it is equivalent to saying that its local potential \( \nu \) satisfies the following
Theorem (Guedj, 2003)

Let \( f : \mathbb{P}^k \) be a holomorphic endomorphism of algebraic degree \( d \geq 2 \). Assume that \( S \in \mathcal{C}_1 \) has zero Lelong number everywhere on \( \mathbb{P}^k \). Then, we have

\[
d^{-n}(f^n)^*S \rightarrow T.
\]
Skoda’s integrability theorem

Let \( \nu \) be a psh function on the unit ball \( B \subset \mathbb{C}^k \) with \( \sup_B \nu = 0 \). Suppose that \( \dd c \nu \) has zero Lelong number. Skoda’s integrability theorem implies that \( \exp(-\nu) \) is integrable. Then, we can do the following. Let \( K \subset B \). Then, for any \( c > 0 \), we have

\[
\text{Vol}(\{ \nu < -c \} \cap K) \leq \int_K \exp(-c - \nu) \leq \exp(-c) \int_K \exp(-\nu).
\]
Let $S \in \mathbb{C}_p$ be such that the Lelong number $\nu(S, a) = 0$ for every $a \in \mathbb{P}^k$. Then, we have

$$d^{-pn}(f^n)^*S \to T^p$$

in the sense of currents.

It is not clear whether the Lelong number works well with super-potentials for $1 < p < k$. 
Definition (Guedj-Zeriahi, 2007)

Let $\text{PSH}(\mathbb{P}^k, \omega) := \{ q \in \mathcal{L}^1(\mathbb{P}^k) : w + dd^c q \geq 0 \}$. We define

$$
\mu_\varphi := \lim_{j \to +\infty} 1_{\{ \varphi > -j \}}[\omega + dd^c \max\{ \varphi, -j \}]^k
$$

and

$$
\mathcal{E}(X, \omega) := \left\{ \varphi \in \text{PSH}(\mathbb{P}^k, \omega) : \mu_\varphi(\mathbb{P}^k) = \int_{\mathbb{P}^k} \omega^k \right\}.
$$

This is the largest class in $\text{PSH}(\mathbb{P}^k, \omega)$ on which the complex Monge-Ampère operator is well defined and the comparison principle is valid. Also, the Lelong number of $\omega + dd^c \varphi$ for this class is 0.
Theorem (A.-Nguyen, in preparation)

For every $\phi \in \mathcal{E}(X, \omega)$ with $\phi \in L^1(\|\omega + dd^c \phi\|^{p-1})$,

$$d^{-pn}(f^n)^*(\omega + dd^c \phi) \to T^p$$

exponentially fast in the sense of currents.
Idea of Proof

We use the comparison principle and localization by a difference of two psh functions. Write $\omega_\varphi := \omega + \dd c \varphi$.

\[
\langle d^{-p}n(f^n)^*(\omega + \dd c \varphi) - T^p, \varphi \rangle \\
= \cdots \langle (d^{-n}(f^n)^* \omega_\varphi - T) \wedge [d^{-n}(f^n)^* \omega_\varphi]^l \wedge T^{p-l-1}, \varphi \rangle \cdots \\
= \cdots \langle d^{-n} \nu \circ f^n [d^{-n}(f^n)^* \omega_\varphi]^l \wedge T^{p-l-1}, \dd c \varphi \rangle \cdots 
\]

where $\omega_\varphi = T + \dd c \nu$ with $\sup_k \nu = 0$. 

Idea of Proof

\[
\int d^{-n} \nu \circ f^n [d^{-n}(f^n)^* \omega_\varphi] \wedge T^{p-l-1} \wedge \omega^{k-p+1}
\]

\[
= \int \{d^{-n} \circ f^n \geq \epsilon_n\} \quad d^{-n} \nu \circ f^n [d^{-n}(f^n)^* \omega_\varphi] \wedge T^{p-l-1} \wedge \omega^{k-p+1}
\]

\[
+ \int \{d^{-n} \circ f^n < \epsilon_n\} \quad d^{-n} \nu \circ f^n [d^{-n}(f^n)^* \omega_\varphi] \wedge T^{p-l-1} \wedge \omega^{k-p+1}
\]

We can write

\[
d^{-n} \int \{\varphi \circ f^n < -C\} \quad \varphi \circ f^n [d^{-n}(f^n)^* \omega_\varphi] \wedge T^{p-l-1} \wedge \omega^{k-p+1}
\]

\[
= -d^n C \int_{\varphi \circ f^n < -C} \quad [d^{-n}(f^n)^* \omega_\varphi] \wedge T^{p-l-1} \wedge \omega^{k-p+1}
\]

\[
+ d^{-n} \int_{-\infty}^{-C} \quad [d^{-n}(f^n)^* \omega_\varphi] \wedge T^{p-l-1} \wedge \omega^{k-p+1}(\{\varphi \circ f^n < t\})dt.
\]
Eventually, we use the following boundedness:
For every $\varphi \in \mathcal{E}(\mathbb{P}^k, \omega)$, there exists a convex increasing function $\varphi : \mathbb{R}^- \to \mathbb{R}^-$ such that

$$\int \chi \circ \varphi \omega^k_\varphi < +\infty.$$ 

Together with our assumption and the comparison principle, we have

$$\int \varphi \omega^{p-1}_\varphi \wedge \omega^{k-p+1}_{\chi \circ \varphi} < +\infty.$$
Thank you!