Monotone Lagrangian tori in cotangent bundles

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A submanifold $L$ of a symplectic manifold $(X^{2n}, \omega)$ is called Lagrangian if $\omega|_L = 0$ and $\dim L = n$

eg. $(\mathbb{C}^n = \mathbb{R}^{2n}, \sum_{i=1}^{n} dx_i \wedge dy_i), \ S^1(r_1) \times \cdots \times S^1(r_n)$

Every Lagrangian $L$ in $(X, \omega)$ comes with two invariants.

1. (Symplectic area) $\omega(\beta)$ for $\beta \in \pi_2(X, L)$.
2. (Maslov index) $\mu(\beta)$ for $\beta \in \pi_2(X, L)$.

A Lagrangian submanifold $L$ is monotone if $\exists \ a > 0$ such that

$$\omega(\beta) = a \cdot \mu(\beta)$$

for all $\beta \in \pi_2(X, L)$. 
Monotone Lagrangian submanifolds

e.g. \((\mathbb{C}^n = \mathbb{R}^{2n}, \sum_{i=1}^{n} dx_i \wedge dy_i)\)

- \(S^1(r_1) \times \cdots \times S^1(r_n)\) is monotone if \(r_1 = \cdots = r_n\).

\[\pi_2(\mathbb{C}^2, S(r_1) \times S(r_2)) \simeq \mathbb{Z}^2,\] generated by two discs bounded by

- \((S(r_1)) \times (\text{a point in } S(r_2))\)
- \((\text{a point in } S(r_1)) \times (S(r_2))\)
Motivation

• (Lagrangian Floer theory)
  • (Oh) Good condition for Lagrangian Floer theory.
  • A candidate for non-displaceable Lag. in a symplectic manifold.
    (Arnold conjecture)
  • A candidate for a generating set for Fukaya category.
    (Homological mirror symmetry conjectured by Kontsevich)

• (Classification of Lagrangian submanifolds)
  • (Chekanov) \( n \) monotone. Lag. tori in \( T^*\mathbb{R}^n \cong \mathbb{R}^{2n} \)
  • (Albers-Frauenfelder) monotone Lag. tori in \( T^*S^2 \)
  • (Oakley-Usher) monotone Lag. tori in \( T^*S^3 \)
  • (Auroux) infinitely many monotone Lag. tori in \( T^*\mathbb{R}^n \) \((n \geq 3)\).
  • (Chekanov-Schlenk, Biran-Cornea, Entov-Polterovich, Fukaya-Oh-Ohta-Ono, Wu) mono. Lag. torus in \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) or \( \mathbb{C}P^2 \)
  • (Vianna) infinitely many monotone Lag. tori in \( \mathbb{C}P^2 \), del Pezzo surf.
Theorem (Cho-K.-Oh)

Let $X$ be one of the followings

- **Spheres**
- **Unitary groups,**
- **Special unitary groups**
- **a product of them**

The cotangent bundle $T^*X$ admits monotone Lagrangian tori.
Outline of construction

- How to produce a monotone Lagrangian torus in $T^*X$?
- Find a completely integrable system on a symplectic manifold $M$ having $X$ as a Lagrangian fiber.
- Pass nearby Lagrangian tori to $T^*X$. 
Coadjoint orbits

\begin{align*}
\begin{cases}
Ad: U(n) \times u(n) \to u(n) & (U, X) \mapsto UXU^{-1} \\
Ad^*: U(n) \times u(n)^* \to u(n)^* & (U, \xi) \mapsto \xi_U; \quad \xi_U(X) = \xi(U^{-1}XU)
\end{cases}
\Rightarrow O_\lambda = \text{the orbit of } \lambda \in u(n)^* \text{ under the coadjoint action } Ad^*.
\end{align*}

\begin{align*}
u(n) &= \{(n \times n) \text{ skew Hermitian matrices.}\} \\
\sqrt{-1}u(n) &= \{(n \times n) \text{ Hermitian matrices.}\}
\end{align*}

- A Killing form on \(\sqrt{-1}u(n)\): \((X, Y) \mapsto \text{tr}(XY) = \text{tr}(X^tY)\)

- Have a \(U(n)\)-equivariant \(\mathbb{R}\)-vector space isom

\begin{align*}
\sqrt{-1}u(n) &\mapsto u(n)^* \quad X \mapsto (X, \cdot).
\end{align*}

\Rightarrow (\text{adjoint orbit of } \lambda \in \sqrt{-1}u(n)) = (\text{coadjoint orbit of } \lambda \in u(n)^*)
Under the identification, fixing a sequence of real numbers
\[ \lambda_1 = \cdots = \lambda_{n_1} > \lambda_{n_1+1} = \cdots = \lambda_{n_2} > \cdots > \lambda_{n_r+1} = \cdots = \lambda_n, \]
we may take
\[ \lambda = \text{diag}(\lambda_1, \cdots, \lambda_n) \in \sqrt{-1}u(n) \]
and then
\[ \mathcal{O}_\lambda = \{ U \text{ diag}(\lambda_1, \cdots, \lambda_n) U^{-1} : U \in U(n) \} \]
\[ = \{ A \in \sqrt{-1}u(n) : \text{spec}(A) = \{ \lambda_1, \cdots, \lambda_n \} \}. \]

We obtain
\[ \mathcal{O}_\lambda \simeq \frac{U(n)}{U(n_1) \times U(n_2 - n_1) \times \cdots \times U(n_r - n_{r-1}) \times U(n - n_r)} \]
\[ \simeq \mathcal{F}(n_1, n_2, \cdots, n_r; n). \]
Gelfand-Cetlin Systems

Consider

\[ U(n) \supset U(n-1) \supset \cdots \supset U(1), \quad U(k) \cong \begin{bmatrix} U(k) & O \\ O & I_{n-k} \end{bmatrix} \]

A moment map of \( U(k) \)-action is

\[ \Phi^{(k)} : O_{\lambda} \rightarrow \sqrt{-1}u(k), \quad X \mapsto X^{(k)}. \]

where \( X^{(k)} \) is the \((k \times k)\)-principal minor of \( X \).

Let \( \lambda^{(k)}_i : \sqrt{-1}u(k) \rightarrow \mathbb{R}, \quad A \mapsto \) (the \( i \)-th largest eigenvalue of \( A \)).

Theorem (Guillemin-Sternberg)

\[ \Phi_{\lambda} = \{ \Phi^{(k)}_j := \lambda^{(k)}_j \circ \Phi^{(k)} \} \] forms a completely integrable system on \( O_{\lambda} \).

It is called a Gelfand-Cetlin system.
Lemma (The min-max theorem)

Let $A^{(k-1)}$ be the $(k - 1) \times (k - 1)$ principal minor of $A \in \sqrt{-1}U(k)$. Then, $\lambda_i^{(k)} \geq \lambda_i^{(k-1)} \geq \lambda_{i+1}^{(k)}$.

e.g. $\mathcal{O}_\lambda \simeq F(1, 2; 3)$ where $\lambda = \text{diag}(\lambda_1 > \lambda_2 > \lambda_3)$.

\[ \begin{align*}
\lambda_1 \\
|V| \\
\phi_1^{(2)} \geq \lambda_2 \\
|V| \\
|V| \\
\phi_1^{(1)} \geq \phi_2^{(2)} \geq \lambda_3
\end{align*} \]

- $\triangle_\lambda = (\text{The Gelfand-Cetlin polytope})$
  - := (the polytope given by the half planes of the above inequalities)
  - := (the image of $\mathcal{O}_\lambda$ under $\Phi_\lambda$)
To understand the face structure of $\triangle_\lambda$, consider a ladder diagram. eg. $\Gamma_\lambda$ where $\lambda = \text{diag}(3, 1, -1, -3)$ or $\lambda = \text{diag}(2, 2, -2, -2)$.

A ladder diag. consists of containers of non-constant comp. of $\Phi_\lambda$.

The red point is called the origin, a blue point is called a top vertex.

A positive path is a shortest path from the origin to a top vertex in $\Gamma_\lambda$. (It can move $\rightarrow$ or $\uparrow$)
Diagram-Face Correspondence

An admissible diagram is a subgraph of a ladder diagram $\Gamma_\lambda$ if

1. it can be expressed as a union of positive paths.
2. it contains all top vertices

For example, $\mathcal{F}(3) \simeq \mathcal{O}_\lambda$ where $\lambda = \text{diag}(2, 0, -2)$.

\[
\begin{array}{cccc}
2 & 0 & -2 \\
2 & 0 & -2 \\
0 & 0 & -2 \\
2 & 0 & -2 \\
2 & 0 & -2 \\
\end{array}
\]

Theorem (An-Cho-Kim)

For given $\lambda$, there is a one-to-one correspondence

$$\Psi : \{\text{admissible diagrams of } \Gamma_\lambda\} \simeq \{\text{faces of } \triangle_\lambda\}$$

satisfying

- \textbf{(Order-preserving)} $\Gamma_1 \subset \Gamma_2$ if and only if $\Psi(\Gamma_1) \subset \Psi(\Gamma_2)$.
- \textbf{(Dimension)} $\dim \Psi(\Gamma)$ is the first betti number of $\Gamma$. 
e.g. $\mathcal{F}(3) \simeq O\lambda$ where $\lambda = \text{diag}(2, 0, -2)$.

$x\Rightarrow \psi(\Gamma)$ is contained in the planes defined by equating two adjacent variables not divided by paths.

For a given point in $\triangle\lambda$, can tell which face contains it.
• Understand the topology of Gelfand-Cetlin fibers.

• Gelfand-Cetlin system vs Toric moment map

<table>
<thead>
<tr>
<th>Fiber</th>
<th>Gelfand-Cetlin systems</th>
<th>Toric integrable systems</th>
</tr>
</thead>
<tbody>
<tr>
<td>The interior</td>
<td>Lagrangian torus</td>
<td>Lagrangian torus</td>
</tr>
<tr>
<td>Not in the interior</td>
<td>Isotropic smooth mfld</td>
<td>Isotropic smooth mfld</td>
</tr>
<tr>
<td>A $k$-dim face</td>
<td>Not necessarily torus</td>
<td>$k$-dim’l torus</td>
</tr>
<tr>
<td>Not in the interior</td>
<td>Can be Lagrangian</td>
<td>Cannot be Lagrangian</td>
</tr>
</tbody>
</table>

• A submanifold $Y$ of $(X^{2n}, \omega)$ is called *isotropic* if $\omega|_Y = 0$. 
Theorem (Cho-K.-Oh)

Let $\lambda = \text{diag}(\lambda_1, \cdots, \lambda_n)$. Let $\Phi_\lambda$ be a Gelfand-Cetlin system on $O_\lambda$. For any $u \in \triangle_\lambda$, $\Phi^{-1}_\lambda(u)$ is the total space $E_{n-1}$ of an iterated bundle

$$\Phi^{-1}_\lambda(u) \longrightarrow E_{n-1} \longrightarrow E_{n-2} \longrightarrow \cdots \longrightarrow E_1 \longrightarrow \{\text{pt}\}$$

$$F_{n-1} \quad F_{n-2} \quad \cdots \quad F_1$$

where $F_k$ is either a point or a product of odd dimensional spheres.

Moreover, $\Phi^{-1}_\lambda(u)$ is an isotropic submanifold of dimension $\sum_{k=1}^{n-1} \dim F_k$. 
$W$-blocks and $M$-blocks

To construct such a bundle, need the following LEGO® kit.

- **$W$-blocks**
  
  $W_1$, $W_2$, $W_3$, \ldots
  
  The origin
  
  Bottom vertex

- **$M$-blocks**
  
  $M_1$, $M_2$, $M_3$, \ldots
  
  $S^1$, $S^3$, $S^5$, \ldots
Combinatorial Procedure

- Suppose that \( u \) is in the relative interior of a face \( f \).

\( \Gamma_f \): the admissible diagram corresponding to \( f \)

1. Cut \( W_k \) along the positive paths of \( \Gamma_f \).
2. Throw away all cut blocks not containing a bottom vertex of \( W_k \).
3. Read \( F_k \) by counting \( M \)-blocks.

eg. \( \lambda = \text{diag}(2, 0, -2) \), \( O_\lambda \cong F(1, 2, 3) \), \( \Phi^{-1}_\lambda(0, 0, 0) \)? \( f = \{(0, 0, 0)\} \).
\[ W_1 \quad \Gamma_f \quad W_2 \]

\[ F_1 = \text{pt} \]

\[ F_2 = S^3 \]

\[ \Phi^{-1}_\lambda(0,0,0) \rightarrow E_2 \rightarrow E_1 \rightarrow \text{pt} \]

\[ F_2 = S^3 \]

\[ F_1 = \text{pt} \]

\[ \Rightarrow \Phi^{-1}_\lambda(0,0,0) \simeq S^3 \]
eg. Let $\lambda = \text{diag}(3, 3, 0, -3, -3)$. Then, $O_\lambda \simeq F(2, 3; 5)$.

\[ \begin{pmatrix} 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \]

$W_1$ $F_1 = \text{pt}$

$W_2$ $F_2 = S^3$

$W_3$ $F_3 = S^1 \times S^1$

$W_4$ $F_4 = S^3$

$\Phi^{-1}_\lambda(u) = E_4 \rightarrow E_3 \rightarrow E_2 \rightarrow E_1 \rightarrow \text{pt}$

$F_4 = S^3$ $F_3 = S^1 \times S^1$ $F_2 = S^3$ $F_1 = \text{pt}$

$\Rightarrow \Phi^{-1}_\lambda(u) \simeq (S^1)^2 \times (S^3)^2$
For a combinatorial test for detecting Lagrangian fibers, we define
- Boards and Extended boards

![Diagram of boards and extended boards, including...](image)

- Symmetric blocks

![Diagram of symmetric blocks, including...](image)
Theorem (Cho-K.-Oh)

The followings are equivalent.

1. the extended board $\tilde{B}_\lambda$ is cut into symmetric blocks along the positive paths of $\Gamma_f$.
2. the fiber over a point in the relative interior of $f$ is Lagrangian.
3. the fiber over any point in the relative interior of $f$ is Lagrangian.

eg. $\lambda = \text{diag}(5, 5, 2, 0, -3, -3, -6), \mathcal{O}_\lambda \simeq \mathcal{F}(2, 3, 4, 6; 7)$
$S^n$-fibers, $U(n)$-fibers

- $S^n$-fibers

- $U(n)$-fibers, $SU(n)$-fibers
**More $S^n$-fibers**

- Get more variety of fibers if looking at G.-C. fibers of other types.

Let

$$K_2(\lambda_k) := \begin{bmatrix} 0 & -\lambda_k \\ \lambda_k & 0 \end{bmatrix}$$

$$O^B_\lambda = \{ A \cdot \text{diag}(K_2(\lambda_1), \cdots, K_2(\lambda_n), 0) \cdot A^{-1} : A \in SO(2n + 1) \}$$

$$O^D_\lambda = \{ A \cdot \text{diag}(K_2(\lambda_1), \cdots, K_2(\lambda_n)) \cdot A^{-1} : A \in SO(2n) \}$$

- $S^n$-fibers

```
2 0
S^2-fiber

3 0 0
S^3-fiber

4 0 0 0
S^4-fiber

5 0 0 0 0
S^5-fiber

...```

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Consider the fibers over the line segment connecting centers to obtain monotone Lag. tori.

Apply the Darboux-Weinstein Theorem in order to carry Lag. tori to cotangent bundles.
Theorem (Nishinou-Nohara-Ueda)

There is one-to-one correspondence between holomorphic discs of Maslov index two bounded by a G.-C. toric fiber and facets of a G.-C. polytope $\Delta_\lambda$

By the monotonicity Lemma, holomorphic discs of small areas are fully contained in the D.-W. nbd.
\[\begin{aligned}
\pi_2(L) &= 0 \\
\pi_1(L) &= \mathbb{Z}^{df} \\
\pi_2(T^*L, T) &\simeq \mathbb{Z}^{N-d_f} \text{ where } \dim_{\mathbb{R}} \mathcal{O}_\lambda = 2N.
\end{aligned}\]

- Use the Diagram-Face correspondence.
  
  eg. Let \(\lambda = \text{diag}(2, 2, -2, -2)\). Then, \(\mathcal{O}_\lambda \simeq \text{Gr}(2, 4)\).
  
  \(\Rightarrow\) Has Lag. \(U(2)\)-fibers over 1-dimensional edge \(f\).

  \(\Rightarrow\) Four facets containing the edge.

- \(\pi_2(T^*L, T)\) is gen. by the discs corresponding to such facets.
Comparision of monotone Lagrangian tori in $T^*S^3$

- Consider two Lag. fibers in different faces that are diffeomorphic.
- What are relation between monotone Lag. tori?
  
  eg. $\mathcal{O}_A^\lambda (\lambda = \text{diag}(2,0,-2))$ vs $\mathcal{O}_B^{\lambda'} (\lambda' = \text{diag}(K_2(3), K_2(0), 0))$
Comparision of monotone Lagrangian tori in $T^*S^{2n+1}$


Assume that they have the same monotonicity constant.

Theorem (Cho-K.-Oh)

They are not related by any symplectomorphisms.

Why? Let

$$\begin{align*}
\widehat{\pi}_2(T^*L, T) &= \{\text{classes of Maslov index 2 realized by holo. discs.}\} \\
\widehat{\pi}_1(T) &= \langle\{\partial\beta : \beta \in \widehat{\pi}_2(T^*L, T)\}\rangle.
\end{align*}$$

If there were a symplectomorphism $\phi$, $\exists$ one-to-one correspondence

$$\widehat{\pi}_2(T^*L, T^A) \simeq \widehat{\pi}_2(T^*L, T^B), \quad \widehat{\pi}_1(T^A) \simeq \widehat{\pi}_1(T^B)$$

$\widehat{\pi}_1(T^B)$ cannot gen $\pi_1(T^B)$, but $\phi_\ast\widehat{\pi}_1(T^A)$ can gen. as a $\mathbb{Z}$-module.
Comparision of monotone Lagrangian tori

eg.

VS

- The first gives monotone Lag tori in $T^*(S^5 \times S^5 \times S^1 \times S^1)$
- The second gives monotone Lag tori in $T^*S^5 \times T^*S^5 \times T^*S^1 \times T^*S^1$

$\Rightarrow$ These two monotone Lag. tori are not related by symplectomorphisms. We can distinguish them by comparing open Gromov-Witten invariants.

- A monotone Lag. torus from the first bounds 14 holomorphic discs of Maslov index 2.
- A monotone Lag. torus from the second bounds 12 holomorphic discs of Maslov index 2.
Theorem (Cho-K.-Oh)

Let $u$ be in the relative interior of a face $f$ in $\Delta_\lambda$. If the board $\mathcal{B}_\lambda$ is cut into solvable blocks along the positive paths of $\Gamma_f$, then there exists a non-unitary line bundle $\rho$ such that

$$HF(\Phi_{-1}^{-1}(u), \rho) \cong H(T; \mathbb{C}).$$

In particular, $\Phi_{-1}^{-1}(u)$ is non-displaceable.
eg. $\lambda = \text{diag}(5, 5, 2, 0, -3, -3, -6)$. $\mathcal{O}_\lambda \simeq \mathcal{F}(2, 3, 4, 6; 7)$. 

\begin{align*}
\text{board } B_\lambda & \quad \text{diagram } \Gamma_f \\
\text{diagram } \Gamma_g & \quad \text{diagram } \Gamma_g
\end{align*}