Pseudoconcavity of flag domains

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Flag domains

Let $Z = Gr(d, \mathbb{C}^n)$ be the Grassmannian of d-dimensional linear subspaces of $\mathbb{C}^n$.

Then $G = SL(n, \mathbb{C})$ acts on Z transitively.

For $p, q \geq 0$ with $n = p + q$,

consider a Hermitian inner product $\langle \, , \, \rangle$ of signature (p, q)

defined by $\langle u, v \rangle = \sum_{i=1}^{p} u_i \overline{v}_i - \sum_{j=1}^{q} u_{p+j} \overline{v}_{p+j}$.

$G_0 = SU(p, q) = \{g \in SL(n, \mathbb{C}) : g \text{ preserves } \langle \, , \, \rangle\}$.

Then G_0 has only finitely many orbits in Z

Flag domains

$(d \leq p \leq q)$

• Open orbits $D_{d,0}, D_{d-1,1}, \ldots, D_{0,d}$

$D_{a,b} = \{[E] \in Gr(d, \mathbb{C}^n) : \langle\ ,\ \rangle|_E$ has signature $(a, b)\}$, $a + b = d$

$D_{d,0} = \{[E] : \langle\ ,\ \rangle$-positive definite $\}$

$D_{0,d} = \{[E] : \langle\ ,\ \rangle$-negative definite $\}$

• $\exists$ a unique closed orbit

$\Sigma = \{[E] \in Gr(d, \mathbb{C}^n) | E$ is $\langle\ ,\ \rangle$-isotropic $\}$

Flag domains

G, a complex semisimple

$Q \subset G$, a parabolic subgroup

$Z = G/Q$ a flag variety

$G_0 \subset G$, a real form

i.e., the Lie algebra $\mathfrak{g}_0$ of G_0 is defined as the fixed point set of an antilinear involution $\tau : \mathfrak{g} \to \mathfrak{g}$.

Theorem (Wolf)

There exist only finitely many G_0-orbits in $Z = G/Q$ and a unique closed G_0-orbit.

Open G_0-orbits are called flag domains

Flag domains

$K_0 \subset G_0$ a maximal compact subgroup

$K \subset G$ its complexification

Then there is a duality

$$\begin{array}{rcl} \{G_0\text{-orbits in } Z = G/Q\} & \longleftrightarrow & \{K\text{-orbits in } Z = G/Q\} \\ \text{open} & \longleftrightarrow & \text{closed} \\ \text{closed} & \longleftrightarrow & \text{open} \end{array}$$

When $G_0(z)$ is open, we have $K_0(z) = K(z)$ (compact complex manifold). In this case,

$C = K(z)$ is called the base cycle of the flag domain $D = G_0(z)$.

(Matsuki) by combinatorial methods.

(Bremigan-Lorch) K_0-momentum geometry of Z.

$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$, Cartan decomposition,
$\mathfrak{g}_u = \mathfrak{k}_0 + i\mathfrak{s}_0$ compact real form of $\mathfrak{g}$

Let ω be a G_u-invariant Kähler form on Z, and
let $\mu_{K_0} : Z \to \mathfrak{k}_0^*$ be the associated K_0-momentum map.
($d\mu_\xi = \iota_{\widehat{\xi}}\omega$ where $\mu_\xi = \xi \circ \mu_{K_0}$ and $\xi \in \mathfrak{k}_0$)

Define $E := ||\mu_{K_0}||^2$ to be the energy function computed with respect to a K_0-invariant Killing norm, and ∇E its gradient field computed with respect to the associated Kähler metric.

1. The critical set $C_Z(K_0) = \{z \in Z : \nabla E = 0\}$ is a finite union of K_0-orbits κ_0.

2. For a K_0-orbit κ_0 in $C_Z(K_0)$, define $\kappa(\kappa_0) := K_0(z)$ and $\gamma(\kappa_0) := G_0(z)$ for $z \in \kappa_0$. Then $\kappa = \kappa(\kappa_0)$ and $\gamma = \gamma(\kappa_0)$ are dual to each other:

$$T_z\kappa + T_z\gamma = T_zZ \text{ and } T_z\kappa \cap T_z\gamma = T_z\kappa_0.$$

3. The degeneracy of the Hessian of E is $T_z\kappa_0$ and the induced form on $T_z\kappa/T_z\kappa_0$ (respectively, $T_z\gamma/T_z\kappa_0$) is positive (respectively, negative) definite.

The flow of ∇E realizes κ_0 as a strong deformation retract to γ and the flow of $-\nabla E$ realizes κ_0 as a strong deformation retract of κ.

Flag domains

Example

$G = SL(n, \mathbb{C})$, $G_0 = SU(p, q)$

$\langle v, w \rangle = \sum_{i=1}^{p} v_i \overline{w}_i - \sum_{j=1}^{q} v_{p+j} \overline{w}_{p+j}$

$\mathbb{C}^n = \mathbb{C}^p_+ \oplus \mathbb{C}^q_-$

$Z = Gr(d, \mathbb{C}^n)$

$D = \{[E] \in Gr(d, \mathbb{C}^n) : \langle \, , \rangle|_E$ has signature $(a, b)\}$, $a + b = d$

$C = \{[E] \in Z : \dim E \cap \mathbb{C}^p_+ = a, \ \dim E \cap \mathbb{C}^q_- = b\}$

Flag domains

Example

$G = Sp(n, \mathbb{C})$, $G_0 = Sp(n, \mathbb{R})$

$\omega(v, w) = \sum_{j=1}^{n} (v_j w_{n+j} - v_{n+j} w_j)$

$\langle v, w \rangle = \sum_{j=1}^{n} v_j \overline{w}_j - \sum_{j=1}^{n} v_{n+j} \overline{w}_{n+j}$

$\rightsquigarrow G_0 = G \cap U(n, n)$

$Z = Gr_\omega(d, \mathbb{C}^{2n}) = \{[E] \in Gr_\omega(d, \mathbb{C}^{2n}) : E, \omega\text{-isotropic}\}$

$D = \{[E] \in Gr_\omega(d, \mathbb{C}^{2n}) : \langle\, , \rangle|_E \text{ has signature} (a, b)\}$, $a + b = d$

$C = \{[E] \in Z : \dim E \cap \mathbb{C}^n_+ = a,\ \dim E \cap \mathbb{C}^n_- = b\}$

Flag domains

Example

$Z = \mathbb{C}P^n \curvearrowleft G = SL(n+1, \mathbb{C})$

$G_0 = SL(n+1, \mathbb{R})$, $K = SO(n+1, \mathbb{C})$

$D = \mathbb{C}P^n \backslash \mathbb{R}P^n$

$C = \mathbb{Q}^{n-1}$

Example

$Z = \mathbb{Q}^{20} \curvearrowleft G = SO(n, \mathbb{C})$

$G_0 = SO(p, q)$, $b(z, z) = \sum_{i=1}^{p} z_j^2 - \sum_{j=1}^{q} z_{p+j}^2$

$D = \{z : b(z, z) = 0, b(z, \overline{z}) > 0\}$

$C = \mathbb{C}P^{p-1} \cap Z$

$D = G_0/H_0$, a flag domain in $Z = G/Q$

$C \subset D$ the base cycle

We say that D is cycle connected if for any two points $x, y \in D$, there exists a chain $C_1 \cup \cdots \cup C_m$ (connected), $C_i = g_i C$, $g_i \in G_0$, such that $x \in C_1$ and $y \in C_m$

Theorem (Huckleberry)

For a flag domain $D = G_0/H_0$ in $Z = G/Q$, the followings are equivalent.

(1) D is pseudoconvex, i.e., there exists an exhaustion $\rho : D \to \mathbb{R}^{\geq 0}$ which is plurisubharmonic outside of a compact set.

(2) D is not cycle connected.

(3) there is a nontrivial G_0-equivariant holomorphic or antiholomorphic map of $D \subset Z$ to a hermitian symmetric space of noncompact type $\widehat{D} = G_0/K_0 \subset \widehat{Z}$ and the neutral fiber K_0/H_0 is the base cycle C itself.

(Kollar) (Griffiths-Robles-Toledo)

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$D = G_0/H_0$, not cycle connected $\Longrightarrow$ pseudoconvex

$D = G_0/H_0$, cycle connected $\Longrightarrow$ not pseudoconvex $\Longrightarrow$?

Levi geometry

A connected complex manifold D is said to be pseudoconcave

if $\exists$ nonempty open $Y \subset D$ such that

(1) Y is relatively compact in D

(2) $\partial Y = \overline{Y} \backslash Y$ is smooth and the Levi form of ∂Y has at least one negative eigenvalue at each point of ∂Y,

or, equivalently,

(2)' for every $p \in \mathrm{cl}(Y)$, there exists a holomorphic map $\phi : \Delta \to \mathrm{cl}(Y)$ with $\phi(0) = p$ and $\phi(\mathrm{bd}(\Delta)) \subset Y$, where Δ is the unit disc in $\mathbb{C}$

$Y \cap U = \{x \in U : \phi(x) < 0\}$ for some $\phi : U \to \mathbb{R}$.

Assume that $\phi(z_0) = 0$ and $d\phi(z_0) \neq 0$.
The real tangent plane to Y at the origin is given by

$$\sum \frac{\partial \phi}{\partial x_\alpha}(0) x_\alpha + \sum \frac{\partial \phi}{\partial y_\alpha}(0) y_\alpha = 0.$$

This plane contains the $(n-1)$-dimensional complex plane with the equation

$$\sum \frac{\partial \phi}{\partial z_\alpha} \phi(0) z_\alpha = 0.$$

This is called the analytic tangent plane to ∂Y at z_0.

Then at any $z_0 \in \partial Y \cap U$ the Levi-form $\mathcal{L}\phi$ of ϕ restricted to the analytic tangent space to ∂Y at z_0 has a signature which is independent of local holomorphic coordinates and of the choice of the defining function ϕ.

Main Theorem

Question. $D = G_0/H_0$, a flag domain

D, not pseudoconvex (or equivalently, cycle connected)

$\Longrightarrow D$, pseudoconcave?

Theorem (Huckleberry)

If D is 1-connected, then D is pseudoconcave.

Theorem (H-Huckleberry-Seo)

G, simple Lie group
$G_0 \subset G$ a real form
$D = G_0/H_0$, a flag domain

Then D is either pseudoconvex or pseudoconcave.

q-convexity and q-concavity

Let U be an open subset of $\mathbb{C}^n$
$\phi : U \to \mathbb{R}$, a C^∞ function
We say that ϕ is strongly q-pseudoconvex at $z_0 \in U$ if the Levi-form

$$\mathcal{L}\phi_{z_0}(u) = \sum \frac{\partial^2 \phi}{\partial z_\alpha \partial \overline{z}_\beta}|_{z_0} u_\alpha \overline{u}_\beta,$$

where $z_1, \ldots, z_n$ are local coordinates at z_0,
has at least $n - q$ positive eigenvalues.

• A strongly 0-pseudoconvex function is strongly plurisubharmonic function.

• independent of the choice of local coordinates, and could be formulated as follows:

there exists a $(n - q)$-complex dimensional plane E through z_0 such that $\phi|_E$ is strongly plurisubharmonic.

For practical reasons it is sometime convenient to release the assumption that ϕ is C^∞ and require only that

(1) in a neighborhood od z_0 there is a finite number of C^∞ functions $\phi_1, \ldots, \phi_r$ such that $\phi = \sup(\phi_1, \ldots, \phi_r)$

(2) there exists an $(n-q)$-dimensional plane E through z_0 such that $\phi_i|_E$, $i \leq i \leq r$, is strongly plurisubharmonic in a neighborhood of z_0 in E

q-convexity and q-concavity

X, a connected complex manifold of complex dimension n
We say that X is q-pseudoconcave if a C^∞ function $\phi : X \to \mathbb{R}$ is given such that

(1) for every $c > \inf \phi$, the sets

$$B_c = \{x \in X : \phi(x) > c\}$$

are relatively compact in X
(2) there exists a compact subset K of X such that on $X \backslash K$ ϕ is strongly q-pseudoconvex, i.e., the Levi form $\mathcal{L}\phi$ has at least $n - q$ positive eigenvalues.

q-convexity and q-concavity

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Remark. For $d < \sup \psi$, $\{x \in X : \psi(x) < d\}$ is relatively compact and on $X \backslash K$ $\mathcal{L}\psi$ has at least $n - q$ negative eigenvalues.

q-convexity and q-concavity

Example

1. Every compact connected manifold X is q-pseudoconcave for any q. Take $K = X$, $\phi = 1$.

2. Let Z be a connected compact complex manifold of complex dimension $n \geq 2$. Let $\{x_1, \ldots, x_k\}$ be a finite subset of Z. Then $X = Z \backslash \{x_1, \ldots, x_k\}$ is 0-pseudoconcave.

3. Let Z be a compact connected manifold of complex dimension $n \geq 2$. Let Y be a compact complex submanifold of Z with $\dim_{\mathbb{C}} Y = q \leq n - 2$.
Then $X = Z \backslash Y$ is q-pseudoconcave.

q-convexity and q-concavity

2. Indeed, if $z_j^{(i)}, 1 \le j \le n$, are local coordinates at x_i, vanishing at x_i, we can select $\varepsilon > 0$ such that the coordinate balls

$$B_i = \left\{ \sum_{j=1}^{n} |z_j^{(i)}|^2 < \varepsilon \right\}, 1 \le i \le k$$

are relatively compact in their coordinate patch and are disjoint.

Take $K = Z \setminus \cup_{i=1}^{k} B_i$ and for ϕ a C^∞ function such that $\phi \ge \varepsilon$ on K,

$$\phi|_{B_i} = \sum_{j=1}^{n} |z_j^{(i)}|^2, 1 \le i \le k.$$

q-convexity and q-concavity

Remark.

For all $c > \inf \phi$, except for a set of measure zero, the boundary $\partial B_c = \overline{B}_c \backslash B_c$ is smooth (By Sard's Theorem).

The Levi form restricted to the analytic tangent plane at $z_0 \in \partial B_c$ to ∂B_c, $c < \inf_K \phi$, has at least $n - q - 1$ positive eigenvalues. Therefore, if $n - q - 1 \geq 1$, these manifolds are a special case of the pseudoconcave manifolds.

we will assume that $0 \leq q \leq n - 2$ and $n \geq 2$, if X is not compact

q-convexity and q-concavity

A complex manifold X is called q-pseudoconvex if there exists on X a C^∞ function $\phi : X \to \mathbb{R}$ and a compact set K such that

(1) the sets $B_c = \{x \in X : \phi(x) < c\}$
are relatively compact in X for every $c \in \mathbb{R}$
(2) on $X \backslash K$ the Levi form $\mathcal{L}\phi$ has at least $n - q$ positive eigenvalues.

If we can choose $K = \emptyset$, the manifold X is called q-complete.

q-convexity and q-concavity

Theorem (Andreotti-Grauert)

If X is q-pseudoconvex, then for any coherent shesf $\mathcal{F}$ on X we have

$$\dim_{\mathbb{C}} H^r(X, \mathcal{F}) < \infty \quad \textit{if } r > q.$$

Moreover, if X is q-complete, then for $r > q$ $H^r(X, \mathcal{F}) = 0$.

Theorem (Andreotti-Grauert)

If X is q-pseudoconcave, for any locally free sheaf $\mathcal{F}$ on X, we have

$$\dim_{\mathbb{C}} H^r(X, \mathcal{F}) < \infty \quad \textit{if } r < n - q - 1.$$

Measurable open orbits

A flag domain is called measurable if it carries a G_0-invariant volume element.

Theorem (Wolf)

Let $G_0(z)$ be an open orbit in the complex flag manifold $Z = G/Q$. Then the following conditions are equivalent:

(1) The orbit $G_0(z)$ is measurable
(2) $G_0 \cap Q_z$ is the G_0-centralizer of a (compact) torus subgroup of G_0
(3) D has a G_0-invariant, possibly indefinite, Kähler metric, thus a G_0-invariant measure obtained from the volume form of the metric
(4) $\tau\Phi^r = \Phi^r$ and $\tau\Phi^n = -\Phi^n$, where $\mathfrak{q}_z = \mathfrak{q}_\Phi$
(5) $\mathfrak{q}_z \cap \tau\mathfrak{q}_z$ is reductive, i.e., $\mathfrak{q}_z \cap \tau\mathfrak{q}_z = \mathfrak{q}_z^r \cap \tau\mathfrak{q}_z^r$
(6) $\mathfrak{q}_z \cap \tau\mathfrak{q}_z = \mathfrak{q}_z^r$
(7) $\tau\mathfrak{q}$ is G-conjugate to the parabolic subalgebra $\mathfrak{q}^r + \mathfrak{q}^n$ opposite to $\mathfrak{q}$

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(7) $\tau\mathfrak{q}$ is G-conjugate to the parabolic subalgebra $\mathfrak{q}^r + \mathfrak{q}^n$ opposite to $\mathfrak{q}$

In particular, (7) is independent of the choice of z. It follows that if one open G_0-orbit on$Z$ is measurable, then all open G_0-orbits

Measurable open orbits

Example

1. If G_0 has a compact Cartan subgroup, for example, if G_0 is hermitian, then every open G_0-orbit on Z is measurable.

2. $Z = \mathbb{C}P^n$ and $G_0 = Sl(n+1, \mathbb{R})$. Then there are two G_0-orbits, the closed orbit $G_0([e_{n+1}])$ and the open orbit $G_0([e_1 + \sqrt{-1}e_{n+1}])$, where $\{e_1, \ldots, e_{n+1}\}$ is a basis in which G_0 consists of the real matrices of determinant 1.
The open orbit is not measurable.

Measurable open orbits

Theorem (Schmid-Wolf)

Let $Z = G/Q$ be a complex flag manifold, G semisimple and simply connected, and let G_0 be a real form of G.
Let $D = G_0(z) \subset Z$ be a measurable open orbit.
Let $C = K_0(z)$, maximally compact subvariety of D.

Then D is q-complete, where $q = \dim_{\mathbb{C}} C$.

In particular, for any coherent analytic sheaf $\mathcal{F}$ on D, we have $H^r(D, \mathcal{F}) = 0$ for $r > q$.

Proof. (Schmid-Wolf)

Assume that $D = G_0(z) \subset Z = G/Q$ is measurable
Then $D = G_0/V_0$, where V_0 is the real form $G_0 \cap Q_z$ of Q_z^r
$V := Q_z^r$, the complexification of V_0
Then $Z = G_U/V_u$, where G_u is the compact real form of G and V_u is the compact real form $G_u \cap Q_z$ of $Q_z^r = V$.

Let L_Z be the dual of the canonical line bundle on Z.
Let L_D be the dual of the canonical line bundle on D.
The Killing form and complex structure of $\mathfrak{g}$ define indefinite-hermitian metrics on the holomorphic tangent spaces.
Thus $L_Z \to Z = G_u/V_u$ has a G_u-invariant hermitian metric h_u and $L_D \to D = G_0/V_0$ has a G_0-invariant hermitian metric h_0.

Define $\phi : D \to \mathbb{R}$ by $\phi = \log(h_0/h_u)$.
Then the sets $B_c = \{x \in D : \phi(x) < c\}$ are relatively compact and the Levi form $\mathcal{L}\phi$ has at least $n - q$ positive eigenvalues at every point of D. □

Recall that the curvature form of the Chern connection of a holomorphic line bundle L with a hermitian metric h is $-\partial\overline{\partial}\log h$.

The Levi form $\mathcal{L}\phi = \sqrt{-1}\partial\overline{\partial}\phi = \sqrt{-1}\partial\overline{\partial}\log h_0 - \sqrt{-1}\partial\overline{\partial}\log h_u$.

Compute:

$\sqrt{-1}\partial\overline{\partial}\log h_0$ has signature $(n-q, q)$

$\sqrt{-1}\partial\overline{\partial}\log h_u$ has signature $(0, n)$.

Therefore, the Levi form $\mathcal{L}\phi$ has at least $n-q$ positive eigenvalues.

Measurable open orbits

(Wolf-Schmid)
A measurable open obit D is q-complete, where $q = \dim_{\mathbb{C}} C$.

Example

$Z = \mathbb{C}P^n$ and $G_0 = SU(p, q+1)$, where $n + 1 = p + q + 1$.
$D = \{E \subset \mathbb{C}^{n+1} : \dim_{\mathbb{C}} E = 1, \text{ negative definite } \}$
$C = \mathbb{P}(\mathbb{C}^{q+1}_{-}) \simeq \mathbb{C}P^q$
$D = \{[Z] \in \mathbb{C}P^n : |Z_1|^2 + \cdots + |Z_p|^2 - |Z_{p+1}|^2 - \cdots - |Z_{p+q+1}|^2 < 0\}$
$D \cap U_{p+q+1} = \{z : |z_1|^2 + \cdots + |z_p|^2 - |z_{p+1}|^2 - \cdots - |z_{p+q}|^2 < 1\}$
where $z_i = Z_i/Z_{p+q+1}$, $i = 1, \ldots, p+q$, coordinates on $U_{p+q+1} = \{Z_{p+q+1} \neq 0\}$.
$\phi(Z) = |Z_1|^2 + \cdots + |Z_p|^2 - |Z_{p+1}|^2 - \cdots - |Z_{p+q+1}|^2$ has $n - q$-positive eigenvalues and q-negative eigenvalues.
Therefore D is q-complete and $(n-q)$-pseudoconcave,
$H^r(D, \mathcal{F}) = 0$ for $r > q$ and $\dim H^s(D, \mathcal{F}) < \infty$ for $s < q - 1$.

Is this true in general?

Question. Assume that D is not pseudoconvex. Is D $n-q$ pseudoconcave? Here, q is the dimension of the base cycle C of D.

By Theorem (H-Huckleberry-Seo), if D is not pseudoconvex, then D is $(n-2)$-pseudoconcave and thus $\dim_{\mathbb{C}} H^0(D, \mathcal{F}) < \infty$ for any locally free sheaf $\mathcal{F}$ on D.

Theorem (H-Huckleberry-Seo)

Let G be a simple Lie group and $G_0 \subset G$ a real form

$D = G_0/H_0$, a flag domain

Then D is either pseudoconvex or pseudoconcave.

Theorem (Kollar)

Measure of pseudoconcavity

- Compute the number of negative eigenvalues of the Wolf-Schmid exhaustion ϕ defined on measurable open orbit D

- Compute the number of positive eigenvalues of the curvature of the Chern connection on the normal bundle of C in D

Tubular neighborhoods

Definition and Notation

E, a holomorphic vector bundle over a complex manifold X

$\langle\, , \rangle$, a hermitian metric on E

$\rightsquigarrow$ $\exists$ a unique connection $D : A^0(E) \to A^1(E)$

compactible with $\langle\, , \rangle$ and the holomorphic structure of E

(called the Chern connection)

$\Theta \in A^{1,1}(End(E))$, the curvature of the Chern connection D
(real (1,1)-form on X)

Tubular neighborhoods

(e_α), local unitary frame for $E|_U$
(z^i), local coordinate on $U \subset X$

For $\xi = \sum \xi_\alpha e_\alpha \in E$

$$\begin{aligned} \Theta(\xi) &:= \sqrt{-1} \sum \Theta_{\alpha,\beta} \xi_\alpha \bar{\xi}_\beta \\ &= \sum \lambda_j \omega^j \wedge \overline{\omega}^j \end{aligned}$$

for some basis $\{\omega^j\}$ of (1,0)-form on X

We sat that $\Theta(\xi)$ has signature (s,t), $s + t \leq \dim X$,
if s of the λ_j are positive and t of the λ_j are negative.

Tubular neighborhoods

We say that E is at least s positive (respectively, t negative) if E has a Hermitian metric such that for any $\xi \neq 0 \in E$, $\Theta(\xi)$ has at least s positive (respectively, t negative) eigenvalues.

Tubular neighborhoods

$\langle\, , \,\rangle$ a hermitian metric on a holomorphic vector bundle E on X,

Θ, the curvature of the Chern connection of $(E, \langle\, , \,\rangle)$

Define $\phi : E \to \mathbb{R}^{\geq 0}$ by $\phi(x, \xi) = \langle \xi, \xi \rangle$ for $\xi \in E_x$.

$W = \{(x, \xi) \in E : \phi(x, \xi) < 1\}$, a tubular neighborhood of the zero section of E in E

Lemma

(1) If $\xi_0 \in E_{x_0}$, $\Theta(\xi_0)$ has signature (s, t)

then $\mathcal{L}\phi(\xi_0)$ has signature $(t + r, s)$.

(2) If $\xi_0 \in E_{x_0} \cap \partial W$, $\Theta(\xi_0)$ has signature (s, t)

then $\mathcal{L}\phi(\xi_0)$ has signature $(t + r - 1, s)$.

Tubular neighborhoods

Proposition 1. (Griffiths)

D, a complex manifold
$X \subset D$, a compact complex submanifold
$\dim X = n$, $\dim D = n + 1$
If the normal bundle $N_{X|D}$ is at least s positive and t negative

then $\exists$ a (relatively compact) neighborhood Y of X in D such that the Levi form of ∂Y has at least t positive eigenvalues and s negative eigenvalues at each point of ∂Y.

Tubular neighborhoods

Proposition 2.

D, a complex manifold
$X \subset D$, a compact complex submanifold
$\dim X = n$, $\dim D = n + r$
the normal bundle $N_{X|D}$ is at least s positive and t negative

$\Longrightarrow \exists$ a (relatively compact) neighborhood Y of X in D such that the Levi form of ∂Y has at least $t + r - 1$ positive eigenvalues and s negative eigenvalues at each point of ∂Y.

Lemma

If $E \to X$ is s positive and t negative, then the tautological line bundle on $\mathbb{P}(E)$ is s positive and $t + r - 1$ negative.

Tubular neighborhoods

Let $X \subset D$, a compct complex submanifold of codimension r
$E = \pi^{-1}(X) \subset \widetilde{D} = Bl_X(D)$
Then $E = \mathbb{P}(N_{X|D})$ and
$N_{E|\widetilde{D}} =$ the tautological line bundle on $\mathbb{P}(N_{X|D})$.

Thus $N_{E|\widetilde{D}}$ is at least s positive and $t + r - 1$ negative because $N_{X|D}$ is s positive and t negative.

Apply Proposition 1(the case when $r = 1$) to get the desired result. □

Flatness and ampleness

$X = G/P$ a flag manifold (base cycle C)
E_0, a P-module
$E = G \times_P E_0$, a homogeneous vector bundle (normal bundle $N_{C|D}$)

Assume that E is generated by global sections
i.e. $H^0(X, E) \to E_x$ is surjective for all $x \in X$.

Then
$H^0(X, E) \twoheadrightarrow E_0$, where $0 \in X$ the base point
G-module $\quad$ P-module

h_E, invariant Hermitian metric on E
Θ_E the curvature of the Chern connection of (E, h_E)

Since E is globally generated, Θ_E is semipositive.

(We will show that for any $\xi \neq 0 \in N_{C|D}$, there exists at least one positive eigenvalue of $\Theta(\xi)$.)

Flatness and ampleness

For $\xi \in E$, $\operatorname{Ker}\Theta_E(\xi) := \{\eta \in T_X : \Theta_E(\xi)(\eta, \overline{\eta}) = 0\}$

$$fl\Theta_E := \max_{\xi \in E_0 \setminus 0} \dim \operatorname{Ker}\Theta_E(\xi)$$

called the flatness of Θ_E (E, h_E)

Flatness and ampleness

• L, a line bundle on X

L is k-ample if

L^m is globally generated for some m and
the fiber of $X \to \mathbb{P}(H^0(X, L^m)^*)$ has dimension at most k.

• E, a vector bundle on X

E is k-ample if $\xi_E = \mathcal{O}(1)_{\mathbb{P}(E^*)} \to \mathbb{P}(E^*)$ is k-ample

• the ampleness of E, $a(E)$, is defined to be the minimum k such that E is k-ample.

Flatness and ampleness

Proposition 3. (Snow)

$E = G \times_P E_0 \to X = G/P$, a homogeneous vector bundle

Assume that E is generated by global sections.

Then

$$fl(\Theta_E) \leq a(E) = \dim X - \operatorname{ind}(-\Lambda_{ext}(E_0)) \leq \dim X.$$

(representation theoretic)

For a weight λ, the index $\operatorname{ind}(\lambda)$ of λ is the number of positive roots α such that $(\lambda, \alpha) < 0$.

For a set Λ of weight, $\operatorname{ind}(\Lambda)$ is the minimum of the indexes of weights in Λ.

$$\Lambda_{ext}(E_0) := W(\Lambda_{max}(E_0)) \cap \Lambda(E_0)$$

Flatness and ampleness

Proposition 4

$E = G \times_P E_0 \to X = G/P$, a homogeneous vector bundle

Assume that E is generated by global sections.

Then
E contains a nonzero trivial G-homogeneous subbundle

$\iff a(E) = \dim X$

Corollary

$fl(\Theta_E) = \dim X$

$\implies E$ contains a nonzero trivial G-homogeneous subbundle.

Proof of Main Theorem.

Let $D = G_0/H_0$ be a flag domain in $Z = G/Q$.
$C \subset D$, the base cycle
N, the normal bundle of C in D
Then N is globally generated because TZ is globally generated.

Case 1. $\Theta_N(\xi) \neq 0$ for any $\xi \neq 0 \in N$
$\Longrightarrow \Theta(\xi)$ has at least one positive eigenvalue at each $\xi \neq 0 \in E$
$\Longrightarrow D$ is pseudoconcave (Proposition 2)

Proof of Main Theorem.

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$\Longrightarrow D$ is pseudoconcave (Proposition 2)

Case 2. $\Theta_N(\xi) = 0$ for some $\xi \neq 0 \in N$
$\Longrightarrow \exists$ a trivial K-subbundle $N' \subset N$ (Proposition 4)
$\overset{\star}{\Longrightarrow} G_0$ is of Hermitian type and there is a nontrivial G_0-equivariant holomorphic or antiholomorphic map from D to a Hermitian symmetric space of noncompact type $\widehat{D} = G_0/K_0 \subset \widehat{Z}$ and the neutral fiber K_0/H_0 is the base cycle itself.
$\Longrightarrow D$ is not cycle connected
$\Longrightarrow D$ is pseudoconvex $\square$

$\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$, Cartan decomposition, and $\theta : \mathfrak{g} \to \mathfrak{g}$, Cartan involution.

$$N = K \times_{Q \cap K} (\mathfrak{s}/((\mathfrak{q} + \theta\mathfrak{q}) \cap \mathfrak{s}))$$

If G_0 is not of Hermitian type, then the K-representation on $\mathfrak{s}$ is irreducible, its nontrivial quotient $\mathfrak{s}/((\mathfrak{q} + \theta\mathfrak{q}) \cap \mathfrak{s})$ cannot have an irreducible K-module, a contradiction. Thus G_0 is of Hermitian type.

Since G_0 is of Hermitian type, D is measurable so that

$$N = K \times_{Q \cap K} (\mathfrak{s}/\mathfrak{q} \cap \mathfrak{s}).$$

Thus N' is either $K \times_{Q \cap K} \mathfrak{s}^+$ or $K \times_{Q \cap K} \mathfrak{s}^-$.

$\mathfrak{g} = \mathfrak{k} + \mathfrak{s}^+ + \mathfrak{s}^-$
$\mathfrak{g} = \mathfrak{q} + \mathfrak{q}^{-u} = (\mathfrak{q} \cap \mathfrak{k}) + (\mathfrak{q} \cap \mathfrak{s}) + (\mathfrak{q}^{-u} \cap \mathfrak{k}) + (\mathfrak{q}^{-u} \cap \mathfrak{s})$
so that $\mathfrak{s}/\mathfrak{q} \cap \mathfrak{s} = \mathfrak{q}^{-u} \cap \mathfrak{s}$.

If $N' = K \times_{Q \cap K} \mathfrak{s}^-$, then $\mathfrak{s}^- \subset \mathfrak{q}^{-u} \cap \mathfrak{s}$ and thus $\mathfrak{k} + \mathfrak{s}^+ \supset \mathfrak{q}$. Therefore, there is a nontrivial G_0-equivariant holomorphic or antiholomorphic map from D to a Hermitian symmetric space of noncompact type $\widehat{D} = G_0/K_0 \subset \widehat{Z}$ and the neutral fiber K_0/H_0 is the base cycle itself. ★

Measure of pseudoconcavity (open)

- Compute the number of negative eigenvalues of the Levi form $\mathcal{L}\phi$ of the Wolf-Schmid exhaustion $\phi = \log(h_0/h_u)$ defined on measurable open orbit D

($\mathcal{L}\phi$ has at least $n - q$ positive eigenvalue)

- Compute the number of positive eigenvalues of the curvature Θ_N of the Chern connection on the normal bundle of C in D

(Θ_N has at least 1 positive eigenvalue)

$\rightsquigarrow$ get the finiteness of higher cohomology space $H^r(D, \mathcal{F})$