The fifth Issue 2024

IBS Center for Geometry and Physics

CGP Walk

Beyond the horizon

ibs 기초과학연구원
Institute for Basic Science

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CGP Walk

-Beyond the horizon-

The fifth Issue
2024

기초과학연구원
Institute for Basic Science

Contents

Director’s Note

The year 2024 has been the busiest year since the founding of CGP. After the pandemic, it happened that all the joint conferences and workshops with CGP's MOU institutions have been put on the shoulder of CGP. Meeting our old alumni, who arrived from all of the places, in the CGP Alumni Reunion workshop is particularly memorable. All events were big success. Of course, this kind of success does not come easy. It would not have been possible without our capable administrative staff's devotion and hard works. As the Director, I take this chance to express my deep gratitude to all the CGP members, and wish for Happy New Year.

I have had made several attempts (I am still trying!) to recruit the second Associate Director of CGP, who could help me to mentor our youngsters as well as ease my administrative duty. All failed one way or the other. It disappointed me, but it forced me to change my priority to focusing more on the recruitment of quality post-doctoral members into CGP and making CGP a place that foster innovative and adventurous pursuit by ambitious young researchers. As the Director, I will keep asking ‘Is the current CGP accomplishing this mission well?’ and make sure that the answer is affirmative.

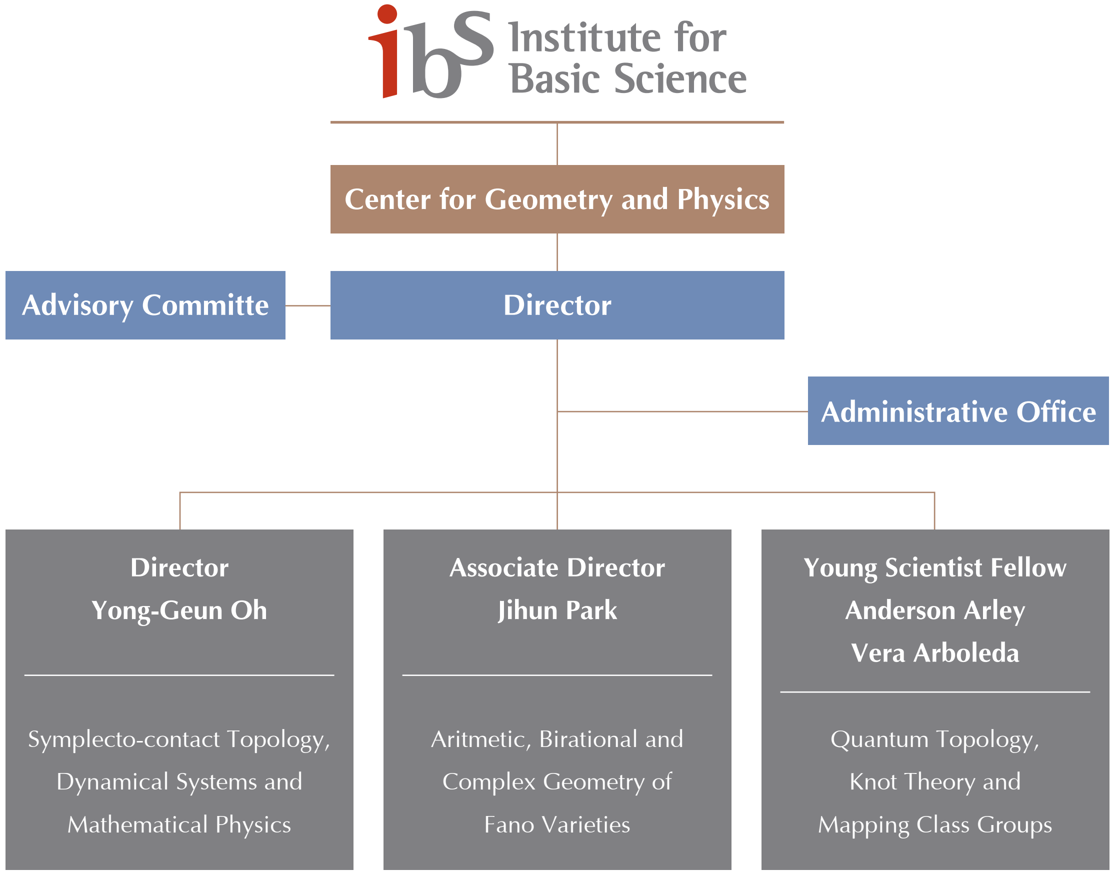
Personally, talking with young mathematicians invigorates my brain activity. It also reminds me of my post-doc years with full of intellectual curiosity and passion. It makes my mental age younger. Come to think of it, all of my own main mathematical works are initiated by talking with my graduate students and other young people. Talking with them ignites my own urge to find out what would happen to the question I propose to them. I feel grateful to IBS and our young dreamers for me to enable to enjoy this luxury of maintaining the pursuit of the journey beyond the horizon.

The Year 2024 is ending. Living in Korea is like living inside one of the K-dramas. I am anxious to know what the final picture of this volatile status of Korea's future would be.

Organization

One arching research theme of the CGP is to promote interaction between symplectic geometry, algebraic geometry and mathematical physics in the study of symplectic topology and homological mirror symmetry and their applications to theoretical physics.

The CGP is organized into multi research groups, each of which comprises a senior scholar and several researchers whose areas of expertise and interest overlap synergistically.



Research Groups

Symplecto-contact Topology, Dynamical Systems and Mathematical Physics

PI: Yong-Geun Oh

[Symplectic topology and mirror symmetry]

- **Dongwook Choa** (Lagrangian Floer theory, Matrix factorization)
- **Norton Lee** (Supersymmetry, Integrable Systems, Quantum Field Theories, Mathematical Physics)
- **Sukjoo Lee** (Mirror symmetry and Hodge theory)
- **Yan-Lung Li** (Symplectic geometry, Lagrangian Floer Theory, 2d and 3d mirror symmetry)
- **Yong-Geun Oh** (Symplectic topology, Hamiltonian dynamics and mirror symmetry)
- **Sam Bardwell-Evans** (Symplectic geometry, Moduli spaces, Pseudoholomorphic curves and discs, Mirror symmetry)
- **Jaekwan Jeon** (Deformations of rational surface singularities and related topics in symplectic geometry)

[Mathematical physics]

- **Alexander Alexandrov** (Mathematical physics, random matrix models, integrable systems, enumera tive geometry)
- **Jorge Valcarcel** (Metric-affine geometry, gauge theories of gravity, cosmology and black hole physics)
- **Dmytro Voloshyn** (Mathematical physics, cluster algebras, Poisson geometry, quantum groups, inte grable systems)

Arithmetic, Birational and Complex Geometry of Fano Varieties

PI: Jihun Park

[Symplectic topology and mirror symmetry]

- **Igor Krylov** (Birational Geometry)
- **Luca Rizzi** (Complex Algebraic Geometry, families of varieties, deformations and Torelli-type prob lems, Hermitian metrics)
- **Jihun Park** (Birational and complex geometry of Fano varieties)
- **Shizhuo Zhang** (Derived categories of coherent sheaves, Fano varieties, stability conditions, moduli spaces and wall-crossing, Kuznetsov components and categorical Torelli problems)

Quantum Topology, Knot Theory and Mapping Class Groups

PI: Anderson Arley Vera Arboleda

[Symplectic topology and mirror symmetry]

- **Anderson Arley Vera Arboleda** (Quantum Topology, Knot Theory and Mapping Class Groups)

CGP Advisory Committee

The CGP Advisory Committee consists of eight distinguished scholars from Korea and abroad. The committee meets once a year and provides advice and input on the operations of the Center.

The current members of the Advisory Committee are (as of December 2024):

Alexander Givental
Professor at University of California, Berkeley

Sergei Gukov
Professor at California Institute of Technology

Bo-Hae Im
Professor at Korea Advanced Institute of Science and Technology (KAIST)

Jae-Hun Jung
Professor at Pohang University of Science and Technology (POSTECH)

Ludmil Katzarkov
Professor at University of Miami & Institute of Mathematics and Informatics (Bulgarian Academy of Sciences)

Young-Hoon Kiem
Professor at Korea Institute for Advanced Study (KIAS) Director of June E Huh Center for Mathematical Challenges

Jongil Park
President at Korean Mathematical Society (KMS)
Professor at Seoul National University

Herman Verlinde
Professor at Princeton University

Scientific Activities

Statistics

- **7** conferences
- **7** colloquium talks and **54** seminar talks
- **3** lecture series

CGP at a Glance 2024

FEB
New Research Member **Luca Rizzi**

MAR
New Research Member **Jorge Valcarcel**

JUN
Event **MATRIX-IBSCGP Workshop on Symplectic and Low-dimensional Topology** (June 3–14)

JUL
Event **Pacific Rim Complex and Symplectic Geometry Conference** (July 29 – August 2)

AUG
Event **Summer Mini-school on Algebraic Geometry** (August 20–23)
BICMR-IBSCGP Conference on Gromov-Witten Theory and Related Topics (August 26–30)

SEP
Event **Autumn School on Low-dimensional Topology and Related Topics** (September 30 – October 4)
New Research Member **Yan-Lung Li, Dongwook Choa, Sukjoo Lee**

OCT
Event **IBS-CGP Reunion Workshop on Geometry and Physics** (October 21–23)
New Research Member **Shizhuo Zhang**

NOV
Event **RIMS-IBSCGP Conference on Recent Developments in Symplectic Topology** (November 4–8)

Conferences

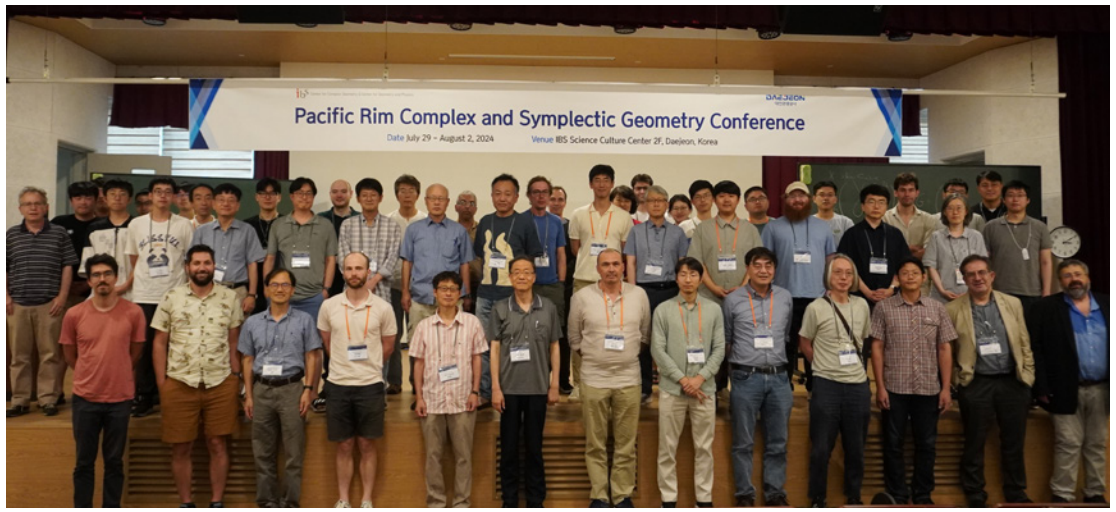
- **MATRIX-IBSCGP Workshop on Symplectic and Low-dimensional Topology;** June 3–14, 2024

This workshop aims to explore the rich interface between symplectic/contact geometry and low dimensional topology. The program is tailored for both young researchers and established mathematicians, providing a platform for collaboration, exchange of ideas, and the advancement of knowledge. By bringing together experts in these areas, this workshop seeks to highlight the profound connections between these areas and inspire further breakthroughs at their intersection.

- Organizers: Yong-Geun Oh (IBS-CGP & POSTECH)
  Brett Parker (Australian National University)
  Anderson Vera (IBS-CGP)
- Invited Lecturers: Jae Choon Cha (POSTECH)
  Tobias Ekholm (Uppsala University)
  Ian Zemke (Princeton University)
- Invited Speakers: Hanwool Bae (Seoul National University)
  Youngjin Bae (Incheon National University)
  Dylan Cant (University of Montreal)
  Georgios Dimitroglou Rizell (Uppsala University)
  Jonathan Hanselman (Princeton University)
  Kristen Hendricks (Rutgers University)
  Jungsoo Kang (Seoul National University)
  Sungkyung Kang (University of Oxford)
  Min Hoon Kim (Ewha Womans University)
  Otto van Koert (Seoul National University)
  Sangjin Lee (KIAS)
  Jianfeng Lin (Tsinghua University)
  Rak-Kyeong Seong (UNIST)
  Seung-ook Yu (POSTECH)
  Jun Zhang (University of Science and Technology of China)

• **Pacific Rim Complex and Symplectic Geometry Conference;** July 29 – August 2, 2024

- Organizing Committee: Jun-Muk Hwang (IBS-CCG)
  Sungyeon Kim (IBS-CCG)
  Yong-Geun Oh (IBS-CGP & POSTECH)
- Scientific Committee: Xiuxiong Chen (Stony Brook University)
  Kengo Hirachi (University of Tokyo)
  Jun-Muk Hwang (IBS-CCG)
  Yong-Geun Oh (IBS-CGP & POSTECH)
  Kaoru Ono (RIMS)
  Yongbin Ruan (Zhejiang University)
- Invited Speakers: Dongwook Choa (KIAS)
  Young-Jun Choi (Pusan National University)
  Siarhei Finski (Йcole Polytechnique)
  Hervй Gaussier (University of Grenoble-Alpes)
  Masafumi Hattori (Kyoto University)
  Siqi He (AMSS)
  Ludmil Katzarkov (University of Miami)
  Yusuke Kawamoto (ETH)
  Takayuki Koike (Osaka Metropolitan University)
  Yu-Shen Lin (Boston University)
  George Marinescu (University of Kuln)
  Yuichi Nohara (Meiji University)
  Semon Rezchikov (Princeton University)
  Harish Seshadri (Indian Institute of Science)
  Rasul Shafikov (University Western Ontario)
  Li-Sheng Tseng (U.C. Irvine)
  Seungook Yu (POSTECH)
  Jihun Yum (Gyeongsang National University)
  Ruobing Zhang (Princeton University)
  Andrew Zimmer (University of Wisconsin)



• **Summer Mini-school on Algebraic Geometry;** August 20–23, 2024

- Organizers: Kyoung-Seog Lee (POSTECH)
  Jihun Park (IBS-CGP & POSTECH)
- Invited Lecturers: Sung Rak Choi (Yonsei University)
  Seung-Jo Jung (Jeonbuk National University)
  In-kyun Kim (KIAS)
  Luca Rizzi (IBS-CGP)
  Joonyeong Won (Ewha Womans University)

• **BICMR-IBSCGP Conference on Gromov-Witten Theory and Related Topics;** August 26–30, 2024

- Organizers: Alexander Aleksandrov (IBS-CGP)
  Xiaobo Liu (Peking University)
  Yong-Geun Oh (IBS-CGP & POSTECH)
  Gang Tian (Peking University)
- Invited Speakers: Alexander Aleksandrov (IBS-CGP)
  Bohan Fang (Peking University)
  Honghuai Fang (Peking University)
  Shuai Guo (Peking University)
  Jianxun Hu (Sun Yat-Sen University)
  Kai Hugtenberg (Lancaster University)
  Eleny Ionel (Stanford University)
  Hiroshi Iritani (Kyoto University)
  Young-Hoon Kiem (KIAS)
  Sukjoo Lee (The University of Edinburgh)
  Yuan-Pin Lee (Academia Sinica and University of Utah)
  Changzheng Li (Sun Yat-Sen University)
  Jeongseok Oh (Seoul National University)
  Chongyu Wang (Peking University)
  Jian Zhou (Tsinghua University)

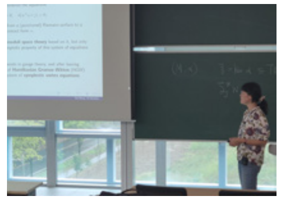
- **Autumn School on Low-dimensional Topology and Related Topics;** September 30 – October 4, 2024
  - Organizers: Yong-Geun Oh (IBS-CGP & POSTECH)
    Anderson Vera (IBS-CGP)
  - Invited Lecturers: Vladimir Fock (University of Strasbourg)
    Jean-Baptiste Meilhan (Grenoble Alpes University)
    Helen Wong (Claremont McKenna College)
  - Contributed Speakers: Eric Dolores Cuenca (Pusan National University)
    Hayato Imori (KAIST)
    Tsukasa Ishibashi (Tohoku University)
    Hongtaek Jung (Seoul National University)
    Shunsuke Kano (Tohoku University)
    Hiroaki Karuo (Gakushuin University)
    Seongjeong Kim (Jilin University)

- **IBS-CGP Reunion Workshop on Geometry and Physics;** October 21–23, 2024
  - Organizers: Byung Hee An (Kyungpook National University)
    Eunjeong Lee (Chungbuk National University)
    Yong-Geun Oh (IBS-CGP & POSTECH)
    Jihun Park (IBS-CGP & POSTECH)
  - Invited Speakers: Yunhyung Cho (Sungkyunkwan University)
    Sung Rak Choi (Yonsei University)
    Morimichi Kawasaki (Hokkaido University)
    Yoosik Kim (Pusan National University)
    Eunjeong Lee (Chungbuk National University)
    Kyoung-Seog Lee (POSTECH)
    Changzheng Li (Sun Yat-Sen University)
    Yat-Hin Suen (National Cheng Kung University)
    Rui Wang (University of California, Berkeley)
    Hwajong Yoo (Seoul National University)

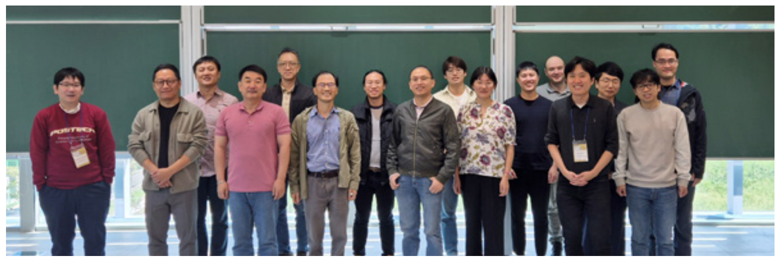




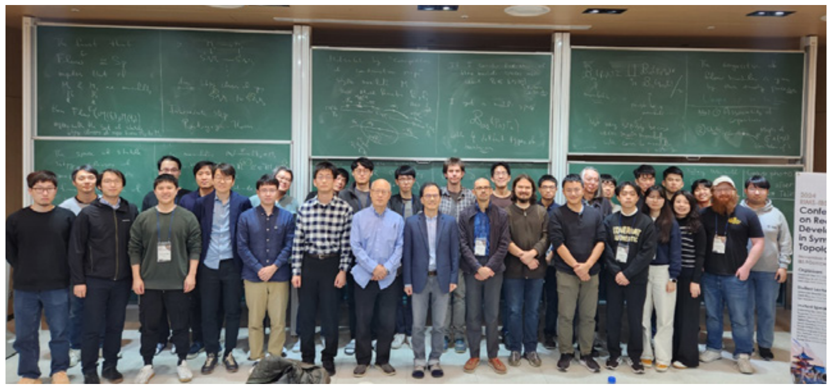




- **RIMS-IBSCGP Conference on Recent Developments in Symplectic Topology;** November 4–8, 2024
  - Organizers: Dongwook Choa (IBS-CGP)
    Yong-Geun Oh (IBS-CGP & POSTECH)
    Kaoru Ono (RIMS, Kyoto University)
  - Invited Lecturers: Mohammed Abouzaid (Stanford University)
    Cheol-Hyun Cho (Seoul National University)
    Egor Shelukhin (Universitй de Montrйal)
  - Invited Speakers: Jongmyeong Kim (Seoul National University)
    Yoosik Kim (Pusan National University)
    Takahiro Oba (Osaka University)
    Yukihiro Okamoto (RIMS, Kyoto University)
    Semon Rezchikov (Princeton University)
    Taisuke Shibata (RIMS, Kyoto University)
    Yoshihiro Sugimoto (Nagasaki University)
    Toru Yoshiyasu (Kyoto University of Education)
    Jun Zhang (University of Science and Technology of China)
    Zhengyi Zhou (Chinese Academy of Science)



Seminars

Integrable systems associated to cohomological field theories
Sergey Shadrin (University of Amsterdam)
December 13, 2024

Unit normal correspondence of a smooth divisor complement-continue
Dongwook Choa (IBS-CGP)
December 11, 2024

[2024-2 IBS-CGP&POSTECH-Math Colloquium] Analysis of Conformal Fields
Nam-Gyu Kang (KIAS)
December 6, 2024

Analogues of hyperlogarithm functions on affine complex curves
Benjamin Enriquez (University of Strasbourg)
December 5, 2024

Unit normal correspondence of a smooth divisor complement
Dongwook Choa (IBS-CGP)
November 27, 2024

Coregularity
Victor Przyjalkowski (Steklov Mathematical Institute & HSE University)
November 26, 2024

3d Mirror Symmetry is Mirror Symmetry I–II
Kifung Chan (The Chinese University of Hong Kong)
November 19–20, 2024

An introduction to isolated singular points of complete intersections-continue
Kyoung-Seog Lee (POSTECH)
October 16, 2024

Recent advances on categorical Torelli problemses
Shizhuo Zhang (IBS-CGP)
October 15, 2024

On Legendre-type transformations of a Frobenius manifold
Di Yang (University of Science and Technology of China)
October 11, 2024

[IBS-CGP&POSTECH-Math Colloquium] Poincaré duality in non-archimedean geometry
ShiZhang Li (Academy of Mathematics and Systems Science, CAS)
September 27, 2024

An introduction to isolated singular points of complete intersections
Kyoung-Seog Lee (POSTECH)
September 26, 2024

K-stability of Fano 3-fold hypersurfaces of index 1
Takuzo Okada (Kyushu University)
September 26, 2024

Equivariant Lagrangian correspondence and Teleman's conjectures
Yan-Lung Li (IBS-CGP)
September 23, 2024

Kontsevich's invariants of disk fiber bundles, and a "product formula"
Xujia Chen (Max-Planck Institute for Mathematics)
August 22, 2024

A spectral analog of a formula of Bezrukavnikov-Kaledin
Semon Rezchikov (Institute for Advanced Study)
August 5, 2024

F-bundles and blowups I-III
Tony Yue Yu (California Institute of Technology)
July 24–26, 2024

Recent advances in the Langlands program
SugWoo Shin (UC Berkely)
July 25, 2024

Two topics in general relativity
SungJin Oh (UC Berkely)
July 23, 2024

Floer theory of higher rank spectral networks
Yoon Jae Nho (University of Cambridge)
July 18, 2024

Operator algebras in holographic spacetime
Monica Jinwoo Kang (University of Pennsylvania)
July 15, 2024

Finding isomorphic quantum field theories
Monica Jinwoo Kang (University of Pennsylvania)
July 11, 2024

Family Floer Program and a Mathematical Formulation of the SYZ Conjecture
Yuan Hang (Northwestern University)
July 8, 2024

Cluster structure on the moduli space of toric vector bundles over toric surfaces via mirror symmetry
Yat-Hin Suen (KIAS)
July 4, 2024

Sheridan's homological mirror symmetry for pairs of pants, part 2
Sam Bardwell-Evans (IBS-CGP)
July 3, 2024

Contact geometry and black hole thermodynamics of AdS spacetimes
Yong-Geun Oh (IBS-CGP & POSTECH)
June 28, 2024

Sheridan's homological mirror symmetry for pairs of pants, part 1
Sam Bardwell-Evans (IBS-CGP)
June 26, 2024

Central derivative values of Rankin-Selberg L-functions as periods and metaplectic Fourier coefficients
Jeanine Van Order (Pontifнcia Universidade Catyli-ca do Rio de Janeiro)
June 18, 2024

On the ideals and syzygies of some Gaussian Graphical Models in Statistics
Kangjin HAN (DGIST)
June 18, 2024

Some topological properties of Milnor fibers of sandwiched singularities
Sam Bardwell-Evans (IBS-CGP)
May 29, 2024

Tetrahedron and 3D reflection equations in integrable systems and quantum cluster algebras
Atsuo Kuniba (University of Tokyo)
May 22, 2024

Automorphisms and deformations of regular semisimple Hessenberg varieties
Donggun Lee (IBS-CCG)
May 21, 2024

Polyhedral parametrization of canonical bases
Gleb Koshevoi (The Institute for Information Transmission Problems Russian Academy of Sciences)
May 16, 2024

Classification of neighbourhoods around leaves of a singular foliation
Simon-Raphael Fischer (National Center for Theoretical Sciences)
May 2, 2024

Curvature of Direct Images and Infinitesimal Deformations
Luca Rizzi (IBS-CGP)
April 30, 2024

Symplectic approach to Milnor fibers of surface singularities
Sam Bardwell-Evans (IBS-CGP)
April 29, 2024

[IBS-CGP Colloquium] The Movement of Carbon Atoms in Metals
Rodney Ruoff (IBS Center for Multidimensional Carbon Materials, UNIST)
April 25, 2024

Equivariant Lagrangian correspondences and applications
Nai Chung Conan Leung (The Chinese University of Hong Kong)
April 24, 2024

Anticanonical minimal models and Zariski decomposition
Sungwook Jang (Yonsei University)
April 23, 2024

Adjoint Asymptotic Multiplier Ideal Sheaves
Sung Rak Choi (Yonsei University)
April 23, 2024

Cluster algebras and Poisson geometry I–III
Dmytro Voloshyn (IBS-CGP)
April 1–4, 2024

Three dimensional mirror symmetry and degenerations of Riemann surface
Dan Xie (Tsinghua University)
March 27, 2024

Castelnuovo Curves
Gerriet Martens (University of Erlangen-Nьrnberg)
March 26, 2024

[IBS CGP-POSTECH Math Colloquium] Group theoretic understanding of manifold diffeomorphism groups
Sang-hyun Kim (KIAS)
March 22, 2024

Joint ergodicity of piecewise monotone maps
Younghwan Son (POSTECH)
March 21, 2024

Deformations of weighted homogeneous surface singularities
Sam Bardwell-Evans (IBS-CGP)
March 19, 2024

Enumerative Geometry and Algebraic cycles
Ramesh Sreekantan (Indian Statistical Institute)
March 14, 2024

Dimers, clusters, and mirrors
Kazushi Ueda (University of Tokyo)
March 13, 2024

[IBS CGP-POSTECH Math Colloquium] Enumerative geometry of curves and surfaces
Young-Hoon Kiem (KIAS)
March 8, 2024

KP integrability through the x-y swap relation
Alexander Aleksandrov (IBS-CGP)
February 28, 2024

Intersection theory on derived stacks I–III
Adeel A. Khan (Academia Sinica)
February 19–22, 2024

Geometry from categorical enumerative invariants
Junwu Tu (ShanghaiTech University)
February 14, 2024

Mapping the Phase Space of Supersymmetric Gauge Theories using Explainable Machine Learning
Rak-Kyeong Seong (UNIST)
January 31, 2024

Algebraic engineering and (q,t)-deformed integrable hierarchies
Jean-Emile Bourgine (The University of Melbourne)
January 5, 2024

Floer theory for the variation operator of an isolated singularity
Dongwook Choa (KIAS)
January 3, 2024

Visitors

[December]
Benjamin Enriquez (University of Strasbourg)
Nam-Gyu Kang (KIAS)
Boris Bychkov (University of Haifa)
Petr Dunin-Barkowski (National Research University Higher School of Economics)
Sergey Shadrin (University of Amsterdam)

[November]
Mohammed Abouzaid (Stanford University)
Cheol-Hyun Cho (Seoul National University)
Octav Cornea (CRM, Universitй de Montrйal)
Jongmyeong Kim (Seoul National University)
Yoosik Kim (Pusan National University)
Takahiro Oba (Osaka University)
Yukihiro Okamoto (RIMS, Kyoto University)
Kaoru Ono (RIMS, Kyoto University)
Semon Rezchikov (Princeton University)
Egor Shelukhin (Universitй de Montrйal)
Taisuke Shibata (RIMS, Kyoto University)
Yoshihiro Sugimoto (Nagasaki University)
Toru Yoshiyasu (Kyoto University of Education)
Jun Zhang (University of Science and Technology of China)
Zhengyi Zhou (Chinese Academy of Science)
Kifung Chan (The Chinese University of Hong Kong)
Jinhyung Park (KAIST)
Victor Przyjalkowski (Steklov Mathematical Institute & HSE University)

[October]
Di Yang (University of Science and Technology of China)
Byung Hee An (Kyungpook National University)
Yunhyung Cho (Sungkyunkwan University)
Sung Rak Choi (Yonsei University)
Morimichi Kawasaki (Hokkaido University)
Yoosik Kim (Pusan National University)
Eunjeong Lee (Chungbuk National University)
Kyoung-Seog Lee (POSTECH)
Changzheng Li (Sun Yat-Sen University)
Yat-Hin Suen (National Cheng Kung University)
Rui Wang (UC Berkeley)
Hwajong Yoo (Seoul National University)

[September]
Takuzo Okada (Kyushu University)
Kyoung-Seog Lee (POSTECH)
ShiZhang Li (Academy of Mathematics and Systems Science)
Vladimir Fock (University of Strasbourg)
Jean-Baptiste Meilhan (Grenoble-Alpes University)
Helen Wong (Claremont Mckenna College)

[August]
Semon Rezchikov (Institute for Advanced Study)
Sung Rak Choi (Yonsei University)
Seung-Jo Jung (Jeonbuk National University)
Kyoung-Seog Lee (POSTECH)
Joonyeong Won (Ewha Womans University)
Xujia Chen (Max-Planck Institute for Mathematics)
In-kyun Kim (KIAS)
Honghuai Fang (Peking University)
Chongyu Wang (Peking University)
Bohan Fang (Peking University)
Shuai Guo (Peking University)
Jianxun Hu (Sun Yat-Sen University)
Kai Hugtenberg (Lancaster University)
Eleny Ionel (Stanford University)
Hiroshi Iritani (Kyoto University)
Young-Hoon Kiem (KIAS)
Sukjoo Lee (The University of Edinburgh)
Yuan-Pin Lee (Academia Sinica and University of Utah)
Changzheng Li (Sun Yat-Sen University)
Xiaobo Liu (Peking University)
Jeongseok Oh (Seoul National University)
Mohammad Tehrani (The University of Iowa)
Gang Tian (Peking University)
Jian Zhou (Tsinghua University)

[July]
Yat-Hin Suen (KIAS)
Yuan Hang (Northwestern University)
Monica Jinwoo Kang (University of Pennsylvania)
Kyoung-Seog Lee (POSTECH)
Yoon Jae Nho (University of Cambridge)
SungJin Oh (UC Berkely)
Tony Yue Yu (Califonia Institute of Technology)
SugWoo Shin (UC Berkely)
Seonhwa Kim (University of Seoul)

[June]
Hanwool Bae (Seoul National University)
Youngjin Bae (Incheon National University)
Dylan Cant (University of Montreal)
Jae Choon Cha (POSTECH)
Georgios Dimitroglou Rizell (Uppsala University)
Tobias Ekholm (Uppsala University)
Jonathan Hanselman (Princeton University)
Kristen Hendricks (Rutgers University)
Jungsoo Kang (Seoul National University)
Sungkyung Kang (University of Oxford)
Min Hoon Kim (Ewha Womans University)
Otto van Koert (Seoul National University)
Sangjin Lee (KIAS)
Jianfeng Lin (Tsinghua University)
Brett Parker (Australian National University)
Ian Zemke (Princeton University)
Jun Zhang (University of Science and Technology of China)
Rak-Kyeong Seong (UNIST)
Kangjin HAN (DGIST)
Jeanine Van Order (Pontifнcia Universidade Catylica do Rio de Janeiro)
Donggun Lee (IBS-CCG)
Yunhyung Cho (Sunkyunkwan University)
Seonhwa Kim (University of Seoul)

[May]
Gleb Koshevoi (The Institute for Information Transmission Problems Russian Academy of Sciences)
Donggun Lee (IBS-CCG)
Atsuo Kuniba (University of Tokyo)
Seonhwa Kim (University of Seoul)

[April]
Sung Rak Choi (Yonsei University)
Sungwook Jang (Yonsei University)
Nai Chung Conan Leung (The Chinese University of Hong Kong)
Rodney Ruoff (IBS Center for Multidimensional Carbon Materials, UNIST)
Seonhwa Kim (University of Seoul)
Simon-Raphael Fischer (National Center for Theoretical Sciences)

[March]
Kifung Chan (KIAS)
Ramesh Sreekantan (Indian Statistical Institute)
Kazushi Ueda (University of Tokyo)
Yat-Hin Suen (KIAS)
Seonhwa Kim (University of Seoul)
Sang-hyun Kim (KIAS)
Changho Keem (Seoul National University)
Seonhwa Kim (University of Seoul)
Gerriet Martens (University of Erlangen-Nьrnberg)
Dan Xie (Tsinghua University)

[February]
Junwu Tu (ShanghaiTech University)
Adeel A. Khan (Academia Sinica)

[January]
Dongwook Choa (KIAS)
Jean-Emile Bourgine (The University of Melbourne)
Jongmyeong Kim
Rak-Kyeong Seong (UNIST)

MOUs

The CGP has signed several MOUs for active research collaborations and academic exchanges with the mathematics community.

Beijing International Center for Mathematical Research (BICMR), China

November 2015 – November 2026 (renewed in 2021)

- 4th IBS-CGP & BICMR Joint Conference on Gromov-Witten Theory and Related Topics (August 26–30, 2024 @ Jeju Uni Hotel)
- 3rd BICMR&IBS-CGP Joint Symplectic Geometry Workshop (September 24–26, 2019 @ POSTECH)
- Silk Road Geometry Conference 2018 (June 4–8, 2018 @ Gukova Geometry/Topology Institute)
- 2nd BICMR&IBS-CGP Joint Symplectic Geometry Workshop (September 18–22, 2017 @ BICMR)
- 1st BICMR&IBS-CGP Joint Symplectic Geometry Workshop (October 31 – November 4, 2016 @ Jeju KAL Hotel)

Research Institute for Mathematical Sciences (RIMS), Japan

August 2017 – July 2025 (renewed in 2020)

- RIMS-IBSCGP Conference on Recent Developments in Symplectic Topology (November 4–8, 2024 @ POSTECH)
- 2021 Pacific Rim Complex & Symplectic Geometry Conference (July 12–16, 2021 @ Online)
  RIMS & IBS-CGP Joint Symplectic Geometry Workshop (December 2–4, 2019 @ Kyoto University)
- Wall-crossing Formula, Open Gromov-Witten Invariants and Related Areas (October 29–31, 2018 @ POSTECH)
- Pacific Rim Complex-Symplectic Geometry Conference (July 31 – August 4, 2017 @ POSTECH)

Mathematical Research Institute (MATRIX), Australia

December 2018 – November 2026 (renewed in 2021)

- MATRIX-IBSCGP workshop on Symplectic and Low-dimensional Topology (June 3–14, 2024 @ POSTECH)
- IBS-CGP and MATRIX workshop on Symplectic Topology (December 5–16, 2022 @ University of Melbourne)

The Centre de recherches mathйmatiques (CRM), Canada

December 2023 – December 2026

Research Highlights

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On universal structures in Cohomological Field Thoeries

Alexander Aleksandrov
CGP Research member since February 2017

Introduction

Recently the so-called cut-and-join operators were constructed for several interesting models in combinatorics and mathematical physics, including classical matrix models and various families of weighted Hurwitz numbers. This success rises a question about origin and universality of the cut-and-join description. In [Ale24] we address this question in the setup of topological recursion and cohomological field theory – two very close constructions, unifying many interesting examples. We prove that in the semi-simple case the partition functions can always be described by cubic cut-and-join operators, independent of the topological expansion parameter. The cut-and-join description leads to an algebraic version of topological recursion, that allows us to construct all correlation functions recursively in a polynomial way. For the same partition functions we also derive N families of the Virasoro constraints and prove that these constraints, supplemented by a deformed dimension constraint, completely specify the partition function and, in particular, imply the cut-and-join description.

To describe this construction we start from two archetypal examples of partition functions: the Kontsevich–Witten and Brézin–Gross–Witten tau-functions.

Cut-and-join operators for KW and BGW tau-functions

Denote by $\overline{\mathcal{M}}_{g,n}$ the Deligne–Mumford compactification of the moduli space $\mathcal{M}_{g,n}$ of all compact Riemann surfaces of genus g with n distinct marked points. It is a non-singular complex orbifold of dimension $3g-3+n$, which is empty unless the *stability condition* $2g-2+n>0$ is satisfied.
In his seminal paper [Wit91], Witten initiated new directions in the study of $\overline{\mathcal{M}}_{g,n}$. For each marking index i consider the cotangent line bundle $\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}$, whose fiber over a point $[\Sigma, z_1, \dots, z_n] \in \overline{\mathcal{M}}_{g,n}$ is the complex cotangent space $T^*_{z_i}\Sigma$ of Σ at z_i. Let $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ denote the first Chern class of $\mathbb{L}_i$. We consider the intersection numbers

(1) $$< \tau_{k_1}\tau_{k_2}\cdots\tau_{k_n} >_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\psi_2^{k_2}\cdots\psi_n^{k_n} \in \mathbb{Q}.$$

Let T_i, $i \geq 0$, be formal variables and let

(2) $$\tau_{KW} := \exp\left(\sum_{g,n} \hbar^{2g-2+n} \sum_{k_1,\dots,k_n\geq 0} \langle\tau_{k_1}\tau_{k_2}\cdots\tau_{k_n}\rangle_g \frac{\prod T_{k_i}}{n!}\right).$$

A parameter $\hbar$ here is introduced to trace the Euler characteristic of the punctured curve Σ, therefore it is associated with the *topological expansion* of the generating function.

Witten's conjecture [Wit91], proved by Kontsevich [Kon92], states that the partition function τ_{KW} is a tau-function of the Korteweg–De Vries (KdV) hierarchy.

Theorem 1 ([Kon92]). *The generating function τ_{KW} is a tau-function of the KdV hierarchy in the variables t_k with $T_n = (2n+1)!!t_{2n+1}$.*

Below we call it the Kontsevich–Witten (KW) tau-function. Integrability of the KW tau-function immediately follows from Kontsevich's matrix integral representation.
Another KdV tau-function we consider is the Brézin–Gross–Witten (BGW) tau-function. This tau-function also governs the intersection theory on the moduli spaces $\overline{\mathcal{M}}_{g,n}$, but this time with the insertions of the fascinating Norbury's Θ-classes, $\Theta_{g,n} \in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n})$, related to the super Riemann surfaces [Nor23, Nor20]. Consider the generating function of the intersection numbers of Θ and ψ classes

(3) $$\tau_\Theta = \exp\left(\sum_{g,n} \hbar^{2g-2+n} \sum_{k_1,\dots,k_n\geq 0} \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n}\psi_1^{k_1}\psi_2^{k_2}\cdots\psi_n^{k_n} \frac{\prod T_{k_i}}{n!}\right).$$

For this generating function Norbury has suggested a direct analog of Witten's conjecture [Nor20, Conjecture 1], recently proven by Chidambaram, Garcia–Failde, and Giacchetto.

Theorem 2 ([CGFG22]). *The generating function τ_Θ is the BGW tau-function of the KdV hierarchy in the variables t_k,*

(4) $$\tau_\Theta = \tau_{BGW}.$$

The KW and BGW tau-functions can be described in terms of the Virasoro constraints which completely specify the generating functions and are indirectly equivalent to the integrability. Let us consider the Virasoro algebra. The Virasoro operators are given

(5) $$\widehat{L}_m = \frac{1}{2}\sum_{i+j=-m-1} \frac{T_iT_j}{(2i-1)!!(2j-1)!!} + \sum_{k=0}^{\infty} \frac{(2k+2m+1)!!}{(2k-1)!!} T_k \frac{\partial}{\partial T_{k+m}} + \frac{1}{2}\sum_{i+j=m-1} (2i+1)!!(2j+1)!! \frac{\partial^2}{\partial T_i \partial T_j}.$$

The operators $\widehat{L}_m$ satisfy the commutation relations of the Virasoro algebra.
Let us denote

(6) $$\tau_1 = \tau_{KW}, \qquad \tau_0 = \tau_{BGW}.$$

Below we always assume $\alpha \in \{0,1\}$. The tau-functions τ_α satisfy the *Virasoro constraints*

(7) $$\left(\frac{1}{2}\widehat{L}_k - \frac{(2k+2\alpha+1)!!}{2\hbar}\frac{\partial}{\partial T_{k+\alpha}} + \frac{\delta_{k,0}}{16}\right)\cdot \tau_\alpha = 0, \quad k \geq -\alpha.$$

Let

$$(8)\qquad \widehat{J}^a_k := \begin{cases} (2k-1)!!\dfrac{\partial}{\partial T^a_{k-1}} & \text{for} \quad k>0, \\ \dfrac{T^a_{|k|}}{(2|k|-1)!!} & \text{for} \quad k\leq 0. \end{cases}$$

Combining (7) with dimension constraints one arrives at

$$(9)\qquad \frac{\partial}{\partial \hbar}\tau_\alpha = \widehat{W}_\alpha \cdot \tau_\alpha,$$

where

$$(10)\qquad \widehat{W}_0 = \sum_{k,m=0}^{\infty}\left(\frac{1}{2}\widehat{J}_{-k-m-1}\widehat{J}_{k+1}\widehat{J}_{m+1}\right)+\frac{\widehat{J}_0}{8},$$

$$\widehat{W}_1 = \frac{1}{3}\sum_{k,m=0}^{\infty}\left(\frac{1}{2}\widehat{J}_{-k-m-2}\widehat{J}_{k+1}\widehat{J}_{m+1}+\widehat{J}_{-k}\widehat{J}_{-m}\widehat{J}_{-k-m}\right)+\frac{\widehat{J}_0^3}{6}+\frac{\widehat{J}_{-1}}{24}.$$

For $\alpha\in\{0,1\}$ equation (9) has a unique solution.

Theorem 3 ([Ale11, Ale18]).

$$(11)\qquad \tau_\alpha = \exp\left(\hbar\,\widehat{W}_\alpha\right)\cdot 1.$$

Let us consider the topological expansion of the tau-functions

$$(12)\qquad \tau_\alpha = \sum_{k=0}^{\infty}\tau_\alpha^{(k)}\hbar^k.$$

The coefficients $\tau_\alpha^{(k)}$ are homogenous polynomials in $\mathbf{T}$ of degree $(2\alpha+1)k$. By Theorem 3 they satisfy a linear recursion relation.

Corollary 2.1.

$$(13)\qquad \tau_\alpha^{(k)} = \frac{\widehat{W}_\alpha}{k}\cdot\tau_\alpha^{(k-1)}.$$

We call it the *algebraic topological recursion* to distinguish it from the Chekhov–Eynard–Orantin topological recursion. Operators $\widehat{W}_\alpha$ describe the change of topology, that's why slightly abusing notation we call such operators the *cut-and-join operators*. These operators are given by infinite sums (10), however, on each step of the recursion (13) only a finite number of terms contribute. Hence, the recursion is given by the action of the polynomial differential operators on polynomials.

It appears that there exists a generalization of Theorem 3 and Corollary 2.1 for a huge family of partition functions, related to cohomological field theories and the Chekhov–Eynard–Orantin topological recursion.

Cohomological field theories and topological recursion

Cohomological field theories were introduced by Kontsevich and Manin [KM94] for axiomatic description of universal properties of Gromov–Witten theory, and many interesting geometrical and combinatorial models can naturally be studied in this setup. A cohomological field theory (CohFT) is given by a family of tensors $\Omega_{g,n}$. These tensors satisfy the compatibility conditions and are labeled by the number of the marked points n and genus g for all stable combinations $2g-2+n>0$. For any CohFT Ω let us consider its partition function, dependent on the formal variables $\mathbf{T}=\{T^a_k; 1\leq a\leq N, k\geq 0\}$,

$$(14)\qquad Z_\Omega = \exp\left(\sum_{g,n}\frac{\hbar^{2g-2+n}}{n!}\sum_{k_j}\int_{\overline{\mathcal{M}}_{g,n}}\Omega_{g,n}(e_{a_1}\otimes\ldots\otimes e_{a_n})\prod_{j=1}^{n}\psi_j^{k_j}T^{a_j}_{k_j}\right)\in\mathbb{C}[\mathbf{T}][[\hbar]].$$

We also call it the (total) ancestor potential, because for CohFTs associated with the Gromov–Witten theory on some variety, function Z_Ω is the generating function of the so-called ancestor Gromov–Witten invariants. The most important example here is the Kontsevich–Witten tau-function associated with the Gromov–Witten theory of a point, considered in the previous section.

Investigation of the Gromov–Witten invariants and, more generally, cohomological field theories, in particular in the higher genera, is a challenging problem. In his seminal works, Givental [Giv01a, Giv01b] introduced a group action on the CohFT partition functions. In [Tel12] Teleman proves that an extension of the Givental group, which we call the Givental–Teleman group, acts transitively on the space of semi-simple CohFTs. CohFTs here are not required to be conformal or have a flat unit. It implies that the partition function of an arbitrary semi-simple CohFT can be described by the Givental decomposition formula,

$$(15)\qquad Z_\Omega = \widehat{R}\,\widehat{T}\,\widehat{\Delta}\cdot\prod_{a=1}^{N}\tau_1(\hbar,\mathbf{T}^{\mathbf{a}}).$$

Here $\widehat{R}$ is a Givental group operator given by a quantization of the twisted loop group element, $\widehat{T}$ is a translation operator, $\widehat{\Delta}$ rescales the topological expansion parameter $\hbar$, and τ_1 is the Kontsevich–Witten tau-function.

The Chekhov–Eynard–Orantin topological recursion [EO07, EO09] is a universal procedure, which allows us to construct a family of symmetric differentials on a spectral curve with additional structures on it. For many known examples these differentials encode interesting invariants of enumerative geometry and mathematical physics. These differentials can also be combined in a canonical way to create a generating function Z_S.

As it was shown by Dunin-Barkowski, Orantin, Shadrin, and Spitz [DBOSS14], partition functions of the semi-simple cohomological field theories are closely related to the Chekhov–Eynard–Orantin topological recursion. Namely, the ancestor potential of a semi-simple CohFT with flat unit can be identified with the partition function of the Chekhov–Eynard–Orantin topological recursion on a certain local spectral curve,

$Z_S = Z_\Omega$. It means, in particular, that the partition function of the associated Chekhov–Eynard–Orantin topological recursion is given by the decomposition formula (15).

This relation was generalized by Chekhov and Norbury [CN19]. This generalization describes Givental's decomposition formula for the partition functions on all, possibly irregular, local spectral curves with simple ramifications, that is, the curves which near ramification points are similar to the Airy or Bessel curves. This generalization also describes some degenerate CohFTs. To describe this generalization one has to consider a slightly more general Givental decomposition formula. Namely, for a Givental group operator $\widehat{R}$, which in the most general case is not a quantization of the twisted loop group element, a translation operator $\widehat{T}$, and an $\hbar$-rescaling operator $\widehat{\Delta}$ we consider the generalized ancestor potential

(16) $$Z := \widehat{R}\,\widehat{T}\,\widehat{\Delta}\cdot\prod_{a=1}^{N}\tau_{\alpha_a}(\hbar,\mathbf{T}^{\mathbf{a}}).$$

Here $\alpha_a \in \{0,1\}$, and τ_0 is the BGW tau-function. For a local, possibly irregular, spectral curve with N simple ramification points the partition function Z_S of the Chekhov–Eynard–Orantin topological recursion is given by this Givental decomposition formula. The operators $\widehat{R}$, $\widehat{T}$, $\widehat{\Delta}$ can be described in terms of the topological recursion data. The functions τ_{α_a} are associated with N ramifications points, and $\alpha_a = 1$ ($\alpha_a = 0$) for regular (irregular) ramification points.

Cut-and-join description and Virasoro constraints

We prove that for an arbitrary Givental operator $\widehat{R}$, a translation operator $\widehat{T}$, and a $\hbar$-rescaling operator $\widehat{\Delta}$ the generalized ancestor potential (16) is described by a cubic cut-and-join operator. In particular, it means that this cut-and-join description is valid for the partition functions of semi-simple CohFTs and Chekhov–Eynard–Orantin topological recursion for local, possibly irregular, spectral curves with simple ramifications. This allows us to describe a generalized ancestor potential by an algebraic version of the topological recursion, which, unlike the Chekhov–Eynard–Orantin topological recursion, is linear and is given in terms of polynomials and differential operators.

For a generalized ancestor potential, given by (16), we construct a cubic operator

(17) $$\widehat{W} = \sum_{\substack{i\le j\le k,\\ i+j+k\ge 0}} A^{i,j,k}_{a,b,c}\,\widehat{J}^a_i\widehat{J}^b_j\widehat{J}^c_k + \sum_{j=-1}^{\infty} B^j_a\widehat{J}^a_j,$$

where the coefficients $A^{i,j,k}_{a,b,c}$ and B^j_a are independent of $\mathbf{T}$ and $\hbar$. We call it the cut-and-join operator.

Theorem 4 ([Ale24]). A generalized ancestor potential satisfies

(18) $$Z = \exp\left(\hbar\,\widehat{W}\right)\cdot 1.$$

Formula (18) follows from the properties of the operators $\widehat{\Delta}$, $\widehat{T}$, and $\widehat{R}$ and the cut-and-join description of the KW and BGW tau-functions [Ale11, Ale18]. By this theorem a generalized ancestor potential is a solution of the cut-and-join equation

(19) $$\frac{\partial}{\partial\hbar}Z = \widehat{W}\cdot Z.$$

Formula (18) gives a unique solution of this equation in $\mathbb{C}[[\mathbf{T}]][[\hbar]]$ with $Z\big|_{\hbar=0} = 1$.

The cut-and-join description allows us to reconstruct a generalized ancestor potential by a linear recursion. Consider the topological $\hbar$-expansion of Z. Its coefficients $Z^{(k)}$ are polynomials in $\mathbf{T}$. From Theorem 4 we have the following corollary.

Corollary 3.1. The coefficients of the topological expansion of a generalized ancestor potential satisfy the algebraic topological recursion

(20) $$Z^{(k)} = \frac{1}{k}\widehat{W}\cdot Z^{(k-1)}$$

with the initial condition $Z^{(0)} = 1$.

The operator $\widehat{W}$ describes a change of topology. This is why we call it the cut-and-join operator. Cut-and-join formulas should be very convenient computationally because of the linear structure of the algebraic topological recursion.

With any generalized ancestor potential for $a = 1\ldots, N$, $m \ge -\alpha_a$ we can associate the operators $\widehat{L}^a_m$. By construction, the operators $\widehat{L}^a_m$ satisfy the commutation relations of a subalgebra of the direct sum of N copies of the Virasoro algebra.

Theorem 5 ([Ale24]). A generalized ancestor potential satisfies the Virasoro constraints

(21) $$\widehat{L}^a_m\cdot Z = 0, \qquad a = 1\ldots, N, \quad m \ge -\alpha_a.$$

To arrive at Theorem 4 we need to supplement these Virasoro constraints with an auxillary equation

(22) $$\hbar\frac{\partial}{\partial\hbar}Z = \sum_{\substack{i<j,\\ i+j\ge 1}} H^{i,j}_{a,b}\,\widehat{J}^a_i\widehat{J}^b_j\cdot Z,$$

where the coefficients $H^{i,j}_{a,b}$ do not depend on $\mathbf{T}$ and $\hbar$. This equation is a deformation of the dimension constraint for the product of the tau-functions τ_α.

The basic ingredients of the decomposition construction, namely, the KW and the BGW tau-functions have classical matrix model descriptions of Kontsevich type. It would be interesting to understand when generalized ancestor potentials can be represented by matrix integrals. We expect that this should be possible for the cases described by topological recursion with almost trivial $x - y$ dual side.

It is very challenging to generalize the construction to non-semi-simple CohFTs corresponding to the topological recursion with higher ramification points. In this case the partition functions should satisfy the higher W-constraints and we expect them to be described by the families of the higher degree cut-and-join operators. But, even for the semi-simple CohFTs geometric meaning of the constructed operators is not clear yet.

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$2n^2$-inequality and birational rigidity

Igor Krylov

CGP Research member since August 2022

Introduction

My research interests lie in the field of algebraic geometry, more precisely, birational geometry. Algebraic geometry is a subject that studies algebraic varieties: zero sets of homogeneous polynomials in a projective space. Classification of algebraic varieties has been one of the central topics in algebraic geometry since the very beginning. Classifying all algebraic varieties up to isomorphism is not a realistic proposition due to the existence of blow ups.

Blow ups are the type of algebraic surgery operation in which one cuts out an algebraic subvariety and instead inserts a variety classifying tangent directions as a subvariety. The simplest example is the blow up of a disk at a point, the tangent to a point directions on a disk are classified by a circle and the operation of blowing up a disk at a point replaces that point with a circle. As a result, the disk becomes a Moebius band. For any algebraic surface we can perform blow ups as many time as we want, which gives us infinitely many non-isomorphic algebraic varieties.

This leads us to the notion of *birational equivalence*. We say that two varieties are birational to each other if they have open dense subsets on which they are isomorphic. In particular, all varieties related by a sequence of blow ups are birationally equivalent to each other. Classifying algebraic varieties up to birational equivalence is a much more manageable problem even though it is far from being solved.

Since there are so many different varieties in one birational equivalence class, one would like to find good representatives within a birational class. The *Minimal Model Program* was introduced to solve this problem. When we run Minimal Model Program on a given variety we contract as many subvarieties to subvarieties of lower dimension as we can, essentially performing operations opposite of blow ups.

The output of a Minimal Model Program is either a minimal model, or a *Mori Fiber Space*. The most well known examples of Mori Fiber Spaces are *Fano* varieties, that is varieties with everywhere positive Ricci curvature. One can view Fano varieties as higher dimension analogues of a sphere. Other Mori Fiber Spaces are fibrations $\pi : X \to B$ such that fibers of the π are Fano varieties.

Minimal models are not unique, but they are the isomorphic in codimension one. That is for any two birational minimal models X and Y there are closed subvarieties $Z \subset X$ and $S \subset Y$ of codimension $\geqslant 2$ such that $X \setminus Z$ and $Y \setminus S$ are isomorphic. On the other hand Mori fiber spaces in the same birational class can differ a lot, for example the projective space $\mathbb{P}^n$ has many different Mori Fiber spaces in its birational class. In dimension three these include $\mathbb{P}^3$ itself, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, quadric hypersurface $X_2 \subset \mathbb{P}^4$, complete intersection of two quadrics $X_{2,2} \subset \mathbb{P}^5$, and many others. This serves as a motivation for the following problem:

intersection of two quadrics $X_{2,2} \subset \mathbb{P}^5$, and many others. This serves as a motivation for the following problem:

Problem 1. Let $\pi : X \to B$ be a Mori Fiber Space, which other Mori Fiber Spaces is X birational to?

Birational rigidity

Birationally rigid varieties are varieties with a trivial answer to Problem 1:

Definition 2. We say that a Fano variety X is birationally rigid if X is the unique Mori Fiber Space in its birational class.

We can also generalize this definition to all Mori Fiber Spaces, although if they are not Fano varieties, then the precise definition becomes a bit involved. But the underlying meaning is the same: $\pi : X \to Z$ is a birationally rigid Mori Fiber Space if $\pi : X \to Z$ is the unique Mori Fiber Space "structure" in the birational class.

The primary method of studying birational rigidity is Noether-Fano method. To explain the method I first explain the notion of a birational map. Let X and Y be birationally equivalent varieties, then by definition there is an injective map $\chi : U \to Y$ from an open dense subset $U \subset X$. We can extend this map to a map $\chi' : U' \to Y$ from a larger open subset U' such that $X \setminus U'$ is a closed subvariety of codimension 2. The map χ' might no longer be injective, it usually contracts some closed subvarieties into subvarieties of lower dimension. We say that $\chi' : X \dashrightarrow Y$ is a *birational map*. Note that the extension χ' of χ is unique, more generally birational map is uniquely determined by its restriction to any open dense subset, hence we can also say that $\chi : X \to Y$ is a birational map.

I will illustrate Noether-Fano method with the following example. Let $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ be a quadric in $\mathbb{P}^3$. Consider the birational map $\chi : \mathbb{P}^2 \dashrightarrow Q$ given by

$$(x : y : z) \mapsto (xy : xz : yz : z^2)$$

This map is undefined at the points $(0:1:0)$ and $(1:0:0)$ and contracts the line $z = 0$ to a point. If we take a family of hyperplane sections on Q and consider closures of their preimages on $\mathbb{P}^2$, we get a 4 -dimensional family of conics $\mathcal{M}$. The basis of this family of conics is $\langle xy, xz, yz, z^2 \rangle$, and we can see that the conics in $\mathcal{M}$ have two base points: $(0:1:0)$ and $(1:0:0)$. Scaling $\mathcal{M}$ to match canonical class of $\mathbb{P}^2$ which is a degree 3 curve we observe that the pair $(\mathbb{P}^2, \frac{3}{2}\mathcal{M})$ is not canonical at these two points. The canonicity is in dimension two is determined by multiplicity at base points, namely $(X, \lambda\mathcal{M})$ is not canonical at P if P is a base point and $\mathrm{mult}_P \mathcal{M} > \frac{1}{\lambda}$.

More generally for any birational map $\chi : X \dashrightarrow Y$ we can take very ample divisors on Y, that is an analogue of hyperplane sections of Y, take closure of their preimages on X and get a family of hypersurfaces $\mathcal{M}$ with base locus in the points of indeterminacy of χ. The family $\mathcal{M}$ is called a linear

system and it is mobile, i.e. its base locus has codimension $\geqslant 2$ since we got $\mathcal{M}$ from a birational map. The following theorem provides the core of the method.

Theorem 3 (Noether-Fano inequality)**.** Let X be a non-rigid Fano variety of Picard rank 1. Then there is a mobile linear system $\mathcal{M}$ such that the the pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical.

In order to prove birational rigidity of a three-dimensional variety we need ways to interpret non-canonicity of the pair $(X, \frac{1}{n}\mathcal{M})$ at some center B geometrically. As we saw, this is simple in dimension two, or more generally when B is of codimension two, but in codimension three it is very difficult.

$4n^2$-inequality and known results

In the past three decades most of the proofs of birational rigidity relied on so-called $4n^2$-inequality.

Theorem 4 ($4n^2$-inequality, [7, Theorem 2.1])**.** Let $p \in X$ be the germ of a smooth 3-fold. Let $\mathcal{M}$ be a mobile linear system on X and let n be a positive rational number. If p is a center of non-canonical singularities of the pair $(X, \frac{1}{n}\mathcal{M})$, then for general members D_1, D_2 in $\mathcal{M}$ we have

$$\mathrm{mult}_p D_1 \cdot D_2 > 4n^2 .$$

This inequality allows us to translate the information about singularities of linear systems into information about intersections of hypersurfaces on X. It enables us to use the geometry of X to derive a contradiction and prove birational rigidity. The sketch of the proof of rigidity of the quartic threefold $X_4 \subset \mathbb{P}^4$ highlights the technique very well. Noether-Fano inequality tells us that if X_4 is not birationally rigid, then there is a mobile linear system $\mathcal{M} \subset \left| -nK_X \right|$ such that the pair $(X_4, \frac{1}{n}\mathcal{M})$ is canonical at some curve C or a point P. The linear system $\mathcal{M}$ is essentially a family of some sections of X_4 by hypersurfaces of degree n. Non-canonicity at a curve C means that all the the hypersurface sections in the family have multiplicity at least n along C, a clever geometrical argument can show that this never happens. Then we use $4n^2$-inequality and conclude that there are general hypersurface sections D_1, D_2 such that their intersection has multiplicity at more $4n^2$ at P. It follows that if we take a general hyperplane section H through P, then intersection $(D_1 \cdot D_2 \cdot H)$ has multiplicity $> 4n^2$ at P, but this obviously never happens, since $(D_1 \cdot D_2 \cdot H)$ has multiplicity $4n^2$ total across all points. This argumentation is sufficient to prove the following classical result.

Theorem 5 ([4])**.** A smooth quartic hypersurface $X_4 \subset \mathbb{P}^4$ is birationally rigid.

If one were to add a couple more tricks to the arsenal for the purpose of dealing with orbifold singularities, one can prove the following:

Theorem 6 ([3] and [1])**.** Let X be a quasismooth weighted Fano hypersurface of index one, then X is birationally rigid.

One can further generalize Theorem 4 to work with a boundary, this results in Corti's inequality ([2]). This inequality can be used to prove birational rigidity of del Pezzo fibrations of low degree which are a certain type of Mori Fiber Spaces:

Theorem 7 ([6]). Let $X \to \mathbb{P}^1$ be a del Pezzo fibration of degree 1, 2, or 3. Suppose X is smooth and if the degree of the fibrations is three suppose further that X satisfies certain generality conditions. Suppose X satisfies the K^2-condition, then X is birationally rigid.

$2n^2$-inequality

Unfortunately it is insufficient to work with smooth or even quasismooth varieties since contractions performed in Minimal Model Program can result in new variety being singular. In particular minimal models and Mori Fiber Spaces may have so called compound du Val singularities: cA_n, cD_n, cE_n. I study Problem 1 for singular Mori Fiber Spaces. In [5] my collaborators and I proved an analogue of Theorem 4 for mildly singular varieties.

Theorem 8 ($2n^2$-inequality for most points ([5]). Let $p \in X$ be the germ of a cA_1 singularity. Let $\mathcal{M}$ be a mobile linear system on X and let n be a positive rational number. If p is a center of non-canonical singularities of the pair $(X, \frac{1}{n}\mathcal{M})$, then for general members D_1, D_2 in $\mathcal{M}$ we have

$$\mathrm{mult}_p(D_1 \cdot D_2) > 2n^2 .$$

This allowed us to extend birational rigidity results to varieties admitting mild Gorenstein singularities for 78 out of 95 families.

Theorem 9 ([5]). Let X be a weighted Fano hypersurface of index one with either quotient or cA_1 -singularities. Suppose further X belongs to one of 78 families specified in the table (see [5] for the table), then X is birationally rigid.

We can also make a version of Theorem 8 with a boundary and it allows us to prove birational rigidity of del Pezzo fibrations:

Theorem 10 ([5]). Let $X \to \mathbb{P}^1$ be a del Pezzo fibration of degree 1 with at worst cA_1 singularities. Suppose X satisfies the K^2-condition, then X is birationally rigid.

Naturally, these Theorems 9 and 10 are only the beginning of application of Theorem 8. With additional hard work and utilization of some clever tricks one can prove more.

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Generalized Calogero-Moser system and supergroup gauge origami

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Introduction

Different branches of theoretical physics which seems unrelated can found connection in the context of string theory. My research focus on studying these connections. One example is the relation between the four-dimensional $\mathcal{N}=2$ supersymmetric gauge theory and algebraic integrable systems. Since the groundbreaking work of Seiberg and Witten and later Nekrasov and Shatashvili, it has been a fruitful research area for the last decade.

The elliptic Calgero-Moser system is an algebraic integrable system closely related to the quantum Hall effect in condense matter. At the same time it has an intrintisic connection to the 4d $\mathcal{N}=2^*$ theory. In this note we will explore the generaliation of these correspondences in [1].

Generalized elliptic Calogero-Moser system

The elliptic Calogero-Moser (eCM) system is defined for any finite-dimenaional Lie algebra $\mathfrak{g}$ by the Hamiltonian [2]:

$$\mathrm{H}_{\mathfrak{g}} = -\frac{1}{2}\Delta_{\mathfrak{g}} + \sum_{\alpha\in\mathscr{R}^+(\mathfrak{g})} g_\alpha(g_\alpha - 1)(\alpha,\alpha)\wp(\alpha\cdot x) \tag{1}$$

where $\mathscr{R}^+(\mathfrak{g})$ is the set of all positive roots of Lie algebra $\mathfrak{g}$, $\Delta_{\mathfrak{g}}$ is the Laplace operator defined on the Lie algebra $\mathfrak{g}$. The Weierstrass $\wp$-function $\wp(z;2\ell,2\mathrm{i}\delta)=\wp(z)$ has half-period $(\ell,\mathrm{i}\delta)$. g_α is the potential coupling that depends on the length of the fundamental root α. The eCM system defined based on the Lie algebra A_{N-1}, which is a one-dimensional quantum mechanics system of N particles with the Hamiltonian:

$$\mathrm{H}_{N;g}(x) = -\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + g(g-1)\sum_{1\le j<k\le N}\wp(x_j - x_k), \tag{2}$$

is closely related to four dimensional $\mathcal{N}=2^*$ $SU(N)$gauge theory. The coupling g corresponds to the adjoint mass of the gauge theory (with an overall scaling of Planck constant).

The algebraic integrable system can be defined on the Lie superalgebra*. The elliptic Calogero-Moser system associated with the root system of the superalgebra $\mathfrak{g}=\mathfrak{gl}(N\,|\,M)$, called elliptic double Calogero-Moser (edCM) system, is defined by:

$$\mathrm{H}_{N,M;g}(x,y) = \mathrm{H}_{N;g}(x) - g\,\mathrm{H}_{M;1/g}(y) + \sum_{j=1}^{N}\sum_{k=1}^{M}(1-g)\wp(x_j - y_k)\,. \tag{3}$$

The two types of particles of different mass corresponds to electrons and holes respectively in the condense matter system [3]. The integrability of the edCM system was proven in [4]. It is related to two gauge theories: the four dimensional $\mathcal{N}=2^*$ $SU(N\,|\,M)$ supergroup gauge theory [5] and folded instanton [4, 6].

A generalization of the edCM system, which we call elliptic quadruple Calogero-Moser system (eqCM), is proposed in [7]. It is a one-dimensional quantum mechanics system of $L=N_+ + N_- + M_+ + M_-$ particles governed by the Hamiltonian:

$$\begin{aligned}
&\mathrm{H}_{N_+,M_+,N_-,M_-;g}(x^+,y^+,x^-,y^-;g)\\
&= \mathrm{H}_{N_+,M_+;g}(x^+,y^+) + \mathrm{H}_{N_-,M_-;g}(x^-,y^-)\\
&\quad + \sum_{j=1}^{N_+}\sum_{k=1}^{N_-} g(g-1)\wp(x_j^+ - x_k^-\mathrm{i}\delta + \sum_{j=1}^{M_+}\sum_{k=1}^{M_-}\left(1-\frac{1}{g}\right)\wp(y_j^+ - x_k^- + \mathrm{i}\delta)\\
&\quad + \sum_{j=1}^{N_+}\sum_{k=1}^{M_-}(1-g)\wp(x_j^+ - y_k^- + \mathrm{i}\delta) + \sum_{j=1}^{N_-}\sum_{k=1}^{M_+}(1-g)\wp(y_j^+ - x_k^- + \mathrm{i}\delta)\,.
\end{aligned} \tag{4}$$

The eqCM (4) describes a 1+1 dimensional system with four kinds particles interact through two-body interaction based on their physical distance $|x|$. The particles can be labeled by the mass (1 or $1/g$) and chirality $\pm$. The particles of the same chirality interact through the repulsive potential proportional to $\wp(x)$, while the particles of opposite chirality interact through attractive potential proportional to $\wp(x+\mathrm{i}\delta)$.

$$\lim_{x\to 0}\wp(x) = \frac{(\pi/\delta)^2}{\sinh^2\frac{\pi x}{\delta}},\qquad \lim_{x\to 0}\wp(x+\mathrm{i}\delta) = -\frac{(\pi/\delta)^2}{\cosh^2\frac{\pi x}{\delta}}\,. \tag{5}$$

The integrability of eqCM system is proven in [1] through explicit construction of the Dunkl operator.

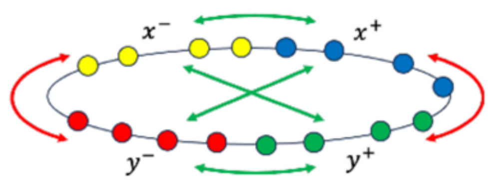


Figure 1: An eqCM of $4+4+4+4=16$ particles on a ring. The attractive potential in green arrow and repulsive potential in red arrow.

* Require deformation of inner product to have non-zero interaction between particles and holes.

The Gauge theory

D-branes act as boundary conditions on the strings world sheet. In the presence of multiple D-branes (say N D-branes), a Chan-Paton factor is assigned to each of the string to identify which D-brane it ends on.
The Chan-Paton factor lives in the vector space $\mathbb{C}^N$. The Dirac quantization requires N to be an integer. However this does not rule out the case N being negative. A negative Chan-Paton factor can be realized by introducing additional minus sign to certain subset of boundary states, i.e. the Chan-Paton factor now takes a value in a graded vector space $\mathbb{C}^{N_+|N_-}$ with the mixed signatures:

$$\begin{pmatrix} +\mathbf{1}_{N_+} & 0 \\ 0 & -\mathbf{1}_{N_-} \end{pmatrix}. \tag{6}$$

Let us consider the scenario of N_+ positive and N_- negative D3 branes place in parallel. A string that connects between one positive brane and one negative brane have the opposite statistics comparing to a string having both ends on positive brane. This is caused by the additional minus sign on the negative brane. An open string with both ends on the negative brane will have usual statistics since the minus signs from both ends cancel each other. This enhances $U(N_+) \times U(N_-)$ quiver gauge group to a $U(N_+|N_-)$ supergroup,

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(N_+|N_-), \quad U^{-1} = U^\dagger. \tag{7}$$

See Fig. 2 for an illustration. Here A and D are $N_+ \times N_+$ and $N_- \times N_-$ even graded matrix with complex number entries, while B and C are $N_+ \times N_-$ and $N_- \times N_+$ odd graded matrix whose entries are Grassmannian. An odd graded matrix means its entries obey anti-commutation relation instead of commutation relation,

$$B_{ij}C_{kl} = (-1)C_{kl}B_{ij}. \tag{8}$$

We consider three sets of D-branes in IIB string theory in Table 1.

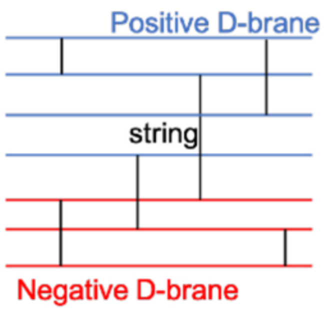


Figure 2: D-brane realization of $U(4|3)$ supergroup gauge theory.

IIB brane	$\mathbb{C}_1$		$\mathbb{C}_2$		$\mathbb{C}_3$		$\mathbb{C}_4$		$\mathbb{R}_9$	$\mathbb{R}_{10}$
	1	2	3	4	5	6	7	8	9	10
$\mathrm{D}(-1)_\pm$										
$\mathrm{D3}_{(12),\pm}$	x	x	x	x						
$\mathrm{D3}_{(23),\pm}$			x	x	x	x				

Table 1: D-brane setup, inspired by [8].

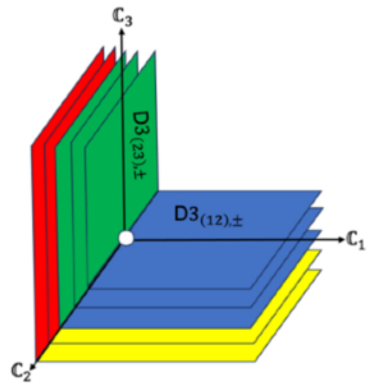
When staying far away from the shared complex plan $\mathbb{C}_2$, the $\mathrm{D3}_{(12),\pm}$ branes give rise to a four dimensional $\mathcal{N} = 2^*$ $U(N_+|N_-)$ theory with adjoint mass g. The $\mathrm{D3}_{(23),\pm}$ branes give rise to a four dimensional $\mathcal{N} = 2^*$ $U(M_+|M_-)$ theory with adjoint mass 1. The supergroup gauge theory on the two set of D3-branes shares a single complexfied gauge coupling $\mathfrak{q}$. The D(-1) branes acts as the instanton of both set of D3 branes. The non-perturbative contribution to the coupled D3-brane gauge theory.

The final element is a $\mathbb{Z}_L$ orbifold on the geometry $\mathbb{C}^4 = \mathbb{C}_1 \times \mathbb{C}_2 \times \mathbb{C}_3 \times \mathbb{C}_4$ induced by the discrete action $\mathbb{Z}_L : (\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4) \mapsto (\mathbf{z}_1, \eta\mathbf{z}_2, \mathbf{z}_3, \eta^{-1}\mathbf{z}_4)$, where $\eta = e^{\frac{2\pi i}{L}}$ is the L-th root of unity. The total $L = N_+ + N_- + M_+ + M_-$ D3 branes are assigned to a representation $\mathscr{R}_\omega$ of the $\mathbb{Z}_L$ orbifold. This assignment is characterized by a (bijective) *coloring function* $c : \{\mathrm{D3-branes}\} \to [N_+] \cup [N_-] \cup [M_+] \cup [M_-] \subset \mathbb{Z}_L$.

The orbifold also splits the instanton counting parameter $\mathfrak{q}$ into $L = N_+ + N_- + M_+ + M_-$ fractional counting parameters $\mathfrak{q}_\omega$ for each representation $\mathscr{R}_\omega$ of $\mathbb{Z}_L$. The fractional couplings $\{\mathfrak{q}_\omega\}$ are related to the bulk gauge coupling by

$$\mathfrak{q} = \prod_{\omega=0}^{L-1} \mathfrak{q}_\omega, \quad \mathfrak{q}_\omega = \frac{\mathfrak{z}_\omega}{\mathfrak{z}_{\omega-1}} \tag{9}$$

The gauge theory on a orbifold space an be identified as gauge theory on the flat space with a co-dimensional two defect Ψ wrapping $\mathbb{C}_1 \times \mathbb{C}_3$.



The D-brane setup in Table 1. The white node in the center of coordinate represents D(-1) brane.

Result

We prove, via supersymmetric localization and carefully breaking the supergroups, that the expectation value of the defect $\langle\Psi\rangle$ is the eigenfunction of eqCM Hamiltonian (4).

$$\mathrm{H}_{N_+,N_-,M_+,M_-}(x^+, y^+, x^-, y^-; g)\langle\Psi(\mathfrak{z};\mathfrak{q})\rangle = E\langle\Psi(\mathfrak{z};\mathfrak{q})\rangle \tag{10}$$

The complex coupling of the gauge theory τ is identified with Weierstrass $\wp$-function modolus. The particle coordinates of eqCM system is related to the fractional coupling $\mathfrak{z}_\omega$:

$$\log \mathfrak{z}_\omega = \begin{cases} x^+_\alpha & \omega \in [N_+],\ c(\alpha) = \omega \\ x^-_\alpha & \omega \in [N_-],\ c(\alpha) = \omega \\ y^+_\beta & \omega \in [M_+],\ c(\beta) = \omega \\ y^-_\beta & \omega \in [M_-],\ c(\beta) = \omega \end{cases}. \tag{11}$$

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Diffeomorphic, but not symplectomorphic manifolds

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A Motivating Question

In this article, we explore symplectic manifolds and symplectic topology. Since symplectic topology has its origins in physics—particularly in classical mechanics—we'll begin by considering a question from classical mechanics.

Classical mechanics, in its earliest form, was established by Sir Isaac Newton to describe the *motion* of objects under the influence of forces. Over time, this field evolved significantly through the contributions of researchers like Leonhard Euler, Joseph-Louis Lagrange, and William Hamilton, among others.

To delve deeper, let us consider a simple setup. In classical mechanics, the primary focus is the motion of a moving object. The *state* of motion is captured by two pieces of data: the object's *position* and *velocity*. Since the object is moving, both its position and velocity vary with time. Let $p(t)$ and $v(t)$ represent the position and velocity of a particle at time t.

A natural question in classical mechanics then arises: **If we know the initial state at time $t = 0$, i.e., the pair $(p(0), v(0))$, can we determine the state of motion, $(p(t), v(t))$, for all future and past times t?** Classical mechanics provides a framework to answer this question by describing the motion $(p(t), v(t))$ if the governing *rule* of the space in which the motion occurs is known.

For example, consider a body (with mass 1) moving on Earth. On Earth, the potential and kinetic energies at time t are determined by the position $p(t)$ and velocity $v(t)$:

$$\text{Potential energy: } E_{po}(t) = 9.8\,|p(t)|, \quad \text{Kinetic energy: } E_{ki}(t) = \frac{1}{2}|v(t)|^2.$$

Here, the constant 9.8 represents the standard gravitational acceleration on Earth. The governing rule on Earth can then be expressed as:

(Rule on Earth) $\quad E_{tot}(t) := E_{po}(t) + E_{ki}(t)$ is constant over time.

Additionally, by definition, the velocity $v(t)$ is the time derivative of the position $p(t)$. Substituting this into (Rule on Earth), we obtain a differential equation. Solving this equation for the initial data $(p(0), v(0))$ allows us to describe the motion at any time t.

Now, consider another planet, which we'll call P. Suppose P is identical to Earth in every way except for its mass. Because of this difference, the rule governing motion on P differs slightly. For example, if the mass of P is twice of that of Earth,

(Rule on P) $$E'_{tot}(t) := E'_{po}(t) + E'_{ki}(t) = \mathbf{19.6}\,|\,p(t)\,| + \frac{1}{2}\,|\,v(t)\,|^2 \text{ is constant.}$$

Here, the coefficient 19.6 reflects the stronger gravitational pull due to the increased mass of P.

These examples reveal that the governing rules of motion are not *universal*; they depend on the physical properties of the space, such as mass. However, despite their differences, the rules for Earth and P are strikingly similar. In fact, the rule for one planet can be continuously deformed into the rule for the other. In this sense, we might say that the two planets have *almost identical* rules.

This leads us to a natural, motivational question: Suppose we know the shape of a planet. From its shape, we can describe all possible positions and velocities. However, determining the explicit governing rule would require additional information, such as the planet's mass. Can we determine the rule—at least up to continuous deformations—without knowing this extra information? Or, to phrase it more succinctly:

Question. Does the shape of a space determine the classical mechanics within it?

Translation to a Mathematical Question

So far, we have presented a motivational question in the language of physics. Let us now reframe it mathematically.

In classical mechanics, the primary focus is on the motion of bodies. If we know the state of a moving body at a particular moment and the rule governing its motion, we can predict its state at all times. Mathematically, this involves considering the space of all possible states, commonly referred to as the *phase space*. The rules of classical mechanics, which all motion must follow, can be expressed as a *2-form* on the phase space.*

This framework can be generalized into a purely mathematical setting. The notion of phase spaces is extended to even-dimensional smooth manifolds. The even dimension condition is natural, as each state of motion consists of two pieces of information: position and velocity. The rule, represented by a 2-form in classical mechanics, generalizes to a *closed* and *nondegenerate* 2-form on the manifold. A pair (X, ω), where X is the manifold and ω is the 2-form, is called a *symplectic manifold*. For simplicity, we often say X is a symplectic manifold, and ω is its *symplectic structure*.

A crucial observation is that a single smooth manifold X can admit multiple symplectic structures; recall that the Earth and the imaginary planet P have different rules, even though they have the same phase spaces. However, their different rules are almost identical. Similarly, two symplectic structures are said to

* Strictly speaking, this 2-form represents only part of the rule. To keep the essay accessible, we omit the technical details.

be *equivalent* if one can be continuously deformed into the other. With these notions in place, we can restate the motivational question mathematically:

Question (Mathematically restated.). Is there an even-dimensional manifold X that admits two *non-equivalent* symplectic structures?

Diffeomorphic but Not Symplectomorphic Manifolds

The above question is profound from a mathematical perspective, even without reference to classical mechanics. It seeks to uncover the existence of smooth manifolds X that can support two entirely different symplectic structures. Such an example would reveal a fundamental distinction between the smooth and symplectic structures on a manifold.

Given its importance, this question has been addressed in several works, including [McL09, MS10, CKL24]. Below, we summarize the common strategy employed in these studies before delving into their specific results.

The general strategy: To construct examples, two key conditions must be met: First, the manifolds must have the same smooth structure. Second, the manifolds must have different symplectic structures.

For the first condition, concerning smooth structures, one can employ the notion of *handle attachment*. In topology, handle attachment is a well-known method of constructing manifolds. Starting with a manifold M (with boundary), one can attach a basic building block, called a *handle*, to obtain a new manifold. The process of attachment requires *attaching information*, which dictates how the handle is attached.

Interestingly, it is well established that smoothly deforming the attaching information does not alter the smooth structure of the resulting manifold. For instance, suppose X is obtained by attaching a handle to M, and Y is obtained by attaching a handle to M with smoothly deformed attaching information. Then X and Y are diffeomorphic.

We note that one can put a specific symplectic structure on the building block, i.e., handle. If M admits a symplectic structure and if the attaching information glues two symplectic structures on M and the handle, the resulting manifold also admits the induced symplectic structure. Based on this, we assume that the resulting manifolds X and Y admit symplectic structures. However, unlike smooth structures, symplectic structures are **not** necessarily preserved under smooth deformations of attaching information. In other words, X and Y may have different symplectic structures and now we need to discuss the second condition concerning difference of symplectic structures.

To prove that X and Y have different symplectic structures, one examines their *symplectic invariants*. Symplectic invariants are properties of symplectic manifolds that remain unchanged under continuous deformations of the symplectic structure. Thus, if X and Y have different symplectic invariants, they cannot be equivalent to each other, i.e., symplectomorphic.

Examples from the Literature

Several examples answering the above question can be found in [McL09, MS10, CKL24]. Below, we summarize their key results and methods:

McLean's Construction [McL09]: McLean constructed infinitely many pairwise distinct symplectic structures on $\mathbb{R}^{2n}$ for $n > 3$. Notably, even the simple manifold $\mathbb{R}^{2n}$ can host multiple symplectic structures. To distinguish these structures, McLean computed their *symplectic homology* and showed that they should be pairwise different.

Maydanskiy and Seidel's Result [MS10]: For any odd $n \geq 3$, Maydanskiy and Seidel demonstrated that the cotangent bundle T^*S^n admits at least two different symplectic structures. By advancing ideas from [May09], they showed that T^*S^n can be constructed via handle attachment, yielding infinitely many symplectic structures. However, we still do not know how many non-equivalent symplectic structure T^*S^n can have. But, they proved that the symplectic structures fall into two distinct classes: Those equivalent to the standard symplectic structure on T^*S^n. Those without any *Lagrangian spheres*. Since the standard structure on T^*S^n has Lagrangian spheres, the two classes are not equivalent.

Choa-Karabas-Lee's Construction [CKL24]: For odd $n \geq 3$, my collaborators and I constructed infinitely many families of symplectic manifolds, each consisting of pairwise non-equivalent symplectic structures on the same smooth manifold. These manifolds are plumbings of T^*S^n along trees, including well-known examples like Milnor fibers of Dynkin types.

To distinguish the symplectic structures, we computed their *wrapped Fukaya categories*. Note that wrapped Fukaya category is a powerful symplectic invariant. For example, one can recover symplectic (co)homology, which is used by McLean [McL09], from it. However, there was no general framework for its computation, until a recent foundational work by Ganatra, Pardon, and Shende [GPS20, GPS24]. Our results revealed that the wrapped Fukaya categories differ between members of each family, proving that they are pairwise non-equivalent as symplectic manifolds, even if they are equivalent as smooth manifolds.

IBS-CGP RESEARCH HIGHLIGHTS

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Lagrangian multi-sections and their toric equivariant mirror

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Mirror symmetry

Mirror symmetry is a duality between symplectic (A-model) and complex (B-model) geometry discovered by string theorists. The first mathematical approach to mirror symmetry is proposed by Kontsevich [4] in '94, which is now known as the *homological mirror symmetry (HMS)*. It states that the Fukaya category of a symplectic manifold (X, ω) is quasi-equivalent to the derived category of coherent sheaves of its mirror partner $(\check{X}, \check{J})$:

$$\mathscr{F}uk(X, \omega) \cong D^b Coh(\check{X}, \check{J}).$$

Roughly speaking, the Fukaya category of (X, ω) consists of objects being Lagrangian submanifolds in X and the morphism space is given by the so-called *Floer cohomology*. Two years after Kontsevich's proposal, Strominger-Yau-Zaslow [9] proposed an entire geometric approach to mirror symmetry, which is now known as the *SYZ proposal* or *SYZ mirror symmetry*. Their proposal suggested that mirror pairs can be obtained by taking dual torus fibration over a common base B. See Figure 1 for pictorial illustration.

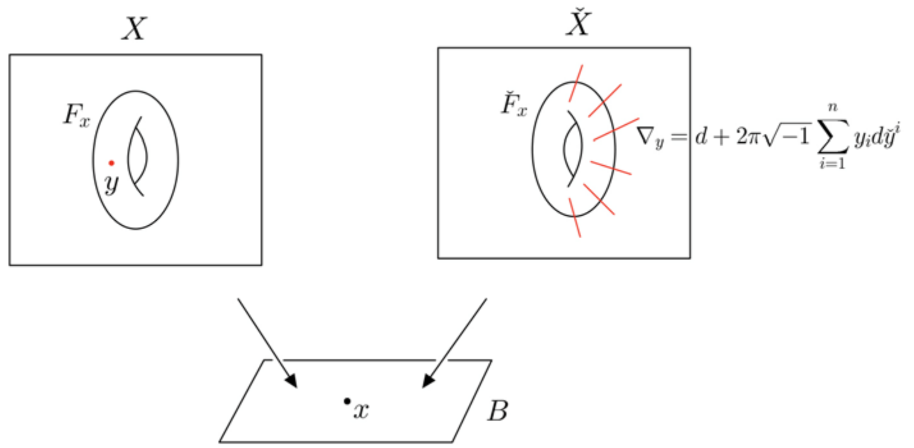


Figure 1. Mirror pairs can be obtained by taking dual torus fibrations. A point $y \in F_x$ is identified with the flat connection ∇_y on the dual fiber $\check{F}_x$.

The SYZ proposal also beautifully explains why homological mirror symmetry is true. For a pair of dual torus fibers $F_x \subset X$, $\check{F}_x \subset \check{X}$, there is a well-known correspondence

$$F_x \cong \mathrm{Hom}(\check{F}_x, U(1)) \cong \{\text{flat } U(1)\text{-connections on } \check{F}_x\}/\text{guage}.$$

This shows that a section L of $X \to B$ can be transformed into a line bundle $\mathscr{L}$ on X with a connection ∇_L. A once-in-a-lifetime calculation shows that L is a Lagrangian if and only if $(\mathscr{L}, \nabla_L^{0,1})$ is a holomorphic line bundle, that is,

$$\omega_{std}|_L = 0 \iff (\nabla_L^{0,1})^2 = 0.$$

It is then very natural to expect *Lagrangian multi-sections* of $X \to B$ correspond to holomorphic vector bundles on $\check{X}$. By a Lagrangian multi-section we mean a Lagrangian immersion/embedding $i : L \to X$ such that the composition $L \to X \to B$ is a possibly branched covering map of finite degree. However, due to the present of branch locus, such correspondence is not that easy to establish.

Toric equivariant mirror symmetry

Let $N = \mathbb{Z}^n$, $N_{\mathbb{R}} = \mathbb{R}^n$ and Σ be a complete rational fan on $N_{\mathbb{R}}$. Let M and $M_{\mathbb{R}}$ be the dual of N and $N_{\mathbb{R}}$, respectively. There is a canonical Lagrangian subset

$$\Lambda_\Sigma := \bigcup_{\sigma \in \Sigma} (\sigma^\perp + M) \times (-\sigma) \subset M_{\mathbb{R}} \times N_{\mathbb{R}} = T^*M_{\mathbb{R}},$$

called the *Fang-Liu-Treumann-Zaslow (FLTZ) skeleton*. The FLTZ skeleton determines a Legendrian subset Λ_Σ^∞ in the infinity sphere bundle $M_{\mathbb{R}} \times S^\infty N_{\mathbb{R}}$ of $T^*M_{\mathbb{R}}$.

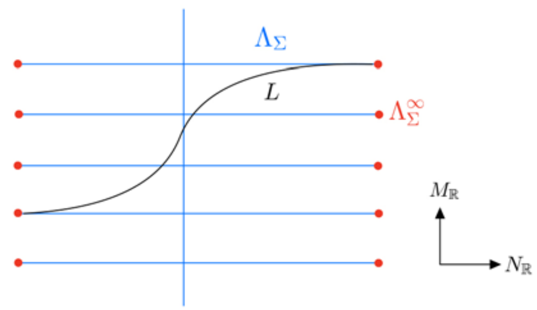
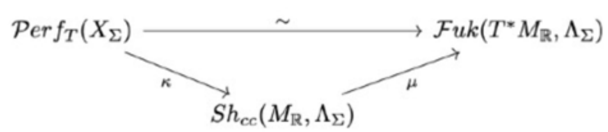


Figure 2. This figure depicts Λ_Σ and Λ_Σ^∞ when Σ is the fan of $\mathbb{P}^1$. The union of all blue lines are the FLTZ skeleton $\Lambda_\Sigma \subset T^*M_{\mathbb{R}} = N_{\mathbb{R}} \times M_{\mathbb{R}} \cong \mathbb{R}^2$ and the union of all the red dots are the Legendrian Λ_Σ^∞ at infinity.

In [2], FLTZ proved an equivariant version for HMS for toric varieties by using the *microlocalization functor* μ [6, 5] and *coherent-constructible correspondence* κ [1]:



Here $\mathscr{P}erf_T(X_\Sigma)$ is the category of toric equivariant perfect complexes and $\mathscr{F}uk(T^*M_{\mathbb{R}}, \Lambda_\Sigma)$ is the Fukaya category generated by exact embedded Lagrangian submanifolds $L \subset T^*M_{\mathbb{R}}$ such that the asymptotic condition

$$L^\infty \subset \Lambda_\Sigma^\infty \subset M_\mathbb{R} \times S^\infty N_\mathbb{R}$$

holds. The category $Sh_{cc}(M_\mathbb{R}, \Lambda_\Sigma)$ is the category of cohomologically compactly supported constructible sheaves on $M_\mathbb{R}$ with the so-called *microlocal support* laying inside Λ_Σ. FLTZ further showed that the inverse functor

$$\mathscr{F}_{FLTZ} : \mathscr{F}uk(T^*M_\mathbb{R}, \Lambda_\Sigma) \xrightarrow{\sim} \mathscr{P}erf_T(X_\Sigma)\,.$$

is compatible with SYZ in the sense that it carries a Lagrangian section of the fibration $T^*M_\mathbb{R} \to N_\mathbb{R}$ to a toric equivariant line bundle. In our work [7], Oh and I proved that $\mathscr{F}_{FLTZ}$ is also compatible with SYZ on the level of Lagrangian multi-sections. Let me emphasis that we are using the fibration $T^*M_\mathbb{R} \to N_\mathbb{R}$ to define Lagrangian multi-sections.

Theorem 2.1 (Oh-S.[7]). *The functor $\mathscr{F}_{FLTZ}$ carries a Lagrangian multi-section in $\mathscr{F}uk(T^*M_\mathbb{R}, \Lambda_\Sigma)$ to a toric equivariant vector bundle over X_Σ.*

This theorem is proved by checking the microlocal criterion provided by Treumann [12]. However, this theorem does not tell us how to construct Lagrangian multi-sections. This is the next question we want to address.

Realization problems

In [8], Payne associated to every rank r toric equivariant vector bundle $\mathscr{E}$ over a toric variety X_Σ a combinatorial data $\mathbb{L}_{\mathscr{E}}^{\text{trop}}$, which consists of an r-fold branched covering map $p : (L, \Sigma_L) \to (N_\mathbb{R}, \Sigma)$ between cone complexes and a piecewise linear function $\varphi^{\text{trop}} : L \to \mathbb{R}$ on L with respect to the cone structure Σ_L. See Figure 3 for an example of such data. His construction generalizes the famous correspondence between toric equivariant line bundles and piecewise linear functions on $(N_\mathbb{R}, \Sigma)$. Motivated by Payne's work, I abstracted this idea and introduced the notion of *tropical Lagrangian multi-section over* Σ in [11]. Those tropical Lagrangian multi-sections that arise from a toric equivariant vector bundle always satisfy the so-called *slope separated* condition, namely, for any cone $\sigma \in \Sigma$, if $\sigma', \sigma'' \in \Sigma_L$ are two distinct lifts of σ, then $\varphi^{\text{trop}}|_{\sigma'} \neq \varphi^{\text{trop}}|_{\sigma''}$. I then formulated the *B-realization problem* as follows:

Question 3.1 (B-realization problem). Given a slope-separated tropical Lagrangian multi-section $\mathbb{L}^{\text{trop}}$ over a complete fan Σ. Is there a toric equivariant vector bundle $\mathscr{E}$ over X_Σ such that $\mathbb{L}_{\mathscr{E}}^{\text{trop}} = \mathbb{L}^{\text{trop}}$?

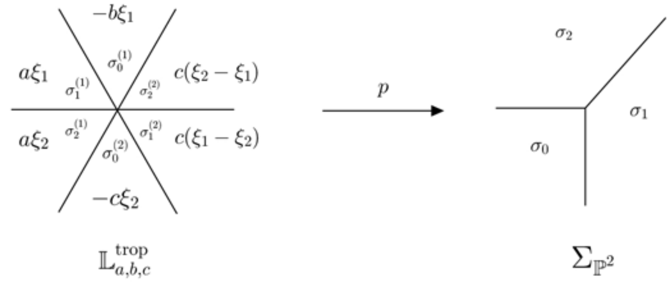


Figure 3. An example of a 2-fold tropical Lagrangian multi-section over the fan $\Sigma_{\mathbb{P}2}$ of $\mathbb{P}^2$. Here ξ_1, ξ_2 are affine coordinates on $N_\mathbb{R} = \mathbb{R}^2$. This tropical Lagrangian multi-section can actually be realized by a rank 2 toric equivariant vector bundle on $\mathbb{P}^2$.

It turns out that B-realizability is not always possible even in dimension 2. Nevertheless, there is a necessary and sufficient condition for B-realizability provided in one of my previous work [11] when X_Σ is a surface and $\mathbb{L}^{\text{trop}}$ is 2-folded. Based on mirror symmetry, a natural question is how to formulate and prove the *A-realization problem*. To each tropical Lagrangian multi-section $\mathbb{L}^{\text{trop}}$, we can associate a Lagrangian subset

$$\Lambda_{\mathbb{L}}^{\text{trop}} := \bigcup_{\sigma' \in \Sigma_L} m(\sigma') \times (-p(\sigma')) \subset \Lambda_\Sigma,$$

where $m(\sigma') \in M/\sigma^\perp \cap M$ is the slope of φ^{trop} along the cone $\sigma' \in \Sigma_L$. This Lagrangian subset determines a Legendrian subset $\Lambda_{\mathbb{L}^{\text{trop}}}^\infty \subset M_\mathbb{R} \times S^\infty N_\mathbb{R}$. As we have seen in the SYZ picture, holomorphic vector bundles are supposed to be mirror to Lagrangian multi-sections, so we formulate our A-realization problem as follows:

Question 3.2 (A-realization problem). Given a slope-separated tropical Lagrangian multi-section $\mathbb{L}^{\text{trop}}$ over a complete fan Σ, does there exist a Lagrangian multi-section $\mathbb{L}$ in $\mathscr{F}uk(T^*M_\mathbb{R}, \Lambda_\Sigma)$ such that $\mathbb{L}^\infty \subset \Lambda_{\mathbb{L}^{\text{trop}}}^\infty$?

By applying Theorem 2.1 and a little bit of microlocal calculations, we can show that A-realizability implies B-realizability. Since B-realizability is in general impossible, this means A-realizability is also in general impossible! Therefore, we need to find a condition on $\mathbb{L}^{\text{trop}}$ for which A-realizability has an affirmative answer.

The first non-trivial scenario is again when $\dim(M_\mathbb{R}) = 2$ and $\mathbb{L}^{\text{trop}}$ is 2-folded. By simple moduli dimension counting, we know that any spin, graded, immersed Lagrangian submanifolds with only index 1 immersed double points are always objects of $\mathscr{F}uk(T^*M_\mathbb{R}, \Lambda_\Sigma)$, so our goal is to hunt for such realization. In [7], we cooked up an integer $N_{\mathbb{L}^{\text{trop}}}$ as follows: Let $S^1 \subset N_\mathbb{R}$ be a circle centered at the origin. There are two cases:

(O) When $p : L \to N_{\mathbb{R}}$ is a non-trivial 2-fold branched covering map. In this case, p is topologically the map $z \mapsto z^2$. The preimage $C := p^{-1}(S^1)$ has one connected component and we parametrized it by $[0,2\pi)$. The unique non-trivial deck transformation $\gamma : C \to C$ is given by $\gamma : \theta \mapsto \theta + \pi$.

(E) When $p : L \to N_{\mathbb{R}}$ is the trivial 2-fold covering map. In this case, $L = N_{\mathbb{R}} \sqcup N_{\mathbb{R}}$ and the preimage $C := p^{-1}(S^1)$ has two connected component C_1, C_2. We parametrized them by $[0,2\pi)$. The unique non-trivial deck transformation $\gamma : C \to C$ is given by switching the two components.

We then define

$$N_{\mathbb{L}^{\mathrm{trop}}} := \begin{cases} \#\Big(\mathrm{Graph}(\varphi^{\mathrm{trop}}|_{[0,\pi)}) \cap \mathrm{Graph}(\varphi^{\mathrm{trop}} \circ \gamma|_{[0,\pi)})\Big) & \text{in Case (O)} \\ \#\Big(\mathrm{Graph}(\varphi^{\mathrm{trop}}|_{[0,2\pi)}) \cap \mathrm{Graph}(\varphi^{\mathrm{trop}} \circ \gamma|_{[0,2\pi)})\Big) & \text{in Case (E)}. \end{cases}$$

By topological reason $N_{\mathbb{L}^{\mathrm{trop}}}$ is odd in Case (O) while $N_{\mathbb{L}^{\mathrm{trop}}}$ is even in Case (E).

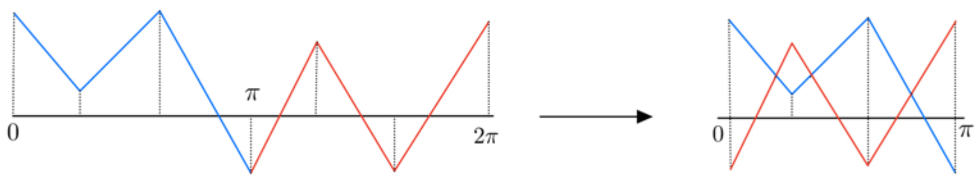


Figure 4. By taking the intersection of $\mathrm{Graph}(\varphi^{\mathrm{trop}}|_{[0,\pi)})$ and $\mathrm{Graph}(\varphi^{\mathrm{trop}} \circ \gamma|_{[0,\pi)})$, we obtain the number $N_{\mathbb{L}^{\mathrm{trop}}}$. In this figure, $N_{\mathbb{L}^{\mathrm{trop}}} = 3$.

It turns out that $N_{\mathbb{L}^{\mathrm{trop}}}$ tells us everything about realizability.

Theorem 3.3 (Oh-S.[7]). *Let $\mathbb{L}^{\mathrm{trop}}$ be a slope-separated 2-fold tropical Lagrangian multi-section over a complete 2-dimensional fan Σ with $N_{\mathbb{L}^{\mathrm{trop}}} \geq 3$. Then $\mathbb{L}^{\mathrm{trop}}$ can be realized by a spin, graded, immersed 2-fold Lagrangian multi-section $\mathbb{L}_{\varphi_{\mathbb{L}^{\mathrm{trop}}}}$ whose immersed sector is concentrated at index 1. In particular, $\mathbb{L}_{\varphi_{\mathbb{L}^{\mathrm{trop}}}}$ is tautologically unobstructed. Moreover, when $\mathbb{L}_{\varphi_{\mathbb{L}^{\mathrm{trop}}}}$ is embedded, the topology of $\mathbb{L}_{\varphi_{\mathbb{L}^{\mathrm{trop}}}}$ is determined by the Betti numbers $b_0 = 1, b_1 = N_{\mathbb{L}^{\mathrm{trop}}} - 3, b_2 = 0$.*

Remark 3.4. I would like to mention that the condition $N_{\mathbb{L}^{\mathrm{trop}}} \geq 3$ is equivalent to the slope condition given in [11] which is equivalent to the 2-dimensional rank 2 B-realizability.

Heuristically,

$$\{((p^*)^{-1}(d\varphi^{\mathrm{trop}}(l)), p(l)) \in M_{\mathbb{R}} \times N_{\mathbb{R}} : l \in L\}$$

is the "Lagrangian" realizing $\mathbb{L}^{\mathrm{trop}}$. However, as φ^{trop} is just piecewise linear, $d\varphi^{\mathrm{trop}}$ does not make sense everywhere and even if it makes sense, $(p^*)^{-1}$ has singularity around the branch point.

To prove Theorem 3.5, we first remove a disk $D_R \subset N_{\mathbb{R}}$ around the branch point and construct a nice smoothing $\varphi_{\geq R}$ of φ^{trop} outside $p^{-1}(D_R) \subset L$ so that the Lagrangian submanifold

$$\mathbb{L}_{\geq R} := \{(p^*)^{-1}(d\varphi_{\geq R}(l)), p(l)) \in M_{\mathbb{R}} \times N_{\mathbb{R}} : l \in L \backslash p^{-1}(D_R)\}$$

satisfies the desired asymptotic condition $\mathbb{L}^{\infty}_{\geq R} \subset \Lambda^{\infty}_{\mathbb{L}^{\mathrm{trop}}}$. Next, we need to choose a good local model over the disk $D_R \subset N_{\mathbb{R}}$ carefully. It turns out that hyper-elliptic curves are the proper choices.

(O) If $N_{\mathbb{L}^{\mathrm{trop}}} = 2k+1 \geq 3$, we put $g = k-1 \geq 0$. We consider the hyper-elliptic curve

$$L_{f_{2g+1}} := \{(x,\xi) \in M_{\mathbb{R}} \times N_{\mathbb{R}} : x^2 = f_{2g+1}(\xi)\},$$

where the complex structure on $M_{\mathbb{R}} \times N_{\mathbb{R}}$ is given by $x := x_1 - \sqrt{-1}x_2, \xi := \xi_1 + \sqrt{-1}\xi_2$ and

$$f_{2g+1}(\xi) := a_{2g+1}(\xi^{2g+1} + a_{2g}\xi^{2g} + \cdots + a_1\xi) \in \mathbb{R}[\xi],$$

is a polynomial with $a_{2g+1} > 0$ and has at most roots of multiplicity 2. The second projection $p_{N_{\mathbb{R}}} : L_{f_{2g+1}} \to N_{\mathbb{R}}$ is a 2-fold branched covering map. The underlying topological surface of $L_{f_{2g+1}}$ has arithmetic genus g and 1 puncture.

(E) If $N_{\mathbb{L}^{\mathrm{trop}}} = 2k+2 \geq 4$, we put $g = k-1 \geq 0$. Consider the hyper-elliptic curve

$$L_{f_{2g+2}} := \{(x,\xi) \in M_{\mathbb{R}} \times N_{\mathbb{R}} : x^2 = f_{2g+2}(\xi)\}$$

where

$$f_{2g+2}(\xi) := a_{2g+2}(\xi^{2g+2} + a_{2g+1}\xi^{2g+1} + \cdots + a_1\xi) \in \mathbb{R}[\xi]$$

is a polynomial with $a_{2g+2} > 0$ and has at most roots of multiplicity 2. The second projection $p_{N_{\mathbb{R}}} : L_{f_{2g+2}} \to N_{\mathbb{R}}$ is a 2-fold branched covering map. The underlying topological surface of $L_{f_{2g+2}}$ has arithmetic genus g and 2 punctures.

In both cases, we can write them in the standard form

$$L_{\geq R} = \{((p^*_{std})^{-1})(d\varphi_{f_d}(l)), p_{std}(l)) \in M_{\mathbb{R}} \times N_{\mathbb{R}} : l \in C_{\geq R}\}$$

around the punctures. The key observation is that φ_{f_d} satisfies the same intersection property as φ^{trop}, namely,

$$\#\Big(\mathrm{Graph}(\varphi_{f_d}|_I) \cap \mathrm{Graph}(\varphi_{f_d} \circ \gamma|_I)\Big) = N_{\mathbb{L}^{\mathrm{trop}}},$$

for $I = [0,\pi)$ or $[0,2\pi)$ depends on whether we are in Case (O) or Case (E). This allows us to glue the local model to $\mathbb{L}_{\geq R}$ without creating extra immersed points on the gluing cylinders.

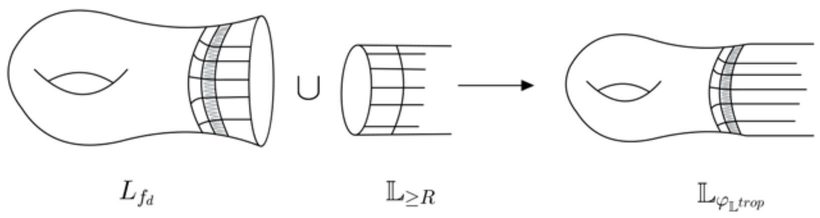


Figure 5. We need to modify the local model L_{f_d} and glue it with $\mathbb{L}_{\geq R}$ carefully so that there are no immersed points created along the gluing cylinder. This results in an embedded Lagrangian multi-section $\mathbb{L}_{\varphi_{\mathbb{L}^{\text{trop}}}}$ with $\mathbb{L}^{\infty}_{\varphi_{\mathbb{L}^{\text{trop}}}} \subset \Lambda^{\infty}_{\mathbb{L}^{\text{trop}}}$, solving the A-realization problem.

We have several applications of Theorem 3.3. First of all, by combining with my previous work on B-realizability and Theorem 2.1, we have

Theorem 3.5 (Oh-S.[7]). When $N_{\mathbb{L}^{\text{trop}}}$ is odd, a slope-separated 2-fold tropical Lagrangian multi-section $\mathbb{L}^{\text{trop}}$ over a complete 2-dimensional rational fan Σ can be realized by an embedded Lagrangian multi-section if $N_{\mathbb{L}^{\text{trop}}} \geq 3$.

By applying the FLTZ equivariant HMS of toric varieties, we get our second application: a stronger version of the B-realization problem.

Theorem 3.6 (Oh-S.[7]). Suppose $\mathbb{L}^{\text{trop}}$ is a slope-separated 2-dimensional 2-fold tropical Lagrangian multi-section over a complete rational fan Σ with $N_{\mathbb{L}^{\text{trop}}} \geq 3$. Then there is an indecomposable rank 2 toric equivariant vector bundle $\mathscr{E}$ on X_Σ such that

$$\dim \mathrm{Ext}^0_T(\mathscr{E},\mathscr{E}) = 1,$$
$$\dim \mathrm{Ext}^1_T(\mathscr{E},\mathscr{E}) = N_{\mathbb{L}^{\text{trop}}} - 3,$$
$$\dim \mathrm{Ext}^2_T(\mathscr{E},\mathscr{E}) = 0,$$

and $\mathbb{L}^{\text{trop}}_{\mathscr{E}} = \mathbb{L}^{\text{trop}}$.

Finally, since rank 2 indecomposable toric equivariant vector bundle over $\mathbb{P}^2$ is classified by Kaneyama [3], our A-realization theorem also can be used to describe their mirror Lagrangian:

Theorem 3.7 (Oh-S.[7]). The mirror of any indecomposable rank 2 toric equivariant vector bundle over $\mathbb{P}^2$ can be represented by an embedded 2-fold Lagrangian multi-section that is diffeomorphic to $\mathbb{R}^2$.

Our work is published in *Advances in Mathematics in April 2024*. Later, I used spectral networks to prove that $N_{\mathbb{L}^{\text{trop}}}$ completely determines the embedded A-realizability of $\mathbb{L}^{\text{trop}}$, namely, in [10], I proved that a slope-separated 2-fold tropical Lagrangian multi-section $\mathbb{L}^{\text{trop}}$ over a complete 2-dimensional rational fan Σ can be realized by an exact, spin, graded, embedded, connected Lagrangian multi-section if and only if $N_{\mathbb{L}^{\text{trop}}} \geq 3$.

Acknowledgment

In July 2023, Prof. Yong-Geun Oh invited Dr. Yoon Jae Nho from the University of Cambridge to speak about his work on spectral networks and family Floer theory. I was so lucky that I just came back from Cambridge and was able to attend Nho's talk. After listening to Nho's presentation, I can smell that there should be a close connection between spectral networks and the two realization problems! The proof of $N_{\mathbb{L}^{\text{trop}}} \geq 3$ being necessary for A-realizability is a beautiful application of spectral networks. On the B-side, spectral networks also provide us with a sufficient condition on B-realizability for arbitrary rank, and also leads me to discover that the moduli space of rank 2 toric equivariant vector bundles over a toric surface is a cluster variety. Furthermore, just recently, I also discovered the role of spectral networks in the so-called *open Gross-Siebert program*, which is a long-term project that I have been working on for years. This gives me so many new research directions! I, therefore, would like to thank Yong-Geun for his effort on the development of CGP and I wish CGP will continue to have more and more great achievements in the future!

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Institute for
Basic Science

Interviews

- Jongmyeong Kim
- Dongwook Choa
- Sam Bardwell-Evans

Jongmyeong Kim

Research Fellow at QSMS, Seoul National University
(CGP Research member from January 2019 to January 2024)

How's your life and work after CGP?

I worked at CGP from January 2019 to January 2024, and I am currently working at QSMS (Center for Quantum Structures in Modules and Spaces), a mathematics research center at Seoul National University. QSMS is a research center composed of algebraists who study topics such as representation theory and number theory, and geometers who mainly study symplectic geometry. We make an effort to find common ground between the two fields, for example, by holding monthly seminars during the semester where we discuss our ongoing research. I think that the environment of interacting with researchers from such diverse fields stimulates the emergence of new ideas.

What made you decide to be a mathematician?

When I was in high school, I wanted to become a string theorist. I was fascinated by string theory, which attempts to explain the universe with the radical idea that everything is made of strings. So, I decided to major in mathematics in university as I had heard that advanced mathematical knowledge is essential for studying string theory. During my undergraduate years, in addition to math courses, I took almost all the physics courses (except the experimental ones), and in my senior year, I joined the theoretical physics research group at the university to conduct my graduation research. That year, I came across the paper "Supersymmetry and Morse theory" written by Edward Witten. I was mesmerized by the paper, which proved a mathematical theorem through physical reasoning. That is the moment I fell in love with mathematics. In the end, I decided to pursue graduate studies in mathematics and spent five years studying Floer theory, which was greatly influenced by Witten's paper.

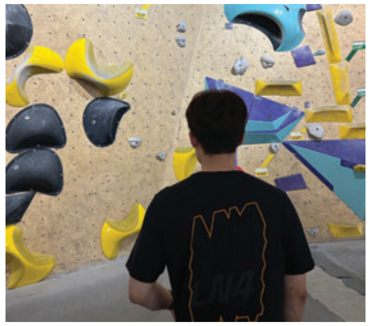
What is your current research-related interest? Please tell us about your research.

I am interested in Fukaya categories, which are categorical invariants of a symplectic manifold that encode Floer theory. My current research focuses on the algebraic structure of the stable Fukaya category of a Liouville manifold, which is defined by quotienting the wrapped Fukaya category by the compact Fukaya category (it is sometimes equivalent to the Rabinowitz Fukaya category). Recently, my collaborators and I observed that the stable Fukaya category often admits a cluster structure.

However, our understanding of this structure remains limited. In particular, we do not yet know how to interpret it in terms of symplectic geometry of the Liouville manifold (or contact geometry of the contact boundary). Our next goal is thus to provide a symplectogeometric interpretation of this structure.

What are you interested in recently? Please share something about you.

After I moved to Seoul, I started bouldering. It is a climbing discipline in which you climb from the bottom to the top using only the designated holds. It is very similar to solving problems in mathematics. Every boulder problem has standard solutions, but there is no need to follow them. Maybe you could find a better way! When I keep attempting a tough boulder problem and finally manage to solve it, I feel a sense of accomplishment similar to the one I get from solving a challenging math problem.



Is there anything you want to tell to younger researchers?

I am not sure if I am qualified to say something to younger researchers, but if you allow me, I would like to say this to them (and to myself as well): Life is short. Do not waste your time.

Dongwook Choa

Research member since Sepetember 2024

How's your life in CGP / Pohang / Korea?

It is a great privilege to being a member of CGP because it offers enormous opportunities with infinite support for carrying out mathematical activities. One of the things that I like most is its calm and peaceful atmosphere. Whenever I walk into the CGP office in the morning, its feels like I become a monk entering the temple of mathematics, preparing my daily mental discipline in the middle of trees, winds, and lots of coffee beans. I also love Pohang very much, especially its shores and delicious Mool-hui.

What made you decide to be a mathematician?

I never thought I would become a professional mathematician because I was neither particularly good nor fast at understanding math. But there was a moment I vividly remember. It was during a winter break in junior high school. I was trying to understand why circular motion needs forces perpendicular to its direction of motion. As usual, I could not grasp what the book was saying at all. I decided to figure out it by my own, which took me a few days of clumsy experiments. I felt stupid, but also satisfied because I felt that I thoroughly understand what was going on form the ground up. Three months later, I realized that what I had done was actually calculating derivatives and integrals of trigonometric functions, without even knowing what they are! That was the first time I truly dip my feet into the realm of math and physics and I couldn't get enough of it.

What is your current research-related interest? Please tell us about your research.

My current interest is exploring singularities using symplectic geometry.
A singularity of space is a point where the space becomes "wild", like a black hole in the universe. The problem with singularities is that many of the effective techniques used to study ordinary spaces break down. Just as studying a black hole by observing the light around it can reveal important information, one successful approach to understanding singularities is to examine the space surrounding them and measure the indirect effects of the singularity. For example, if you move around a singularity, you will never return to your original position. We can talk about how "large" or "chaotic" this change is, and by repeating this process, we can extract a lot of information about the singularity. Symplectic topology is a very powerful tool that is sensitive to these changes, and I am using it to distinguish between various singularities and understand their interactions.

What are you interested in recently? Please share something about you.

Recently, I am exploring my strong interest in coffee. I always enjoyed espresso-based drinks, but now I am drawn to the vast world of pour-over coffee. I love exploring the wide range of flavors that come from different beans and various processing methods. Outside of coffee, I also enjoy cycling around the campus shores or through a nearby forest in Pohang to take in the beauty of the autumn leaves.

We'd like to hear about your dream and future plans.
Do you have a role model or a philosophy of life?

We are witnessing unbelievably rapid developments in AI, even challenging human reason at an unprecedented level. I believe we are on the verge of uncovering one of the greatest mysteries of all time: the human brain. I don't know whether the outcome will be for the best or the worst, but I do believe there is no turning back this time. I feel incredibly fortunate to be someone who can understand what is going on.
On a more philosophical note, I am drawn to the ideas in Albert Camus' work. His response to nihilism resonates deeply with me. As he writes in the [Myth of Sisyphus]:

"The struggle itself [...] is enough to fill a man's heart. One must imagine Sisyphus happy."

I also think the lives of mathematicians closely resemble Camus' happy Sisyphus. We struggle every day to understand how the world works, often without much recognition. After enduring a great deal of effort and frustration, a moment of clarity shines through, and we say, "Well, that was tautological," before starting the climbing again.

Sam Bardwell-Evans

Research member since September 2023

How's your life in CGP / Pohang / Korea?

Life is good here in Korea! The quality of food is very high, public services and the healthcare system are very good, and prices for everything (especially healthcare) are much more reasonable than in the United States, where I grew up. The climate, weather, topography etc. are remarkably similar to where I grew up in the US (central Virginia). The temperature is about the same, though a little cooler here in summer, the humidity is about the same, though a little higher, and the hills are about the same height, though they are steeper here. The trees are similar, though I'm sure someone who actually knows about plants would disagree with me. Overall, I am very happy here.

What made you decide to be a mathematician?

I have always loved mathematics, for as long as I can remember. Something about holding mathematical objects in my head and working out new things about them has always felt nice, in a way that is tricky to articulate. It's a bit like becoming a painter because you like paint. I had always planned to go into some sort of scientific or technical field, but I decided to become a mathematician specifically when I was about 15. I had been very absorbed in my Euclidean geometry class the previous year, and had spent a lot of time trying to use a compass and straightedge to trisect an angle. I didn't succeed of course, since it's impossible, although I did find a nice approximation, and I found trying to understand the problem by playing around with the compass and straightedge very satisfying. I was frustrated with how slowly the geometry course itself had moved, so over the next year I taught myself the remaining three years of high school math courses, and I found that really I wanted to do as much math as possible. If you want to do lots of math, "mathematician" is in many ways the best possible job.

What is your current research-related interest? Please tell us about your research.

I currently work in symplectic geometry, specifically in the study of Kuranishi structures on moduli spaces of pseudoholomorphic curves. Pseudoholomorphic curves are essentially certain well-behaved two-dimensional objects living inside some higher-dimensional space, like surfaces in a ten-dimensional space. One interpretations of these objects is as world lines of strings in string theory, although my work is fairly far removed from the Physics context. A moduli space of these objects is a separate geometric object where each point of the new object represents a pseudoholomorphic curve.

These moduli spaces are extremely important tools in symplectic geometry, but they are in general quite strange and nasty spaces. In order to work with these moduli spaces, we have to put them inside larger, nicer spaces. This is essentially what a Kuranishi structure for a moduli space is.

What are you interested in recently? Please share something about you.

I have been (slowly) learning Korean, and I would like to increase my focus on that in the coming year. It's been surprisingly easy to get by just knowing English and being able to sound out Hangeul (as there are many English loanwords), which has let me be lazy, but the language itself is fascinating and I would like to learn it properly.

I've also been getting back into visual art. I dabble in a few things, including drawing and pixel art, but my favorite is probably wire sculpture. I have left my wire tools and wire back in the United States, but I plan to get some more here in Korea and get back into it. Beyond that, I am also a big fan of strategy video games and TTRPGs. Recently I have been playing a lot of Balatro and the second Pathfinder video game, as well as being in two remote TTRPG campaigns with friends in the US (one Pathfinder and one Starfinder).

We'd like to hear about your dream and future plans. Do you have a role model or a philosophy of life?

My primary personal goal, at least for the near-term, is to be able to keep doing math professionally for the foreseeable future. I have more specific research goals, problems I'd like to solve, and things I'd like to understand, but just being able to continue doing math is the important thing.

Further down the line, I would like to help restructure mathematics education. I feel that there ought to be significantly more effective and efficient ways to approach teaching people, and the current system seems to ultimately achieve very little for most students. At present, vast amounts of time, energy, and emotion, from both teachers and students, are sunk into general mathematics education, and it seems that many, many students come away from over a decade of classes with very little to show for it except for negative feelings about math and/or themselves.

Lastly, on a more personal mathematical/physical level, I would like to see a more satisfying mathematical description of spacetime at small scales than we currently have. There are so many fundamental aspects of the real numbers/manifolds that have to be ignored when trying to use them to model spacetime, and so many major issues relating our understanding of spacetime to the rest of physics. It would be very nice if humanity could develop a satisfying resolution to this problem in my lifetime.

CGP Walk

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