Introduction to Invariant Theory

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Abstract.

The group action of the general linear group $GL(n,\mathbb{C})$, which is a group of $n \times n$ invertible matrices under matrix multiplication, on the polynomial ring $S = \mathbb{C}[x_1, \ldots, x_n]$ is given as follows. We may regard $(x_1, \ldots, x_n)$ as an $n \times 1$ matrix, so $A \cdot (x_1, \ldots, x_n)$ is an $n \times 1$ matrix for all $A \in GL(n,\mathbb{C})$. Then the action is given by $A \cdot f(x_1, \ldots, x_n) = f(A \cdot (x_1, \ldots, x_n))$. This group action induces a group action of a subgroup $G \leq GL(n,\mathbb{C})$ on S.

The purpose of the invariant theory is to study the ring of invariance S^G, which consists of polynomials $f(x_1, \ldots, x_n)$ in S such that $A \cdot f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$ for all $A \in G$. One of the most fundamental problems in the invariant theory is Hilbert's 14th problem, which asks whether S^G is always finitely generated. Hilbert proved that the answer is yes when G is a reductive group (this result is known as the Hilbert finiteness theorem). Note that every finite group is reductive.

In this lecture, we will show that S^G is finitely generated when G is a finite group. We closely follow Hilbert's original approach; we first prove the Hilbert basis theorem, and then, deduce the Hilbert finiteness theorem for finite groups. In fact, the proof for the reductive group case is similar to the finite group case. Even though we can prove that S^G is finitely generated, the proof does not tell us how to find finite generators of S^G. To obtain the finite generators, we consider the Hilbert series of S^G, which help us find finite generators of S^G. In the second lecture, I introduce a powerful technique to compute the Hilbert series of S^G. Finally, note that a counterexample of Hilbert's 14th problem was constructed by Nagata in 1959. In the final lecture, I briefly explain Nagata's counterexample. His example is coming from algebraic geometry.

References.

- S.Mukai - An Introduction to Invariants and Moduli
- D. Cox, J. Little, and D. O'Shea - Ideals, Varieties, and Algorithms

Problems.

1. Let $S := \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring.

– A **monomial** is a polynomial of the form $cx_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ for some $a_i \in \mathbb{Z}_{\geq 0}$ with $1 \leq i \leq n$ and $c \in \mathbb{C}$. An ideal $I \subseteq S$ is called a **monomial ideal** if it is generated by monomials.

– For two different monomials $x_1^{a_1}\cdots x_n^{a_n}, x_1^{b_1}\cdots x_n^{b_n} \in S$, we write $x_1^{a_1}\cdots x_n^{a_n} > x_1^{b_1}\cdots x_n^{b_n}$ if $a_1 + \cdots + a_n > b_1 + \cdots + b_n$ or $a_1 = b_1, \ldots, a_i = b_i, a_{i+1} > b_{i+1}$ when $a_1 + \cdots + a_n = b_1 + \cdots + b_n$. For any nonzero polynomial $f \in S$, we can define the **leading monomial**, denoted by $lm(f)$, to be the largest term of f. (e.g., if $f = x^4 + x^2y^2 + x^2 + xy$, then $lm(f) = x^4$).

– For an ideal $I \subseteq S$, let $in(I)$ be the monomial ideal generated by $lm(f)$ for all $f \in I$.

(1) Prove that every monomial ideal I in S is generated by finitely many monomials.
(Hint: Proceed by the induction on n. For this purpose, consider the ideal J in $\mathbb{C}[x_1, \ldots, x_{n-1}]$ generated by monomials $x_1^{a_1}x_2^{a_2}\cdots x_{n-1}^{a_{n-1}}$ for all $x_1^{a_1}x_2^{a_2}\cdots x_{n-1}^{a_{n-1}}x_n^{a_n} \in I$.)

(2) Using (1), prove that every ideal I in S is generated by finitely many polynomials.
(Hint: By (1), $in(I)$ is generated by $lm(g_1), \ldots, lm(g_t)$ for some $g_1, \ldots, g_t \in I$. Show that I is generated by $g_1, \ldots, g_t$.)

2. Let $G \leq GL(n, \mathbb{C})$ be a finite group acting on $\mathbb{C}[x_1, \ldots, x_n]$. The **Reynolds operator** of G is the map

$$R_G \colon \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n] \text{ defined by } R_G(f)(x_1, \ldots, x_n) = \frac{1}{|G|} \sum_{A \in G} f(A \cdot (x_1, \ldots, x_n)).$$

Prove that $\mathbb{C}[x_1, \ldots, x_n]^G = \mathbb{C}[R_G(x_1^{a_1} \cdots x_n^{a_n}) : a_1 + \cdots + a_n \leq |G|]$.
(Hint: Show first that $R_G(f) \in \mathbb{C}[x_1, \ldots, x_n]^G$ for all $f \in \mathbb{C}[x_1, \ldots, x_n]$ and if $f \in \mathbb{C}[x_1, \ldots, x_n]^G$, then $R_G(f) = f$.)

3. Let G be a subgroup of $GL(2, \mathbb{C})$ generated by $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and G act on $\mathbb{C}[x, y]$.
Prove that $\mathbb{C}[x, y]^G \cong \mathbb{C}[A, B, C]/(C^2 - A^2B + 4B^3)$.
(Hint: $A = x^4 + y^4, B = x^2y^2, C = xy(x^4 - y^4) \in \mathbb{C}[x, y]^G$.)

4. Let G be a subgroup of $GL(2, \mathbb{C})$ generated by $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^a \end{pmatrix}$, where $\epsilon = \exp\left(\frac{2\pi i}{r}\right)$ is a primitive r-th root of 1 and a is an integer with $1 \leq a < r$ and $gcd(a, r) = 1$, and G act on $\mathbb{C}[x, y]$. Prove that the ring of invariance $\mathbb{C}[x, y]^G$ is generated by monomials

$$u_0 = x^r, \ u_1 = x^{r-a}y, \ u_2, \ldots, u_k, \ u_{k+1} = y^r$$

such that $u_{i-1}u_{i+1} = u_i^{a_i}$ for $i = 1, \ldots, k$, where $\frac{r}{r-a} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \cdots}}$.